AD-A158 939 STOCHASTIC INTEGRALS AND PROCESSES WITH INDEPENDENT INCREMENTS(U) NORTH CAROLINA UNIV AT CHAPEL HILL CENTER FOR STOCHASTIC PROCESSES W N HUDSON MAR 85 TR-98 1/1

UNCLASSIFIED AFOSR-TR-85-0679 F49620-82-C-0009 F/G 12/1 NL

END

FILMED

DTIC

1.0
2.8
2.5
3.15
2.2
3.5
2.0
1.1
4.0
1.8
1.25
1.4
1.6

NATIONAL BUREAU OF STANDARDS
MICROCOPY RESOLUTION TEST CHART

# CENTER FOR STOCHASTIC PROCESSES



Department of Statistics
University of North Carolina
Chapel Hill, North Carolina



STOCHASTIC INTEGRALS AND PROCESSES WITH INDEPENDENT INCREMENTS

by

William N. Hudson

Technical Report No. 98

March 1985

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

AD-A158 939

# REPORT DOCUMENTATION PAGE

**1a. REPORT SECURITY CLASSIFICATION:** UNCLASSIFIED

**1b. RESTRICTIVE MARKINGS:**

**2a. SECURITY CLASSIFICATION AUTHORITY:**

**2b. DECLASSIFICATION/DOWNGRADING SCHEDULE:**

**3. DISTRIBUTION/AVAILABILITY OF REPORT:** Unlimited [illegible]

**4. PERFORMING ORGANIZATION REPORT NUMBER(S):** Technical Report No. 98

**5. MONITORING ORGANIZATION REPORT NUMBER(S):** AFOSR-TR- 85-0679

**6a. NAME OF PERFORMING ORGANIZATION:** Center for Stochastic Processes

**6b. OFFICE SYMBOL (If applicable):**

**7a. NAME OF MONITORING ORGANIZATION:** Air Force Office of Scientific Research

**6c. ADDRESS (City, State and ZIP Code):** Dept. of Statistics, University of North Carolina, Chapel Hill, NC 27514

**7b. ADDRESS (City, State and ZIP Code):** Bolling Air Force Base, Washington, DC 20332

**8a. NAME OF FUNDING/SPONSORING ORGANIZATION:** AFOSR

**8b. OFFICE SYMBOL (If applicable):** NM

**9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER:** F49620-82-C-0009

**8c. ADDRESS (City, State and ZIP Code):** Bolling Air Force Base, Washington, DC 20332

**10. SOURCE OF FUNDING NOS.**

| PROGRAM ELEMENT NO. | PROJECT NO. | TASK NO. | WORK UNIT NO. |
|---|---|---|---|
| 61102F | 2304 | A5 | |

**11. TITLE (Include Security Classification):** "Stochastic integrals and processes with independent increments"

**12. PERSONAL AUTHOR(S):** W.N. Hudson

**13a. TYPE OF REPORT:** technical

**13b. TIME COVERED:** FROM 9/84 TO 8/85

**14. DATE OF REPORT (Yr., Mo., Day):** March 1985

**15. PAGE COUNT:** 57

**16. SUPPLEMENTARY NOTATION:**

**17. COSATI CODES**

| FIELD | GROUP | SUB. GR. |
|---|---|---|
| | | |
| | | |

**18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number):** Keywords: Stochastic integrals, processes with independent increments

**19. ABSTRACT (Continue on reverse if necessary and identify by block number)**


Stochatic integrals are defined using processes with independent increments as integrators. A simple and perhaps new method is given for obtaining approximating simple integrands. In the special case where the integrand is a stable motion of index $p \in (1,2)$, the integrand may have pthas in $L_p$. Basic properties are established. Then the characteristic functions of integrals involving nonrandom integrands are computed and used to establish necessary and sufficient conditions forthe independence of such integrals.


**20. DISTRIBUTION/AVAILABILITY OF ABSTRACT:** UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS ☐

**21. ABSTRACT SECURITY CLASSIFICATION:** Unlimited

**22a. NAME OF RESPONSIBLE INDIVIDUAL:** Major Brian Woodruff

**22b. TELEPHONE NUMBER (Include Area Code):** (202) 767-5027

**22c. OFFICE SYMBOL:** NM

DD FORM 1473, 83 APR — EDITION OF 1 JAN 73 IS OBSOLETE.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

STOCHASTIC INTEGRALS AND PROCESSES WITH INDEPENDENT INCREMENTS

William N. Hudson

Center for Stochastic Processes
University of North Carolina at Chapel Hill
and
Department of Mathematics
Auburn University
Auburn, AL 36849



Abstract

Stochastic integrals are defined using processes with independent increments as integrators. A simple and perhaps new method is given for obtaining approximating simple integrands. In the special case where the integrand is a stable motion of index p ε (1,2), the integrand may have paths in $L_p$. Basic properties are established. Then the characteristic functions of integrals involving nonrandom integrands are computed and used to establish necessary and sufficient conditions for the independence of such integrals.

Keywords: Stochastic Integrals, Processes with Independent Increments

AMS 1980 Subject Classification: 60H05

This research was partially funded by the Air Force Office of Scientific Research Grant no. F49620 82 C 0009.

## 1. Introduction

In this paper stochastic integrals are defined with respect to processes with independent but not necessarily stationary increments. Sufficient conditions are given for a process to be integrable. In the special case where the integrator is a stable motion of index $p \in [1,2)$, the integrand may have sample paths in $L_p$. Basic properties of these integrals are established and then attention is restricted to integrals involving nonrandom integrands. For this special case, characteristic functions are computed and used to establish necessary and sufficient conditions for the independence of such integrals.

Throughout this paper $\zeta(t)$, $t \geq 0$, will denote a stochastically continuous process having independent increments and not having a Gaussian component. The reason for the assumption of no gaussian component is that such a component is obtained by a nonrandom time change from a Brownian motion and leads to a simple variation of the extensively studied Ito integral. We further assume that the sample paths of $\zeta(t)$ lie in $D[0,\infty)$ and that $\zeta(0) = 0$. It is well-known that for every $\alpha > 0$ such a process may be written as a sum,

$$\zeta(t) = b_\alpha(t) + \zeta_\alpha(t) + \zeta'_\alpha(t),$$

where $b_\alpha(t)$ is a continuous nonrandom function such that $b_\alpha(0)=0$, and where $\zeta_\alpha(t)$ and $\zeta'_\alpha(t)$ are stochastically continuous independent processes with independent increments having paths in $D[0,\infty)$ and such that $\zeta_\alpha(0) = 0 = \zeta'_\alpha(0)$. Moreover, the sample paths of $\zeta_\alpha$ have jumps of absolute values less than or equal to $\alpha$ while those of $\zeta'_\alpha$ have jumps of absolute values greater than $\alpha$.

The jump-time Levy measure will be denoted by $M$; $M(A \times B)$ is the expected number of jumps of size in $A$ which occur at a time in $B$. The notation $M_t$ will be used to denote the Levy measure of $\zeta(t)$; then $M_t(A) = M(A \times [0,t))$, $t \geq 0$, $M_t(\{0\}) = 0$, and if $J = [-1,1]$, $\int_J x^2 \, dM_t < \infty$, and $M_t(J^c) < \infty$. The jump-time Levy measures of $\zeta_\alpha$ and of $\zeta'_\alpha$ are the restrictions of $M$ to $\alpha J \times [0,\infty)$ and $\alpha J^c \times [0,\infty)$ respectively. The centering for $\zeta_\alpha(t)$ will be chosen so that $\zeta_\alpha(t)$ has the ch. f. $\exp\{\int_{\alpha J} \Psi_0(ux) \, dM_t(x)\}$ where $\Psi_0(x) = e^{ix} - 1 - ix$. The ch. f. of $\zeta'_\alpha(t)$ will be $\exp\{\int_{\alpha J^c} e^{iux} - 1 \, dM_t(x)\}$ and this process will have step functions as sample paths. We will assume throughout that for some $\alpha > 0$ the nonrandom function $b_\alpha(t)$ is of bounded variation over every finite interval. This will be case iff it is the case for all $\alpha > 0$.

The process $\zeta(t)$, $t \geq 0$, will be assumed to be adapted to a nondecreasing family $\{\mathcal{A}_t: \ t \geq 0\}$ of $\sigma$-fields, such that for each $t \geq 0$, $\mathcal{A}_t$ and $\sigma\{\zeta(t+h) - \zeta(t): \ h \geq 0\}$ are independent. The integrands will be stochastic processes $V(t)$, $t \geq 0$, which are adapted to $\{\mathcal{A}_t\}$. The term "adapted process" will always mean adapted to these same $\sigma$-fields $\mathcal{A}_t$. An adapted process $V(t)$ will be said to be <u>simple</u> if it is of the form

$$V = \sum_{k=0}^{\infty} V_k \, I_{(t_k, t_{k+1}]} + W \, I_{\{0\}}$$

where $0 = t_0 < t_1, t_2 < \ldots$, $\lim t_n = \infty$, and for all $k$, $V_k$ is $\mathcal{A}_{t_k}$ - measurable. Note that a simple process $V$ has left- continuous paths. For such a process the stochastic integral is defined as usual to be

$$\int_0^t V d\zeta = \sum_{j=1}^{k(t)} V_j \, \Delta\zeta_j + V_{k(t)} \, (\zeta(t) - \zeta(k(t)))$$

where $\Delta\zeta_j = \zeta(t_{j+1}) - \zeta(t_j)$ and $t_{k(t)} \leq t < t_{k(t)+1}$. (Since $\zeta(0+) = \zeta(0) = 0$ , the definition of $\int_0^t V d\zeta$ does not involve W.) Observe that for simple V the sample paths of $\int_0^t V d\zeta$ lie in D $[0,\infty)$. In sections two and three the class of integrable processes is extended to adapted predictable processes V(t) such that for some $\alpha > 0$, $\int_{\alpha J \times [0,T]} n(xV(s)) \, dM(x,s) < \infty$ a.s., $T > 0$ where $n(x) = |x| \wedge x^2 = \min(|x|, x^2)$. The idea of using this criterion in the case of stationary increments is Kallenberg's (see Theorem 3.1 of [2]). The resulting class of integrable processes is larger than the usual class of processes which have paths in $L_2[0,T]$ a.s. and so generalizes Millar's work in [5]. In the nonstationary increment case, we require that the integrands V(t) be predictable; that is, V(t) is measurable with respect to the $\sigma$-field of subsets of $\Omega \times [0,\infty]$ generated by the left-continuous adapted processes. In section 2 the stochastic integral is defined with respect to $\zeta_\alpha$ and in section 3 the definition is extended to $\zeta$. This definition contains that of Kallenberg in the special case of stationary increments and predictable integrands. Throughout section 2 we use Kallenberg's techniques to define the integral; in particular we use his extension of an inequality due to Dubins and Savage [1]. In section 4 we construct a complex-valued exponential martingale and in section 5 we consider stochastic integrals with nonrandom integrands.

In addition to the work of Kallenberg [2] generalizing that of Millar [5], we should mention that of Rosinski and Woyczynski [9], Urbanik and

Woyczynski [10], and Prekopa [6], [7], and [8]. Rosinski and Woyczynski studied integrals with respect to p-stable motions and established several interesting results. Urbanik and Woyczynski considered nonrandom integrals with respect to symmetric processes with stationary independent increments. They established conditions for stochastic integrability in terms of Orclicz spaces. Prekopa made an extensive study of random measures.

## 2. Construction of the Integral

In this section the existence of simple adapted processes which approximate a given predictable process V is established. Then the stochastic integral is shown to exist as a suitable limit. The given process V is assumed to satisfy the Kallenberg condition

$$\int_{\alpha J\times[0,t]} n(x\, V(s))\, dM(x,s) < \infty \quad \text{a.s.}$$

for some $\alpha > 0$ and a fixed $t$. The smoothing technique used by Kallenberg and others to construct these simple processs fails to work in general for nonstationary increments and we use another method here.

Let $t$ denote an arbitrary fixed positive number. Partition the interval $[0,t]$ into the subsets $\{0\}$, $I_{r1} = (0,2^{-r}t]$, $I_{r2}(2^{-r}t, (2)2^{-r}t]$, ..., $I_{rj} = ((j-1)2^{-r}t, j2^{-r}t]$, ... $I_{r,2^r} = ((2^r-1)\, 2^{-r}t,\ t]$. Let $\mathcal{G}_r$ be the family of all finite disjoint unions of sets of the form $A_0 \times \{0\}$, $A_j \times I_{r,j}$, $1 \leq j \leq 2^r$, where $A_j \in \mathring{A}_{j2^{-r}}$.

<u>Lemma</u> 2.1. $\mathcal{G}_r$ is a $\sigma$-field of subsets of $\Omega \times [0,t]$.

<u>Proof</u>. The proof is standard and is omitted.

<u>Lemma</u> 2.2. If a process V is $\mathcal{G}_r$-measurable, then it is simple and

adapted to the σ-fields $\mathcal{A}_s$.

Proof. Assume $V$ is $\mathcal{B}_r$-measurable and let $V_j$ be the restriction of $V$ to $\Omega \times I_{rj}$. The σ-field on $\Omega \times I_{rj}$ induced by $\mathcal{B}_r$ is the product σ-field $\mathcal{A}_{j2^{-r}} \times \{\emptyset, I_{rj}\}$ and so $V_j$ is measurable with respect to this product σ-field. Consequently, for each $\omega \in \Omega$, $V_j(\cdot,\omega)$ is constant on $I_{r,j}$ and for each $s \in I_{r,j}$, $V_j(s,\cdot)$ is $\mathcal{A}_{j2^{-r}}$ - measurable. Thus $V(s,\omega) = V_j$, a fixed $\mathcal{A}_{j2^{-r}}$ - measurable r.v. on $I_{rj}$. This proves that $V$ is simple and adapted.

Q.E.D.

Now let $\tilde{\mu}_\alpha$ be a finite Borel measure on $[0,t]$ defined for Borel sets $D$ by the equation

$$\tilde{\mu}_\alpha(D) = \int_{\alpha J \times D} x^2 \, dM(x,s) + M(\alpha J^c \times D).$$

Let $\mu_\alpha$ be the probability measure on $[0,t]$ determined by $\tilde{\mu}_\alpha$; i.e. $\mu_\alpha(D) = \tilde{\mu}_\alpha(D)/\tilde{\mu}_\alpha([0,t])$. Then $P \times \mu_\alpha$ is a probability measure on the product space: $(\Omega \times [0,t], \mathcal{A}_t \times B[0,t])$ where $B[0,t]$ denotes the Borel σ-field of subsets of $[0,t]$. We denote expectation relative to this probability measure $P \times \mu_\alpha$ by $E_{P\times\mu_\alpha}(\cdot)$. A predictable measurable process is a r.v. on this space.

Theorem 2.1 Let $V$ be a predictable measurable process which is bounded over $\Omega \times [0,t]$. Define $V_r = E_{P\times\mu_\alpha}[V|\mathcal{B}_r]$. Then $\{V_r\}$ is a sequence of simple adapted processes such that

(a) $P \times \mu_\alpha\{\lim V_r = V\} = 1$,

(b) $\lim_{r\to\infty} E \int_{\alpha J\times[0,t]} x^2 |V_r(s) - V(s)| dM(x,s) = 0$, and

(c) $\lim_{r\to\infty} E \int_{\alpha J^c\times[0,t]} |V_r(s) - V(s)| dM(x,s) = 0$.

Proof. Since $\{\mathcal{G}_r\}$ is a nested family of σ-fields, $\{V_r\}$ is a martingale. Also any upper bound for $V$ is a upper bound for $V_r$ so by the Martingale Convergence Theorem there exists $\tilde{V}$ such that $\{V_r\}$ converges to $\tilde{V}$ $P\times\mu_\alpha$ a.s. and in $L_1(P\times\mu_\alpha)$. The statements (a), (b) and (c) will follow if $\tilde{V} = V$. Let $\mathcal{G} = \sigma\{\mathcal{G}_r: \ r \geq 1\}$. Then every left-continuous process over $\Omega \times [0,t]$ is $\mathcal{G}$-measurable since if $j_r 2^{-r} t < s \leq (j_r+1)2^{-r}t$, $V(s) = \lim_{r\to\infty} V(j_r 2^{-r}t)$. Consequently the predictable process $V$ is $\mathcal{G}$-measurable and $V = \tilde{V}$. Q.E.D.

Corollary 1. Let $\alpha > 0$ and suppose that $V(s)$, $0 \leq s \leq t$, is an adapted predictable measurable process which is bounded over $\Omega \times [0,t]$. Define $V_r = E(V|\mathcal{G}_r)$ as in Theorem 2.1. Then

$$\lim_{r\to\infty} E \int_{\alpha J\times[0,t]} n(xV(s)-xV_r(s))\, dM(x,s) = 0.$$

Proof. Let $c_0 = \sup\{|V(s,\omega)|: \ 0 \leq s \leq t, \ \omega \in \Omega\}$. Then $|V_r(s,\omega)| \leq c_0$ for $(\omega,s) \in \Omega \times [0,t]$. First suppose that $|xV(s) - xV_r(s)| \leq 1$. In this case

$$\begin{aligned} n(xV(s) - xV_r(s)) &= x^2|V(s) - V_r(s)|^2 \\ &\leq 2c_0x^2|V(s) - V_r(s)|. \end{aligned}$$

On the other hand if $|x\,V(s) - x\,V_r(s)|>1$, then $|x| \geq 1/2c_0$ and so

$$\begin{aligned} n(x\,V(s) - x\,V_r(s)) &= |x|\ |V(s) - V_r(s)| \\ &\leq 2\,c_0|x|^2\ |V(s) - V_r(s)|. \end{aligned}$$

That is, in either case

$$n(x\,V(s) - x\,V_r(s)) \leq 2\,c_0\ |x|^2\ |V(s) - V_r(s)|,$$

and by (b) of Theorem 2.1,

$$\lim_{r\to\infty} E \int_{\alpha J\times[0,t]} n(x\,V(s) - x\,V_r(s))\, dM(x,s) = 0.$$

Q.E.D.

Lemma 2.3 (Kallenberg) For all $x$ and $y$,

$$n(x + y) \le 4n(x) + 4\,n(y).$$

Proof Note that $\sup_x \dfrac{n(2x)}{n(x)} \le 4$ and hence

$$\begin{aligned} n(x + y) &= n(|x + y|) && \text{(n is even)} \\ &\le n(|x| + |y|) && \text{(n is nondecreasing in } [0,\infty)) \\ &\le n\,(2\max(|x|, |y|)) \\ &\le 4\,n\,(\max(|x|, |y|)) \\ &\le 4\,n\,(|x|) + 4\,n(|y|) = 4\,n(x) + 4\,n\,(y). \end{aligned}$$ Q.E.D.

Corollary 2. Let $\alpha > 0$ and suppose that $V(s)$, $0 \le s \le t$, is an adapted predictable process such that

$$\int_{\alpha J\times[0,t]} n(x\,V(s))\, dM(x,s) < \infty \text{ a.s}$$

Then there exist simple adapted process $V_r$ such that

$$\lim_{r\to\infty} \int_{\alpha J\times[0,t]} n(x\,V(s) - x\,V_r(s))\, dM(x,s) = 0 \text{ a.s.}$$

Proof Define processes $V^{(k)}(r,\omega)$, $0 \le s \le t$, by

$$V^{(k)}(s,\omega) = \begin{cases} k & \text{if } V(s,\omega) > k \\ V(s,\omega) & \text{if } -k \le V(s,\omega) \le k \\ -k & \text{if } v(s,\omega) < -k. \end{cases}$$

Then by the Dominated Convergence Theorem

$$\lim_{k\to\infty} \int_{\alpha J\times[0,t]} n(x\, V(s) - x\, V^{(k)}(s))\, dM(x,s) = 0 \quad \text{a.s.}$$

Thus for each positive integer m, there exists $k_m$ such that

$$P\{\int_{\alpha J\times[0,t]} n(x\, V(s) - x\, V^{(k_m)}(s))\, dM(x,s) > \frac{1}{8m}\} \le \frac{1}{m^2}.$$

From Cor. 1 it follows that for each $k_m$ there exist simple adapted processes $V_r$ such that

$$P\text{-}\lim_{r\to\infty} \int_{\alpha J\times[0,t]} n(x\, V^{(k_m)} - x\, V_r(s))dM(x,s) = 0.$$

Thus for each m we may choose $r_m$ so that

$$P\{\int_{\alpha J\times[0,t]} n(x\, V^{(k_m)}(s) - x\, V_{r_m}(s)\, dM(x,s) > \frac{1}{8m}\} < 1/m^2.$$

Since by Lemma 2.3, $n(x + y) \le 4\, n(x) + 4\, n(y)$,

$$P\{\int_{\alpha J\times[0,t]} n\,(x\, V(s) - x\, V_{r_m}(s))\, dM(x,s) > \frac{1}{m}\}$$

$$\le P\{4\int_{\alpha J\times[0,t]} n(x\, V(s) - X\, V^{(k_m)}(s))\, dM(x,s) > \frac{1}{2m}\}$$

$$+ P\{4\int_{\alpha J\times[0,t]} n(x\, V^{(k_m)}(s) - x\, V_{r_m}(s))\, dM(x,s) > \frac{1}{2m}\}$$

$$\le \frac{2}{m^2}.$$

By the Borel-Cantelli Lemma,

$$\lim_{m\to\infty} \int_{\alpha J\times[0,t]} n(x\, V(s) - x\, V_{r_m}(s)\, dM(x,s) = 0 \quad \text{a.s.}$$

Q.E.D.

Corollary 3. Let $\alpha > 0$ and let $V(s)$, $s \ge 0$, be a predictable process adapted to the $\sigma$-fields $\mathring{A}_t$ such that for every $t > 0$

$$\int_{\alpha J\times[0,t]} n(x\ V(s))\ dM(x,s) < \infty \quad \text{a.s.}$$

Then there exist simple adapted processes $V_r(s)$ such that

(a) $V_r(s) = 0$ for $s > r$, and

(b) for every $t > 0$ $\lim_{r\to\infty} \int_{\alpha J\times[0,t]} n(x\ V(s)-x\ V_r(s))dM(x,s) = 0$

a.s.

Proof. By Cor. 2, for each positive integer $r > 0$ there exists a simple adapted process $V_r(s)$ satisfying (a) and also the condition that

$$P\{\int_{\alpha J\times[0,t]} n(x\ V(s)-x\ V_r(s))\ dM(x,s) > \frac{1}{r}\} \le 1/r^2.$$

It follows from the Borel-Cantelli Lemma that

$$\lim_{r\to\infty} \int_{\alpha J\times[0,t]} n(x\ V(s)-x\ V_r(s))\ dM(x,s) = 0 \qquad \text{a.s.}$$

which implies (b) since $n(\cdot) \ge 0$. Q.E.D.

Corollary 4. Let $\alpha > 0$ and assume that $V(s)$, $0 \le s \le t$ is a bounded nonrandom real-valued measurable function on $[0,t]$. Then there exist step functions $V_r$ of the form

$$V_r = \bar{v}_0\ I_{\{0\}} + \sum_{j=0}^{k-1} v_j\ I_{(t_{rj},\ t_{rj+1}]}$$

where $0 = t_0 < t_{r1} < \dots < t_{k_r} = t$ such that

(a) $\|V_r\|_\infty \le \|V\|_\infty$,

(b) $\lim_{r\to\infty} V_r(s) = V(s)$ $\mu_\alpha$ - a.e.,

(c) $\lim_{r\to\infty} \int_{\alpha J\times[0,t]} x^2 \left| V(s)-V_r(s)\right|\ dM(x,s) = 0$,

(d) $\lim_{r\to\infty} \int_{\alpha J^c\times[0,t]} \left|V(s) - V_r(s)\right|\ dM(x,s) = 0$, and

$$\text{P-}\lim_{m\to\infty} \int_{\alpha J\times[0,t]} n(x\, V_m(s) - x\, V(s))\, dM = 0.$$

Define $V'_m$ by the equation

$$V'_m(s) = \max(-(m \wedge \|V\|_\infty), \min(V_m(s), m \wedge \|V\|_\infty)).$$

Then for $m > K$, $|V(s) - V'_m(s)| \leq |V(s) - V_m(s)|$ and hence

$$\text{P-}\lim_{m\to\infty} \int_{\alpha J\times[0,t]} n(x\, V'_m(s) - xV(s))\, dM = 0.$$

Now if $|x| \leq \alpha$ there exists a positive constant $K_1$ such that

$$K_1\, x^2\, (V'_m(s) - V(s))^2 \leq n\,(xV'_m(s) - x\, V(s)).$$

It follows by the Dominated Convergence Theorem that

$$\lim_{m\to\infty} E \int_{\alpha J\times[0,t]} x^2 (V'_m(s) - V(s))^2\, dM = 0$$

which proves the Lemma for bounded $V$.

For general $V$ define

$$V_k(s) = \max(-k, \min(V(s), k))$$

and again use the Dominated Convergence Theorem to see that

$$\lim_{k\to\infty} E \int_{\alpha J\times[0,t]} x^2\, (V(s) - V_k(s))^2\, dM = 0.$$

Since each $V_k$ is bounded, there exist simple processes $V'_k$ such that for each $k$, $\|V'_k\|_\infty \leq \|V_k\|_\infty$, and

$$E \int_{\alpha J\times(0,t]} x^2\, (V_k(s) - V'_k(s))^2\, dM < 1/k.$$

(This follows from the first part of the proof.) From the elementary inequality $(a+b)^2 \leq 2a^2 + 2b^2$, we obtain the inequality and since both terms on the right converge in probability to zero,

$$\underset{k\to\infty}{\text{P-lim}} \sup_{0\le s\le t} \left| \int_0^s V_k \, d\zeta - \int_0^s V d\zeta \right| = 0.$$

To prove the second part of Property 3 assume that

$$V_k(u) = \begin{cases} k \text{ if } V(u) > k \\ V(u) \text{ if } -k \le V(u) \le k \\ -k \text{ of } V(u) < -k. \end{cases}$$

Then by the Dominated Convergence Theorem

$$\lim_{k\to\infty} \int_{\alpha J\times[0,t]} n(x\, V_k(u) - x\, V(u))\, dM(x,u) = 0 \qquad \text{a.s.}$$

Clearly the $V_k$'s are integrable since $V$ is integrable and $n(xV_k(u)) \le n(x\, V(u))$. It follows by the first part of the proof that

$$\underset{k\to\infty}{\text{P-lim}} \sup_{0<s<t} \left| \int_0^s V_k \, d\zeta - \int_0^s V \, d\zeta \right| = 0. \qquad \text{Q.E.D.}$$

<u>Lemma</u> 2.6: Let $V$ be integrable over $[0,t]$ with respect to $\zeta \in C_\alpha$. If

$$E \int_{\alpha J\times[0,t]} x^2\, V(s)^2 \, dM < \infty,$$

then there exist simple processes $V_m$ such that

$$\|V_m\|_\infty \equiv \sup \{ |V_m(s,\omega)| : \; 0 \le s \le t, \; \omega \in \Omega \} \le \|V\|_\infty \wedge m$$

and

$$\lim_{m\to\infty} E \int_{\alpha J\times[0,t]} x^2\, (V(s) - V_m(s))^2 \, dM = 0.$$

<u>Proof</u>. First assume that $\|V\|_\infty \le K < \infty$ where $K$ is some finite constant. Since $V$ is integrable there exist simple processes $V_m$ such that

$$\underset{m\to\infty}{P\text{-}\lim} \int_{\alpha J\times[0,t]} n(xV_{km}(u) - x\,V_k(u))\, dM(x,u) = 0.$$

The existence of such sequences is guaranteed by Cor. 2 to Theorem 2.1. Use Theorem 2.2 to see that

$$\underset{m\to\infty}{P\text{-}\lim} \sup_{0\le s\le t} \left|\int_0^s V_{km}\, d\zeta - \int_0^s V_k\, d\zeta\right| = 0.$$

Choose positive integers $m_k$ tending to infinity such that

$$P\,\{\int_{\alpha J\times[0,t]} n(x\,V_{km_k}(u) - x\,V_k(u))\, dM(x,u) > 1/k\} < 1/k,$$

and

$$P\,\{\sup_{0\le s\le t} \left|\int_0^s V_{km_k}\, d\zeta - \int_0^s V_k\, d\zeta\right| > 1/k\} < 1/k.$$

Now by hypothesis

$$\underset{k\to\infty}{P\text{-}\lim} \int_{\alpha J\times[0,t]} n(xV_k(u) - x\,V(u))\, dM(x,u) = 0.$$

But by Lemma 2.3

$$n(x\,V_{km_k}(u) - x\,V(u)) \le 4\, n(x\,V_{km_k}(u) - x\,V_k(u)) + 4\, n(x\,V_k(u) - x\,V(u)).$$

From the above it follows that

$$\underset{k\to\infty}{P\text{-}\lim} \int_{\alpha J\times[0,t]} n(x\,V_{km_k}(u) - x\,V(u))\, dM(x,u) = 0$$

Use Theorem 2.2 again to see that

$$\underset{k\to\infty}{P\text{-}\lim} \sup_{0\le s\le t} \left|\int_0^s V_{km_k}\, d\zeta - \int_0^s V\, d\zeta\right| = 0$$

Since

$$\sup_{0\le s\le t} \left|\int_0^s V_k\, d\zeta - \int_0^s V d\zeta\right| \le \sup_{0\le s\le t} \left|\int_0^s V_k\, d\zeta - \int_0^s V_{km_k}\, d\zeta\right| + \sup_{0<s<t} \left|\int_0^s V_{km_k} d\zeta - \int_0^s V d\zeta\right|$$

$i=1,2$. Thus $V$ is integrable with respect to $\zeta_1$ and $\zeta_2$. Furthermore by Theorem 2.2

$$\underset{m\to\infty}{\text{P-lim}} \int_0^s V_m d\zeta_i = \int_0^s V\, d\zeta_i,\ i=1,2.$$

for $0 \leq s \leq t$. Now since the $V_m$'s are simple, it is immediate that

$$\int_0^s V_m\, d\zeta_3 = \int_0^s V_m d\zeta_1 + \int_0^s V_m\, d\zeta_2, \quad \text{a.s.}$$

$0 \leq s \leq t$. Let $m\to\infty$ and use the right-continuity of the sample paths to complete the proof. Q.E.D.

Property 3. Let $V$ be integrable with respect to $\zeta \in C_\alpha$ over $[0,t]$ and suppose that $\{V_k\}$ is a sequence of integrable but not necessarily simple processes such that

$$\underset{k\to\infty}{\text{P-lim}} \int_{\alpha J\times[0,t]} n(xV_k(u)-x\,V(u))\, dM(x,u) = 0.$$

Then

$$\underset{k\to\infty}{\text{P-lim}} \sup_{0\leq s\leq t} \left|\int_0^s V_k\, d\zeta - \int_0^s Vd\zeta\right| = 0.$$

In particular if

$$V_k(u) = \begin{cases} k \text{ if } V(u) > k \\ V(u) \text{ if } -k \leq V(u) \leq k \\ -k \text{ if } V(u) < -k \end{cases},$$

then

$$\underset{k\to\infty}{\text{P-lim}} \sup_{0\leq s\leq t} \left|\int_0^t V_k\, d\zeta - \int_0^t V\, d\zeta\right| = 0.$$

Proof. For each $k$ let $V_{km}$, $m=1,2,\ldots$ be a sequence of simple processes such that

$$\text{P-lim}_{m\to 0} \int_0^s V_{im}\, d\zeta = \int_0^s V_i d\zeta \quad \text{a.s.}, \qquad 0\le s\le t$$

and hence

$$\int_0^s c_1 V_1 + c_2 V_2 d\zeta = c_1 \int_0^s V_1\, d\zeta + c_2 \int_0^s V_2 d\zeta \qquad \text{a.s.}$$

Since the integrals have right-continuous paths, equality holds simultaneously for all $s \in [0,t]$. Q.E.D.

Property 2. Let $\zeta_1$ and $\zeta_2$ be two independent processes in $C_\alpha$. Then $\zeta_1 + \zeta_2$ is in $C_\alpha$. If $V$ is integrable over $[0,t]$ with respect to $\zeta_3 = \zeta_1 + \zeta_2$, then $V$ is integrable with respect to both $\zeta_1$ and to $\zeta_2$. Furthermore

$$P\{ \int_0^s Vd\zeta_3 = \int_0^s Vd\zeta_1 + \int_0^s Vd\zeta_2, \ 0 \le s\le t\} = 1.$$

Proof. Let $M_i$ denote the jump-time Levy measure of $\zeta_i$, $i = 1,2,3$. Then $M_3 = M_1 + M_2$. Now by hypothesis,

$$\int_{\alpha J\times[0,t]} n(xV(s))\, dM_3 < \infty \quad \text{a.s.},$$

and there exist simple processes $V_m$ such that

$$\text{P-lim}_{m\to\infty} \int_{\alpha J\times[0,t]} n(xV_m(s) - xV(s))\, dM_3 = 0.$$

Since $n(x)$ is nonnegative,

$$\int_{\alpha J\times[0,t]} n(xV(s))\, dM_i < \infty \quad \text{a.s.}$$

and

$$\text{P-lim}_{m\to\infty} \int_{\alpha J\times[0,t]} n(xV_m(s) - xV(s))\, dM_i = 0 \quad \text{a.s.}$$

if $c_1$ and $c_2$ are any two real numbers, then $c_1 V_1 + c_2 V_2$ is integrable with respect to $\zeta$ and

$$P \{ \int_0^s c_1 V_1 + c_2 V_2 \, d\zeta = c_1 \int_0^s V_1 \, d\zeta + c_2 \int_0^s V_2 d\zeta, \; 0<s<t\} = 1.$$

Proof. If $V_1$ and $V_2$ are both simple, this property is obvious. To prove it in general, let $V_{1m}$, and $V_{2m}$, be sequences of simple processes such that

$$\underset{m\to\infty}{P\text{-lim}} \int_{\alpha J\times[0,t]} n(xV_{im}(s) - xV_i(s)) \, dM(x,s) = 0$$

for $i = 1, 2$. Then by Lemma 2.3 and the definition of $n(x)$,

$$n((x\, c_1 V_1 + x\, c_2 V_2) - x\, c_1 V_{1m} + x\, c_2 V_{2m}))$$

$$\leq 4 \max(c_1^2, |c_1|) \, n(xV_1 - x\, V_{1m})$$

$$+ 4 \max(c_2^2, |c_2|) \, n(xV_2 - xV_{2m}),$$

so

$$\underset{m\to\infty}{P\text{-lim}} \int_{\alpha J\times[0,t]} n((xc_1V_1(s) + xc_2V_2(s)) - (xc_1V_{1m}(s) + xc_2 V_{2m}(s))) \, dM = 0.$$

Also by the integrability of $V_1$ and $V_2$

$$\int_{\alpha J\times[0,t]} n(x\, c_1 V_1(s) + x\, c_2 V_2(s)) \, dM(x,s) < \infty \quad \text{a.s.}$$

Thus $c_1 V_1 + c_2 V_2$ is integrable, and furthermore by Theorem 2.2

$$\int_0^s c_1 V_1 + c_2 V_2 \, d\zeta = \underset{m\to\infty}{P\text{-lim}} \int_0^s c_1 V_{1m} + c_2 V_{2m} \, d\zeta.$$

But since $V_{1m}$ and $V_{2m}$ are simple,

$$\int_0^s c_1 V_{1m} + c_2 V_{2m} \, d\zeta = c_1 \int_0^s V_{1m} \, d\zeta + c_2 \int_0^s V_{2m} \, d\zeta$$

Again it follows from Theorem 2.2 that for $i=1, 2$

Yet $P\text{-}\lim_n \int_0^t V_n d\zeta$ may not exist. For instance, take $V_{2n} \equiv 1$ and $V_{2n+1} \equiv 0$.

The proof of Theorem 2.2 is the same as the proof given by Kallenberg for his theorem. The only change necessary was the adaptation of Millar's inequalities to the present setting.

Let $C_\alpha$ denote the class of all stochastically continuous adapted processes $\zeta(t)$ with independent increments, with paths in $D[0,\infty)$, without a Gaussian component and satisfying the conditions

(i) $\zeta(0) = 0$,

(ii) $E\ \zeta(t) = 0$ for all $t \geq 0$,

(iii) $\sigma\ \{\zeta(t+h) - \zeta(t);\ \ h \geq 0\}$ is independent of $A_t$ for all $t \geq 0$,

and

(iv) the jump-time Levy masure $M$ of $\zeta$ is concentrated in $\alpha J \times [0,\infty)$.

<u>Definition</u>. A predictable process $V(s)$, $s \geq 0$, is <u>integrable</u> over $[0,t]$ with respect to $\zeta \in C_\alpha$ if

$$\int_{\alpha J \times [0,t]} n(xV(s))dM(x,s) < \infty \quad \text{a.s.}$$

If $V$ is integrable, the limiting process $\int_0^s V d\zeta$ will be called the <u>stochastic integral</u> of $V$ with respect to $\zeta$ and will always be chosen so that the sample paths of the integral lie in $D[0,t]$.

We now give some basic properties of the stochastic integral $\int_0^t V\, d\zeta$.

<u>Property</u> 1. If $V_1$ and $V_2$ are integrable with respect to $\zeta_\alpha \in C_\alpha$ and

(2.2) and let $(\int_0^t V\, d\zeta_\alpha)'$ be the limiting process having paths in $D[0,\infty)$. By considering the mixed sequence $V_1, V'_1, V_2, V'_2, \ldots$ we see that $\int_0^t V d\zeta_\alpha = (\int_0^t V\, d\zeta_\alpha)'$ a.s. for each $t \geq 0$. But since the paths of $(\int_0^t V\, d\zeta_\alpha)'$ are right continuous,

$$P\ \{\ (\int_0^t V\, d\zeta_\alpha)' = \int_0^t V d\zeta_\alpha,\quad t \geq 0\} = 1$$

Remark. Theorem 2.2 is an extension of Theorem 3.1 of Kallenberg [2]. In his theorem Kallenberg states the existence of the stochastic integral, $\int_0^t V d\zeta$, under one of the following two conditions:

(i) $\int_{J\times[0,t]} n(x\, V(s))\ dM(x,s) < \infty$ a.s., and

(ii) $\int_{\mathbb{R}\times[0,t]} \left|x\, V(s)\right| \wedge 1\ dM\ (x,s) < \infty$ a.s.

In Kallenberg's setting $\zeta$ has stationary independent increments so $dM(x,s) = dM_1(x)\ ds$. Under condition (ii) the sample paths of $\zeta$ are of bounded variation over $[0,t]$ with probability one, and so, as Kallenberg noted, one may simple use the Lebesgue-Stieltjes integral. For this reason condition (ii) was not extended to the present setting.

Theorem 2.2 differs from Kallenberg's theorem in another respect. In Theorem 2.2 the integrating process $\zeta_\alpha$ has jumps of absolute value not greater than $\alpha$. In Kallenberg's theorem, no such restriction is needed. For example Kallenberg's hypotheses are automatically satisfied if $M$ is concentrated in $J^c \times [0,\infty)$. In particular if $V$ is any process and if $V_n$ is any sequence of simple processes, then (i) holds and

$$\int_{J\times[0,t]} n(xV_n(s) - xV(s))\ dM(x,s) = 0.$$

But $g$ is continuous and strictly increasing on $[0,\infty)$ and $g(0) = 0$, so $g^{-1}$ is continuous and strictly increasing on $[0,\infty)$ and $g^{-1}(0) = 0$. Thus

$$P\{\sup_{0\leq t\leq T} \left|\int_0^t V_m - V_n \, d\zeta_\alpha\right| \geq g^{-1}(\varepsilon^2)\} < \varepsilon, \qquad n,m \geq n_0.$$

It follows that for all $\varepsilon > 0$ and all $T > 0$ that

$$\lim_{m,n\to\infty} P\{\sup_{0<t<T} \left|\int_0^t V_m - V_n d\zeta_\alpha\right| > \varepsilon\} = 0$$

Pick an increasing sequence $\{n_k\}$, positive integers such that if $m,n \geq n_k$, then

$$P\{\sup_{0\leq t\leq k} \left|\int_0^t V_m - V_n \, d\zeta_\alpha\right| > 2^{-k}\} < 2^{-k},$$

and so by the Borel-Cantelli Lemma

$$P\{\liminf_{k\to\infty} [\sup_{0<t<k} \left|\int_0^t V_{n_{k+1}} - V_{n_k} \, d\zeta_\alpha\right| \leq 2^{-k}]\} = 1$$

Therefore with probability one

$$\int_0^t V_{n_k} \, d\zeta_\alpha = \sum_{j=1}^{k-1} \int_0^t V_{n_{j+1}} - V_{n_j} \, d\zeta_\alpha + \int_0^t V_{n_1} \, d\zeta_\alpha$$

converges uniformly on each bounded interval to $\int_0^t V d\zeta_\alpha$. But since $V_{n_k}$ is simple, the paths of $\int_0^t V_n \, d\zeta_\alpha$ lie in $D[0,\infty)$ and consequently so do the paths of $\int_0^t V d\zeta_\alpha$. Furthermore since

$$\sup_{0<t<T}\left|\int_0^t V_m d\zeta_\alpha - \int_0^t V d\zeta_\alpha\right| < \sup_{0<t<T}\left|\int_0^t V_m - V_{n_k} d\zeta_\alpha\right| + \sup_{0<t<T}\left|\int_0^t V_{n_k} d\zeta_\alpha - \int_0^t V d\zeta_\alpha\right|,$$

$$P - \lim_{m\to\infty} \sup_{0<t<T} \left|\int_0^t V_m d\zeta_\alpha - \int_0^t V \, d\zeta_\alpha\right| = 0$$

which proves (2.3).

Finally, let $V_n'(t)$ be another sequence of adapted processes satisfying

$$\underset{n\to\infty}{\text{P-lim}} \int_{\alpha J\times[0,t]} g(x\, V_n(s) - xV(s))\, dM(x,s) = 0.$$

Now $g(x+y) \leq 4\, g(x) + 4g(y)$ (Lemma 2.3) so

$$\int_{\alpha J\times[0,t]} g(xV_n(s) - xV_m(s))\, d\, M(x,s)$$

$$\leq 4 \int_{\alpha J\times[0,t]} g(x\, V_n(s) - xV(s))\, dM(x,s)$$

$$+4 \int_{\alpha J\times[0,t]} g(xV(s) - x\, V_m(s))\, dM(x,s).$$

Thus

$$\underset{n,m\to\infty}{\text{P-lim}} \int_{\alpha J\times[0,t]} g(xV_n(s) - xV_m(s))\, dM(x,s) = 0$$

Now let $\varepsilon$ be any positive number and choose $n_0$ so large that if $n,m \geq n_0$, then

$$P\{ \int_{\alpha J\times[0,t]} g\, (x\, V_n(s) - xV_m(s))\, dM(x,s) > \varepsilon^3/32\} \leq \varepsilon/2$$

Consequently for $m,n \geq n_0$

$$P\{ \sup_{0\leq t\leq T} \left| g(\textstyle\int_0^t V_m - V_n d\zeta_\alpha) \right|^{1/2} \geq \varepsilon\}$$

$$\leq P\{ \sup_{0\leq t\leq T} \left| g(\textstyle\int_0^t V_m - V_n\, d\zeta_\alpha \right|^{1/2} \geq \varepsilon, \int_{\alpha J\times[0,T]} g(xV_n(s)-xV_m(s)dM \leq \varepsilon^3/32\}$$

$$+ P\{ \int_{\alpha J\times[0,T]} g(xV_n(s) - x\, V_m(s))\, dM > \varepsilon^3/32\}$$

$$\leq P\{ \sup_{0\leq t\leq T} \left|\textstyle\int_0^t V_m - V_n\, d\zeta_\alpha\right|^{1/2} \geq \frac{16}{\varepsilon^2} \int_{\alpha J\times[0,t]} g(xV_n(s) - x\, V_m(s)dM + \frac{\varepsilon}{2} \}$$

$$+ \frac{}{2}\, .$$

By (2.4) with $\rho = 8/\varepsilon^2$ and $\beta = \varepsilon/2$

$$P\, \{ \sup_{0\leq t\leq T} \left| g( \textstyle\int_0^t V_m - V_n\, d\zeta_\alpha) \right|^{1/2} \geq \varepsilon\, \} < \varepsilon, \quad n,\, m \geq n_0.$$

Now let $T > 0$ be arbitrary and suppose that $V(t)$ is a simple adapted process of the form

$$V(t) = \sum_{i=0}^{j-1} V_i \, I_{(t_i, t_{i+1}]}(t) + v_0 \, I_{\{0\}}(t)$$

where $0 = t_0 < t_1, < \ldots < t_j = T$, where $V_i$ is $\mathcal{A}_{t_i}$-measurable and $v_0$ is $\mathcal{A}_0$-measurable. For brevity put $\Delta\zeta_i = \zeta_\alpha(t_{i+1}) - \zeta_\alpha(t_i)$. We have

$$\sum_{i=0}^{j-1} E\left[g(V_i \Delta\zeta_i) \,\middle|\, \mathcal{A}_{t_i}\right] \leq 2 \sum_{i=0}^{j-1} \int_{\alpha J \times (t_i, t_{i+1}]} g(V_i x) \, dM(x,s)$$

$$= 2 \int_{\alpha J \times [0,T]} g(V(s)x) \, dM(x,s).$$

Since $g'$ is concave and increasing on $[0,\infty)$, Lemma 2.3 of Kallenberg [2] is applicable and says

$$(2.4) \qquad P\{\sup_{t \in D} \left| g\left(\int_0^t V d\zeta_\alpha\right)\right|^{1/2} > 2\rho \int_{\alpha J \times [0,T]} g(xV(s)) \, dM(x,s) + \beta\} \leq \frac{2}{\beta\rho}$$

where $D$ is any finite subset of $[0,T]$ and where $\rho$ and $\beta$ are any positive numbers. Let $\{D_n\}$ be an increasing sequence of finite subsets of $[0,T]$ such that $D' \equiv \bigcup D_n$ is dense in $[0,T]$. By monotone convergence (2.4) holds with $D$ replaced by $D'$. But for simple $V$ the sample paths of $\int_0^t V d\zeta_\alpha$ are right-continuous so (2.4) holds with $D$ replaced by $[0,T]$.

Let $V(s)$ be any adapted process such that (2.1) holds, and let $V_n(s)$ be a sequence of simple adapted processes such that (2.2) holds. Since $g(x) \leq C_2 \, n(x)$, If $\{V_n(s):\ n=1,2,\ldots\}$ is any sequence of simple adapted processes such that for every $t > 0$

$$(2.2)\qquad P-\lim_{n\to\infty} \int_{\alpha J\times[0,t]} n(x\,V_n(s) - xV(s))\, dM(x,s) = 0,$$

then there exists a stochastic process, $\int_0^t V d\zeta_\alpha$, with sample paths in $D[0,\infty]$ such that for every $T > 0$

$$(2.3)\qquad P-\lim_{n\to\infty} \sup\{\left|\int_0^t V_n\, d\zeta_\alpha - \int_0^t V d\zeta_\alpha\right| : 0 \le t \le T\} = 0.$$

Furthermore, if $\{V_n'(s)\}$ is another sequence of simple processes satisfying (2.2) and if $(\int_0^t V d\zeta_\alpha)'$ denotes the corresponding limit having paths in $D[0,\infty]$, then with probability one, the paths of $\int_0^t V d\zeta_\alpha$ and of $(\int_0^t V d\zeta_\alpha)'$ are identical.

Proof. Let $V_n(t)$ be a sequence of simple adapted processes which satisfies (2.2). Put $g(x) = x^2/(1+|x|)$ then according to Lemma 2.5 $g(x) \le C_2\, n(x)$ and so for every $T > 0$

$$\int_{\alpha J\times[0,T]} g(xV(s))dM(x,s) < \infty \quad \text{a.s.}$$

Now $g(\sqrt{x})$ is concave and strictly increasing on $[0,\infty)$ while $g(x)$ is convex on $[0,\infty)$. So by Lemma 2.4a if $G(u) \equiv E\{g(u[\zeta_\alpha(t+h) - \zeta_\alpha(t)])\,|\,\mathcal{A}_t\}$ then $G(u) \le 2\int_{\alpha J} g(ux)\ d(M_{t+h}(x) - M_t(x))$. Hence if $Y$ is any $\mathcal{A}_t$-measurable r.v., then

$$E\,\{g(Y[\zeta_\alpha(t+h) - \zeta_\alpha(t)])\,|\,\mathcal{A}_t\} = G(Y)$$
$$\le 2\int_{\alpha J\times(t,t+h]} g(xY)\ dM(x,s).$$

It is necessary to apply this result without the assumption of stationary increments. Such an application may be justified by observing that if $\mu$ is any infinitely divisible distribution with Levy measure $\nu$, then there exists a stochastic process $X(t)$ with stationary independent increments etc such that the distribution of $X(1)$ is $\mu$. Since $Eh(X(1))$ and $Eh(X(1)^2)$ depend only on the distribution of $X(1)$, we may restate Lemma 2.4 in the following alternative form.

Lemma 2.4a Let $Y$ be an infinitely divisible r.v. with Levy measure $\nu$ and without a Gaussian component. Let $h$ be a concave strictly increasing function from $[0,\infty)$ onto $[0,\infty)$ such that $h(0) = 0$. Assume that the support of $\nu$ is contained in $[-a,a]$. If $h(x^2)$ is convex on $[0,\infty)$ and if $EY = 0$, then

$$Eh(Y^2) \leq 2 \int h(x^2)\, \nu(dx).$$

Lemma 2.5 If $g(x) = \dfrac{x^2}{1+|x|}$, then there exist positive constants $C_1$ and $C_2$ such that

$$C_1\, n(x) \leq g(x) \leq C_2\, n(x) \qquad x \in \mathbf{R}$$

Proof. Since $n(x)$ and $g(x)$ are even functions, it suffices to prove the Lemma for $x \geq 0$. Consider the ratio, $r(x) = \dfrac{g(x)}{n(x)}$. Obviously, $\lim_{x\to 0} r(x) = 1 = \lim_{x\to\infty} r(x)$.

Define $C_1 \equiv \inf\{r(x)\colon\ 0 < x < \infty\}$ and $C_2 \equiv \sup\{r(x)\colon\ 0 < x < \infty\}$. Since $r(x)$ is continuous and never zero on $(0,\infty)$, $0 < C_1 \leq C_2 < \infty$. Thus

$$C_1 \leq \frac{g(x)}{n(x)} \leq C_2, \quad 0 \leq x < \infty.$$

Theorem 2.2. Let $V(s)$ be an adapted process satisfying the condition

$$\text{(2.1)} \qquad \int_{\alpha J\times[0,t]} n(xV(s))\, dM(x,s) < \infty \quad \text{a.s., } t>0.$$

(e) $$\lim_{r\to\infty} \int_{\alpha J\times[0,t]} n(x\, V(s) - x\, V_r(s))\, dM(x,s) = 0.$$

Proof. This is an obvious special case of Theorem 2.1 and Cor 1, so no proof is given.

The stochastic integral of a simple process has been defined in the introduction. We now consider the problem of defining the stochastic integral of a process V(t) which satisfies the condition

$$\int_{\alpha J\times[0,t]} n(x\, V(s))\, dM(x,s) < \infty \quad \text{a.s.}$$

We will show that if $V_r$ is a sequence of simple processes such that as $r \to \infty$

$$\int_{\alpha J\times[0,t]} n(x\, V(s) - x\, V_r(s))\, dM(x,s) \to 0,$$

then $\lim_{r\to\infty} \int_0^t V_r d\zeta_\alpha$ exists a.s. Of course we define $\int_0^t V d\zeta_\alpha = \lim_{r\to\infty} \int_0^t V_r d\zeta_\alpha$.

We will need a result of Millar (Theorem 4.1 of [4]) which he obtained for stochastically continuous processes X(t) with stationary independent increments and without a Gaussian component. Assume that X(0) = 0 and that the sample paths of X lie in $D[0,\infty]$. Let $\nu$ denote the Levy measure of X(1). Millar proved the following.

Lemma 2.4 Let h be a concave strictly increasing function from $[0,\infty)$ onto $[0,\infty)$ such that h(0) = 0. Assume that the support of $\nu$ is contained in $aJ = [-a,a]$. If EX(t) = 0 for all t and if $h(X^2)$ is convex on $[0,\infty)$, then

$$Eh(X(t)^2) \leq 2t \int h(x^2)\, \nu(dx).$$

In the above statement we have inserted the actual bound obtained by Millar in his proofs.

$$E \int_{\alpha J\times[0,t]} x^2 \, (V(s) - V'_k(s))^2 \, dM$$

$$\leq 2 E \int_{\alpha J\times[0,t]} x^2 (V(s) - V_k(s))^2 \, dM$$

$$+ 2 E \int_{\alpha J\times[0,t]} x^2 (V_k(s) - V'_k(s))^2 \, dM$$

and hence

$$\lim_{k\to\infty} E \int_{\alpha J\times[0,t]} x^2 \, (V(s) - V'_k(s))^2 \, dM = 0.$$

Since by the definition of $V_k(s)$, $\|V_k\|_\infty < \|V\|_\infty$, $\|V'_k\|_\infty \leq k \wedge \|V\|_\infty$. Q.E.D.

Property 4. Let $\zeta \in C_\alpha$ and assume that $V$ is integrable with respect to $\zeta$ over $[0,t]$. If

$$E \int_{\alpha J\times[0,t]} x^2 \, V(s)^2 \, dM < \infty,$$

then

$$E \, \{\textstyle\int_0^t V \, d\zeta\} = 0, \qquad \text{and}$$

$$E \, \{\textstyle\int_0^t V \, d\zeta\}^2 = E \, \{ \int_{\alpha J\times[0,t]} x^2 \, V(s) \, dM\}.$$

Proof. Let $\phi(u) = E \exp\{iu\zeta(t)\}$ denote the ch. f. of $\zeta(t)$. Then $\phi(u) = \exp\{ \int_{\alpha J} e^{iux} - 1 - iux \, dM_t\}$. Differentiate twice and evaluate at $u = 0$ to see that $E\{\zeta(t)^2\} = \int_{\alpha J} x^2 \, dM_t$. We use this to first prove the property in the case that $V$ is simple. Suppose that $0 = t_0 < t_1 < t_2 < \ldots < t_n = t$ and that

$$V(s) = \sum_{i=0}^{n-1} V_i \, I_{(t_i, t_{i+1}]}(s) + W_0 \, I_{\{0\}}(s),$$

where $V_i$ is $\mathcal{A}_{t_i}$ - measurable and $W_0$ is $\mathcal{A}_0$- measurable. Then

$$E\{\int_0^t V\,d\zeta\} = E\{\sum_{i=0}^{n-1} V_i\,(\zeta(t_{i+1}) - \zeta(t_i))$$

$$= \sum_{i=0}^{n-1} E\{V_i\,E\{\zeta(t_{i+1}) - \zeta(t_i) \mid \mathcal{A}_{t_i}\}\}$$

$$= \sum_{i=0}^{n-1} E\{V_i\,E\{\zeta(t_{i+1}) - \zeta(t_i)\}\}$$

since $\zeta(t_{i+1}) - \zeta(t_i)$ is independent of $\mathcal{A}_{t_i}$. But $\zeta \in C_\alpha$ so $E\zeta(s) = 0$, $0 \le s \le t$, and thus

$$E\{\int_0^t V\,d\zeta\} = 0.$$

Similarly, if $\Delta\zeta_i = \zeta(t_{i+1}) - \zeta(t_i)$,

$$E\{\int_0^t V d\zeta\}^2 = \sum_{i=0}^{n-1} E\{V_i^2\,(\Delta\zeta_i)^2\} + 2 \sum_{i<k} E\{V_i V_k \Delta\zeta_i \Delta\zeta_k\}$$

$$= \sum_{i=0}^{n-1} E\{V_i^2(\Delta\zeta_i)^2\}$$

$$= \sum_{i=0}^{n-1} E\{V_i^2\,E\{(\Delta\zeta_i)^2 \mid \mathcal{A}_{t_i}\}\}$$

$$= \sum_{i=0}^{n-1} E\{V_i^2 \int_{\alpha J\times(t_i,t_{i+1}]} x^2\,dM\}$$

$$= E \int_{\alpha J\times[0,t]} V(s)^2\,x^2\,dM.$$

This proves Property 4 for simple V.

Now let V be any integrable process satisfying the condition $E\{\int_{\alpha J\times[0,t]} x^2\,V(s)^2 dM\} < \infty$. Then according to Lemma 2.5 there exist simple processes $V_m(s)$ such that

$$\lim_{m\to\infty} E \int_{\alpha J\times[0,t]} x^2 \, [V_m(s) - V(s)]^2 \, dM = 0.$$

Since $n(x) = x^2 \wedge |x| \leq x^2$,

$$P\text{-}\lim_{m\to\infty} \int_{\alpha J\times[0,t]} n(x\, V_m(s) - x\, V(s)) \, dM = 0,$$

so by Theorem 2.2 it follows that

$$P\text{-}\lim_{m\to\infty} \int_0^t V_m \, d\zeta = \int_0^t V \, d\zeta.$$

Now $V_m(s) - V_k(s)$ is simple so by the first part of the proof

$$E\{ \int_0^t V_m d\zeta - \int_0^t V_k \, d\zeta\}^2 = E\{ \int_{\alpha J\times[0,t]} x^2 \, [V_m(s) - V_k(s)]^2 \, dM.\}$$

The right side tends to zero as $m,k\to\infty$ so $\int_0^t V_m \, d\zeta$ is Cauchy and converges in $L^2$ norm. Since this sequence converges to $\int_0^t V \, d\zeta$ in probability, it converges to $\int_0^t V \, d\zeta$ in $L_2$ also. Thus

$$\lim_{m\to\infty} E \{\int_0^t V_m d\zeta\} = E \{\int_0^t V \, d\zeta\} = 0, \text{ and}$$

$$\lim_{m\to\infty} E \{\int_0^t V_m d\zeta\}^2 = E \{\int_0^t V \, d\zeta\}^2.$$

Therefore,

$$E\{\int_0^t V \, d\zeta\}^2 = \lim_{m\to\infty} E \int_{\alpha J\times[0,t]} x^2 \, V_m(s)^2 dM = E \int_{\alpha J\times[0,t]} x^2 V(s)^2 \, dM.$$

Q.E.D.

For Property 5 below we need to introduce the notation $L\text{-}\int_0^t V \, d\zeta$ for the pathwise Lebesgue-Stieltjes integral of $V$ with respect to $\zeta$ over $[0,t]$ Such an integral is defined when the path of $\zeta$ is of bounded variation over $[0,t]$, and this is the case a.s. provided $\int_{\alpha J\times[0,t]} |x| \, dM(x.s) < \infty$.

Also in Property 5 we use the assumption that the integrand V is predictable. The proof of Property 5 below is essentially that given by Millar for his Property 6 in [5].

Property 5. Assume that V is integrable with respect to $\zeta \in C_\alpha$ over [0,t]. Assume further that $\int_{\alpha J\times[0,t]} |x|\, dM(x,s) < \infty$, i.e. that almost every path of $\zeta$ is of bounded variation over [0,t]. Let $d|\zeta|$ denote the total variation measure determined pathwise by $\zeta$. If $L\text{-}\int_0^t |V|\,d|\zeta| < \infty$ a.s., then

$$P\{\int_0^s V\, d\zeta = L\text{-}\int_0^s V\, d\zeta,\ 0 \le s \le t\} = 1.$$

Proof. Without loss of generality we may assume that M is concentrated on $(0,\alpha]\times[0,t)$. Let S denote the class of all bounded simple processes and let H denote the class of all bounded integrable processes satisfying Property 5. Then S is a vector space with the property that if f and g are any two elements of s, then $f \wedge g \in S$. H is also a vector space of functions defined over $\Omega \times (0,t]$ and contains S. Let $V_m$ be a bounded nondecreasing sequence of functions in H which converges to V. Then by the Dominated Convergence Theorem

$$\lim_{m\to\infty} E\{\int_{\alpha J\times[0,t]} x^2\, (V(s) - V_m(s))^2\, dM(x,s)\} = 0.$$

From Property 4 it follows for $0 \le s \le t$ that

$$E\{(\int_0^s V_m\, d\zeta - \int_0^s V\, d\zeta)^2\} \le E\{\int_{\alpha J\times[0,t]} x^2\, (V_m(s) - V(s))^2\, dM(x,s)\}$$
$$\to 0 \quad \text{as} \quad m \to \infty.$$

That is, $\int_0^s V_m\, d\zeta$ converges to $\int_0^s V\, d\zeta$ in $L_2(\Omega)$ for each $s \in [0,t]$. On the other hand

$$\lim_{m\to\infty} L\text{-}\int_0^s V_m \, d\zeta = L\text{-}\int_0^s V \, d\zeta \quad \text{a.s.,} \quad 0 \le s \le t. \quad \text{Thus}$$

$$\int_0^s V \, d\zeta = L\text{-}\int_0^s V \, d\zeta, \quad 0 \le s \le t,$$

and since both integrals have right - continuous paths, equality holds simultaneously in $s \in [0,t]$ outside a null set. This proves that H is closed under bounded nondecreasing limits. By the remark following T 20 of Meyer [3] (Chapter I) H contains all bounded predictable V. Now let V be nonnegative and predictable. Define

$$V^{(p)}(s) = \begin{cases} p \text{ if } V(s) > p \\ V(s) \text{ if } 0 \le V(s) \le p. \end{cases}$$

Then for each p, $V^{(p)} \in H$, and so

$$\int_0^s V^{(p)} \, d\zeta = L\text{-}\int_0^s V^{(p)} \, d\zeta.$$

Now according to Property 3

$$P\text{-}\lim_{p\to\infty} \sup_{0\le s\le t} \left| \int_0^s V^{(p)} \, d\zeta - \int_0^s V \, d\zeta \right| = 0.$$

Write $\zeta(t') = \overline{\zeta}(t') - m(t')$ where the sample paths of $\overline{\zeta}$ are nondecreasing and where $m(t') = \int_{[0,\alpha]\times[0,t']} x \, dM(x,s)$. By the Monotone Convergence Theorem

$$\lim_{p\to\infty} L\text{-}\int_0^s V^{(p)} \, d\overline{\zeta} = L\text{-}\int_0^s V \, d\overline{\zeta} \quad \text{a.s. and}$$

$$\lim_{p\to\infty} L\text{-}\int_0^s V^{(p)} dm = L\text{-}\int_0^s V \, dm \quad \text{a.s.}$$

Since $d|\zeta| = \overline{\zeta} + dm$ and since by hypothesis

$$L\text{-}\int_0^t V\, d|\zeta| < \infty \quad \text{a.s.},$$

$$L\text{-}\int_0^t V\, d\overline{\zeta} < \infty \text{ a.s. and } L\text{-}\int_0^t V\, dm < \infty \quad \text{a.s.}$$

Hence $\lim_{p\to\infty} L\text{-}\int_0^s V^{(p)}\, d\zeta = L\text{-}\int_0^s V\, d\zeta$, $0 \leq s \leq t$. Consequently

$$\begin{aligned} \int_0^s V\, d\zeta &= \lim_{p\to\infty} \int_0^s V^{(p)} d\zeta \\ &= \lim_{p\to\infty} L\text{-}\int_0^s V^{(p)} d\zeta \\ &= L\text{-}\int_0^s V\, d\zeta, \quad 0 \leq s \leq t. \end{aligned}$$

Since both $\int_0^s V\, d\zeta$ and $L\text{-}\int_0^s V\, d\zeta$ have right-continuous paths

$$P\left\{\int_0^s V\, dS = L\text{-}\int_0^s V\, d\zeta, \quad 0 \leq s \leq t\right\} = 1.$$

This proves Property 5 for nonnegative $V$. For general $V$, the Property follows from the decomposition $V = V^+ - V^-$. Q.E.D.

<u>Property</u> 6. If $\zeta \in C_\alpha$, if $V$ is integrable with respect to $\zeta$, and if $E \int_{\alpha J\times[0,\infty]} x^2\, V(s)^2\, dM(x,s) < \infty$, then $\int_0^t V d\zeta$ and $(\int_0^t V d\zeta)^2 - \int_{\alpha J\times[0,t]} x^2\, V(s)^2\, dM(x,s)$ are martingales.

<u>Proof</u>. We first prove that property 6 holds for simple $V$. Let $0 \leq r < t$. Since $V$ is simple, there exists a partition $0 = t_0 < t_1 < \ldots < t_k = t$ of $[0,t]$ such that $V(p) = \sum_{i=0}^{k-1} V_i\, I_{(t_i,t_{i+1}]}(p) + v_0\, I$ for $0 \leq p \leq t$. Here $V_i$ is $\mathcal{A}_{t_i}$-measurable. We add $r$ to the partition if necessary so that $r = t_j$ for some $j < k$. Set $\Delta\zeta_i = \zeta(t_{i+1}) - \zeta(t_i)$, $E_i(\cdot) = E(\cdot\, |\mathcal{A}_{t_i})$, and $T_i = (t_i, t_{i+1}]$.

$$\int_0^t V\, d\zeta = \sum_{i=0}^{k-1} V_i \,\Delta\zeta_i .$$

Since $V_i$ and $\Delta\zeta_i$ are independent, and since $\zeta \in C_\alpha$ $E\Delta\zeta_i = 0$, and $E_i(V_i\Delta\zeta_i) = 0$. Thus for $i \geq j$, $E_j(V_i\Delta\zeta_i) = E_jE_i(V_i\Delta\zeta_i) = 0$, while for $i<j$, $E_j(V_i\Delta\zeta_i) = V_i\Delta\zeta_i$ since $V_i\Delta\zeta_i$ is $A_{t_{i+1}}$ -measurable.

It follows that $E_j(\int_0^t Vd\zeta) = \sum_{i=0}^{j-1} V_i\,\Delta\zeta_i = \int_0^r Vd\zeta$ which proves that $\int_0^t Vd\zeta$ is is a martingale.

Next calculate

$$(\int_0^t Vd\zeta)^2 = \sum_{i=0}^{k-1} V_i^{\,2}\,(\Delta\zeta_i)^2 + 2\sum_{m=1}^{k-1}\ \sum_{i=0}^{m-1} (V_i\Delta\zeta_i)(V_m\Delta\zeta_m).$$

For $m \geq j$,

$$\begin{aligned} E_j\,(V_m^2\,(\Delta\zeta_m)^2) &= E_j\,E_m\ (V_m^2\,(\Delta\zeta_m)^2) \\ &= E_j\,[V_m^2\,E_m(\Delta\zeta_m)^2] \\ &= E_j \int_{\alpha J\times(t_m,t_{m+1}]} x^2\,V_m^2\,dM(x,s). \end{aligned}$$

Thus

$$\begin{aligned} E_j(\sum_{i=0}^{k-1} V_i^2\,(\Delta\zeta_i)^2) &= E_j \int_{\alpha J\times(r,t]} x^2\,V(s)^2 dM(x,s) \\ &+ \sum_{i=0}^{j-1} V_i^2\,(\Delta\zeta_i)^2 \end{aligned}$$

Also if $m \geq j$ and $m > i$,

$$\begin{aligned} E_j(V_i\,\Delta\zeta_i\,V_m\,\Delta\zeta_m) &= E_j\,E_m(V_i\Delta\zeta_i\,V_m\,\Delta\zeta_m) \\ &= E_j\,[V_i\Delta\zeta_i\,V_m\,E_m(\Delta\zeta_m)] \\ &= 0 \end{aligned}$$

and for $i<m<j$,

$$E_j(V_i\,\Delta\zeta_i\,V_m\,\Delta\zeta_m) = V_i\,\Delta\zeta_i\,V_m\,\Delta\zeta_m.$$

Thus

$$E_j\Big(\sum_{m=1}^{k-1}\sum_{i=0}^{m-1} V_i\,\Delta\zeta_i\,V_m\,\Delta\zeta_m\Big) = \sum_{m=1}^{j-1}\sum_{i=0}^{m-1} V_i\,\Delta\zeta_i\,V_m\,\Delta\zeta_m.$$

From the above we get the equation

$$\begin{aligned} E_j\Big(\int_0^t V d\zeta\Big)^2 &= E_j \int_{\alpha J\times(r,t]} x^2\, V(s)^2 dM(x,s) \\ &\quad + \sum_{i=0}^{j-1} V_i^2\,(\Delta\zeta_i)^2 + 2\sum_{m=1}^{j-1}\sum_{i=0}^{m-1} V_i\,\Delta\zeta_i\,V_m\Delta\zeta_m \\ &= E_j \int_{\alpha J\times(r,t]} x^2\, V(s)^2\, dM(x,s) \\ &\quad + \Big(\int_0^r V d\zeta\Big)^2. \end{aligned}$$

From this we see that

$$E_j\Big[\Big(\int_0^t V d\zeta\Big)^2 - \int_{\alpha J\times[0,t]} x^2 V(s)^2 dM(x,s)\Big] = \Big(\int_0^r V d\zeta\Big)^2 - \int_{\alpha J\times[0,r]} x^2 V(s)^2 dM(x,s)$$

which proves that $(\int_0^t V d\zeta)^2 - \int_{\alpha J\times[0,t]} x^2\, V(s)^2\, dM(x,s)$ is a martingale.

Now consider a general process $V(t)$ such that

$$E \int_{\alpha J\times[0,\infty)} x^2 V(s)^2\, dM(x,s) < \infty.$$

Then according to Lemma 2.6 there exist simple adapted processes $V_m(t)$ such that

$$\lim_{m\to\infty} E \int_{\alpha J\times[0,\infty)} x^2\, [V(s) - V_m(s)]^2\, dM(x,s) = 0.$$

By Property 4,

$$E\Big(\int_0^t V d\zeta - \int_0^t V_m d\zeta\Big)^2 = E \int_{\alpha J\times[0,t]} x^2\, [V(s) - V_m(s)]^2\, dM(x,s),$$

and consequently

$$\lim_{m\to\infty} E\left(\int_0^t V d\zeta - \int_0^t V_m d\zeta\right)^2 = 0.$$

So for $r < t$

$$E\left(\int_0^t V d\zeta \,\middle|\, \mathcal{A}_r\right) = \lim_{m\to\infty} E\left(\int_0^t V_m \, d\zeta \,\middle|\, \mathcal{A}_r\right)$$

$$= \lim_{m\to\infty} \int_0^r V_m d\zeta$$

$$= \int_0^r V d\zeta.$$

Also,

$$E\left\{ \left(\int_0^t V d\zeta\right)^2 - \int_{\alpha J\times[0,t]} x^2\, V(s)^2 dM(x,s) \,\middle|\, \mathcal{A}_r \right\}$$

$$= \lim_{m\to\infty} E\left\{ \left(\int_0^t V_m d\zeta\right)^2 - \int_{\alpha J\times[0,t]} x^2\, V_m(s)^2\, dM(x,s) \,\middle|\, \mathcal{A}_r \right\}$$

$$= \lim_{m\to\infty} \left\{ \left(\int_0^r V_m d\zeta\right)^2 - \int_{\alpha J\times[0,r]} x^2\, V_m(s)^2\, dM(x,s)\right\}.$$

$$= \left(\int_0^r V\, d\zeta\right)^2 - \int_{\alpha J\times[0,r]} x^2\, V(s)^2\, dM(x,s).$$

(To see that $E\left\{ \int_{\alpha J\times[0,t]} x^2 V_m(s)^2\, dM\,(x,s) - \int_{\alpha J\times[0,t]} x^2\, V(s)^2 dM(x,s)\right\} \to 0.$

as $m \to \infty$ observe that by Cauchy - Schwarz

$$E \int_{\alpha J\times[0,t]} x^2\, V_m(s)^2 - x^2\, V(s)^2\, dM(x,s)$$

$$= E \int_{\alpha J\times[0,t]} (xV_m(s) - xV(s))\, (xV_m(s) + xV(s))\, dM(x,s)$$

$$\leq \left\{E \int_{\alpha J\times[0,t]} x^2(V_m(s) - V(s))^2\, dM(x,s)\right\}^{1/2} \left\{E \int_{\alpha J\times[0,t]} x^2(V_m(s) + V(s))^2\, dM(x,s)\right\}^{1/2}$$

and while the first term on the last line tends to zero, the second term is not greater than

$$2\,E \int_{\alpha J\times[0,t]} x^2\, V_m(s)^{\,2} + x^2\, V(s)^2\, dM(x,s)$$

which is bounded since it converges as $m \to \infty$.).

3. The Integral.

In this section the stochastic integral $\int_0^t V d\zeta$ is defined for general processes $\zeta$ using the decomposition $\zeta = b_\alpha + \zeta_\alpha + \zeta'_\alpha$. This decomposition is not unique so it is necessary to show that the integral is independent of the choice of the decomposition. (See Theorem 3.1.)

Definition. Let $V(s)$, $s \geq 0$, be a measurable predictable stochastic process adapted to the $\sigma$-fields $\{\mathcal{A}_t:\ t \geq 0\}$. We say that $V$ is integrable (with respect to $\zeta$) over $[0,T]$ if for some $\alpha > 0$

(a) $\int_{\alpha J\times[0,T]} n(x\, V(s))\, dM(x,s) < \infty$ a.s., and

(b) $\int_{[0,T]} \left|V(s)\right|\, db_\alpha(s) < \infty$ a.s..

If $V$ is as above, we define the stochastic integral $\int_0^t V d\zeta$, $0 \leq t \leq T$, by the formula

$$\int_0^t V d\zeta = \int_{[0,t]} V db_\alpha + \lim_{r\to\infty} \int_0^t V_r d\zeta_\alpha + \int_0^t V d\zeta'_\alpha$$

where $V_r$ is a sequence of simple functions satisfying the condition.

$$\lim_{r\to\infty} \int_{\alpha J\times[0,T]} n(xV(s) - x\, V_r(s))\, dM(x,s) = 0.$$

According to Cor. 3 of Theorem 2.1 such a sequence exists, and according to Theorem 2.2, any two such sequences lead to the same limit, $\int_0^t V d\zeta_\alpha$, except for a set of paths having probability zero. The third term, $\int_0^t V d\zeta'_\alpha$, is the usual Lebesgue-Stieltjes integral. It exists a.s. since with probability one the paths of $\zeta'$ are step functions with finitely many jumps in every bounded interval.

As defined above the integral depends on $\alpha > 0$. Clearly, if $V$ satisfies condition (a) of the definition for some particular $\alpha > 0$, then for any $\beta \in (0,\alpha)$, it will satisfy condition (a) with $\alpha$ replaced by $\beta$. It is therefore necessary to show that the integral does not depend on the choice of $\alpha > 0$ for which (a) holds. Temporarily we denote the stochastic integral as defined above by $\alpha - \int_0^t Vd\zeta$ to indicate the possible dependence on the choice of $\alpha$.

<u>Theorem</u> 3.1 Suppose that $V$ is both $\alpha$- integrable and $\beta$- integrable over $[0,T]$. Then

$$\alpha - \int_0^t Vd\zeta = \beta - \int_0^t Vd\zeta, \qquad 0 \le t \le T, \quad \text{a.s.}$$

The exceptional set does not depend on $t$.

<u>Proof.</u> By the definition of the integral,

$$\beta - \int_0^t Vd\zeta = \int_0^t Vdb_\beta + \int_0^t Vd\zeta_\beta + \int_0^t Vd\zeta'_\beta$$

Now $b_\beta(t) = b_\alpha(t) + \int_{\alpha<|x|<\beta} x\, dM_t(x)$. By Property 2,

$$\int_0^t Vd\zeta_\beta = \int_0^t Vd\zeta_\alpha + \int_0^t Vd(\zeta_\beta - \zeta_\alpha).$$

But according to Property 5,

$$\int_0^t Vd(\zeta_\beta - \zeta_\alpha) = L - \int_0^t Vd(\zeta_\beta - \zeta_\alpha).$$

Since $\zeta_\beta(t) - \zeta_\alpha(t) = \zeta'_\alpha(t) - \zeta'_\beta(t) - \int_{\alpha<|x|<\beta} x\, dM_t(x)$

$$= \zeta'_\alpha(t) - \zeta'_\beta(t) - (b_\beta(t) - b_\alpha(t)),$$

$L - \int_0^t Vd(\zeta_\beta - \zeta_\alpha) = \int_0^t Vd(\zeta'_\alpha - \zeta'_\beta) - \int_0^t Vd(b_\beta - b_\alpha)$. Thus,

$$\int_0^t Vd(\zeta_\beta - \zeta_\alpha) = \int_0^t Vd(\zeta'_\alpha - \zeta'_\beta) - \int_0^t Vd(b_\beta - b_\alpha) \text{ and hence}$$

$$\beta - \int_0^t Vd\zeta = \int_0^t Vdb_\alpha + \int_0^t Vd\zeta_\alpha + \int_0^t Vd\zeta'_\alpha = \alpha - \int_0^t Vd\zeta \qquad \text{a.s.}$$

Since both sides are right-continuous in $t$,

$$P\left\{\beta - \int_0^t Vd\zeta = \alpha - \int_0^t Vd\zeta,\ 0 \le t \le T\right\} = 1.$$

## 4 A Complex-valued Martingale.

Let $V$ be an integrable process and let

$$Z(t) = \exp\left\{iu\int_0^t Vd\zeta - \int_{R\times[0,t]} \Psi_\alpha(x,uV(s))\, dM(x,s) - iu\int_0^t Vdb_\alpha\right\}$$

where $\alpha > 0$ and $\Psi_\alpha(x,w) = e^{ixw} - 1 - i\, I_{\alpha J}(x)\, xw$. In this section we show that $Z(t)$ is a martingale under a certain moment condition (Theorem 4.1.). This result generalizes Prop 3.1 of Rosinski and Woyczynski [9] which they used to establish an "inner clock" for symmetric p-stable motion.

Lemma 4.1. Let $V$ be a integrable process bounded on $\Omega \times [0,t]$ and adapted to the $\sigma$-fields $A_t$. If $\int_{\alpha J^c\times[0,t]} |x|\, dM(x,s) < \infty$, then

$$E\int_0^t Vd\zeta'_\alpha = E\int_{\alpha J^c\times[0,t]} xV(s)\, dM(x,s).$$

Proof. Let $\beta > \alpha$ and set $A = [-\beta, -\alpha) \cup (\alpha,\beta]$. Put $\zeta'_{\alpha\beta}(t) = \zeta'_\alpha(t) - \zeta'_\beta(t)$ and $X(t) = \zeta'_{\alpha\beta}(t) - \int_{A\times[0,t]} x\, dM(x,s)$ so that $X \in C_\beta$. Since $V$ is bounded, $V$ isintegrable with respect to $X$. Furthermore the Levy measure of $X$ is the restriction of the Levy measure of $\zeta$ to $A\times[0,t]$. The hypotheses of Property 5 are satisfied and so

$L - \int_0^t VdX = \int_0^t VdX.$

But $L - \int_0^t VdX = \int_0^t Vd\zeta'_{\alpha\beta} - \int_{\Lambda\times[0,t]} xV(s)\, dM(x,s).$ Hence

$$E\{\int_0^t VdX\} = E\{\int_0^t Vd\zeta'_{\alpha\beta}\} - E\{\int_{\Lambda\times[0,t]} xV(s)dM(x,s)\}.$$

But by Property 4, $E\int_0^t VdX = 0$, and thus

$$E\int_0^t Vd\zeta'_{\alpha\beta} = E\int_{A\times[0,t]} xV(s)\, dM(x,s).$$

Since the paths of $\zeta'_\alpha$ have at most finitely many jumps in $[0,t]$,

$$\zeta'_\alpha(\cdot,\omega) = \zeta'_{\alpha\beta}(\cdot,\omega)$$

for sufficiently large $\beta$ depending on $\omega$. Since $\int_0^t Vd\zeta'_\alpha$ and $\int_0^t Vd\zeta'_{\alpha\beta}$ are defined pathwise,

$$\int_0^t Vd\zeta'_\alpha(\omega) = \int_0^t Vd\zeta'_{\alpha\beta}(\omega)$$

for sufficiently large $\beta$. Thus

$$\lim_{\beta\to\infty} \int_0^t Vd\zeta'_{\alpha\beta} = \int_0^t Vd\zeta'_\alpha.$$

But also if $|\zeta'_{\alpha\beta}|$ and $|\zeta'_\alpha|$ denote the total variation measures of $\zeta'_{\alpha\beta}$ and $\zeta'_\alpha$ respectively, and if $C$ is an upper bound for $|V|$,

$$\left|\int_0^t Vd\zeta'_{\alpha\beta}\right| \le C\, |\zeta'_{\alpha\beta}|([0,t]) \le C\, |\zeta'_\alpha|([0,t]).$$ Hence

$$E\left|\int_0^t Vd\zeta'_{\alpha\beta}\right| \le C\, E\{|\zeta'_\alpha|([0,t])\}$$
$$= C\int_{{}_\alpha J^c\times[0,t]} |x|\, dM(x,s) < \infty.$$

Thus by the Dominated Convergence Theorem,

$$E\int_0^t Vd\zeta'_\alpha = \lim_{\beta\to\infty} E\int_0^t Vd\zeta'_{\alpha\beta}$$

$$= \lim_{\beta\to\infty} \int_{A\times[0,t]} xV(s)\, dM(x,s)$$

$$= \int_{\alpha J^c\times[0,t]} xV(s)\, dM(x,s).$$

Q.E.D.

Corollary. Let $V$ be a bounded integrable process.

If $\int_{\alpha J^c\times[0,t]} |x|\, dM(x,s) < \infty$, then $E\left|\int_0^t Vd\zeta'_\alpha\right| \leq E \int_{\alpha J^c\times[0,t]} |xV(s)|\, dM(x,s)$.

Proof. For each $\omega$ $\int_0^t Vd\zeta'_\alpha$ is an ordinary Lebesgue-Stieltjes integral and so

$$\left|\int_0^t Vd\zeta'_\alpha\right| \leq \int_0^t |V|\; d\,|\zeta'_\alpha|$$

where $d|\zeta'_\alpha|$ denotes the total variation of $d\zeta'_\alpha$. Let $Y^+(s)$ and $-Y^-(s)$ denote the sum of the jumps of $\zeta'_\alpha$ which occur in $[0,s]$ and have positive and negative magnitudes respectively. Then $\zeta'_\alpha = Y^+ - Y^-$ and as is well-known $Y^+$ and $Y^-$ are both stochastic processes with independent increments and right-continuous paths which are step functions. The jump-time Levy measures of $Y^+$ and $Y^-$ are $M|_{(\alpha,\infty)\times[0,\infty)}$ and $M^-|_{(\alpha,\infty]\times[0,\infty)}$ where $M^-$ is defined by the equation $M^-(D_1\times D_2) = M((-D_1)\times D_2)$. Furthermore, $d|\zeta'_\alpha| = dY^+ + dY^-$. Thus by Lemma 4.1,

$$E\left|\int_0^t Vd\zeta'_\alpha\right| \leq E\int_0^t |V|\; dY^+ + E\int_0^t |V|\; dY^-$$

$$= E \int_{(\alpha,\infty)\times[0,t]} x|V(s)|\; dM(x,s)$$

$$+ E \int_{(\alpha,\infty)\times[0,t]} x|V(s)|\; dM^-(x,s)$$

$$= E \int_{\alpha J^c\times[0,t]} |x|\; |V(s)|\; dM(x,s)$$

Q.E.D.

<u>Proof.</u> We first prove the lemma in the special case that $V$ is simple, say

$$V(s) = w_0 I_{\{0\}} + \sum_{j=0}^{k-1} v_j I_{(t_j, t_{j+1}]},$$

where $0 = t_0 < t_1 < \ldots < t_k = t$. By independence of increments

$$E\, e^{i\langle u, \int_0^t V d\zeta_\alpha\rangle} = E\, e^{\sum_j i\langle u, v_j\rangle \Delta\zeta_j}$$

$$= \prod_j E\, e^{i\langle u, v_j\rangle \Delta\zeta_j}$$

$$= \prod_j \exp\{ \int_{\alpha J \times (t_j, t_{j+1}]} \psi_0(x\langle u, v_j\rangle) dM(x,s)$$

$$= \prod_j \exp\{ \int_{\alpha J \times (t_j, t_{j+1}]} \psi_0(x\langle u, V(s)\rangle) dM(x,s)$$

$$= \exp\{ \int_{\alpha J \times [0,t]} \psi_0(x\langle u, V(s)\rangle) dM(x,s)\}.$$

This proves the lemma for simple $V$.

Next assume that $V$ is bounded over $[0,t]$ and let $V_r$ be a sequence of simple processes such that $\|V_r\|_\infty \leq \|V\|_\infty$,

$$\lim_{r\to\infty} \int_{\alpha J \times [0,t]} x^2 \left|V(s) - V_r(s)\right| dM(x,s) = 0 \quad \text{a.s., and}$$

$$\lim_{r\to\infty} \int_{\alpha J \times [0,t]} n(xV(s) - xV_r(s)) dM(x,s) = 0.$$

Now a Taylor series expansion shows that if $(x, \theta, \psi)$ is confined to a compact subset of $R^3$, then

$$\left|\psi_0(x\theta) - \psi_0(x\phi)\right| = \left|e^{ix\theta} - ix\theta - (e^{ix\phi} - ix\phi)\right|$$

$$\leq kx^2 \left|\theta - \phi\right|$$

where the constant $k$ only depends on the compact set. Since $\|V_r\|_\infty \leq \|V\|_\infty < \infty$, we obtain for each $u \in \mathbb{R}^p$

$$\int_{\alpha J \times [0,t]} \left|\psi_0(x\langle u, V(s)\rangle) - \psi_0(x\langle u, V_r(s)\rangle)\right| dM(x,s)$$

$$\leq \int_{\alpha J \times [0,t]} kx^2 \|u\| \cdot \|V(s) - V_r(s)\| dM(x,s).$$

(The last equality follows from the Dominated Convergence Theorem since $M(\alpha J^c \times [0,t]) < \infty$.)  Q.E.D.

Prop. 5.3. If $\psi_0(x) = e^{ix} - 1 - ix$, then for each $w \neq 0$, there exist positive constants $c_1$ and $c_2$ such that

$$c_1 n(x) \leq |\psi_0(xw)| \leq c_2 n(x), \qquad x \in \mathbf{R}\ .$$

Proof. Observe that as $x \to 0$ $|\psi_0(xw)| \sim \frac{x^2 w^2}{2}$ and $n(x) \sim x^2$ so

$$\lim_{x \to 0} \frac{|\psi_0(xw)|}{n(x)} = w^2/2.$$

Also observe that as $|x| \to \infty$, $|\psi_0(xw)| \sim |xw|$ and $n(x) \sim |x|$, so

$$\lim_{|x| \to \infty} \frac{|\psi_0(xw)|}{n(x)} = |w|.$$

Furthermore $\psi_0(xw) = 0$ iff $\mathrm{Re}\ \psi_0(xw) = 0$ and $\mathrm{Im}\ \psi_0(xw) = 0$. That is, $\psi_0(xw) = 0$ iff $\cos(xw) = 1$ and $\sin(xw) = xw$. The last two conditions are equivalent to $xw = 0$, i.e. $x = 0$ ($\sin x < x$ if $x > 0$ etc). Thus since $|\psi_0(xw)|$ and $n(x)$ are continuous functions there exist positive constants $c_1 = c_1(w)$ and $c_2 = c_2(w)$ such that

$$c_1 \leq \frac{|\psi_0(xw)|}{n(x)} \leq c_2.$$  Q.E.D.

Lemma 5.4. Let $V(s)$ be a nonrandom Borel measurable $\mathbf{R}^p$-valued function on $[0,t]$ such that

$$\int_{\alpha J \times [0,t]} n(x \| V(s) \|) dM(x,s) < \infty.$$

Then for all $u \in \mathbf{R}^p$

$$E\, e^{i\langle u, \int_0^t V d\zeta_\alpha \rangle} = \exp \left\{ \int_{\alpha J \times [0,t]} \psi_0(x \langle u, V(s) \rangle) dM(x,s) \right\}$$

where $\psi_0(x) = e^{ix} - 1 - ix$.

$$\int_0^t V d\zeta'_\alpha = \lim_{\beta\to\infty} \int_0^t V d\zeta'_{\alpha,\beta}$$

Consequently,

$$E\, e^{i\langle u, \int_0^t V d\zeta'_\alpha\rangle} = \lim_{\beta\to\infty} E\, e^{i\langle u, \int_0^t V d\zeta_{\alpha,\beta}\rangle}$$

$$= \lim_{\beta\to\infty} \exp\{\int_{A_\beta\times[0,t]} e^{ix\langle u,V(s)\rangle}-1\ dM(x,s)\}$$

$$= \exp\{\int_{\alpha J^c\times[0,t]} e^{ix\langle u,V(s)\rangle}-1\ dM(x,s)\}.$$

This proves the lemma for bounded $V$.

Now suppose that $V$ is not bounded. Let

$$V_k(s) = \begin{cases} k & \text{if } V(s) \geq k \\ V(s) & \text{if } -k < V(s) < k \\ -k & \text{if } V(s) \leq -k. \end{cases}$$

Then $\lim V_k(s) = V(s)$ for all $s \in [0,t]$. Let $T_j$ denote the time of the $j^{th}$ jump of $\zeta'_\alpha$ in $[0,t]$ and let $\Delta\zeta'_\alpha(T_j)$ denote the magnitude of the $j^{th}$ jump. Since $\zeta'_\alpha$ has finitely many jumps in $[0,t]$

$$\int_0^t V d\zeta'_\alpha = \sum_j V(T_j)\Delta\zeta'_\alpha(T_j) \qquad \text{(a finite sum)}$$

$$= \lim_{k\to\infty} \sum_j V_k(T_j)\Delta\zeta'_\alpha(T_j)$$

$$= \lim_{k\to\infty} \int_0^t V_k \zeta'_\alpha.$$

Hence,

$$E\, e^{i\langle u, \int_0^t V d\zeta'_\alpha\rangle} = \lim_{k\to\infty} E\, e^{i\langle u, \int_0^t V_k d\zeta'_\alpha\rangle}$$

$$= \lim_{k\to\infty} \exp\{\int_{\alpha J^c\times[0,t]} e^{ix\langle u,V_k(s)\rangle}-1\ dM(x,s)\}$$

$$= \exp\{\int_{\alpha J^c\times[0,t]} e^{ix\langle u,V(s)\rangle}-1\ dM(x,s)\}.$$

It follows that

$$\lim_{n\to\infty} Ee^{i\langle u,\, \int_0^t V_n d\zeta'_{\alpha,\beta}\rangle} = \lim_{n\to\infty} \exp\{\int_{A\times[0,t]} e^{ix\langle u,\, V_n(s)\rangle} - 1 dM(x,s)\}$$

$$= \exp\{\int_{A\times[0,t]} e^{ix\langle u,\, V(s)\rangle} - 1\, dM(x,s)\}.$$

That is,

$$Ee^{i\langle u,\, \int_0^t V d\zeta'_{\alpha,\beta}\rangle} = \exp\{\int_{A\times[0,t]} e^{ix\langle u,\, V(s)\rangle} - 1\, dM(x,s)\}.$$

Q.E.D.

<u>Lemma 5.2.</u> Let $V(s)$ be a nonrandom Borel-measurable $\mathbf{R}^p$-valued function on $[0,t]$. Then for all $u \in \mathbf{R}^p$

$$E\, e^{i\langle u,\, \int_0^t V d\zeta'_\alpha\rangle} = \exp\{\int_{\alpha J^c\times[0,t]} e^{ix\langle u, V(s)\rangle} - 1\, dM(x,s)\}.$$

<u>Proof.</u> We first prove the lemma under the assumption that $V$ is bounded over $[0,t]$. Let $\beta$ be any positive number greater than $\alpha$ and set $A_\beta = [-\beta, -\alpha) \quad (\alpha, \beta]$. Then according to Lemma 5.1

$$Ee^{i\langle u, \int_0^t V d\zeta'_{\alpha,\beta}\rangle} = \exp\{\int_{A_\beta\times[0,t]} e^{ix\langle u, V(s)\rangle} - 1 dM(x,s)\}.$$

Since the sample paths of $\zeta'_\alpha$ have finitely many jumps in $[0,t]$,

$$\zeta'_\alpha(s,\omega) = \zeta'_{\alpha,\beta}(s,\omega),\ 0 \le s \le t$$

for $\beta > \beta_0(\omega)$. It follows from the pathwise definition of $\int_0^t V d\zeta'_\alpha$ and of $\int_0^t V d\zeta'_{\alpha,\beta}$ as the usual Lebesgue-Stieltjes integral, that

$$\int_0^t V d\zeta'_\alpha(\omega) = \int_0^t V d\zeta'_{\alpha,\beta}(\omega),\ \beta > \beta_0(\omega).$$

Hence

so $|\mu|$ is a finite measure.)

Then

$$E\left|e^{i\langle u,\int_0^t V d\zeta'_{\alpha,\beta}\rangle} - e^{i\langle u,\int_0^t V_n d\zeta'_{\alpha,\beta}\rangle}\right|$$

$$\leq E\left|\langle u, \int_0^t V d\zeta_{\alpha,\beta}\rangle - \langle u, \int_0^t V_n d\zeta'_{\alpha,\beta}\rangle\right|$$

$$= E\left|\langle u, \int_0^t V-V_n d\zeta'_{\alpha,\beta}\rangle\right|$$

$$\leq \|u\|\cdot E\left\|\int_0^t V-V_n d\zeta'_{\alpha,\beta}\right\|$$

But according to an obvious extension of the corollary to Lemma 4.1,

$$E\left\|\int_0^t V-V_n d\zeta'_{\alpha,\beta}\right\| \leq \int_{A\times[0,t]} \|x(V(s)-V_n(s))\| dM(x,s),$$

and hence

$$\lim_{n\to\infty} E\left\|\int_0^t V - V_n \, d\zeta'_{\alpha,\beta}\right\| = 0.$$

Thus

$$Ee^{i\langle u,\int_0^t V d\zeta'_{\alpha,\beta}\rangle} = \lim_{n\to\infty} Ee^{i\langle u,\int_0^t V_n d\zeta'_{\alpha,\beta}\rangle} .$$

On the other hand by the first part of the proof

$$Ee^{i\langle u,\int_0^t V_n d\zeta'_{\alpha,\beta}\rangle} = \exp\left\{\int_{A\times[0,t]} e^{ix\langle u,V_n(s)\rangle}-1 dM(x,s)\right\}$$

Now similarly,

$$\left|\int_{A\times[0,t]} e^{ix\langle u,V_n(s)\rangle}-1 \, dM(x,s) - \int_{A\times[0,t]} e^{ix\langle u,V(s)\rangle}-1 \, dM(x,s)\right|$$

$$\leq \int_{A\times[0,t]} \left|e^{ix\langle u,V_n(s)\rangle} - e^{ix\langle u,V(s)\rangle}\right| dM(x,s)$$

$$\leq \int_{A\times[0,t]} |x|\cdot\|u\|\cdot\|V_n(s)-V(s)\| dM(x,s)$$

$$\to 0 \text{ as } n\to\infty.$$

$$\zeta'_{\alpha,\beta}(s) = \sum\{\Delta\zeta'_\alpha(T_j): 0 \le T_j \le s \text{ and } \alpha < |\Delta\zeta'_\alpha(T_j)| \le \beta\}.$$

Lemma 5.1. Let $V(s)$ be a bounded $R^p$-valued nonrandom Borel-measurable function on $[0,t]$. Let $\beta > \alpha$ and put $A = [-\beta,-\alpha,) \cup (\alpha,\beta]$. Then

$$Ee^{i\langle u,\int_0^t V d\zeta_{\alpha,\beta}\rangle} = \exp\{\int_{A\times[0,t]} e^{ix\langle u,V(s)\rangle} - 1 \, dM(x,s)\}.$$

Proof. First suppose that $V(s)$ is a simple process, say

$$V = v_0 I_{\{0\}} + \sum_{j=0}^{k-1} v_j I_{(t_j,t_{j+1}]}$$

where $0 = t_0 < t_1 < t_2 < \ldots < t_k = t$. Since $\zeta_{\alpha,\beta}$ is right-continuous and since $\zeta'_{\alpha,\beta}(0) = 0$,

$$\int_0^t V d\zeta'_{\alpha,\beta} = \sum_{j=0}^{k-1} v_j \Delta_j \zeta'_{\alpha,\beta}$$

where $\Delta_j\zeta'_{\alpha,\beta} = \zeta'_{\alpha,\beta}(t_{j+1}) - \zeta_{\alpha,\beta}(t_j)$. By independence of increments,

$$\begin{aligned} Ee^{i\langle u,\int_0^t V d\zeta'_{\alpha,\beta}\rangle} &= \prod_{j=0}^{k-1} Ee^{i\langle u,v_j\rangle\Delta_j\zeta'_{\alpha,\beta}} \\ &= \prod_{j=0}^{k-1} \exp\{\int_{A\times(t_j,t_{j+1}]} e^{ix\langle u,v_j\rangle} - 1 \, dM(x,s)\} \\ &= \prod_{j=0}^{k-1} \exp\{\int_{A\times(t_j,t_{j+1}]} e^{ix\langle u,V(s)\rangle} - 1 \, dM(x,s)\} \end{aligned}$$

This proves the Lemma for simple $V$.

Now let $V$ be any bounded Borel-measurable function on $[0,t]$ and choose simple $V_n$ such that

$$\lim_{n\to\infty} \int_{A\times[0,t]} |x| \cdot \|V(s) - V_n(s)\| dM(x,s) = 0.$$

(This is possible since step functions are dense in $L^1([0,t],|\mu|)$

where $$|\mu|(D) = \int_{A\times D} |x| dM(x,s)$$

## 5. Characteristic Functions and Independence

Let $V_1(s), \ldots, V_p(s)$ be nonrandom Borel-measurable integrable functions on $[0,\infty)$. In this section we compute the joint ch.f. of $\int_0^t V_1 d\zeta, \ldots, \int_0^t V_p d\zeta$ and establish necessary and sufficient conditions for these integrals to be independent r.v.'s. Let $V(s) = \begin{pmatrix} V_1(s) \\ \cdot \\ \cdot \\ \cdot \\ V_p(s) \end{pmatrix}$, and let $\langle u,v\rangle$ denote the usual inner product of two vectors $u$ and $v$ in $R^p$, and let $\|u\| = \langle u,v\rangle^{1/2}$ be the Euclidean norm. Since each $V_i$ is integrable, there exists $\alpha > 0$, and simple $R^d$-valued functions

$$V^{(r)}(s) = v_0 I_{\{0\}} + \sum_{j=0}^{k_r} v_j I_{(t_{rj}, t_{r,j+1}]}$$

where $0 = t_{r0} < t_{r1} < \ldots < t_{r,k_r} = t$, such that

$$P\text{-}\lim_{r\to\infty} \int_{\alpha J\times[0,t]} n(xV(s) - xV^{(r)}(s))dM(x,s) = 0.$$

Let $\zeta_\alpha(s) = b_\alpha(s) + \zeta_\alpha(s) + \zeta'_\alpha(s)$ denote the corresponding decomposition of the process $\zeta(t)$. Since $V$ is nonrandom and since the processes $\zeta_\alpha$ and $\zeta'_\alpha$ are independent,

$$Ee^{i\langle u, \int_0^t V d\zeta\rangle} = e^{i\langle u, b_\alpha(t)\rangle}\, Ee^{i\langle u, \int_0^t V d\zeta_\alpha\rangle} \cdot E\, e^{i\langle u, \int_0^t V d\zeta'_\alpha\rangle}.$$

We will first compute $Ee^{i\langle u, \int_0^t V d\zeta'_\alpha\rangle}$ and then $Ee^{i\langle u, \int_0^t V d\zeta_\alpha\rangle}$.

To compute $Ee^{i\langle u, \int_0^t V d\zeta'_\alpha\rangle}$ we will use a truncation argument. For $\beta > \alpha$, define $\zeta_{\alpha,\beta}(s)$ to be the sum of those jumps of $\zeta$ which occur in the time interval $[0,s]$ and have absolute value in $(\alpha,\beta]$. That is,

<u>Cor</u>. (Rosinski and Woyczynski) Let $\zeta(t)$ be a symmetric p-stable motion (i.e. $\zeta(t)$ has stationary independent increments and $\zeta(1)$ is symmetric and stable with index p) where $1 < p < 1$. Let V be an integrable process such that

$$E \exp \{r \int_0^t |V(s)|^p ds\} < \infty$$

for every $r > 0$ and $t > 0$. Then for every $\lambda \in \mathbf{R}$

$$Z(t) = \exp\{i\lambda \int_0^t V\, d\zeta + |\lambda|^p \int_0^t |V(s)|^p ds\}$$

is a complex-valued martingale.

<u>Proof</u>. Since $\zeta(t)$ is symmetric $b_\alpha(\cdot) \equiv 0$ and by the well-known formula for the ch.f. of a symmetric stable law $(dM(x,s) = |x|^{-p-1}dxds)$

$$\int_{\mathbf{R}} \psi_\alpha(x, \lambda V(s))|x|^{-p-1}dx = |\lambda|^p|V(s)|^p$$

and so $Z(t)$ above is the same as the $Z(t)$ defined in Theorem 4.1. A straight-forward calculation shows that if

$$E \exp \{r \int_0^t |V(s)|^p ds\} < \infty$$

for every r, then

$$E \exp \{ \int_{\alpha J \times [0,t]} n(xV(s))|x|^{-p-1}dx\, ds\} < \infty$$

for some $\alpha > 0$. Q.E.D.

If $x \in \alpha J^c$,

$$\left|\psi_\alpha(x,uV_p(s))\right| = \left|e^{ixuV_p(s)}-1\right| \leq 2.$$

Thus $\left|\psi_\alpha(x,uV_p(s))\right| \leq n(xuV(s))I_{\alpha J}(x) + 2I_{\alpha J^c}(x)$, and the right side is integrable with respect to $M$ a.s. Then by the Dominated Convergence Theorem,

$$\lim_{p\to\infty} \int_{\mathbf{R}\times[0,t]} \psi_\alpha(x,uV_p(s))dM(x,s) = \int_{\mathbf{R}\times[0,t]} \psi_\alpha(x,uV(s))dM(x,s)$$

Consequently,

$$\lim_{p\to\infty} Z_p(s) = Z(s) \text{ a.s.}$$

In order to conclude that $Z(s)$, $0 \leq s \leq t$ is a martingale, it suffices to show that $\{|Z_p(s)|\}_{p=1}^{\infty}$ is bounded above by a r.v. having finite expectation since we may then again invoke the Dominated Convergence Theorem. But

$$\left|Z_p(s)\right| = \exp\left\{\int_{\mathbf{R}\times[0,s]} 1 - \cos(xuV(r))dM(x,r)\right\}$$

Now $\lim_{\theta\to 0} \frac{1-\cos(u\theta)}{n(\theta)} = \frac{1}{2}u^2$, $\lim_{\theta\to\infty} \frac{1-\cos(u\theta)}{n(\theta)} = 0$, and this ratio is continuous in $\theta$ so there is a constant $c = c(u)$ such that $1 - \cos(u\theta) \leq cn(\theta)$. It follows that

$$\int_{\mathbf{R}\times[0,s]} 1 - \cos(xuV(r))dM(x,r) \leq \int_{\alpha J\times[0,t]} c\, n(xuV(r))dM(x,r) + 2M(\alpha J^c\times[0,t])$$

Thus

$$\left|Z_p(s)\right| \leq e^{2M(\alpha J^c\times[0,t])}\cdot e^{c\int_{\alpha J\times[0,t]} n(xuV(r))dM(x,r)}$$

By hypothesis the right side has finite expectation and so $Z(s)$, $0 \leq s \leq t$ is a martingale. Q.E.D, Again the convergence is bounded since $1 - \cos\theta \leq \min(2, \theta^2/2)$ so $|Z_p(s)| \leq \exp\{\frac{u^2C^2}{2}\int_{J\times[0,s]} x^2 dM(x,s) + 2M(J^c\times[0,s])\}$. It follows that $Z$ is a martingale. Q.E.D.

<u>Theorem 4.1.</u> Let $V$ be an integrable process which satisfies the condition that

$$E\exp\{\int_{\alpha J\times[0,t]} n(xV(s))dM(x,s)\} < \infty$$

for some $\alpha > 0$. Then the random variables

$$Z(s) = \exp\{iu\int_0^t Vd\zeta - \int_{R\times[0,s]} \psi_\alpha(x,uV(s))dM(x,s) - iu\int_0^t Vdb_\alpha\}$$

form a martingale over the σ-fields $\mathcal{A}_s$, $0 \leq s \leq t$.

<u>Proof.</u> Define processes $V_p$ by

$$V_p(s) = \begin{cases} p & \text{if } V(s) > p \\ V(s) & \text{if } -p \leq V(s) \leq p \\ -p & \text{if } V(s) < -p \end{cases}.$$

According to Lemma 4.3

$$Z_p(s) = \exp\{iu\int_0^t V_p d\zeta - \int_{\mathbf{R}\times[0,s]} \psi_\alpha(x,uV_p(s))dM(x,s) - iu\int_0^t Vdb_\alpha\}$$

is a martingale. Now by Property 3, $\lim_{p\to\infty}\int_0^s V_p d\zeta_\alpha = \int_0^s Vd\zeta_\alpha$, a.s. And also $\lim_{p\to\infty}\int_0^s V_p d\zeta_\alpha' = \int_0^s Vd\zeta_\alpha'$ since for every $\omega$ the integrals with respect to $\zeta_\alpha'$ are finite sums. Thus $\lim_{p\to\infty}\int_0^t V_p d\zeta = \int_0^t Vd\zeta$. To see that $\int_{\mathbf{R}\times[0,t]} \psi_\alpha(x,uV_p(s))dM(x,s)$, converges a.s. to $\int_{\mathbf{R}\times[0,t]} \psi_\alpha(x,uV(s))dM(x,s)$, we need to bound the quantity $|\psi_\alpha(x,uV_p(s))|$ by an integrable function relative to $M$. Note that $|e^{i\theta} - 1 - i\theta| \leq (1/2)\theta^2 \leq 2\theta^2$ and also $|e^{i\theta} - 1 - i\theta| \leq |e^{i\theta} - 1| + |\theta| \leq 2|\theta|$. Hence $|e^{i\theta} - 1 - i\theta| \leq 2n(\theta)$, so

$$\begin{aligned}|\psi_\alpha(x,uV_p(s))| &\leq n(xuV_p(s)) \\ &\leq n(xuV(s)) \quad \text{if } x \in \alpha J.\end{aligned}$$

$$\lim_{r\to\infty} E \int_{\beta J\times[0,t]} \left|\psi_\alpha(x,uV(s)) - \psi_\alpha(x,uV_r(s))\right| dM(x,s) = 0$$

and so

$$\text{P-}\lim_{r\to\infty} \int_{\beta J\times[0,t]} \psi_\alpha(x,uV_r(s))dM(x,s)$$

$$= \int_{\beta J\times[0,t]} \psi_\alpha(x,uV(s))dM(x,s).$$

Put $Z_{\beta r}(s) = \exp\{iu \int_0^s V_r d(\zeta_\alpha + \zeta'_{\alpha\beta}) - \int_{\beta J\times[0,s]} \psi_\alpha(x,uV_r(q))dM(x,q)\}$.

Then

$$\text{P-}\lim_{r\to\infty} Z_{\beta r}(s) = Z_\beta(s), \quad 0 \le s \le t,$$

where

$$Z_\beta(s) = \exp\{iu \int_0^s V_r \; d(\zeta_\alpha + \zeta'_{\alpha\beta}) - \int_{\beta J\times[0,s]} \psi_\alpha(x,uV_r(q))dM(x,q)\}$$

But also since $1-\cos\theta \le (1/2)\theta^2$, $\theta \in \mathbf{R}$ ,

$$\left|Z_{\beta r}(s)\right| = \exp\{\int_{\beta J\times[0,s]} 1-\cos(xuV_r(q)) \; dM(x,q)\}$$

$$\le \exp\{\frac{u^2C^2}{2} \int_{\beta J\times[0,s]} x^2 dM(x,q)\} < \infty.$$

Thus if $0 \le s_1 < s_2 \le t$, the Dominated Convergence Theorem together with Lemma 4.2 implies that

$$E\{Z_\beta(s_2)\,|\,\mathring{A}_{s_1}\} = \lim_{r\to\infty} E\{Z_{\beta r}(s_2)\,|\,\mathring{A}_{s_1}\}$$

$$= \lim_{r\to\infty} Z_{\beta r}(s_1)$$

$$= Z_\beta(s_1).$$

Now let $\beta \to \infty$ and note that as in the proof of Lemma 4.1,

$$\lim_{\beta\to\infty} \int_0^t V d\zeta'_{\alpha\beta} = \int_0^t V d\zeta'_\alpha,$$

Thus,

$$\lim_{\beta\to\infty} Z_\beta(s) = Z(s), \quad 0 \le s \le t.$$

$$E\left| \int_0^t V d\zeta'_{\alpha\beta} - \int_0^t V_r d\zeta'_{\alpha\beta} \right| \le E \int_{A\times[0,t]} |x| \cdot |V(s)-V_r(s)| dM(x,s)$$

$$\le \beta E_c \int_{\alpha J^c \times[0,t]} |V(s)-V_r(s)| dM(x,s),$$

and hence $P - \lim_{r\to\infty} \int_0^t V_r d\zeta'_{\alpha\beta} = \int_0^t V d\zeta'_{\alpha\beta}$. It follows that

$$P - \lim_{r\to\infty} \int_0^t V_r d(\zeta_\alpha + \zeta'_{\alpha\beta}) = \int_0^t V d(\zeta_\alpha + \zeta'_{\alpha\beta}).$$

On the other hand,

$$\int_{\beta J\times[0,t]} |\psi_\alpha(x,uV(s)) - \psi_\alpha(x,uV_r(s))| dM(x,s)$$

$$= \int_{\alpha J\times[0,t]} |(e^{ixuV(s)} - ixuV(s)) - (e^{ixuV_r(s)} - ixu V_r(s))| dM(x,s)$$

$$+ \int_{A\times[0,t]} |e^{ixuV(s)} - e^{ixuV_r(s)}| dM(x,s).$$

To bound the first integral on the right use a Taylor series expansion to see that

$$|(e^{ixuV(s)} - ixuV(s)) - (e^{ixuV_r(s)} - ixuV_r(s))| \le Kx^2 |V(s) - V_r(s)|$$

where $K$ is a constant depending only on the bound for $V$ and on $\alpha$. Thus by the choice of $V_r$

$$\lim_{r\to\infty} E \int_{\alpha J\times[0,t]} |(e^{ixuV(s)} - ixuV(s)) - (e^{ixuV_r(s)} - ixuV_r(s))| dM(x,s)$$

$$= 0.$$

Also,

$$\int_{A\times[0,t]} |e^{ixuV(s)} - e^{ixuV_r(s)}| dM(x,s)$$

$$\le \int_{A\times[0,t]} |xuV(s) - xuV_r(s)| dM(x,s)$$

$$\le \beta |u| \int_{A\times[0,t]} |V(s)-V_r(s)| dM(x,s),$$

and consequently, $\lim_{r\to\infty} E \int_{A\times[0,t]} |e^{ixuV(s)} - e^{ixuV_r(s)}| dM(x,s) = 0.$

It follows that

$Z(t_j)$ is $\mathcal{A}_{t_j}$-measurable we obtain the equation

$$EZ(t_{j+1}) = E\{Z(t_j)L_j | \mathcal{A}_{t_j}\}$$
$$= Z(t_j)\, E\{L_j | \mathcal{A}_{t_j}\}$$
$$= Z(t_j).$$

Thus $\{Z(t_j): j = 0, 1, \ldots, k\}$ is a martingale and hence

$$E\{Z(t) | \mathcal{A}_s\} = E\{Z(t_k) | \mathcal{A}_{t_\ell}\} = Z(t_\ell) = Z(s).$$

That is, $Z(t)$ is a martingale for continuous time. Q.E.D,

<u>Lemma 4.3</u>. Let $V$ be a bounded integrable process on $\Omega \times [0,t]$. Let $\alpha$ be any positive number and put

$$Z(s) = \exp\{iu \int_0^s V d\zeta - \int_{R\times[0,s]} \psi_\alpha(x, uV(s))dM(x,s) - iu \int_0^t V db_\alpha\}$$

where $\psi_\alpha(x,w) = e^{ixw} - 1 - iI_{\alpha J}(x)xw$. Then $\{Z(s), \mathcal{A}_s : 0 \leq s \leq t\}$ is a martingale.

<u>Proof</u>. Let $C = \sup\{|V(s,\omega)| : 0 \leq s \leq t,\ \omega \in \Omega\}$. By hypothesis, $C < \infty$. We may assume $b_\alpha \equiv 0$. Select simple adapted processes $\{V_r\}$ as in Theorem 2.1 such that:

(a) $|V_r| \leq C$ on $\Omega \times [0,t]$,

(b) $\lim_{r\to\infty} E \int_{\alpha J\times[0,t]} x^2 |V(s) - V_r(s)| dM(x,s) = 0,$

(c) $\lim_{r\to\infty} E \int_{\alpha J\times[0,t]} n(xV(s) - xV_r(s))dM(x,s) = 0,$ and

(d) $\lim_{r\to\infty} E \int_{\alpha J^c\times[0,t]} |V(s)-V_r(s)| dM(x,s) = 0.$

Condition (c) is the hypothesis of Theorem 2.2 and so

$$\lim_{r\to\infty} \int_0^t V_r d\zeta_\alpha = \int_0^t V d\zeta_\alpha \quad \text{a.s.}$$

Let $\beta$ be any real number greater than $\alpha$. Then according to the Corollary to Lemma 4.1 with $A = [-\beta,\alpha) \cup (\alpha,\beta]$, Lemma 4.2. Let $V(t)$ be a simple process adpated to the $\sigma$-fields $\mathcal{A}_t$. Then for all $u \in \mathbf{R}$, and $\alpha > 0$,

$$Z(t) := \exp\{iu\int_0^t V d\zeta - \int_{\mathbf{R}\times[0,t]} e^{iuxV(s)} - 1 - I_{\alpha J}(x)uxV(s)dM(x,s) - iu\int_0^t V db_\alpha\}$$

is a complex-valued martingale relative to the $\sigma$-fields $\mathcal{A}_t$.

Proof: Let $\alpha > 0$ and $u \in \mathbf{R}$ be arbitrarily chosen and then fixed. Without loss of generality we may (and do) assume that $b_\alpha(\cdot) = 0$. Let $0 \leq s < t$ and choose a partition $0 = t_0 < t_1 < \ldots < t_k = t$ of $[0,t]$ such that

$$V(r) = WI_{\{0\}} + \sum_{j=0}^{k-1} V_j I_{(t_j,t_{j+1}]}, \quad 0 \leq r \leq t$$

where $W$ and $V_0$ are $\mathcal{A}_0$-measurable, and for each $j \geq 1$ $V_j$ is $\mathcal{A}_{t_j}$-measurable. Since if necessary we may enlarge the partition to include s, we may assume that $s = t_\ell$ for some $\ell < k$. Then by the definition of the integral for simple V,

$$\int_0^t V d\zeta = \sum_{i=0}^{k-1} V_i \Delta\zeta_i$$

where $\Delta\zeta_i = \zeta(t_{i+1}) - \zeta(t_i)$. Let $\psi_\alpha(x,w) = e^{ixw} - 1 - iI_{\alpha J}(x)xw$. Then

$$Z(t) = Z(t_k) = \exp\{iu \sum_{j=0}^{k-1} V_j\Delta\zeta_j - \sum_0^{k-1} \int_{\mathbf{R}\times(t_j,t_{j+1}]} \psi_\alpha(x,uV_j)dM(x,s)\}.$$

Set

$$L_j \equiv \exp\{iu V_j\Delta\zeta_j - \int_{\mathbf{R}\times(t_j,t_{j+1}]} \psi_\alpha(x,uV_j)dM(x,s)\}.$$

Since $\Delta\zeta_j$ is independent of the $\sigma$-field $\mathcal{A}_{t_j}$ while $V_j$ is $\mathcal{A}_{t_j}$-measurable

$$E\{e^{iuV_j\Delta\zeta_j} | \mathcal{A}_{t_j}\} = \exp\{\int_{\mathbf{R}\times(t_j,t_{j+1}]} \psi_\alpha(x,uV_j)dM(x,s)\}.$$

It follows that $E\{L_j | \mathcal{A}_{t_j}\} = 1$, $j = 0, 1, \ldots, k-1$. From the fact that

The right side tends to zero as $r \to \infty$, and thus,

$$\lim_{r\to\infty} \int_{\alpha J\times[0,t]} \psi_0(x\langle u, V_r(s)\rangle)dM(x,s)$$
$$= \int_{\alpha J\times[0,t]} \psi_0(x\langle u,V(s)\rangle)dM(x,s).$$

But by Theorem 2.2.

$$\lim_{r\to\infty} \int_0^t V_r d\zeta_\alpha = \int_0^t V d\zeta_\alpha \quad \text{a.s.},$$

so

$$E\exp(i\langle u, \int_0^t V d\zeta_\alpha\rangle) = \lim_{r\to\infty} E\exp(i\langle u, \int_0^t V_r d\zeta_\alpha\rangle)$$
$$= \lim_{r\to\infty} \exp\{\int_{\alpha J\times[0,t]} \psi_0(x\langle u,V_r(s)\rangle)dM(x,s)$$
$$= \exp\{\int_{\alpha J\times[0,t]} \psi_0(x\langle u,V(s)\rangle)dM(x,s),$$

so the lemma holds for bounded $V$.

Now consider the general case where

$$\int_{\alpha J\times[0,t]} n(xV(s))dM(x,s) < \infty.$$

Define

$$V^{(r)}(s) = \begin{cases} r & \text{if } V(s) > r \\ V(s) & \text{if } -r \le V(s) \le r \\ -r & \text{if } V(s) < -r. \end{cases}$$

Then by Property 3, $\text{P-}\lim_{r\to\infty} \int_0^t V^{(r)} d\zeta_\alpha = \int_0^t V d\zeta_\alpha$, so by the Dominated Convergence

$$\lim_{r\to\infty} E\, e^{i\langle u, \int_0^t V^{(r)} d\zeta_\alpha\rangle} = E\, e^{i\langle u, \int_0^t V d\zeta_\alpha\rangle}$$

That is,

$$E\, e^{i\langle u,\, \int_0^t V d\zeta_\alpha\rangle} = \lim_{r\to\infty} \exp\{ \int_{\alpha J\times[0,t]} \psi_0(x\langle u, V^{(r)}(s)\rangle) dM(x,s).$$

But by the preceding Proposition,

$$\left|\psi_0(x\langle u, V^{(r)}(s)\rangle)\right| \le c_2 n(x\langle u, V^{(r)}(s)\rangle).$$

By the definition of $n$, $n(cx) \le \max(|c|, c^2)\, n(x)$, and so by the Cauchy-Schwarz inequality

$$\begin{aligned} n(x\langle u, V^{(r)}(s)\rangle) &\le \max(\|u\|, \|u\|^2) n(x\|V^{(r)}(s)\|) \\ &\le \max(\|u\|, \|u\|^2) n(x\|V(s)\|). \end{aligned}$$

Since $\int_{\alpha J\times[0,t]} n(x\|V(s)\|) dM(x,s) < \infty$ by hypothesis, the Dominated Convergence Theorem implies that

$$\begin{aligned} &\lim_{r\to\infty} \int_{\alpha J\times[0,t]} \psi_0(x\langle u, V^{(r)}(s)\rangle) dM(x,s) \\ &\quad = \int_{\alpha J\times[0,t]} \psi_0(x\langle u, V(s)\rangle) dM(x,s). \end{aligned}$$

This proves the Lemma. Q.E.D.

<u>Theorem 5.1.</u> Let $V$ be a nonrandom Borel-measurable $R^p$-valued function on $[0,t]$ such that for some $\alpha > 0$.

$$\int_{\alpha J\times[0,t]} n(x\|V(s)\|) dM(x,s) < \infty.$$

Then the ch.f. of $\int_0^t V d\zeta$ is given by the expression

$$\exp\{i\langle u, \int_0^t V db_\alpha\rangle + \int_{R\times[0,t]} \psi_\alpha(x, \langle u, v(s)\rangle) dM(x,s)\}$$

where

$$\psi_\alpha(x, \langle u, V(s)\rangle) = e^{ix\langle u, V(s)\rangle} - 1 - iI_{\alpha J}(x) x\langle u, V(s)\rangle.$$

<u>Proof.</u> This Theorem follows immediately from Lemmas 5.2, 5.4, and the independence of $\int_0^t V d\zeta_\alpha$ and $\int_0^t V d\zeta'_\alpha$. Q.E.D.

<u>Corollary 1</u>. Let $V$ be a Borel-measurable nonrandom $\mathbf{R}^p$-valued function on $[0,\infty)$, such that for $\alpha>0$, $\int_{\alpha J\times[0,t]} n(x\|V(s)\|)dM(x,s) < \infty$. Let $T_t\colon \mathbf{R}\times[0,t) \to \mathbf{R}^p$ be defined by the equation $T_t(x,s) = xV(s)$. Then $\int_0^t V d\zeta$, $t \geq 0$, is a stochastic process with independent increments and no Gaussian component. The Levy measure of $\int_0^t V d\zeta$ is $MT_t^{-1}$.

<u>Proof</u>. Note that for $a, h \geq 0$, and $t \geq a+h$, the increment

$$\int_0^{a+h} V d\zeta - \int_0^a V d\zeta = \int_0^t I_{(a,a+h]}(s)V(s)d\zeta$$

has the ch.f. (assume $b_\alpha(t) \equiv 0$ for simplicity)

$$\exp\{\int_{\mathbf{R}\times[0,t]} e^{i\langle u,xI_{(a,a+h]}(s)V(s)\rangle} -1 - iI_{\alpha J}(x)\langle u,x\, I_{(a,a+h]}(s)V(s)\rangle dM(x,s)$$

$$= \exp\{\int_{\mathbf{R}\times(a,a+h]} e^{i\langle u,xV(s)\rangle} -1 - I_{\alpha J}(x)\langle u,xV(s)\rangle dM(x,s)\}$$

The independent increment property thus follows easily from the expression for the ch.f. of $\int_0^t V d\zeta$, given in Theorem 5.1.

Next we check that $MT_t^{-1}$ is a Levy measure. It suffices to show that $\int_{\mathbf{R}^p} \|x\|^2\wedge 1\; dMT_t^{-1} < \infty$. Observe that by a change of variables

$$\int_{\mathbf{R}^p} \|x\|^2\wedge 1\; dMT_t^{-1} = \int_{\mathbf{R}\times[0,t]} \|xV(s)\|^2\wedge 1\; dM(x,s).$$

It is easy to check that $\|xV(s)\|^2\wedge 1 \leq n(xV(s))$. If $\|xV(s)\| \leq 1$, $\|xV(s)\|^2\wedge 1 = n(x\|V(s)\|)$; if $\|xV(s)\| > 1$, $\|xV(s)\|^2\wedge 1 = 1 \leq \|xV(s)\| = n(x\|V(s)\|)$. Hence

$$\int_{\mathbb{R}\times[0,t]} \|xV(s)\|^2 \wedge 1 dM(x,s) \le \int_{\alpha J\times[0,t]} n(x\|V(s)\|)dM(x,s) + \int_{\alpha J^c\times[0,t]} 1dM(x,s)$$

$$< \infty.$$

This proves that $MT_t^{-1}$ is a Levy measure.

To see that $MT_t^{-1}$ is the Levy measure of $\int_0^t Vd\zeta$, write the ch.f. of $\int_0^t Vd\zeta$ in the form

$$\exp\{c_\alpha(t) + \int_{R\times[0,t]} e^{i<u,xV(s)>} - 1 - iI_{\{y:\|y\|\le 1\}}(xV(s))<u,xV(s)>dM(x,s)$$

and then make a change of variables to get the expression

$$\exp\{c_\alpha(t) + \int_{R^p} e^{i<u,y>} - 1 - iI_{\{y:\|y\|\le 1\}}(y)<u,y>dMT_t^{-1}(y)\}$$

for the ch. f.

It is clear from the above expression for the ch.f. that $\int_0^t Vd\zeta$ has no Gaussian component.

Q.E.D.

<u>Lemma 5.5</u>. If $v_1, \ldots, v_p$ are any $p$ real numbers and if $v = \begin{pmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_p \end{pmatrix}$, then there is a positive constant $c_p$ which does not depend on $v$ such that

$$n(\|v\|) \le c_p \sum_1^p n(v_i).$$

<u>Proof</u>. First assume that $\|v\| > \sqrt{p}$. Since $\|v\| \le \sqrt{p} \max |v_i|$, there is some $i_*$ such that $|v_{i_*}| \ge 1$, and $|v_{i_*}| = \max_i |v_i|$. Then

$$\max_i |v_i| = |v_{i_*}| = n(v_{i_*}) \le \sum_1^p n(v_i)$$

so $n(\|v\|) = \|v\| \le \sqrt{p} \sum_1^p n(v_i)$ in this case.

Next observe that if $\|v\| \leq 1$, then for each $i$, $|v_i| \leq 1$. Hence

$n(\|v\|) = \|v\|^2 = \sum_1^p v_i^2 = \sum_1^p n(v_i)$ in this case.

Finally note that the function $r(v) = \sum_1^p n(v_i)/n(\|v\|)$ is continuous on $\mathbb{R}^p \setminus \{0\}$ so if $m = \min\{r(v): 1 \leq \|v\| \leq \sqrt{p}\}$, $m > 0$. Let $c_p^{-1} = \min(\frac{1}{\sqrt{p}}, m)$.

Q.E.D.

<u>Corollary 2</u>. Let $V_1, \ldots, V_p$ be real-valued Borel-measurable nonrandom functions on $[0,t]$ such that for some $\alpha > 0$

$$\int_{\alpha J \times [0,t]} n(xV_i(s))dM(x,s) < \infty, \quad i = 1, \ldots, p.$$

Then the stochastic integrals $\int_0^t V_1 d\zeta, \ldots, \int_0^t V_p d\zeta$ are independent iff $M(\mathbb{R} \times A) = 0$ where $A = \{s \in [0,t]:$ for some $i$ and $j$, $V_i(s)V_j(s) \neq 0\}$.

<u>Proof</u>. Let $V(s) = \begin{pmatrix} V_1(s) \\ \cdot \\ \cdot \\ \cdot \\ V_p(s) \end{pmatrix}$. Then by Lemma 2.6,

$$\int_{\alpha J \times [0,t]} n(x\|V(s)\|)dM(x,s) \leq c_p \sum_1^p \int_{\alpha J \times [0,t]} n(xV_i(s))dM(x,s)$$

$$< \infty.$$

Thus $V$ satisfies the hypothesis of Cor. 1, and hence $\int_0^t V d\zeta$ exists and is infinitely divisible with no Gaussian component and with Levy measure $MT^{-1}$ where $T: R \times [0,t] \to R^p$ is defined by the equation $T(x,s) = xV(s)$. The integrals $\int_0^t V_1 d\zeta, \ldots, \int_0^t V_p d\zeta$ are the marginals of $\int_0^t V d\zeta$, and thus, as is easy to see, they are independent iff $MT^{-1}$ is concentrated on the coordinate axes. That is, if $D = \{v \in \mathbb{R}^p:$ for some $i$ and $j$, $v_i v_j \neq 0\}$, then $MT^{-1}(D) = 0$. But $T(x,s) \in D$ iff for some $i$ and $j$, $x^2 V_i(s) V_j(s) \neq 0$. Since $M_t\{0\} = 0$, $MT^{-1}(D) = 0$ iff $M(\mathbb{R} \times A) = 0$. Q.E.D.

<u>Remark</u>. The expressions given in Theorem 5.1 for the ch.f. seem to be natural extensions of the case of simple V. If $p = 1$, the integral

$\int_{\alpha J \times [0,t]} e^{i\langle u, xV(s)\rangle} - 1 - iI_{\alpha J}(x) \langle u, xV(s)\rangle dM(x,s)$ exists for all $u$ iff

$\int_{\alpha J \times [0,t]} n(xV(s))dM(x,s) < \infty$. (Prop. 5.3) If $p > 1$, then the corresponding

expressions for the marginals of $\int_0^t V d\zeta$ make sense iff

$\int_{\alpha J \times [0,t]} n(xV_i(s))dM(x,s) < \infty$ for each $i$ and some $\alpha > 0$. But according

to Lemma 5.5, the latter condition implies that $\int_{\alpha J \times [0,t]} n(x\|V(s)\|)dM(x,s) < \infty$.

Thus in this connection, at least for nonrandom V, it seems natural to require that

$$\int_{\alpha J \times [0,t]} n(x\|V(s)\|)dM(x,s) < \infty$$

for V to be integrable.

<u>Acknowledgement</u>. The author is grateful to Professor Stamatis Cambanis for suggesting this project and for many valuable comments and suggestions.

## References

1. Dubins, L. E. and Savage, L. J. (1965). A Tchebycheff-like inequality for stochastic processes. Proc. Nat. Acad. Sci. U.S.A. 53, 274-275.

2. Kallenberg, O. (1975). On the existence and path properties of stochastic integrals. Ann. Prob. 3, 262-280.

3. Meyer, P.A. (1966). Probability and Potentials. Blaisdell Publishing Co. Waltham, MA.

4. Millar, P. W. (1971). Path behavior of processes with stationary independent increments. Z. Wharscheinlichkeitstheorie und Verw. Gebiete 17, 53-73.

5. Millar, P. W. (1972). Stochastic integrals and processes with stationary independent increments. Proc. Sixth Berkeley Symposium Math. Statist. Prob. 3, 307-331. Univ. of California Press, Berkeley.

6. Prekopa, A. (1956) Acta. Math. Acad. Scient. Hung. 7, 215-262.

7. Prekopa, A. (1957) Acta. Math. Acad. Scient. Hung. 8, 337-374.

8. Prekopa, A. (1957) Acta. Math. Acad. Scient. Hung. 8, 375-400.

9. Rosinski, J. and Woyczynski, W. A. (1984), On Ito stochastic integration with respect to p-stable motion: inner clock, integrability of sample paths, double and multiple integrals. Preprint.

10. Urbanik, K. and Woyczynski, W.A. (1967) A Random integral and Orlicz spaces. Bull. Acad. Polon. Sci. Ser. Sci. math. astrom. and phys. 15, 161-169.

END

FILMED

10-85

DTIC