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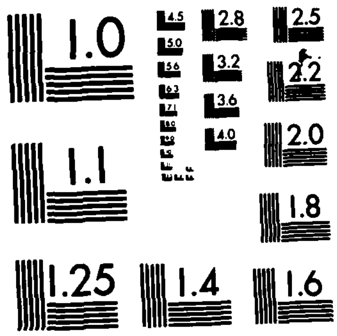

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ON THE MATRIX CONVEXITY OF THE MOORE-PENROSE INVERSE AND SOME APPLICATIONS

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Abstract

It is well known that if A and B are two positive definite matrices of the same order and $0 \le \lambda \le 1$, then

$$[\lambda A + (1-\lambda)B]^{-1} \le \lambda A^{-1} + (1-\lambda)B^{-1}.$$

It is easy to construct an example consisting of two positive semi-definite matrices for which the above inequality is not true when one replaces the inverse operation by Moore-Penrose inverse operation. In this paper, we give necessary and sufficient conditions for the validity of ~~the~~ an inequality Keywords: theorems; random matrices

$$[\lambda A + (1-\lambda)B]^{+} \le \lambda A^{+} + (1-\lambda)B^{+}$$

for every $0 \le \lambda \le 1$. As an application, we give a sufficient condition under which the inequality $(EA)^{+} \le E(A^{+})$ is valid, where A is a square matrix of random variables which is almost surely positive semi-definite, generalizing the well-known result $(EA)^{-1} \le EA^{-1}$ when A is almost surely positive definite.

AMS subject classification: Primary 15A09, 15A45 Secondary 62H99

Key Words and Phrases: g-inverse, Moore-Penrose inverse, optimal designs, random matrices

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1. Introduction:

Fedorov (1972, Theorem 1.1.12, p. 19) states that if A and B are positive definite matrices and $0 \leq \lambda \leq 1$ is any real number, then

$$[\lambda A + (1-\lambda)B]^{-1} \leq \lambda A^{-1} + (1-\lambda)B^{-1}. \tag{1}$$

(We say that, for two square matrices C and D of the same order, $C \leq D$ if $D - C$ is positive semi-definite.) For a proof of this inequality, see Moore (1973, p. 408) or Marshall and Olkin (1979, pp. 469-471).

The above inequality is useful in optimal designs and, especially, in linear optimal designs. This inequality is used in Lemma 2.9.1 of Fedorov (1972, p. 123]. Let $\mathcal{D}$ be the collection of all square matrices of order q and L a real linear functional on $\mathcal{D}$, i.e.,

$$L(A+B) = L(A) + L(B) \text{ for every } A,B \in \mathcal{D}, \tag{2}$$

and

$$L(cA) = cL(A) \text{ for every c real and } A \in \mathcal{D}. \tag{3}$$

Assume, further, that

$$L(A) \geq 0 \text{ if A is positive semi-definite.} \tag{4}$$

Consider an optimal design problem involving q parameters. Let $\mathcal{M}$ be the collection of all information matrices and $\mathcal{M}_1$ the collection of all non-singular information matrices. Let L be a linear functional satisfying the above three conditions. Lemma 2.9.1 of Fedorov [1972, p. 123] says that the function

$$L_1: \mathcal{M}_1 \to R$$

defined by $L_1(M) = L(M^{-1})$, $M \in \mathcal{M}_1$ is a convex function on $\mathcal{M}_1$, i.e.

A-1

$$L_1[\lambda M_1 + (1-\lambda)M_2] \leq \lambda L_1(M_1) + (1-\lambda)L_1(M_2),$$

for any M_1, $M_2 \in \mathcal{M}_1$ and $0 \leq \lambda \leq 1$. This is a simple consequence of the inequality (1) and the conditions (2), (3) and (4).

In Remark 1 on page 124, Fedorov comments that if $M \in \mathcal{M}$ is singular, one can consider Moore-Penrose inverse M^+ of M and define $L_1(M) = L(M^+)$. See also Remark 1 to Theorem 2.7.1. In other words, if we define $L_1: \mathcal{M} \to R$ by

$$L_1(M) = L(M^+),\ M^+ \text{ being the Moore-Penrose of } M,\ M \in \mathcal{M},$$

his remarks seem to mean that L_1 is a convex function on $\mathcal{M}$. We show that this is not true in general.

In this connection, we ask the following question. Let A and B be two positive semi-definite matrices of the same order and $0 \leq \lambda \leq 1$ be any real number. Is the inequality

$$[\lambda A + (1-\lambda)B]^+ \leq \lambda A^+ + (1-\lambda)B^+, \tag{5}$$

analogous to (1), true?

The organization of this paper is as follows. In Section 2, we give a necessary and sufficient condition for (5) to be valid for every $0 \leq \lambda \leq 1$. In Section 3, we we study this inequality in the context of a collection of positive semi-definite matrices indexed by a probability space. In particular, we examine under what conditions $(EA)^+ \leq EA^+$ when A is a symmetric matrix of random variables such that A is almost surely positive semi-definite.

For any matrix A, range of A is defined to be the linear space spanned by the columns of A and it is denoted by $\mathcal{R}(A)$. A^- denotes an arbitrary g-inverse of A, i.e. a matrix satsifying $AA^-A = A$. For basic ideas concerning Moore-Penrose inverse, see Rao and Mitra (1971, pp 50-53).

2. Convexity of the Moore-Penrose Inverse:

The following result gives conditions under which (5) holds for every $0 \leq \lambda \leq 1$.

Theorem 1. Let A and B be two real positive semi-definite matrices of the same order. Then the following are equivalent:

(i) $R(A) = R(B)$.

(ii) There exist positive semi-definite g-inverses A^- and B^- of A and B respectively such that $[\lambda A + (1-\lambda)B]^- \leq \lambda A^- + (1-\lambda)B^-$ for some positive semi-defintie g-inverse $[\lambda A + (1-\lambda)B]^-$ of $\lambda A + (1-\lambda)B$ and for every $0 \leq \lambda \leq 1$.

(iii) $[\lambda A + (1-\lambda)B]^+ \leq \lambda A^+ + (1-\lambda)B^+$ for every $0 \leq \lambda \leq 1$.

Proof: The proof of (i) => (iii) is similar to the proof of the remark in Giovagnoli and Wynn (1985, p. 129). Let P be an orthogonal matrix such that $A = P\,\mathrm{diag}(A_1, 0)P^T$, where A_1 is a diagonal positive definite matrix. When $R(A) = R(B)$, we then have $B = P\,\mathrm{diag}(B_1, 0)P^T$ where B_1 is positive definite. To show that (iii) holds, we have to show that $[\lambda A_1 + (1-\lambda)B_1]^{-1} \leq \lambda A_1^{-1} + (1-\lambda)B_1^{-1}$, which is true since A_1 and B_1 are positive definite. (iii) => (ii) is obvious. We shall now prove (ii) => (i). Suppose $[\lambda A + (1-\lambda)B]^- \leq \lambda A^- + (1-\lambda)B^-$ for positive semi-definite g-inverses A^- and B^- (independent of λ) and a positive semi-definite g-inverse $[\lambda A + (1-\lambda)B]^-$ for every λ as specified in the theorem. Premultiply and postmultiply the above by $\lambda A + (1-\lambda)B$ yielding $\lambda A + (1-\lambda)B \leq [\lambda A + (1-\lambda)B][\lambda A^- + (1-\lambda)B^-][\lambda A + (1-\lambda)B]$. If $R(A) \neq R(B)$, assume without loss of generality that $R(A)$ is not contained in $R(B)$, in which case there exists a vector b satisfying $Ab \neq 0$ and $Bb = 0$. Premultiplying and postmultiplying the above inequality by b^T and b respectively lead to $\lambda b^T Ab \leq \lambda^3 b^T Ab + \lambda^2(1-\lambda)b^T AB^- Ab$ or equivalently $b^T Ab \leq \lambda(b^T AB^- Ab - b^T Ab)$, when $0 < \lambda < 1$. But, since $b^T Ab > 0$, the above inequality cannot hold for all $0 < \lambda < 1$. This completes the proof.

Remark 1. From Theorem 1 it follows that when A is positive semi-definite the function $A \to A^+$ is matrix convex iff A varies over a set of positive semi-definite matrices with the same range. The 'if' part of this assertion is proved in Giovagnoli and Wynn (1985).

Remark 2. For a given λ with $0<\lambda<1$, one can always find positive semi-definite g-inverses $[\lambda A+(1-\lambda)B]^-$, A^- and B^- (depending on λ) and satisfying $[\lambda A+(1-\lambda)B]^- \leq \lambda A^- + (1-\lambda)B^-$ even though $R(A)$ and $R(B)$ are different. This could be seen as follows. Let P be a nonsingular matrix satisfying $A = P\,\mathrm{diag}(\Delta_1,\Delta_2,0,0)P^T$ and $B = P\,\mathrm{diag}(D_1,0,D_2,0)P^T$ where $\Delta_1,\Delta_2,D_1,D_2$ are diagonal positive definite matrices. The existence of such a P is guaranteed by Theorem 6.2.3 in Rao and Mitra (1971). For a given $0<\lambda<1$, consider the g-inverses $[\lambda A+(1-\lambda)B]^- = (P^T)^{-1}\mathrm{diag}[(\lambda\Delta_1+(1-\lambda)D_1)^{-1}, (\lambda\Delta_2)^{-1},((1-\lambda)D_2)^{-1},0]P^{-1}$, $A^- = (P^T)^{-1}\mathrm{diag}(\Delta_1^{-1},\Delta_2^{-1},M,0)P^{-1}$ and $B^- = (P^T)^{-1}\mathrm{diag}(D_1^{-1},N,D_2^{-1},0)P$ where M and N are positive semi-definite matrices satisfying $(\lambda\Delta_2)^{-1} \leq N+\lambda\Delta_2^{-1}$ and $((1-\lambda)D_2)^{-1} \leq M+(1-\lambda)D_2^{-1}$. With such a choice of M and N, it can be verified that $[\lambda A+(1-\lambda)B]^- \leq \lambda A^- + (1-\lambda)B^-$.

Corollary 1. Let $A_1,\ldots,A_k$ be k real positive semi-definite matrices. Then $(\lambda_1A_1+\ldots+\lambda_kA_k)^+ \leq \lambda_1A_1^+ +\ldots+\lambda_kA_k^+$ for every λ_i satisfying $0\leq\lambda_i\leq 1$ $(i=1,2,\ldots,k)$ and $\sum_{i=1}^{k}\lambda_i = 1$ iff $R(A_i) = R(A_j)$ for $i,j = 1,2,\ldots,k$.

Proof. We shall prove the result for $k=3$. The proof in the general case follows along similar lines, by induction. Assume $\lambda_1<1$, $R(A_1)=R(A_2)=R(A_3)$. Then

$$(\lambda_1A_1+\lambda_2A_2+\lambda_3A_3)^+ = [\lambda_1A_1+(1-\lambda_1)(\frac{\lambda_2}{1-\lambda_1}A_2+\frac{\lambda_3}{1-\lambda_1}A_3)]^+ \leq \lambda_1A_1^+ + (1-\lambda_1)(\frac{\lambda_2}{1-\lambda_1}A_2+\frac{\lambda_3}{1-\lambda_1}A_3)^+$$

(applying Theorem 1). Since $\frac{\lambda_2}{1-\lambda_1}+\frac{\lambda_3}{1-\lambda_1} = 1$, applying Theorem 1 again, we get

$$(\lambda_1A_1+\lambda_2A_2+\lambda_3A_3)^+ \leq \lambda_1A_1^+ + (1-\lambda_1)[\frac{\lambda_2}{1-\lambda_1}A_2^+ + \frac{\lambda_3}{1-\lambda_1}A_3^+] = \lambda_1A_1^+ + \lambda_2A_2^+ + \lambda_3A_3^+,$$

which concludes the proof of the 'if' part. To prove the 'only if' part, choose $\lambda_3 = 0$. Then from Theorem 1, we get $R(A_1) = R(A_2)$. Similarly $R(A_i) = R(A_j)$ for all i,j.

Corollary 2. Let A_n, $n\geq 1$ be a sequence of positive semi-definite matrices and λ_n, $n\geq 1$ be a sequence of nonnegative real numbers such that (i) $\sum_{i\geq 1}\lambda_i = 1$ (ii) $\sum_{i\geq 1}\lambda_iA_i$ converges (iii) $\sum_{i\geq 1}\lambda_iA_i^+$ converges. Then $(\sum_{i\geq 1}\lambda_iA_i)^+ \leq \sum_{i\geq 1}\lambda_iA_i^+$ for every such sequence λ_n $n\geq 1$ iff $R(A_n)$ is the same for all $n\geq 1$.

Proof. The 'only if' part is proved as in Corollary 1. To prove the 'if' part, assume that $R(A_i)$ is the same for all i. Let $B_n = \sum_{i=1}^{n}\lambda_i A_i$ and $B = \sum_{i\geq 1}\lambda_i A_i$. Let C be a matrix having the same range as the A_i's and let $\delta_n = 1 - \sum_{i=1}^{n}\lambda_i$. If we assume that at least one λ_i is positive $(1\leq i\leq n)$ in B_n, then $R(B_n+\delta_n C) = R(B_n) = R(B)$. Since $B_n + \delta_n C \to B$ as $n\to\infty$, following the argument given in Stewart (1969, p. 34) (see also Campbell and Meyer, 1979, Chapter 10), we see that $(B_n + \delta_n C)^+ \to B^+$ as $n\to\infty$. Applying Corollary 1, we get $(B_n + \delta_n C)^+ \leq \lambda_1 A_1^+ + \ldots + \lambda_n A_n^+ + \delta_n C^+$. The result now follows by taking limits as $n\to\infty$.

Remark 3. The results in this section proved for real symmetric positive semi-definite matrices are also valid for complex hermitian positive semi-definite matrices, with obvious modifications in the proofs.

3. Some Extensions and Applications:

In this section, we consider the problem of extending the inequality specialized in Seciton 2 for a collection of positive semi-definite matrices indexed by a probability space.

Let $(Y,\mathcal{B},\mu)$ be a probability space and A_y, $y\in Y$ a collection of positive semi-definite matrices of the same order. Let $A_y = ((a_{ijy}))$, $1\leq i, j\leq n$ and $y\in Y$. Assume that a_{ijy} as a function of y is measurable for every $1\leq i, j\leq n$. There are three basic questions one can ask in this connection.

(a) Let $A_y^+ = (b_{ijy})$, $1\leq i, j\leq n$, $y\in Y$. Is b_{ijy} as a function of y measurable for every $1\leq i, j\leq n$?

(b) If the answer to (a) is affirmative and each a_{ijy} is integrable with respect to the measure μ, is each b_{ijy} integrable with respect to μ?

(c) If the answers to (a) and (b) are affirmative, is the inequality

$$\left(\int_Y A_y\mu(dy)\right)^+ \leq \int_Y A_y^+\mu(dy)$$

true?

We, first, tackle (a). We give two sets of sufficient conditions under which A_y^+ as a function of y is measurable.

Theorem 2.

(a) Suppose $R(A_y)$ is the same for all $y \in D \in \mathcal{B}$ with $\mu(D) = 1$. Then A_y^+ as a function of y is measurable.

(b) Suppose Y is a topological space and $\mathcal{B}$ is some σ-field on Y containing all open subsets of Y. Suppose $R(A_y)$ is the same for all y in Y. If A_y as a function of y is continuous, then A_y^+ as a function of y is continuous.

Proof: Let A be any symmetric matrix with $R(A_y) = R(A)$ for every y in D. There exists an orthogonal matrix P such that $PAP^T = \mathrm{diag}(A_*,0)$, where A_* is a diagonal matrix with diagonal entries being the non-zero eigen values of A. Since Range (A_y) = Range (A), $y \in D$, $P A_y P^T = \mathrm{diag}\,(A_{y*},0)$ for some nonsingular matrix A_{y*} which is of same order as A_*. Note that $A_y^+ = P^T \mathrm{diag}((A_{y*})^{-1},0)P$. If A_y as a function of y is measurable (continuous) so is A_{y*} as a function of y. Consequently, $(A_{y*})^{-1}$ as a function of y is measurable (continuous). Hence A_y^+ as a function of y is measurable (continuous).

Theorem 3.

(a) Suppose there exists a set $D \in \mathcal{B}$ such that $\mu(D) = 1$ and $A_{y_1}A_{y_2} = A_{y_2}A_{y_1}$ for every $y_1, y_2 \in D$. Then A_y^+ as a function of y is measurable.

(b) Suppose Y is a topological space and $\mathcal{B}$ is a σ-field on Y containing all open subsets of Y. Suppose $A_{y_1}A_{y_2} = A_{y_2}A_{y_1}$ for all $y_1, y_2 \in Y$. If A_y as a function of y is continuous, then A_y^+ as a function of y is continuous.

We need the following lemma in the proof of the above theorem.

Lemma 1. Let $\{A_y\colon\ y \in Y\}$ be a family of pairwise commuting symmetric matrices of order $n \times n$. Then there exists an orthogonal matrix C such that

$$C^T A_y C = \text{diag}\{\lambda_{1y}, \lambda_{2y}, \ldots, \lambda_{ny}\},$$

where $\lambda_{1y}, \lambda_{2y}, \ldots, \lambda_{ny}$ are the eigenvalues of A_y

Proof: This result is well known when Y is finite. See, for example, Rao (1973, Exercise 15, p. 72). Let Ω be the least ordinal number corresponding to the cardinal number of Y. Ω is obviously a limit ordinal. See Kamke (1950).

Let us identify Y with $[0,\Omega)$. In other words, the given family of matrices can be written down as a generalized sequence $A_0, A_1, A_2, \ldots, A_\alpha, \ldots,\ \alpha < \Omega$.

We claim that there exists a vector $z^{(1)}$ of unit length such that it is an eigenvector for every $A_\alpha,\ \alpha < \Omega$. For this, we proceed as follows.

Let λ_{10} be any eigenvalue of A_0. Let $0 < \alpha < \Omega$ be any ordinal number. Then, there exists a vector $x^{(\alpha)}$ of unit length and real numbers $\lambda_{1\beta}$, $\beta < \alpha$, satisfying the following properties.

(i) $\lambda_{1\beta}$ is an eigenvalue of A_β for every $0 \leq \beta < \alpha$.

(ii) $A_\beta x^{(\alpha)} = \lambda_{1\beta} x^{(\alpha)}$ for every $0 \leq \beta < \alpha$.

(iii) $\lambda_{1\beta}$, $0 \leq \beta < \alpha$ does not depend on α.

This $x^{(\alpha)}$ is obtained by transfinite induction as follows.

There exists a vector $x^{(1)}$ of unit length such that $A_0 x^{(1)} = \lambda_{10} x^{(1)}$. Note that $A_1 x^{(1)}$, $A_1^2 x^{(1)}, \ldots$ are eigenvectors of A_0 corresponding to the same eigenvalue λ_{10}. Consequently, every vector in the linear manifold spanned by $\{x^{(1)}, A_1 x^{(1)}, A_1^2 x^{(1)} \ldots\}$ is an eigenvector of A_0 corresponding to the eigenvalue λ_{10}. This linear manifold contains an eigenvector of A_1. See Rao (1973, p. 39). Let us assume that this eigenvector $x^{(2)}$, say, is of unit length and the corresponding eigenvalue for A_1 be λ_{11}. Thus at the second stage, we have

$$A_0 x^{(2)} = \lambda_{10} x^{(2)}$$

$$A_1 x^{(2)} = \lambda_{11} x^{(2)}.$$

Now, since A_0 and A_2 commute and also A_1 and A_2 commute, every vector in the linear mainfold spanned by $\{x^{(2)}, A_2x^{(2)}, A_2^2x^{(2)}, \ldots\}$ is an eigenvector of A_0 corresponding to the same eigenvalue λ_{10} and also is an eigenvector of A_1 corresponding to the same eigenvalue λ_{11}. This manifold contains an eigenvector $x^{(3)}$ of A_2. Assume $x^{(3)}$ to be of unit length and λ_{12} to be the corresponding eigenvalue of A_2. Thus we have

$$A_0x^{(3)} = \lambda_{10}x^{(3)}$$

$$A_1x^{(3)} = \lambda_{11}x^{(3)}$$

$$A_2x^{(3)} = \lambda_{12}x^{(3)}.$$

Continuing this procedure for every $n < w$, where w is the first infinite ordinal number, we find a sequence $x^{(n)}$, $1 \leq n < w$, of vectors of unit length and a sequence λ_{1k}, $0 \leq k < w$ of real numbers satisfying the following property:

$$A_kx^{(n)} = \lambda_{1k}x^{(n)}, \quad 0 \leq k < n.$$

It is important to note that once an eigenvalue enters into the system, it remains in the system at every stage of the induction process.

Since each $x^{(n)}$, $1 \leq n < w$ is of unit length, by compactness argument, this sequence admits a convergent subsequence converging to $x^{(w)}$, say. Obviously, this vector is of unit length. Further,

$$A_kx^{(w)} = \lambda_{1k}x^{(w)} \text{ for } 0 \leq k < w.$$

Now, every vector in the linear manifold spanned by $\{x^{(w)}, A_wx^{(w)}, A_w^2x^{(w)} \ldots\}$ is an eigenvector of A_k, $0 \leq k < w$, corresponding to the eigenvalue λ_{1k}, $0 \leq k < w$. But this manifold contains an eigenvector $x^{(w+1)}$ of A_w. Let us assume this vector to be of unit length and let the corresponding eigenvalue of A_w be λ_{1w}. Thus we have

$$A_k x^{(w+1)} = \lambda_{1k} x^{(w+1)}, \quad 0 \leq k < w+1.$$

This process is continued arguing separately for the case of limit ordinals and the case of non-limit ordinals.

Now, by compactness argument, $x^{(\alpha)}$, $\alpha < \Omega$ admits a subnet converging to a vector $z^{(1)}$ of unit length. This vector is the desired one.

Now, we claim that there exists a vector $z^{(2)}$ of unit length such that $z^{(2)} \perp z^{(1)}$ and $z^{(2)}$ is a common eigenvector for each A_α, $0 \leq \alpha < \Omega$. Let λ_{20} be an eigenvalue of A_0 admitting an eigenvector $y^{(1)}$ such that $y^{(1)}$ is of unit length and $y^{(1)} \perp z^{(1)}$. Let $0 < \alpha < \Omega$. We claim that there exists a vector $y^{(\alpha)}$ of unit length and real numbers $\lambda_{2\beta}$, $0 \leq \beta < \alpha$, satisfying the following properties.

(i) $\lambda_{2\beta}$ is an eigenvalue of A_β.

(ii) $A_\beta y^{(\alpha)} = \lambda_{2\beta} y^{(\alpha)}$ for every $0 \leq \beta < \alpha$.

(iii) $y^{(\alpha)} \perp z^{(1)}$.

(iv) $\lambda_{2\beta}$, $0 \leq \beta < \alpha$ is independent of α.

The $y^{(\alpha)}$'s and $\lambda_{2\beta}$'s are obtained by transfinite induction as follows. At the first step, for $\alpha = 1$, we have $y^{(1)}$ and λ_{20} satisfying (i) through (iv). Let $\alpha = 2$. The linear manifold spanned by $\{y^{(1)}, A_1 y^{(1)}, A_1^2 y^{(1)}, \ldots\}$ contains an eigenvector $y^{(2)}$ for A_1 with the corresponding eigenvalue, say, λ_{21}. Since A_0 and A_1 commute, every vector in this manifold is an eigenvector of A_0 corresponding to the eigenvalue λ_{20}. Without loss of generality we can assume $y^{(2)}$ to be of unit length. Further, $y^{(2)} \perp z^{(1)}$. To prove this, consider $A_1^n y^{(1)}$. We have

$$(A_1^n y^{(1)})^T z^{(1)} = y^{(1)T} A_1^n z^{(1)} = y^{(1)T} (\lambda_{11})^n z^{(1)} = (\lambda_{11})^n y^{(1)T} z^{(1)} = 0$$

Consequently, every vector in the linear manifold spanned by $\{y^{(1)}, A_1 y^{(1)}, A_1^2 y^{(1)}, \ldots\}$ is orthogonal to $z^{(1)}$. Hence $y^{(2)} \perp z^{(1)}$. Thus, we have a vector $y^{(2)}$ of unit length satisfying

$$A_\beta y^{(2)} = \lambda_{2\beta} y^{(2)}, \quad 0 \leq \beta < \alpha$$

and

$$y^{(2)} \perp z^{(1)}.$$

This process is continued as in the first part of this proof noting that once an eigenvalue $\lambda_{2\beta}$ enters the system it remains in the system. By compactness of the unit ball of R^n, we can find a subnet of $y^{(\alpha)}$, $0 \leq \alpha < \Omega$ converging to a vector, say, $z^{(2)}$. This $z^{(2)}$ is the desired vector.

Thus, we can obtain n vectors $z^{(1)}, z^{(2)}, \ldots, z^{(n)}$ satisfying the following properties

(a) $\| z^{(i)} \| = 1$, $i = 1$ to n.

(b) $z^{(i)} \perp z^{(j)}$, $i \neq j$.

(c) $A_\alpha z^{(i)} = \lambda_{i\alpha} z^{(i)}$, $0 \leq \alpha < \Omega$, $i = 1$ to n.

Define $C = (z^{(1)}, z^{(2)}, \ldots, z^{(n)})$. C is the required orthogonal matrix.

<u>Proof of Theorem 3</u>: By Lemma 1, there exists an orthogonal matrix $C = (z^{(1)}, z^{(2)}, \ldots, z^{(n)})$ such that

$$C^T A_y C = \mathrm{Diag}\{\lambda_{1y}, \lambda_{2y}, \ldots, \lambda_{ny}\}, \ y \in Y,$$

where $\lambda_{1y}, \lambda_{2y}, \ldots, \lambda_{ny}$ are the eigenvalues of A_y. Let $f_i\colon Y \to R$ be defined by

$$f_i(y) = \lambda_{iy}, \ y \in Y, \quad i = 1 \text{ to } n.$$

It is easy to check that each f_i is measurable. For, $f_i(y) = z^{(i)^T} A_y z^{(i)}$, a linear combination of the elements of A_y. Let $g_i\colon Y \to R$ be defined by

$$\begin{aligned} g_i(y) &= \frac{1}{f_i(y)} \quad \text{if } f_i(y) \neq 0 \\ &= 0 \qquad\quad \text{if } f_i(y) = 0, \ y \in Y, \ i = 1 \text{ to } n. \end{aligned}$$

$g_1, g_2, \ldots, g_n$ are, obviously, measurable functions. Now,

$$A_y = C \, \mathrm{Diag}\{f_1(y), f_2(y), \ldots, f_n(y)\} C^T.$$

Then $A_y^+ = C \, \mathrm{Diag}\{g_1(y), g_2(y), \ldots, g_n(y)\} C^T$ (see Rao and Mitra 1971, p. 69). Consequently, the elements of A_y^+ as functions on Y are measurable.

Now, we come to the question raised in (b). A_y^+ as a function of y need not be integrable. The following is a simple example. Let $Y = (0,1)$, $\mathcal{B}$ = Borel σ-field on Y, μ = Lebesgue measure on $\mathcal{B}$, and $A_y = (y)$, $y \in Y$, is of order 1×1. A_y as a function of y is integrable with respect to μ but A_y^+ is not.

The following result generalizes the inequality expounded in Section 2 and answers the query raised in (c).

Theorem 5 Let $\mathcal{R}(A_y)$ be the same for all $y \in D \in \mathcal{B}$ with $\mu(D) = 1$. Suppose A_y and A_y^+ as functions of y are integrable with respect to μ. Then

$$\left[\int_Y A_y \, \mu(dy)\right]^+ \leq \int_Y A_y^+ \, \mu(dy).$$

Proof: Let $\mathcal{D}$ be the collection of all positive semi-definite matrices of the same order $n\times n$ as that of A_y and range the same as that of A_y, $y \in D$. Then $\mathcal{D}$ is a closed convex subset of an appropriate finite-dimensional Euclidean space and the map $y \to A_y$ from Y to $\mathcal{D}$ is measurable. By Theorem 1, the map $A \to A^+$ from $\mathcal{D}$ to $\mathcal{D}$ is convex. Let $C \in R^n$ be an arbitrary but fixed vector. Then the map $f: \mathcal{D} \to R$ defined by $f(A) = C^T A^+ C$ is convex. By Jensen's inequality (see Ferguson (1967, p. 76)),

$$f(EA_{(\cdot)}) \leq Ef(A_{(\cdot)}), \text{ i.e.,}$$

$$C^T\left(\int_Y A_y \mu(dy)\right)^+ C \leq \int_Y C^T A_y^+ C \, \mu(dy)$$

$$= C^T\left(\int_Y A_y^+ \, \mu(dy)\right) C.$$

This implies that, as C is arbitrary,

$$\left(\int_Y A_y \, \mu(dy)\right)^+ \leq \int_Y A_y^+ \, \mu(dy).$$

This completes the proof.

The condition on the range in the above theorem, in a certain sense, is necessary for the inequlaity to be valid. If the above inequality is valid for all probability measures for which the concerned integrals are finite, then the above conditon on the range is necessary.

The above result can be couched in the language of random matrices as follows.

Corollary 3. Let A be a symmetric matrix of random variables such that A is positive semi-definite almost surely and $\mathcal{R}(A)$ is the same almost surely. Assume that EA and EA^+ exist. Then

$$(EA)^+ \leq EA^+.$$

The above inequality is an analogue of the usual Harmonic-Arithmetic inequality, namely, if f is an almost surely positive random variables with Ef and Ef^{-1} finite then $(Ef)^{-1} \leq Ef^{-1}$.

We also obtain as a corollary the following result due to Groves and Rothenberg (1969, p. 690). See also Srivastava (1970, p. 236).

Corollary 4. Let A be a symmetric matrix of random variables such that A is positive definite almost surely, and EA and EA^{-1} exist. Then

$$(EA)^{-1} \leq EA^{-1}.$$

Corollary 5. Let $Y_1, Y_2, \ldots, Y_N$ be a random sample of size N form a multivariate normal distribution with a singular variance covariance matrix Σ. Let $\bar{Y}$ be the sample mean and $S = \sum_{i=1}^{N} (Y_i - \bar{Y})(Y_i - \bar{Y})^T$. If $r = \text{rank}(\Sigma)$ and $N > r$, then $(ES)^+ \leq E(S^+)$.

Proof: It is known that $\mathcal{R}(S) \subset \mathcal{R}(\Sigma)$ and $\text{rank}(S) = \text{rank}(\Sigma)$ with probability 1 when $N > r$. Hence $\mathcal{R}(S) = \mathcal{R}(\Sigma)$ almost surely. The result now follows from Corollary 3.

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It is well known that if A and B are two positive definite matrices of the same order and $0 \le \lambda \le 1$, then $[\lambda A + (1-\lambda)B]^{-1} \le \lambda A^{-1} + (1-\lambda)B^{-1}$. It is easy to construct an example consisting of two positive semi-definite matrices for which the above inequality is not true when one replaces the inverse operation by Moore-Penrose inverse operation. In this paper, we give necessary and sufficient conditions for the validity of the inequality $[\lambda A+(1-\lambda)B]^{+} \le \lambda A^{+} + (1-\lambda)B^{+}$ for every $0 \le \lambda \le 1$. As an application, we give a sufficient condition under which.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) the inequality $(EA)^+ \leq E(A^+)$ is valid, where A is a square matrix of random variables which is almost surely positive semi-definite, generalizing the well-known result $(EA)^{-1} \leq EA^{-1}$ when A is almost surely positive definite.

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