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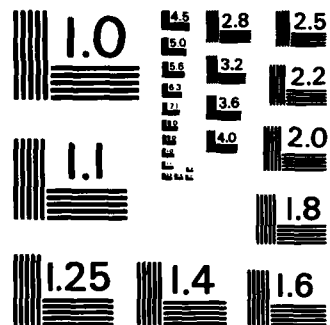
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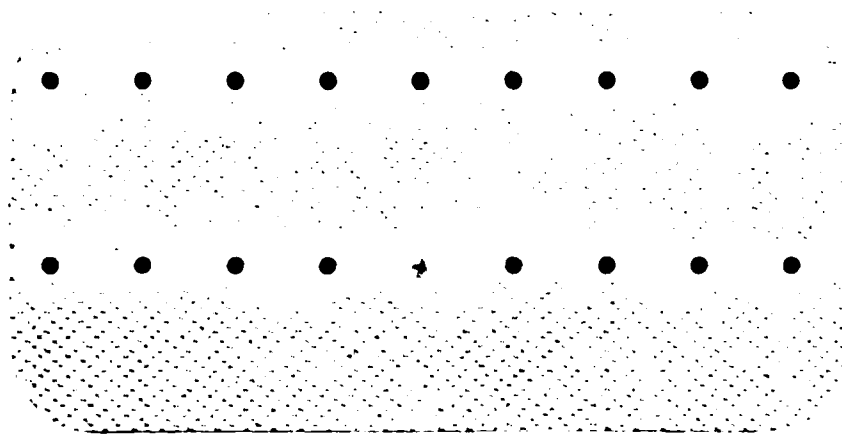
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UNIFIED METHOD FOR DELAY ANALYSIS OF RANDOM
MULTIPLE ACCESS ALGORITHMS

by

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In this paper, we present a unified method for the delay analysis of a large class of random multiple - access algorithms. Our method is based on a powerful theorem referring to regenerative processes, in conjunction with results from the theory of infinite dimensionality linear systems. We apply the method to analyze and compute the per packet expected delays induced by three algorithms, in the presence of the Poisson user model. The considered algorithms are: The controlled ALOHA algorithm, the "0.487" algorithm, and the n-ary stack algorithm.					
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UNIFIED METHOD FOR DELAY ANALYSIS OF RANDOM MULTIPLE ACCESS ALGORITHMS

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Abstract

In this paper, we present a unified method for the delay analysis of a large class of random multiple - access algorithms. ^{This} ~~Our~~ method is based on a powerful theorem referring to regenerative processes, in conjunction with results from the theory of infinite dimensionality linear systems.

~~We apply~~ ^{is applied} the method to analyze and compute the per packet expected delays induced by three algorithms, in the presence of the Poisson user model.

The considered algorithms are: The controlled ALOHA algorithm, the "0.487" algorithm, and the n-ary stack algorithm. *Additional keywords:*

Communication networks; Markovian models; throughput; delay

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1. INTRODUCTION

A key problem in the design of communication networks is the efficient sharing of a common transmission channel, (such as a satellite link, a ground radio channel, a computer bus, a coaxial cable, or an optical fibre) among a large population of network users. This problem is referred to as the multiple-access problem, since many independent users share, and, thus, access a common channel for transmission of information. The solution to the multiple-access problem must incorporate a distributed control scheme, termed multiple-access algorithm, for allocating the channel resources among the network users.

The design and performance of multiple-access algorithms are highly dependent on the nature of the users. When a channel is to support large numbers of bursty (low duty-cycle) users, random multiple-access algorithms (RMAAs) become more efficient than deterministic algorithms. This has been early recognized by the researchers in the field, and a plethora of RMAAs have been proposed during the past fifteen years [1,2].

The key performance measures of a RMAA are its throughput and delay characteristics. The evaluation of such characteristics has been the subject of numerous studies. In most cases, a Markovian model is employed, and the existence of steady state of the random-access system is related to the ergodicity of an underlying Markov process. Depending on the complexity of the state space of such a process, this formulation usually gives sufficient information on the maximum input traffic rate that an algorithm can maintain. However, the evaluation of the delay characteristics is a much harder problem, since they are intimately interwoven with the dynamical behavior of the algorithm's scheduling mechanisms. Due to this fact, it is not surprising that results concerning the delay characteristics are limited, and are obtained after a rather intricate and difficult analysis, which is usually matched to the

peculiarities of the specific algorithm at hand.

In this paper, we show how the delay analysis of RMAAs can be unified and simplified, by the use of some known results from the theory of regenerative processes, and the theory of infinite dimensional systems of linear equations. After outlining the method in section 2, we demonstrate its wide applicability and relative simplicity, by applying it, in sections 3 and 4, to three algorithms that represent different classes of RMAAs, namely:

- 1) the Controlled ALOHA algorithm ("ALOHA-type" class) [6]
- 2) the "0.487" algorithm ("full sensing-blocked access" class) [7,8]
- 3) the n-ary stack algorithm ("limited sensing-free access" class) [15,16,17]

For the above algorithms, we obtain explicit results on the induced mean delay, for the Poisson infinite-user population model. The higher moments of the delay, for the Poisson as well as for an arbitrary memoryless input stream, can be computed using the same method.

2. THE METHOD

In random-access systems, as in virtually every queueing system, many of the involved stochastic processes are regenerative. A regenerative process is a process consisting of a sequence of regeneration cycles such that the probabilistic structure of the sample function of the process during such a cycle is the same for every cycle, and independent of that of previous or future cycles [5]. The beginnings of such cycles are referred to as regeneration times, and form a renewal process. A discrete-time process $\{X_n\}_{n \geq 1}$ is said to be regenerative with respect to the renewal process $\{R_i\}_{i \geq 1}$, if the process $\{X_{R_i+n}\}_{n \geq 1}$ is a probabilistic replica of the process $\{X_n\}_{n \geq 1}$, for every i , $i=1,2,\dots$.

The regenerative probabilistic structure of a regenerative process makes it possible to express its asymptotic behavior in terms of quantities that refer only to one cycle of the process. This is made precise by the following elegant and powerful result, which will be referred to as the regeneration theorem. [3,4,5]

Theorem 1

Let the discrete-time process $\{X_n\}_{n \geq 1}$ be regenerative with respect to the renewal process $\{R_i\}_{i \geq 1}$. Also, let $C_i = R_{i+1} - R_i$, $i=1,2,\dots$, denote the length of the i -th regeneration cycle, and let f be a nonnegative, real valued, measurable function.

If $C = E\{C_1\} < \infty$ and $S = E\left\{\sum_{i=1}^{C_1} f(X_i)\right\} < \infty$, then,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n f(X_i) = \lim_{n \rightarrow \infty} \frac{1}{n} E \left\{ \sum_{i=1}^n f(X_i) \right\} = \frac{S}{C}, \quad \text{w.p. 1}$$

Furthermore, if, in addition to the finiteness of C and S , the distribution of C_1 is not periodic, then X_i converges in distribution to a random variable X_∞ , and

$$E\{f(X_\infty)\} = \frac{S}{C}$$

Thus, under the conditions stated above, the limiting (expected) average, and the mean of the limiting distribution of $\{f(X_n)\}_{n \geq 1}$ exist, coincide, and are finite. Moreover, their common value is then given in terms of the per cycle quantities S and C .

Given a RMAA, let $\{X_n\}_{n \geq 1}$ be the process of interest associated with the random-access system; this process might, for example, be the delay process induced by the algorithm. Then, provided that $\{X_n\}_{n \geq 1}$ can be shown to be regenerative, the regeneration theorem itself shows the way to establish the existence of steady state, and to compute the steady-state moments, and the distribution of $\{X_n\}_{n \geq 1}$, by appropriately selecting the function f .

In virtually all existing RMAAs, it is relatively easy to identify regenerative times (e.g., when the system becomes empty, or when an appropriate Markov chain hits a suitable fixed state), at which the process of interest probabilistically restarts itself. Given a RMAA and a function f , the problem then is to exploit the dynamics of the algorithm, to find those per cycle properties of the sample function of the process, that could be subsequently used to evaluate the quantities C and S .

In section 3, it is shown that for the delay process, and for $f(x) = x$, the computation of S and C are intimately related to the solution of an infinite dimensional system of linear equations. It can be shown that this is the case when $f(x) = x^n$, $n = 2, 3, \dots$, as well [12]. Therefore, the steady-state moments of the delay process induced by a particular algorithm, can be computed from the solution of the corresponding infinite linear system. In Appendix A, we give a number of general results, that are useful in establishing the existence and uniqueness of a solution, and in developing approximations to the solution of such systems. In section 4, we apply these results to the specific infinite linear systems developed for the three algorithms of section 3. This procedure involves the following steps.

- Step 1 Find conditions under which the infinite linear system has a unique, nonnegative solution.

Step 2 Show that the variables of interest coincide with the unique solution.

Step 3 Develop arbitrarily tight upper and lower bounds on the solution.

3. THREE ALGORITHMS AND THEIR RELATED SYSTEMS OF EQUATIONS

For all three algorithms of this section, we assume a collision-type, packet-switched, slotted, broadcast channel. The channel is accessed by a very large (effectively infinite) number of identical, independent, packet-transmitting, bursty users. The cumulative packet generation process is modelled as a Poisson process, with intensity λ packets per slot. However, the proposed method can be applied equally well, when the number of packets per slot are independent and identically distributed (i.i.d) random variables.

We define the delay, D_n , experienced by the n -th arrived packet, as the time difference between its arrival at the transmitter, and the end of its successful transmission. We are interested in evaluating the steady state statistics of the delay process $\{D_n\}_{n \geq 1}$, when they exist. Due to space limitations in this paper, we give explicit results, only for the first moment of the delay process. However, higher moments of the delay, as well as other quantities of interest can be computed, using the same method; (the computation of the delay variance for the third algorithm in this paper can be found in [12]).

3.1 Example 1 : Controlled ALOHA

The earliest and most well known RMAAs belong to the class of the ALOHA techniques [14,6,19]. Here, we analyze a version of the slotted ALOHA algorithm, that operates with each user transmitting a newly arrived packet, in the first slot after its arrival. Should this cause a collision, each involved user independently retransmits its packet in the next slot,

with probability f .

A packet whose transmission is unsuccessful is said to be blocked. Let M_i be the number of blocked packets at the beginning of slot i (time segment $(i, i+1]$). This number will be referred to as the backlog size. Also, let R_i denote the number of blocked packets retransmitted in slot i , and N_i denote the number of new packets transmitted in slot i . Given $M_i = m$, then clearly,

$$P(R_i=r) = B_r^m(f) = \binom{m}{r} f^r (1-f)^{m-r}, \quad r = 0, 1, 2, \dots \quad (1)$$

$$P(N_i=n) = p_n = \frac{e^{-\lambda} \lambda^n}{n!}, \quad i = 0, 1, 2, \dots \quad (2)$$

The delay process induced by the above algorithm "probabilistically restarts itself" at the beginning of each slot T_i , at which $M_{T_i} = 0$, $i = 1, 2, \dots$; this is so because the number of arrivals per slot is an i.i.d. sequence of random variables. Precisely, let $T_1 = 1$, and define T_{i+1} as the first slot after T_i at which $M_{T_{i+1}} = 0$. The interval (T_i, T_{i+1}) , $i = 1, 2, \dots$, will be referred to as the i th session.

Let R_i , $i = 1, 2, \dots$, denote the number of packets successfully transmitted in the interval $(0, T_{i+1}]$ (Note that R_i also represents the number of packets arrived during the interval $(0, T_i - 1)$). Then, $C_i = R_{i+1} - R_i$, $i = 1, 2, \dots$, is the number of packets successfully transmitted in the interval $(T_i, T_{i+1}]$. The sequence $\{R_i\}_{i \geq 1}$ is a renewal process, since $\{C_i\}_{i \geq 1}$ is a sequence of nonnegative i.i.d. random variables. Furthermore, the delay process $\{D_n\}_{n \geq 1}$ is regenerative with respect to the renewal process $\{R_i\}_{i \geq 1}$, with regeneration cycle, C_1 .

From theorem 1, with $f(D_i) = D_i$, we have that if $C = E[C_1] < \infty$, and if

$S = E\{\sum_{i=1}^{C_i} D_i\} < \infty$, then, there exists a real number D , such that,

$$D = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n D_i = \lim_{n \rightarrow \infty} \frac{1}{n} E\left(\sum_{i=1}^n D_i\right) = E\{D_\infty\} = \frac{S}{C} \text{ a.e.}$$

Thus, provided that both S and C are finite, the limiting average, the limiting expected average, and the mean of the limiting distribution of $\{D_i\}_{i \geq 1}$, exist, coincide, and are finite; their common value D will be referred to as the mean packet delay.

Next, we develop two systems of equations, whose solution may be used to compute the mean cycle length C , and the mean cumulative delay S . The properties and the computation of the solution will be postponed until section 4.

1.a Mean Cycle Length

If the mean session length $H = E\{T_{i+1} - T_i\}$, $i \geq 1$, is finite, then by Wald's identity, we have that,

$$C = \lambda H \quad (3)$$

To determine H , we proceed as follows. Let h_i denote the random number of slots needed to return to zero backlog size, starting from a slot j where the backlog size is equal to i , $i \geq 0$. By definition, then, $H = E(h_0)$. The operation of the algorithm yields the following relation for the h_i 's.

$$h_0 = \begin{cases} 1 & \text{if } N_j = 0, 1 \\ 1 + h_{N_j} & \text{if } N_j > 1 \end{cases} \quad (4.a)$$

$$h_i = \begin{cases} 1 + h_i & \text{if } R_j + N_j = 0 \\ 1 + h_{i+N_j-1} & \text{if } R_j + N_j = 1 \\ 1 + h_{i+N_j} & \text{if } R_j + N_j > 1 \end{cases} \quad (4.b)$$

If we let $H_i = E\{h_i\}$, $i \geq 0$, then after taking expectations in (4) we obtain,

$$H_i = b_i + \sum_{k=0}^{\infty} c_{ik} H_k \quad (5)$$

where $b_i = 1$, $i \geq 0$, $c_{00} = c_{01} = 0$, $c_{0i} = p_i$, $i \geq 2$, $c_{ik} = p_{k-i}$, $k > i+1$, $c_{i,i+1} = p_1(1-B_0^i(f))$, $c_{i,i} = p_0(1-B_1^i(f)) + p_1 B_0^i(f)$, $c_{i,i-1} = p_0 B_1^i(f)$, $c_{ik} = 0$, $k < i-1$, and where p_i , $B_j^i(f)$, $i \geq 0$, $0 \leq j \leq i$, are as defined in (1), and (2), respectively.

Note that the mean session length H , can be computed from system (5), since $H = H_0$.

1.b Mean Cumulative Delay

The mean cumulative delay, S' , can be computed using a system of equations similar to system (5). To develop such a system we proceed as follows. Let w_i denote the cumulative delay experienced by all the packets that were successfully transmitted during h_i slots.⁽¹⁾ Also, let $W_i = E\{w_i\}$, $i \geq 0$, and note that $S = W_0$.

The operation of the algorithm yields the following relations for the w_i 's.

$$w_0 = \begin{cases} N_j & \text{if } N_j = 0, \text{ or } 1 \\ N_j + w_{N_j} & \text{if } N_j > 1 \end{cases} \quad (6.a)$$

(1) Here, for convenience, we count the delay of a packet, starting from the beginning of the first slot after its arrival.

$$w_i = \begin{cases} i + N_j + w_i & \text{if } R_j + N_j = 0 \\ i + N_j + w_{i+N_j-1} & \text{if } R_j + N_j = 1 \\ i + N_j + w_{i+N_j} & \text{if } R_j + N_j > 1 \end{cases} \quad (6.b)$$

After taking expectations in (6) we obtain

$$w_i = b_i^! + \sum_{k=0}^{\infty} c_{ik} w_k \quad (7)$$

where $b_0^! = \lambda$, $b_i^! = i + \lambda$, $i > 1$, and c_{ik} are as defined in (5).

3.2 Example 2 : The "0.487" Algorithm

This algorithm is the most efficient RMAA known to date, for the Poisson infinite-user population model and ternary feedback; (it allows a maximum throughput of 0.487 packets per slot). It is assumed that at the end of each slot i , the users receive a feedback $z_i = 0, 1$, or c , if in slot i there were zero, one, or more than one packets transmitted.

The following is a brief description of the algorithm; for details, motivation, and background discussions, the reader is referred to [8], and [18].

Suppose that at the beginning of slot v (time segment $(v, v+1]$), all packets that arrived before time $t_v < v$, have been successfully transmitted, and there is no information concerning the packets that may have arrived in the interval $[t_v, v)$, (i.e., the distribution of the interarrival times of the packets in $[t_v, v)$ is the same as the one assumed originally). The beginning of such a slot v is called a "collision resolution instant". The time difference $d_v = v - t_v$ will be referred to as the "lag at v ". In slot v , the users

that generated packets in the interval $[t_v, t_v + U_v)$, where $U_v = \min(d_v, \Delta)$, are allowed to transmit; Δ is a parameter to be properly chosen for throughput maximization. In this case, we say that the interval $[t_v, t_v + U_v)$ is "transmitted". Depending on the received feedback z_v , the algorithm operates as follows:

If $z_v = 0$, or 1, the transmitted interval is "resolved", and at time $v + 1$ the resolution of another interval starts, where now $t_{v+1} = t_v + U_v$; $v + 1$ is a new collision resolution instant.

If $z_v = c$, the collision in the transmitted interval is resolved according to the following steps:

1. The interval $[t_v, t_v + U_v/2)$ is transmitted in slot $v + 1$.
 - 1.1 If $z_{v+1} = c$, the resolution of the interval $[t_v, t_v + U_v/2)$ starts at $v + 2$ according to step 1, with $t_{v+2} = t_v$, and $U_{v+2} = U_v/2$. The users in $[t_v + U_v/2, t_v + U_v)$ stop participating in the collision resolution process (in this case we say that the interval $[t_v + U_v/2, t_v + U_v)$ "returns" to the unexamined portion of the arrival axis).
 - 1.2 If $z_{v+1} = 0$, the resolution of the interval $[t_v + U_v/2, t_v + U_v)$ starts at $v + 2$ according to step 1, with $t_{v+2} = t_v + U_v/2$, and $U_{v+2} = U_v/2$.
 - 1.3 If $z_{v+1} = 1$, the interval $[t_v + U_v/2, t_v + U_v)$ is transmitted in slot $v + 2$.
 - 1.3.1 If $z_{v+2} = c$, the resolution of the interval $[t_v + U_v/2, t_v + U_v)$ starts at $v + 3$ according to step 1, with $t_{v+3} = t_v + U_v/2$, and $U_{v+3} = U_v/2$.

- 1.3.2 If $z_{v+2} = 1$, the originally transmitted interval has been resolved, and the resolution of another interval starts at $v + 2$, with $t_{v+2} = t_v + U_v/2$; $v + 2$ is a new collision resolution instant.

For the analysis of the above algorithm, we need the following definitions:

- δ : the length of the originally transmitted interval that is not returned to the unexamined portion of the arrival axis; we refer to δ as the "examined portion of U_v "
- l : the number of slots needed for the resolution of the interval $[t_v, t_v + \delta)$.
- N : number of packets in δ
- ω : sum of delays of the N packets, after the resolution process of $[t_v, t_v + U_v)$ begins
- ψ : sum of delays of the N packets, until the instant $t_v + U_v$.
- $E\{X|u\}$: conditional expectation of the random variable X , given that $U_v = u$

Let $\{v_i\}_{i \geq 1}$ be the sequence of successive collision resolution instants, and let d_i be the lag at v_i . It is known, [10], that the sequence $\{d_i\}_{i \geq 1}$ is a Markov chain, with state space, F a denumerable dense subset of the interval $[1, \infty)$. Let $T_1 = 1$, $d_1 = 1$, and define T_{i+1} , as the first slot after T_i , at which $d_{T_{i+1}} = 1$. From the description of the algorithm it can be seen, after a little thought, that the induced delay process probabilistically restarts itself at the beginning of each slot T_i , $i = 1, 2, \dots$. Therefore, using the notation and definitions of example 1, the mean packet delay D is equal to S/C provided that both S and C are finite.

2.a Mean Cycle Length

As in example 1, if the mean session length $H = E\{T_{i+1} - T_i\}$ is finite, then $C = \lambda H$. To evaluate H we proceed as follows.

Let h_d denote the random number of slots needed to return to lag equal to one, starting from a collision resolution instant v_i with $d_i = d$. Note that, by definition, h_1 is the session length. The operation of the algorithm yields the following relations for the h_d 's, $d \in F$.

$$1 \leq d \leq \Delta, \quad h_d = \begin{cases} \ell & \text{if } \ell = 1 \\ \ell + h_{d-\delta+\ell} & \text{if } \ell > 1 \end{cases} \quad (8.a)$$

$$d > \Delta, \quad h_d = \ell + h_{d-\delta+\ell} \quad (8.b)$$

Taking expectations in (8) yields:

$$1 \leq d \leq \Delta, \quad H_d = E\{\ell|d\} + \sum_{\substack{r,s \\ s \neq 1}} p(r,s|d) H_{d-r+s} \quad (9.a)$$

$$d > \Delta, \quad H_d = E\{\ell|\Delta\} + \sum_{s,r} p(r,s|\Delta) H_{d-r+s} \quad (9.b)$$

where $p(r,s|x)$ is the joint conditional probability distribution of δ , and ℓ , at the point values r and s , given that the transmitted interval is of length x . Note that,

$$p(r,\ell|x) = \begin{cases} (1+\lambda x)e^{-\lambda x} & \text{if } r = x \\ 0 & \text{otherwise} \end{cases}$$

System (9) can be written in the form

$$H_d = b_d + \sum_{t \in F} c_{dt} H_t, \quad d \in F \quad (10)$$

where $b_d = E\{\ell|d\}$, $1 \leq d \leq \Delta$, $b_d = E\{\ell|\Delta\}$, $d > \Delta$, and where c_{dt} , $d, t \in F$ are nonnegative coefficients that can be appropriately identified from (9). The conditional expectation $E\{\ell|d\}$, $1 \leq d \leq \Delta$, can be computed as shown in Appendix B.

2.b Mean Cumulative Delay

Let w_d denote the cumulative delay experienced by all the packets that were successfully transmitted during h_d slots. The operation of the algorithm yields the following relation for the w_d 's, $d \in F$.

$$1 \leq d \leq \Delta, w_d = \begin{cases} \omega + \psi & \text{if } \ell = 1 \\ \omega + \psi + w_{d-\delta+\ell} & \text{if } \ell > 1 \end{cases} \quad (10.a)$$

$$d > \Delta, w_d = \begin{cases} \omega + \psi + (d - \Delta)N & \text{if } \ell = 1 \\ \omega + \psi + (d - \Delta)N + w_{d-\delta+\ell} & \text{if } \ell > 1 \end{cases} \quad (10.b)$$

Taking expectations in (10), yields:

$$1 \leq d \leq \Delta, w_d = E\{\omega|d\} + E\{\psi|d\} + \sum_{\substack{r,s \\ s \neq 1}} p(s,r|d) w_{d-r+s} \quad (11.a)$$

$$d > \Delta, w_d = E\{\omega|\Delta\} + E\{\psi|\Delta\} + (d - \Delta)E\{N|\Delta\} + \sum_{s,r} p(s,r|\Delta) w_{d-r+s} \quad (11.b)$$

System (11) can be written in the form

$$w_d = b_d' + \sum_{t \in F} c_{dt} w_t, d \in F \quad (12)$$

where $b_d^* = E\{\omega|d\} + E\{\psi|d\}$, $1 \leq d \leq \Delta$, $b_\Delta^* = E\{\omega|\Delta\} + E\{\psi|\Delta\} + (d-\Delta)E\{N|\Delta\}$, and where the coefficients c_{dt} , $d, t \in F$ are as defined in (10). The conditional expectations $E\{\omega|d\}$, $E\{\psi|d\}$, $1 \leq d \leq \Delta$, and $E\{N|\Delta\}$ can be computed as shown in Appendix B.

3. Example 3 : Stack Algorithm

A new trend towards the design of easy-to-implement RAAs, which combine stability and good performance, with modest feedback requirements, started with the introduction of the "stack" algorithm by Tsybakov and Vvedenskaya [9]. The new class of algorithms has "limited feedback sensing" and "free access" characteristics. Limited feedback sensing algorithms require that users sense the feedback broadcast only while they have a packet to transmit, and, therefore, they have practical advantages over continuous feedback sensing algorithms, such as the algorithm in example 2. The "free access" characteristics of the new algorithms simplify their implementation, since newly arrived packets are transmitted in the first slot after their arrival.

The applicability of the proposed method to the analysis of algorithms from this new class has been demonstrated in [12], where a representative algorithm, called n-ary stack algorithm (SA_n), has been analyzed. The analysis presented in [12] is included here for reasons of completeness.

The SA_n uses binary feedback of the "collision-No Collision" (C-NC) type; that is, at the end of each slot, the users that monitor the channel are informed whether that slot contained a collision or not. Let z_k denote the binary feedback corresponding to slot k , where $z_k = NC$, and $z_k = C$ represent respectively a noncollision versus collision slot, k . Let some packet arrive during slot $k-1$. We then assume that the packet attempts transmission during

slot k , and it observes the feedback, z_k and all the feedbacks after that, until successfully transmitted. We also assume that the packet has no knowledge of the channel feedback history, $z_i: i < k$ (limited feedback sensing). In its effort to be transmitted successfully, the packet utilizes a counter, whose indication at the beginning of slot k is denoted by I_k , and it applies the following set of rules.

1. A packet arrived during slot $(k-1)$, sets $I_k = 1$
2. A packet attempts transmission within slot k , if and only if, $I_k = 1$.
The packet is successfully transmitted within slot k , if and only if, $I_k = 1$ and $z_k = NC$.
3. The updating of the counter indication I_k is done as follows.
 - a) If $z_k = NC$, and $I_k > 1$, then, $I_{k+1} = I_k - 1$
 - b) If $z_k = C$, and $I_k = 1$, then, $I_{k+1} = J$; where J is an integer random variable, uniformly distributed on $\{1, 2, \dots, n\}$, and n is an integer parameter, $n > 2$.
 - c) If $z_k = C$, and $I_k > 1$, then $I_{k+1} = I_k + -1$

The operational characteristics of the SA_n are perhaps better explained by introducing the concept of a "stack", as it was first done in [9]. A stack is an abstract storage device, consisting of an infinite number of cells, labelled $1, 2, 3, \dots$. The number of packets that a cell can accommodate is unrestricted. At the beginning of each slot, k , the m -th cell of the stack contains the packet with $I_k = m$. Packets are eventually successfully transmitted after moving through the cells of the stack in accordance with the rules defined above, as shown in figure 1. To resolve conflicts, the algorithm splits uniformly the group of collided packets into the first n cells of the stack. The integer n

is an algorithmic parameter, whose value may be chosen for performance optimization.

The random access system operates with the SA_n in sessions. A session is a sequence of consecutive slots that begins and ends at two consecutive algorithmic renewal instants, to be defined below. Those instants are denoted by T_i , $i \geq 1$, and are determined by means of a renewal counter. The first session begins at $T_1=1$, with the renewal counter set to "2". Depending on the channel outcomes, the renewal counter updates its indication in accordance with rules 3.a, and 3.c. The second renewal instant, T_2 , is the instant at which the renewal counter drops to "1" for the first time; this signifies the end of the first session. Immediately after T_2 the renewal counter is reset to "2", and the second session begins. This process continues indefinitely, and defines the successive sessions $[T_i, T_{i+1})$, $i=1,2,\dots$.

From the definition of the session given above, it can be easily seen that immediately before the end of a session, all cells of the stack are empty of packets, and that a new session begins with the group of new packets, that arrived during the last slot of the previous session, placed in cell #1. Due to the independent and stationary increments property of the arrival process the session lengths $T_{i+1}-T_i$, $i=1,2,\dots$, are positive i.i.d. random variables. Furthermore, as in the previous two examples, the delay process $\{D_i\}_{i \geq 1}$ is regenerative with respect to the renewal process $\{R_i\}_{i \geq 1}$, with regeneration cycle length, the number of packets successfully transmitted during a session. Therefore, the mean packet delay D induced by the SA_n is equal to S/C , provided that both S and C are finite.

3.1 Mean Cycle Length

As in the previous examples, $C = \lambda H$. To determine the mean session

length, H , we proceed as follows. Let h_i , $i \geq 0$, denote the random length of a session that begins with i packets in the first cell of the stack. After a little thought, it can be seen that the dynamics of the SA_n yield the following relation for the h_i 's:

$$h_0 = h_1 = 1 ; h_i = 1 + \sum_{j=1}^n h_{I_j + N_j} , i \geq 2 \quad (13)$$

where $N_1, N_2, \dots, N_n$ are independent, Poisson random variables with parameter λ , which are also independent of the random variables $I_1, I_2, \dots, I_n$, which are multinomially distributed, with

$$P(I_1=i_1, \dots, I_n=i_n) = \frac{i!}{i_1! \dots i_n!} \left(\frac{1}{n}\right)^i , 0 \leq i_j \leq i, \sum_{j=1}^n i_j = i$$

If we define $H_i = \Delta E\{h_i\} = E\{h_i | I=i\}$, then, after taking expectations in (13), we obtain,

$$H_0 = H_1 = 1$$

$$H_i = b_i + \sum_{k=2}^{\infty} c_{ik} H_k , i \geq 2 \quad (14)$$

where $b_i = 1 + c_{i1} + c_{i0}$, $i \geq 2$, $c_{ik} = n(p_k * B_k^{i(n-1)})$, $i \geq 0$, $k \geq 0$ and where $B_k^{i(\cdot)}$, p_k are as defined in (1), and (2), respectively, and $*$ signifies convolution.

The mean session length is given by

$$H = \sum_{i=0}^{\infty} p_i H_i \quad (15)$$

3.2 Mean Cumulative Delay

Let w_i ; $i \geq 0$, denote the cumulative delay experienced by all the packets successfully transmitted during a session that begins with i packets in the first cell of the stack. The rules of the algorithm yield the following relation for the w_i 's:

$$w_0 = 0; w_1 = 1; w_i = i + \sum_{j=1}^{n-1} Q_j h_{I_j+N_j} + \sum_{j=1}^n w_{I_j+N_j}, \quad i > 1 \quad (16)$$

where $I_j, N_j, i < j \leq n$, are as defined in (13), and

$$Q_j = \sum_{m=j+1}^n I_m, \quad 1 \leq j \leq n-1$$

Taking expectations in (16) yields the following system of equations for the conditional mean cumulative delay $W_i \triangleq E\{w_i\} = E\{w_i | I=i\}$, $i \geq 0$:

$$\begin{aligned} W_0 &= 0, W_1 = 1 \\ W_i &= b_i^1 + \sum_{k=2}^{\infty} c_{ik} W_k, \quad i \geq 2 \end{aligned} \quad (17)$$

where

$$b_i^1 = b_i^1((H_i)_i, \infty) = i + c_{i1} + \sum_{\zeta=0}^{n-2} \sum_{j=0}^i \sum_{m=0}^{\infty} \sum_{k=0}^{i-j} B_j^1 \left(\frac{\zeta}{n} \right) B_k^{i-j} \left(\frac{1}{n-\zeta} \right) p_m(i-j-k) H_{k+m}, \quad i \geq 2,$$

and where $c_{ik}, B_k^1(\cdot), p_k, i \geq 2, k \geq 2$ are as defined in (14), (1), and (2), respectively.

The mean cumulative delay, S , is given by

$$S = \sum_{i=0}^{\infty} p_i W_i \quad (18)$$

4. SYSTEM SOLUTION AND MEAN PACKET DELAY BOUNDS

In this section, we investigate the conditions under which the infinite dimensional linear systems (5), (7), (10), (12), (14), and (17) have unique, nonnegative solutions, and we develop upper and lower bounds on those solutions. These bounds are then used to obtain bounds on the mean packet delay. We proceed, following the steps outlined in section 2.

4.1 Step 1

For convenience, we rewrite an infinite linear system in an operator form. Specifically, let E be the space of sequences $x = \{x(v)\}: A \rightarrow R$, where A is a countable set. Also, let E^L be the subspace of E for which,

$$\sum_{v \in A} |c_{\mu v} x(v)| < \infty, \mu \in A$$

We define the operator $L = \{L_{\mu}(x)\}: E^L \rightarrow E$, as follows.

$$L_{\mu}(x) = b_{\mu}^L + \sum_{v \in A} c_{\mu v}^L x(v), \mu \in A, x \in E^L$$

In this notation, systems (5), (7), (10), (12), (14), and (17), can be written in the form,

$$s^L = L(s^L), s^L \in E^L \quad (19)$$

We are interested in the existence and uniqueness of nonnegative points $s^L \in E^L$, that satisfy (19); such points will be referred to as fixed points of L , and represent solutions to the corresponding infinite linear system of equations. The question of uniqueness of a fixed point s^L , or equivalently of the solution, $\{s^L(i)\}$, to the system that operator L

represents, depends upon what conditions are imposed on the solution. Thus, after the existence of a solution, $\{s^L(i)\}$, has been established, one has to indicate a class of sequences in which the solution is unique. If the algorithmic sequences of interest $\{H_i\}$, or $\{W_i\}$ belong to the indicated class, then they must coincide with the solution $\{s^L(i)\}$. (This will be examined in Step 2).

Appendix A includes a number of results that can be used to establish existence and uniqueness of a fixed point of an operator. Depending on the operator, some are more straightforward to apply than others. Among the results in Appendix A that can be used to establish existence of a solution, Lemma A.2 is usually the most useful. According to Lemma A.2, to establish existence of a nonnegative fixed point, s^L , it suffices to find a point $x^0 \in E^L$, such that,

$$0 \leq L(x^0) \leq x^0 \quad (20)$$

A point x^0 , satisfying (20), also serves as an upper bound on s^L . Furthermore, to establish a lower bound on s^L , it suffices to find a point $y^0 \in E^L$, such that,

$$y^0 \leq L(y^0) \leq x^0 \quad (21)$$

Thus, under (20) and (21), we have that,

$$y^0 \leq s^L \leq x^0 \quad (22)$$

We proceed now with the analysis of the systems developed in section 3.

1. Controleld ALOHA

System (5) -- Existence; System (5) corresponds to an operator L_1 with $b_\mu = b_\mu$, $c_{\mu\nu} = c_{\mu\nu}$, $\mu, \nu \in N_0$, where N_0 is the set of nonnegative integers, and the b_μ 's and $c_{\mu\nu}$'s are as defined in (5). If we let $x^0 = \{x^0(k)\}$ with $x^0(k) = \alpha_u k + \beta_u$, $k \geq 0$, then by straightforward manipulations we have that, for this choice of x_0 , (20) is satisfied if and only if the following inequalities are satisfied.

$$\lambda < \xi_k(f) - p_0 B_1^k(f) + p_1 B_0^k(f), \text{ for every } k \geq 1 \quad (23)$$

$$\alpha_u \geq \sup \left\{ \frac{1}{\xi_k(f) - \lambda}, k \geq 1 \right\} \quad (24)$$

$$\beta_u \geq (1 + \alpha_u(\lambda - p_1))/(p_0 + p_1) \quad (25)$$

It can be readily seen from (23) that if the retransmission probability f is constant in every slot, then there is no $\lambda \geq 0$ for which (23) is satisfied.

If the retransmission probability f_i , at each slot i , were allowed to depend on the current backlog size, M_i , in accordance to a stationary control policy $f = f(M_i)$, then it is of interest to choose $f(\cdot)$ so that it maximizes the set of λ 's for which inequality (23) is satisfied. This is equivalent to maximizing $\xi_k(f)$ with respect to f . It can be easily verified that, for every $k \geq 1$, $\xi_k(f)$ is maximized for $f(k) = f^*(k)$, where (2)

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2. We should mention that, in a distributed environment, the backlog size dependent retransmission probability $f^*(\cdot)$ is nonimplementable, since users are not aware of the current backlog size. However, the control policy given by (26) can be implemented approximately by adaptive control schemes that estimate the current backlog size using observable feedback information from the past activity on the channel [6, 19, 20].

$$f^*(k) = \frac{1-\lambda}{k-\lambda}, \quad k \geq 1 \quad (26)$$

From this point on, we assume that f is chosen as in (26). Under this assumption, inequality (23) is satisfied, provided that,

$$\lambda < \inf \{ \xi_k(f^*), k \geq 1 \} = e^{-1}$$

To satisfy inequalities (24), and (25), we choose,

$$\alpha_u = \sup \left\{ \frac{1}{\xi_k(f^*) - \lambda}, k \geq 1 \right\} = \frac{1}{e^{-1} - \lambda}, \quad \beta_u = \frac{1 + \alpha_u(\lambda - p_1)}{p_0 + p_1} = \frac{1-\lambda}{1+\lambda} \frac{1}{e^{-1} - \lambda} \quad (27)$$

Similarly, it is straightforward to show that if $\lambda < e^{-1}$ then the point y^0 with $y^0(k) = \alpha_\ell k + \beta_\ell$, $k \geq 0$, and

$$\alpha_\ell = \frac{1}{e^{-1} - \lambda}, \quad \beta_\ell = \frac{1-\lambda}{1+\lambda} \frac{1}{e^{-1} - \lambda} \quad (28)$$

satisfies (21). Thus, from (22) and for $\lambda < e^{-1}$ we have that system (5) has a solution, $S^{L_1} = \{s^{L_1}(k)\}$, such that

$$0 < \alpha_\ell k + \beta_\ell \leq s^{L_1}(k) \leq \alpha_u k + \beta_u, \quad k \geq 0 \quad (29)$$

where α_u, β_u are as given by (27), and α_ℓ, β_ℓ are as given by (28).

System (7) -- Existence: System (7) corresponds to an operator L_2 with $b_\mu^{L_2} = b_\mu^1$, $c_{\mu\nu}^{L_2} = c_{\mu\nu}^1$, $\mu, \nu \in N_0$, where the b_μ 's and $c_{\mu\nu}$'s are as defined in (7).

Due to the fact that b_k is a linear function of k , and since

$$\sum_{k=0}^{\infty} c_{ik} = 1, \quad i = 1,$$

it can be easily seen that there is no linear sequence $x^0 = \{x^0(k)\}$ satisfying (20). However, given $\lambda < e^{-1}$, it is straightforward to show that we can choose coefficients $\gamma_u, \delta_u, \zeta_u, \gamma_\ell, \delta_\ell, \zeta_\ell$ such that the point x^0 with $x^0(k) = \gamma_u k^2 + \delta_u k + \zeta_u, k \geq 0$, and the point y^0 with $y^0(k) = \delta_\ell k^2 + \delta_\ell + \zeta_\ell, k \geq 0$, satisfy (20), and (21), respectively. The following is such a choice:

$$\gamma_u = \frac{1}{2(e^{-1}-\lambda)}, \quad \delta_u = \frac{\lambda + \gamma_u(\lambda^2 + \lambda + e^{-1})}{e^{-1}-\lambda}, \quad \zeta_u = \lambda \frac{1 + \gamma_u(1 + \lambda - e^{-\lambda}) + \delta_u(1 - e^{-\lambda})}{(1 + \lambda)e^{-\lambda}} \quad (30)$$

$$\gamma_\ell = \frac{1}{2(e^{-\lambda}-\lambda)}, \quad \delta_\ell = \frac{\lambda + \gamma_\ell(\lambda^2 + \lambda + e^{-\lambda}(1-2\lambda))}{e^{-\lambda}-\lambda}, \quad \zeta_\ell = \lambda \frac{1 + \gamma_\ell(1 + \lambda - e^{-\lambda}) + \delta_\ell(1 - e^{-\lambda})}{(1 + \lambda)e^{-\lambda}} \quad (31)$$

Then, if $\lambda < e^{-1}$, then system (7) has a solution $S^{L_2} = \{s^{L_2}(k)\}$, such that,

$$0 < \gamma_\ell k^2 + \delta_\ell k + \zeta_\ell < s^{L_2}(k) < \delta_u k^2 + \delta_u k + \zeta_u, \quad k \geq 0 \quad (32)$$

where $\gamma_u, \delta_u, \zeta_u$ are as given by (30), and $\gamma_\ell, \delta_\ell, \zeta_\ell$ are as given by (31).

Systems (5) and (7) -- Uniqueness

We will show that both the solution $\{s^{L_1}(i)\}$ of system (5) and the solution $\{s^{L_2}(i)\}$ of system (7) are unique in the class

$$E_2 = \left\{ x : \sup_{i \in N_0} \frac{|x(i)|}{i^{2+c}} < \infty \right\}$$

where c is a positive constant.

We start with system (7). Since L_2 is majorant of itself, from theorem A.1,

we have that L_2 has a principal fixed point $S_{\star}^{L_2}$, such that $0 < S_{\star}^{L_2} < S^{L_2}$.

According to theorem A.2, the fixed point S^{L_2} is unique in the class,

$$E_{\star}^{L_2} = \left\{ x : \sup_{i \in N_0} \frac{|x(i)|}{S_{\star}^{L_2}} < \infty \right\},$$

provided that $y^0 \in E_{\star}^{L_2}$. Since, by definition, $y^0 \in E_2$, S^{L_2} will be unique in E_2 , if we show that $E_{\star}^{L_2} = E_2$. According to lemma A.1, it suffices to show that,

$$\sup_{i \in N_0} \frac{s_{\star}^{L_2}(i)}{i^{2+c}} < \infty \quad (33)$$

and

$$\inf_{i \in N_0} \frac{s_{\star}^{L_2}(i)}{i^{2+c}} > 0 \quad (34)$$

Since $0 < s_{\star}^{L_2}(i) < s^{L_2}(i)$, (33) follows from (32). To show that (34) holds, we use the power sequence, $\{S_n^{L_2}\}_{n \geq 1}$, of L_2 with initial point 0. By definition (see Appendix A), $S_n^{L_2}$ is the point that results after L_2 operates n times on the zero point, (i.e., $S_n^{L_2} = L_2^n(0)$), and $S_n^{L_2} \rightarrow S_{\star}^{L_2}$, as $n \rightarrow \infty$. Due to the fact that $b_i^{L_2} > 0$, $c_{ik}^{L_2} > 0$, $i, k \in N_0$, we have that $0 < s_n^{L_2}(i) < s_{n+1}^{L_2}(i) < s_{\star}^{L_2}(i)$, for every $n \geq 1$, $i \geq 0$. Also, it can be readily shown by induction that, for every $i \geq 1$, $n \geq 1$,

$$s_n^{L_2}(i) \geq ni - n(n-1)/2 \quad (35)$$

From (35) we obtain,

$$\liminf_{i \rightarrow \infty} \frac{L_2 s_{\star}(i)}{i^2+c} > \liminf_{i \rightarrow \infty} \frac{L_2 s_i(i)}{i^2+c} > \frac{1}{2} \frac{i^2+i}{i^2+c} = \frac{1}{2} \quad (36)$$

(34) follows from (36), and the fact that $s_{\star}(i) > 0, i > 0$.

The uniqueness in E_2 of the solution $\{s_{L_1}(i)\}$ of system (5) follows from theorem A.4, part (ii), after one identifies L_1 with θ_2 and L_2 with θ_1 , in the theorem.

2. The "0.487" Algorithm

System (10) -- Existence and Initial Bounds

System (10) corresponds to an operator L_1 with $b_{\mu}^{L_1} = b_{\mu}$, $c_{\mu\nu}^{L_1} = c_{\mu\nu}$, $\mu, \nu \in F$, where the b_{μ} 's and $c_{\mu\nu}$'s are as defined in (10). To establish the existence of a nonnegative solution to system (10), we follow the same procedure as in system (5).

Let $x^0 = \{x^0(d)\}$ with $x(d) = \alpha_u d + \beta_u$, $d \in F$, and let $x' = L_1(x^0)$. After straightforward manipulations, we obtain,

$$x'(d) = x^0(d) + E\{\ell|d\} + \alpha_u(E\{\ell|d\} - E\{\ell|\Delta\}) - (1+\lambda d)e^{-\lambda d} - \beta_u(1+\lambda d)e^{-\lambda d}, \quad 1 \leq d \leq \Delta \quad (37.a)$$

$$x'(d) = x^0(d) + E\{\ell|d\} - \alpha_u(E\{\ell|\Delta\} - E\{\ell|d\}), \quad d > \Delta \quad (37.b)$$

According to Lemma A.2, to establish the existence of a nonnegative fixed point of L_1 , it suffices to show that there exist α_u, β_u , such that,

$$0 < x'(d) < x^0(d), \quad \text{for every } d \in F \quad (38)$$

From (37.b), we see that this is possible only if

$$E\{\delta|\Delta\} > E\{\ell|\Delta\} \quad (39)$$

If (39) holds, then it can be readily seen from (37) that (38) is satisfied, if we choose α_u, β_u as follows:

$$\alpha_u = \frac{E\{\ell|\Delta\}}{E\{\delta|\Delta\} - E\{\ell|\Delta\}} \quad (40.a)$$

$$\beta_u = \max\{-\alpha_u, \sup_{1 \leq d \leq \Delta} (\rho(d))\} \quad (40.b)$$

where

$$\rho(d) = \frac{E\{\ell|d\} + \alpha_u(E\{\ell|\Delta\} - E\{\delta|d\} - (1+\lambda d)\exp(-\lambda d))}{(1+\lambda d)\exp(-\lambda d)}$$

The conditional expectations appearing in the above expressions can be computed as shown in Appendix B.

Similarly, it can be shown that, under (39), the point $Y^0 = \{y^0(d)\}$ with $y^0(d) = \alpha_\ell d + \beta_\ell$, $d \in F$ satisfies the inequality $Y^0 \leq L(Y^0) \leq X^0$, if α_ℓ and β_ℓ are chosen as follows:

$$\alpha_\ell = \alpha_u, \quad \beta_\ell = \inf_{1 \leq d \leq \Delta} (\rho(d)) \quad (41)$$

where $\alpha_u, \rho(d)$ are as given by (40).

Thus, if (39) holds, then from lemma A.2 we have that system (10) has a nonnegative solution S^{L_1} , such that,

$$\alpha_\ell d + \beta_\ell \leq S^{L_1}(d) \leq \alpha_u d + \beta_u, \quad d \in F \quad (42)$$

where α_u, β_u , and α_ℓ, β_ℓ are as given by (40) and (41), respectively.

System (12) -- Existence and Initial Bounds

Let L_2 be the operator that corresponds to system (12). Also, let $x^0 = \{x^0(d)\}$ with $x^0(d) = \gamma_u d^2 + \delta_u d + \zeta_u$, $d \in F$, and $y^0 = \{y^0(d)\}$ with $y^0(d) = \gamma_\ell d^2 + \delta_\ell d + \zeta_\ell$, $d \in F$.

Following the same procedure as for system (10), we can show that if (39) holds, then system (12) has a nonnegative solution $S^{L_2} = \{s^{L_2}(d)\}$, $d \in F$, such that,

$$\gamma_\ell d^2 + \delta_\ell d + \zeta_\ell < s^{L_2}(d) < \gamma_u d^2 + \delta_u d + \zeta_u \quad (43)$$

where,

$$\gamma_u = \gamma_\ell = \frac{E\{N|\Delta\}}{2(E\{\delta|\Delta\} - E\{\ell|\Delta\})}$$

$$\delta_u = \delta_\ell = \frac{E\{\omega|\Delta\} + E\{\psi|\Delta\} - \Delta E\{N|\Delta\} + \gamma_u E\{(\delta-\ell)^2|\Delta\}}{E\{\delta|\Delta\} - E\{\ell|\Delta\}}$$

$$\zeta_u = \sup_{1 \leq d \leq \Delta} (\phi(d)), \quad \zeta_\ell = \inf_{1 \leq d \leq \Delta} (\phi(d))$$

$$\phi(d) = \frac{E\{\omega|d\} + E\{\psi|d\} + \gamma_u (E\{(\delta-\ell)^2|d\} - 2dE\{\delta-\ell|d\} - (1+\lambda d)e^{-\lambda\Delta}) - \delta_u (E\{\delta-\ell|d\} - (1+\lambda d)e^{-\lambda d})}{(1+\lambda d)e^{-\lambda d}}$$

The conditional expectations in the above expressions can be computed as shown in Appendix B.

Remark It is known [7] that inequality (39) is satisfied if $\lambda < \lambda_m(\Delta)$; where $\lambda_m(\Delta)$ is maximized for $\Delta \approx 2.6$, and $\lambda_m(2.6) \approx 0.4871$.

Systems (10) and (12) -- Uniqueness

We will show that both systems (10) and (12) have unique solutions in the class

$$E_2 = \left\{ X : \sup_{d \in F} \frac{|x(d)|}{d^2} < \infty \right\} \quad (44)$$

As in the case of systems (7) and (9) in example 1, if we show uniqueness for system (12), then the uniqueness for system (10) follows from theorem A.4, part (ii).

According to theorem A.2, the fixed point S^{L_2} is unique in the class

$$E_*^{L_2} = \left\{ X : \sup_{d \in F} \frac{|x(d)|}{S_*^{L_2}} < \infty \right\}$$

provided that $Y^0 \in E_*^{L_2}$. Since, by construction, $Y^0 \in E_2$, S^{L_2} will be unique in E_2 , if $E_*^{L_2} = E_2$. To show that the latter equation holds, we proceed as follows.

Let $S_n = L_2^n(0)$, where $L_2^n(0)$ is the n -th power of the operator L_2 , acting on the zero point. Clearly,

$$s_1(d) = b_d^{L_2} \geq \epsilon > 0, \text{ for every } d \in F \quad (45)$$

Also, it can be easily shown by induction that,

$$s_n(d) = n((d-\Delta)E\{N|\Delta\} + E\{w|\Delta\} + E\{\psi|\Delta\}) - \frac{n(n-1)}{2}(E\{\delta|\Delta\} - E\{z|\Delta\})E\{N|\Delta\}$$

(46)

for every $d \in F$, $n \geq 1$, such that $d > n\Delta$. For $d > 2\Delta$, letting⁽³⁾
 $n = \left\lfloor \frac{d}{\Delta} \right\rfloor - 1$ in (46), and using the fact that $\left\lfloor \frac{d}{\Delta} \right\rfloor > \frac{d}{\Delta} - 1$, yields,

$$s_n(d) > \alpha d^2 + \beta d + \gamma, \quad d > 2\Delta \quad (47)$$

where $\alpha > 0$. (The expressions for the coefficients α, β, γ are not of interest and, therefore, are omitted).

If $S_{\star}^{L_2}$ is the principal solution of L_2 , then from theorem A.1 we have,

$$s_{\star}^{L_2}(d) > s_n(d) > 0, \quad \text{for every } d \in F, \quad n \geq 1 \quad (48)$$

From (45) and (48) we have that,

$$s_{\star}^{L_2}(d) > \varepsilon > 0, \quad \text{for every } d \in F \quad (49)$$

From (47) and (49) we conclude that,

$$\inf_{d \in F} \frac{s_{\star}^{L_2}(d)}{d^2} > 0 \quad (50)$$

From (43), and the fact that $S_{\star}^{L_2} < S^{L_2}$, we have,

$$\sup_{d \in F} \frac{s_{\star}^{L_2}(d)}{d^2} < \infty \quad (51)$$

Finally, from (50), (51), and lemma A.1 we have that $E_{\star}^{L_2} = E_2$

3. $\lfloor a \rfloor$ denotes the maximum integer not exceeding a .

3. Stack Algorithms

System (14) -- Existence and Initial Bounds

System (14) corresponds to an operator L_1 with $b_\mu = b_\mu$, $c_{\mu\nu} = c_{\mu\nu}$, $\mu \geq 2$, $\nu \geq 2$, where the b_μ 's and $c_{\mu\nu}$'s are as defined in (14). Let $x^0 = \{x^0(k)\}$ with $x^0(k) = \alpha_u k + \beta_u$, $k \geq 2$, and let $y^0 = \{y^0(k)\}$ with $y^0(k) = \alpha_\ell k + \beta_\ell$, $k \geq 2$. Then, by straightforward manipulations (see [12] for details) we can show that (20), and (21) are satisfied (with $L_1 = L$) if

$$\lambda < \lambda_0(n) = ((8n^3 - 7n^2 + 2n + 1)^{1/2} - 3n + 1) / (2n(n-1)) \quad (52)$$

$$\alpha_u = n(1+n+\lambda(n-1)) / (2(n-1-n\lambda) - \lambda(n-1)(1+\lambda n)), \beta_u = (1+n\lambda\alpha_u) / (n-1) \quad (53)$$

$$\alpha_\ell = n / (n-1-n\lambda), \beta_\ell = (1+n\alpha_\ell\lambda) / (n-1) \quad (54)$$

Thus, if $\lambda < \lambda_0(n)$, then system (14) has a solution S^{L_1} , such that

$$0 < \alpha_\ell k + \beta_\ell < S^{L_1}(k) < \alpha_u k + \beta_u, \quad k \geq 2 \quad (55)$$

Where α_u, β_u are as given by (53), and α_ℓ, β_ℓ are as given by (54).

We note that $\lambda_0(n)$ attains a maximum at $n = 3$, with $\lambda_0(3) = 0.3874$.

System (14) -- Uniqueness:

We will show that S^{L_1} is unique in the class $E_2 \subset E^{L_1}$, which is defined as follows,

$$E_2 = \left\{ x : \sup_{i \geq 2} \frac{|x(i)|}{i^2} < \infty \right\}$$

In order to do so, we use theorem A.4 with $L_1 \equiv 0_1 \equiv 0_2$ and $g(k) = g_1(k)$, where,

$$g_1(k) = k^2 + u_1 k + u_0, \quad k \geq 2 \quad (56)$$

and the coefficients u_1, u_0 are chosen so as to satisfy the conditions stated in the theorem.

Conditions (a) and (c) are obviously satisfied for any u_1, u_2 . To satisfy conditions (b), (d), and (e), it suffices to choose u_1, u_0 so that,

$$g_1(i) > 0, \text{ and } G_1(i) > 0, \text{ for every } i \geq 2 \quad (57)$$

where,

$$G_1(i) \stackrel{\Delta}{=} g_1(i) - \sum_{k=2}^{\infty} c_{ik} g_1(k) = h(i) + c_{i1} + (c_{i1-n\lambda})u_1 + (c_{i0} + c_{i1-n+1})u_2, \quad i \geq 2 \quad (58)$$

and

$$h(i) = (1 - \frac{1}{n})i^2 - (1 - \frac{1}{n} - 2\lambda)i - n\lambda(1+\lambda), \quad i \geq 2$$

Given $\lambda < \lambda_0(n)$, then it is straightforward to show that there exist u_1, u_2 that satisfy (57). The following choice is adopted from [12]:

$$u_1 = \max\{-1, (h_2/(n-1) + v/(n\lambda/(n-1) - v))\} + 1, \quad u_0 = (u_1 + 1)v \quad (59)$$

where $v = (\lambda(n-1)+2)/((1+\lambda)(n-1)+2)$.

With u_1, u_2 chosen as above, it is clear from (55) and (57) that condition (e) is satisfied. Thus, S^{L_1} is the unique solution of system (14) in the class $E_{g_1}^{L_1}$, which is defined as follows,

$$E_{g_1}^{L_1} = \left\{ x : \sup_{s \in L_1} \frac{|x(i)|}{s(i) + g_1(i)} < \infty \right\}$$

Finally, from lemma A.1 we, have that $E_{g_1}^{L_1} = E_2$.

System (17) -- Existence and Initial Bounds

In section 4.2, we will show that the sequence of the conditional mean session lengths, $\{H_i\}_{i \geq 2}$, coincides with the solution S^{L_1} of system (14), if this solution exists, (i.e., if $\lambda < \lambda_0(n)$). With this in mind, system (17) corresponds to an operator L_2 with $b_\mu = b_\mu^{L_1}(S^{L_1})$, $c_{\mu\nu} = c_{\mu\nu}^{L_2}$, $\mu \geq 2$, $\nu \geq 2$, where the $b_\mu^{L_1}(\cdot)$'s and $c_{\mu\nu}^{L_2}$'s are as defined in (17).

Let $x^0 = \{x^0(k)\}$ with $x^0(k) = \gamma_u g_1(k)$, $k \geq 2$, where $g_1(k)$ is as defined in (56) and γ_u is a positive real.

Given $\lambda < \lambda_0(n)$, we choose γ_u so that $x^0 > L_1(x^0) > 0$, or equivalently,

$$\gamma_u g_1(i) > b_i^{L_1}(S^{L_1}) + \sum_{k=2}^{\infty} c_{ik} \gamma_u g_1(k) > 0, \quad i \geq 2 \quad (60)$$

From (55) we have that $b_i^{L_1}(S^{L_1}) < b_i^{L_1}(\{\alpha_u k + \beta_u\}_{k \geq 2})$, $i \geq 2$, thus, (60) holds if, for every $i \geq 2$, we have,

$$g_1(i) > 0, \quad \gamma_u G_1(i) > b_i^{L_1}(\{\alpha_u k + \beta_u\}_{k \geq 2}), \quad i \geq 2 \quad (61)$$

where $G_1(i)$ is as given by (58).

With u_1, u_2 chosen as in (59), it follows from (58) that inequalities (61) are satisfied if,

$$\gamma_u = \sup_{i \geq 2} \frac{b_i^{L_1}(\{\alpha_u k + \beta_u\}_{k \geq 2})}{G_1(i)} \quad (62)$$

(it can be readily shown that $0 < \gamma_u < \infty$).

Thus, from lemma A.2, we have that if $\lambda < \lambda_0(n)$, then system (17) has a solution S^{L_2} such that,

$$0 \leq s^{L_2}(i) \leq \gamma_u i^2 + \delta_u i + \zeta_u, \quad i \geq 2 \quad (63)$$

where $\delta_u = u_1 \gamma_u$, $\zeta_u = u_0 \gamma_u$, and where u_1 , u_0 are as given by (59) and γ_u is given by (62).

Using similar arguments, one can readily establish the following lower bounds on s^{L_2} :

$$0 \leq \gamma_l i^2 + \delta_l i + \zeta_l \leq s^{L_2}(i), \quad i \geq 2 \quad (64)$$

where,

$$\gamma_l = \inf_{i \geq 2} \frac{b_1^i(\{\alpha_l k + \beta_l\}_{k \geq 2})}{G_1(i)}, \quad \delta_l = \delta_u, \quad \zeta_l = \zeta_u \quad (65)$$

and where α_l , β_l are as given by (54).

System (17) -- Uniqueness

In theorem A.3, let $O_1 = L_1$, $O_2 = L_2$, and $g(i) = \gamma_u i^2 + \delta_u i + \zeta_u$, $i \geq 2$, where γ_u , δ_u , ζ_u are as given in (63). Then, all conditions in the theorem are satisfied. Thus, s^{L_2} is unique in the class,

$$E_g^{L_1} = \left\{ x : \sup_{i \geq 2} \frac{|x(i)|}{s^{L_1}(i) + g(i)} < \infty \right\}$$

It is clear from (55) and lemma A.1 that $E_g^{L_1} = E_2$, where the class E_2 is as defined in the proof of uniqueness for system (14).

4.2 Step 2

In step 1, we have established conditions for the existence of nonnegative solutions to the systems of interest, and we have identified classes of

sequences in which these solutions are unique. Here, we show that the algorithmic sequences $\{H_i\}$, $\{W_i\}$, where $H_i = E\{h_i\}$ and $W_i = E\{w_i\}$, belong to the corresponding identified class, and therefore, coincide with the unique solution in the class. The proof is based on theorem A.6, and is the same for all three algorithms.

For the case of the sequence $\{H_i\}$, let, in theorem A.6, $L = L_1$, $X_i = h_i$, and $X_i^n = \min(h_i, n)$, $n = 1, 2, 3, \dots$. By definition, the X_i 's and X_i^n 's satisfy condition (a) in the theorem. Condition (b) follows from the fact that $X_i^n \leq n$ a.e.. Finally, condition (c) follows from the operation of the algorithm. Thus, $\{H_i\} = S^{L_1}$.

Similarly, to show that $\{W_i\} = S^{L_2}$, we apply theorem A.6, with $L = L_2$, $X_i = w_i$, and $X_i^n = \min(w_i, n)$, $n = 1, 2, 3, \dots$.

4.3 Step 3

In step 1, we have already found upper and lower bounds, X^0 and Y^0 , respectively, on the solutions to the systems of interest. These bounds can be improved either by computing the power sequences of the corresponding operators with initial points the bounds X^0 and Y^0 , (lemma A.2), or by solving finite systems of linear equations that are truncations of the original infinite systems, (theorem A.5). Both methods can provide arbitrarily tight upper and lower bounds. We use the first method in the "0.487" algorithm, and the second method in the controlled ALOHA and the n-ary stack algorithm.

1. Controlled ALOHA

For system (5), we apply theorem A.5 with $L = L_1$ and,

$$u^{L_1}(i) = \alpha_u i + \beta_u, \quad i \in N_0$$

$$\ell^{L_1}(i) = \alpha_\ell i + \beta_\ell, \quad i \in N_0$$

$$D_j = \{0, 1, 2, \dots, j\}, \quad j \in N_0$$

where α_u, β_u , and α_ℓ, β_ℓ are as given by (27) and (28), respectively. Note that, for given j , D_j is a finite set and, therefore, all conditions in the theorem are satisfied. Thus, for $\lambda < e^{-1}$,

$$s^{\phi_j}(i) \leq H_i = s^{L_1}(i) \leq s^{\theta_j}(i), \quad 0 \leq i \leq j$$

where $\{s^{\theta_j}(i)\}_{0 \leq i \leq j}$ and $\{s^{\phi_j}(i)\}_{0 \leq i \leq j}$ are the unique solutions of the $(j+1)$ -dimensional systems (66) and (67), respectively.

$$H_i^u = b_i^{\theta_j} + \sum_{k=0}^j c_{ik}^{L_1} H_k^u, \quad i, 0 \leq i \leq j \quad (66)$$

$$H_i^\ell = b_i^{\phi_j} + \sum_{k=0}^j c_{ik}^{L_1} H_k^\ell, \quad i, 0 \leq i \leq j \quad (67)$$

where $b_i^{\theta_j}, b_i^{\phi_j}$ are as defined in the theorem with $\rho_i = \sigma_i = b_i^{L_2}, 0 \leq i \leq j$. therein. We solved systems (66) and (67) for $j = 50$. The resulted upper bound H_0^u and lower bound H_0^ℓ on the mean session length H_0 , can be found in table 1, for different values of $\lambda, (\lambda < e^{-1})$. For system (7) we followed the procedure described above with,

$$L = L_2$$

$$u^{L_2}(i) = \gamma_u i^2 + \delta_u i + \epsilon_u, \quad i \in N_0$$

$$\ell^{L_2}(i) = \gamma_\ell i^2 + \delta_\ell i + \epsilon_\ell, \quad i \in N_0$$

$$\rho_i = \sigma_i = b_i^{L_2}, \quad i \in D_j = \{0, 1, 2, \dots, j\}, \quad j \in N_0$$

where $\gamma_u, \delta_u, \zeta_u$ are as given by (30), and $\gamma_\ell, \delta_\ell, \zeta_\ell$ are as given by (31). The resulting bounds w_o^u, w_o^ℓ on the mean cumulative delay w_o are included in table 1; they were computed using $j = 50$. From the regeneration theorem and (3) we have⁽⁴⁾,

$$D = \frac{w_o}{\lambda H_o} + 0.5 \quad (68)$$

The upper bound $D^u = w_o^u / (\lambda H_o^\ell) + 0.5$, and the lower bound $D^\ell = w_o^\ell / (\lambda H_o^u) + 0.5$ on D are included in table 1. Note that, according to theorem A.5, arbitrarily tight bounds can be obtained by increasing j . From a theoretical view point the bounds become exact as $j \rightarrow \infty$.

2. The "0.487" Algorithm

From section 4.2 we have that, for $\lambda < 0.487$, $H_d = s^{L_1}(d)$, $d \in F$, and $w_d = s^{L_2}(d)$, $d \in F$, where S^{L_1} and S^{L_2} are the fixed points identified in section 4.1. According to Lemma A.2 we have that,

$$L_1^n(y_1^o) < S^{L_1} < L_1^n(x_1^o), \quad n=1,2,\dots, \quad d \in F \quad (69)$$

$$L_2^n(y_2^o) < S^{L_2} < L_2^n(x_2^o), \quad n=1,2,\dots, \quad d \in F \quad (70)$$

where $x_1^o = \{\alpha_u d + \beta_u\}_{d \in F}$, $y_1^o = \{\alpha_\ell d + \beta_\ell\}_{d \in F}$

$$x_2^o = \{\gamma_u d^2 + \delta_u d + \zeta_u\}_{d \in F}, \quad y_2^o = \{\gamma_\ell d^2 + \delta_\ell d + \zeta_\ell\}_{d \in F}$$

and where $\alpha_u, \beta_u, \alpha_\ell, \beta_\ell, \gamma_u, \delta_u, \zeta_u$ are as given by (40), (41), and (43). For $n = 1$, and $d = 1$, (69) yields the following bounds on the mean session length H_1 :

$$H_1^\ell < H_1 < H_1^u$$

4. The additional 0.5 units of time represent the mean delay of a packet, until the beginning of the first slot following its arrival. (See footnote 1).

where

$$H_1^u = E\{\ell|1\} + \alpha_u(1-(1+\lambda)e^{-\lambda}) + E\{\ell|1\} - E\{\delta|1\} + \beta_u(1-(1+\lambda)e^{-\lambda})$$

$$H_1^l = H_1^u - (\beta_u - \beta_l)(1-(1+\lambda)e^{-\lambda})$$

The above bounds can be found in table 2, for different values of λ , ($\lambda < 0.487$). For $n = 1$, and $d = 1$, (70) yields the following bounds on the mean cumulative delay over a session W_1 :

$$W_1^l < W_1 < W_1^u$$

where

$$W_1^u = E\{\omega|1\} + E\{\psi|1\} + \gamma_u(1-(1+\lambda)e^{-\lambda}) + E\{(\delta-\ell^2)|1\} - 2E\{\delta-\lambda|1\} \\ + \delta_u(1-(1+\lambda)e^{-\lambda}) - E\{\delta-\lambda|1\} + \zeta_u(1-(1+\lambda)e^{-\lambda})$$

$$W_1^l = W_1^u - (\zeta_u - \zeta_l)(1-(1+\lambda)e^{-\lambda})$$

The bounds W_1^u and W_1^l are included in table 2. From the regeneration theorem we have $D = W_1/(\lambda H_1)$. The upper bound $D^u = W_1^u/(\lambda H_1^l)$, and the lower bound $D^l = W_1^l/(\lambda H_1^u)$ on the mean packet delay D are included in table 2, and are plotted in figure 2.

Finally, we note that tighter bounds can be obtained either by evaluating the bounds given by (69) and (70) for higher values of n , or by the method of truncated systems used in the previous example. In both methods, however, we must first compute the conditional probabilities $p(\delta, \ell|x)$ defined in (9), which is a computationally complex task. Note that for the found bounds, (i.e., for $n = 1$ in (69) and (70)), such a computation is not required.

3. n-ary Stack Algorithm

For system (14), we apply theorem A.5 with $L = L_1$ and

$$u^{L_1}(i) = \alpha_u i + \beta_u, \quad i \geq 2$$

$$l^{L_1}(i) = \alpha_l i + \beta_l, \quad i \geq 2$$

$$D_j = \{2, 3, \dots, j\}, \quad j \geq 2$$

$$l_i = \sigma_i = b_i^{L_1}, \quad i \geq 2$$

where α_u , β_u and α_l , β_l are as given by (53) and (54), respectively. As in the case of the ALOHA algorithm, the fixed points S^{ϕ_j} and S^{θ_j} , in theorem A.5, are obtained by solving two $(j-1)$ -dimensional systems of linear equations. If we denote $s^{\phi_j}(i)$ by H_i^l , $i \geq 2$, and $S^{\theta_j}(i)$ by H_i^u , $i \geq 2$, then using (15) and (55) we obtain,

$$H^l < H < H^u$$

where

$$H^u = p_0 + p_1 + \sum_{i=2}^j p_i H_i^u + \sum_{i=j+1}^{\infty} p_i (\alpha_u i + \beta_u)$$

$$H^l = p_0 + p_1 + \sum_{i=2}^j p_i H_i^l + \sum_{i=j+1}^{\infty} p_i (\alpha_l i + \beta_l)$$

In table 3, we give the bounds H^u , H^l for different values of λ in the interval $(0, \lambda_0(n))$, and for $n = 2, 3, \dots$, where $\lambda_0(n)$ is as given by (52); these bounds were computed using $j = 15$. For system (17), we use theorem A.5 with $L = L_2$, and

$$u^{L_2}(i) = \gamma_u i^2 + \delta_u i + \zeta_u, \quad i \geq 2$$

$$l^{L_2}(i) = \gamma_l i^2 + \delta_l i + \zeta_l, \quad i \geq 2$$

$$\rho_i = b_i^{L_2}(\{\gamma_k^l\}_{k \geq 0}), \quad i \geq 2$$

$$\sigma_i = b_i^{L_2}(\{\gamma_k^u\}_{k \geq 0}), \quad i \geq 2$$

where

$\gamma_u, \delta_u, \zeta_u, \gamma_\ell, \delta_\ell, \zeta_\ell$ are as given in (62), (63), (65),

$b_i^{L_2}(\cdot) = b_i^L(\cdot)$, with $b_i^L(\cdot)$ as defined in (17)

$$Q_0^\ell = Q_1^\ell = 1; Q_k^\ell = H_k^\ell, 2 \leq k \leq j; Q_k^\ell = \alpha_\ell k + \beta_\ell, k \geq j+1$$

$$Q_0^u = Q_1^u = 1; Q_k^u = H_k^u, 2 \leq k \leq j; Q_k^u = \alpha_u k + \beta_u, k \geq j+1$$

As in the case of system (14), the fixed points S^{Φ_j} and S^{Θ_j} , in theorem A.5, are obtained by solving the corresponding $(j-1)$ -dimensional systems of linear equations. If we denote $s^{\Phi_j}(i)$ by w_i^ℓ , $i \geq 2$, and $s^{\Theta_j}(i)$ by w_i^u , $i \geq 2$, then using (18), (63), and (64) we obtain the following bounds on the mean cumulative delay over a session, S :

$$S^\ell \leq S \leq S^u$$

where

$$S^u = p_1 + \sum_{i=2}^j p_i w_i^u + \sum_{i=j+1}^{\infty} p_i (\gamma_u i^2 + \delta_u i + \zeta_u)$$

$$S^\ell = p_1 + \sum_{i=2}^j p_i w_i^\ell + \sum_{i=j+1}^{\infty} p_i (\gamma_\ell i^2 + \delta_\ell i + \zeta_\ell)$$

The bounds S^u, S^ℓ that correspond to $j = 15$ are included in table 3. From the regeneration theorem we have⁽⁴⁾ $D = S/(\lambda H) + 0.5$. The upper bound $D^u = S^u/(\lambda H^\ell) + 0.5$, and the lower bound $D^\ell = S^\ell/(\lambda H^u) + 0.5$ on the mean packet delay D are included in table 3. The bounds for $n = 3$ are also plotted in figure 2. From the table we see that bounds found with $j = 15$ practically coincide even for λ close to $\lambda_0(n)$. (According to theorem A.5, $D^u \rightarrow D^\ell$ as $j \rightarrow \infty$).

Finally, note that the algorithm with $n = 3$ has uniformly better mean delay characteristics, as compared to the algorithm with $n = 2$. From the operation of the algorithm and the fact that the quantity $\lambda_0(n)$ decreases monotonically for $n > 4$, we have every reason to believe that $n = 3$ is the best choice among all n .

5. CONCLUSIONS AND PRIOR WORK

In this paper we have introduced a method for the delay analysis of RMAAS, in which the induced packet delay process is regenerative, and we have demonstrated its wide applicability by applying it to three specific examples. The method is based on a well known result from the theory of regenerative processes, which relates the asymptotic statistics of such processes to quantities that refer only to one cycle of the process. The per cycle quantities, (e.g., mean cycle length, expectation of the sum of the values of the process over a cycle), are evaluated from the solution of infinite dimensional systems of linear equations. In Appendix A, we have given a number of general results concerning the existence, uniqueness and approximation of the solution of such systems, which are of independent interest. Most of these results are generalizations and extensions of results that can be found in the early reference [21].

In applying the method to the three example-algorithms, we have put emphasis on the methodology and rigorous derivations rather than finding short cuts in the analysis of a particular algorithm. In doing so, the essential simplicity of the method may have been obscured. However, to appreciate the simplicity of the method, we note that only by using Lemma A.2, one can obtain with minimal effort:

- 1) A lower bound on the maximum input rate that an algorithm maintains with finite delay, (i.e., a lower bound on the maximum stable throughput induced by the algorithm). Note that for the first two examples of this paper, the found bound coincides with the maximum stable throughput; for the third example, the found bound is very close to the maximum stable throughput [12], and since the induced delays are already at very high values, determining the exact maximum stable throughput is of theoretical interest only.

- 2) Optimal algorithmic parameter choices (e.g., the retransmission probability policy in the ALOHA algorithm, the window size Δ in the "0.487" algorithm, the splitting parameter n in the stack algorithm).
- 3) Initial bounds on the mean packet delay, that can be used (if so desired) to form finite linear systems, whose solution can yield arbitrarily tight bounds on the mean packet delay.

In this paper, we have given explicit results, only for the mean packet delay, when the cumulative packet arrival process is Poisson. These results can be readily extended to the case of an arbitrary memoryless arrival process, as long as the regenerative character of the delay process is preserved. Moreover, the method can be used to compute higher moments of the delay process. This is due to the fact that if $f(x) = x^n$, $n \geq 1$, in theorem 1, then the resulting per cycle quantities are again related to the solution of the infinite linear systems [12]. Thus, the results in Appendix A are applicable.

The algorithms that served as examples in this paper, have been analyzed in a number of studies. From the literature on ALOHA-type algorithms, we mention the work in [6], where the stability properties of the version of the Controlled ALOHA algorithm considered here have been studied, using a Markovian model. The optimal retransmission policy was derived in [6] using Pake's lemma, but the delay analysis problem was not addressed.

The delay characteristics of the "0.487" algorithm have been studied in [10], using a different approach. In contrast to the method in [10], the method proposed here does not require the computation of steady-state probabilities of the underlying Markov chain and, therefore, it is computationally simpler. Furthermore, since our approach is based on the asymptotic properties of regenerative processes, it yields stronger convergence results.

The delay analysis method of this paper was first applied to the n -ary stack algorithm in [12]. The analysis in [12] was stimulated by the approach taken in

[11], where the delay characteristics of the $n = 2$ algorithm were evaluated, using the regenerative formulation used in this paper, in conjunction with the solution of a functional equation. The $n = 2$ algorithm has also been analyzed in [13], where a Markovian model is adopted, in conjunction with the solution to infinite linear systems. We believe that the method proposed here is simpler than both of the above methods. Finally, we note that the study of infinite linear systems, in conjunction with the throughput analysis of the stack algorithm, was initiated in [9].

REFERENCES

- [1] IEEE Trans. Info. Theory, Special Issue on Random-Access Communications, Vol. IT-31, no. 2, March 1985.
- [2] J. L. Massey, "Collision-resolution algorithms and random-access communications", in Multi-User Communications, G. Longo, Ed. New York: Springer-Verlag, CISM Courses and Lecture Series, no. 265, 1981.
- [3] S. Stidham, Jr., "Regenerative processes in the theory of queues, with applications to the alternating-priority queue", Adv. Appl. Prob., vol. 4, pp. 542-577, 1972.
- [4] J. W. Cohen, On Regenerative Processes in Queueing Theory, New York: Springer-Verlag, 1976.
- [5] W. L. Smith, "Renewal theory and its ramifications", J. Roy. Statist. Soc. B20, pp. 243-302, 1958.
- [6] G. Fayolle, E. Gelenbe and J. Labentoulle, "Stability and optimal control of the packet switching broadcast channel", J. Ass. Comput. Mach., vol. 24, pp. 375-386, July 1977.
- [7] R. G. Gallager, "Conflict resolution in random access broadcast networks", in Proc. AFOSR Workshop Commun. Theory and Appl. Provincetown, MA, Sept. 1978.
- [8] B. S. Tsybakov and V. A. Mihailov, "Random multiple access of packets. Part-and-try algorithm", Probl. Peredachi Inform., vol. 16, pp. 65-79, Oct.-Dec. 1980.
- [9] B. S. Tsybakov and N. D. Vvedenskaya, "Stack-algorithm for random multiple access", Probl. Peredachi Inform., vol. 16, pp. 80-94, July-Sept. 1980.
- [10] J. C. Huang and T. Berger, "Delay analysis of the modified 0.487 contention resolution algorithm", Tech. Rep., School of Electrical Engineering, Cornell Univ., Ithaca, NY; IEEE Trans. Commun., July, 1984, submitted for publication.
- [11] G. Fayolle, P. Flajolet, M. Hofri, and P. Jacquet, "The evaluation of packet transmission characteristics in a multi-access collision detection channel with stack collision resolution protocol", Computer Science Dept., Technion, Israel Institute of technology, Haifa, Tech. Rep. 293, Oct. 1983, also in IEEE Trans. Inform. Theory, vol. IT-31, no. 2, pp. 244-254, March 1985.
- [12] L. Merakos and C. Bisdikian, "Delay analysis of the n-ary stack algorithm for a random-access broadcast channel", in Proc. 22nd Allerton Conf. on Commun. Comp., Univ. of Illinois at Urbana-Champaign, Illinois, Oct. 1984, pp. 385-394.
- [13] N. D. Vvedenskaya and B. S. Tsybakov, "Packet delay in the case of a multiple-access stack algorithm", Probl. Peredachi Inform., vol. 20, no. 2, pp. 77-97, April-June, 1984.
- [14] N. Abramson, "The ALOHA system - Another alternative for computer communications", in Proc. 1970 Fall Joint Computer Conf., AFIPS Press, vol. 37, 1970, pp. 281-285.
- [15] N. D. Vvedenskaya and B. S. Tsybakov, "Random multiple access of packets to a channel with errors", Probl. Peredachi Inform., vol. 19, no. 2, pp. 52-68, Apr.-Jun. 1983.
- [16] L. Merakos, "Limited sensing random multiple access using binary feedback", in Proc. 1982 Conf. Inform. Sciences Syst., Johns Hopkins Univ., Baltimore, MD, March 1983, pp. 557-563.
- [17] P. Mathys and P. Flajolet, "Q-ary collision resolution algorithms in random-access systems with free or blocked channel access", IEEE Trans. Inform. Theory, vol. IT-31, no. 2, pp. 217-243, March 1985.

- [18] R. G. Gallager, "A perspective on multiaccess Channels, IEEE Trans. Inform. Theory, vol. IT-31, no. 2, pp. 124-142, March 1985.
- [19] S. Lam and L. Kleinrock, "Packet switching in a multiaccess broadcast channel: Dynamic control procedures", IEEE Trans. Commun., vol. COM-23, pp. 891-904, Sept. 1975.
- [20] S. C. Thomopoulos, "Decentralized estimation and control in random access communication networks: Stability and performance analysis", in Proc. IEEE INFOCOM'85 Conf., Washington, D.C., March 1985, pp. 246-255.
- [21] L. V. Kantorovich and V. I. Krylov, Approximate Methods of Higher Analysis. New York: Interscience, 1958.
- [22] L. Georgiadis and P. Papantoni-Kazakos, "Limited Feedback Sensing Algorithms for the Packet Broadcast Channel", IEEE Trans. Info. Theory, Special Issue on Random Access Communicaitons, Vol. IT-31, no. 2, pp. 280-294, March 1985.

	H_O	H_O^u	W_O	W_O^u	D	D^u
0.05	1.00547	1.00547	0.05789	0.05789	1.651631	1.651631
0.10	1.02463	1.02463	0.141334	0.141334	1.87936	1.87936
0.15	1.06445	1.06445	0.27824	0.27824	2.24265	2.24265
0.20	1.14015	1.14015	0.54174	0.54174	2.87576	2.87576
0.25	1.29097	1.29097	1.17832	1.17832	4.15097	4.15097
0.30	1.66015	1.66015	3.52360	3.52360	7.57485	7.57485
0.35	3.65080	3.69063	38.27660	41.89753	80.13219	33.28928

Table 1

Delays for the Controlled ALOHA

	H_1	H_1^u	W_1	W_1^u	D	D^u
.01	1.00025	1.0003	.015258	.015258	1.5253	1.5255
.05	1.00395	1.00474	.08234	.082346	1.6348	1.6388
.1	1.025	1.030	.18503	.1859	1.796	1.8130
.15	1.060	1.061	.3212	.3245	2.000	2.040
.2	1.1167	1.11367	.5162	.5254	2.270	2.352
.25	1.2069	1.240	.8243	.8468	2.66	2.80
.3	1.356	1.408	1.381	1.434	3.270	3.525
.35	1.627	1.710	2.6088	2.7423	4.358	4.8151
.40	2.2279	2.374	6.6438	6.8603	6.779	7.670
.45	4.487	4.8536	35.012	37.871	16.030	18.754
.47	9.110	9.916	163.698	178.178	35.125	41.613
.48	21.175	23.122	944.35	1031.12	85.086	101.452

Table 2

Delays for the "0.487" Algorithm

	n = 2		n = 3	
	H ^u H	D ^u D	H ^u H	D ^u D
0.05	1.00565	1.68404	1.00619	1.67669
0.10	1.02622	1.96913	1.02788	1.92764
0.15	1.07113	2.44552	1.07265	2.30451
0.20	1.16167	3.33212	1.15587	2.91389
0.25	1.35801	5.29176	1.31346	4.01133
0.30	1.92055	11.38280	1.65242	6.37571
0.35	8.22892	87.28995	2.69508	13.87780

Table 3Delays for the SA₂ and the SA₃

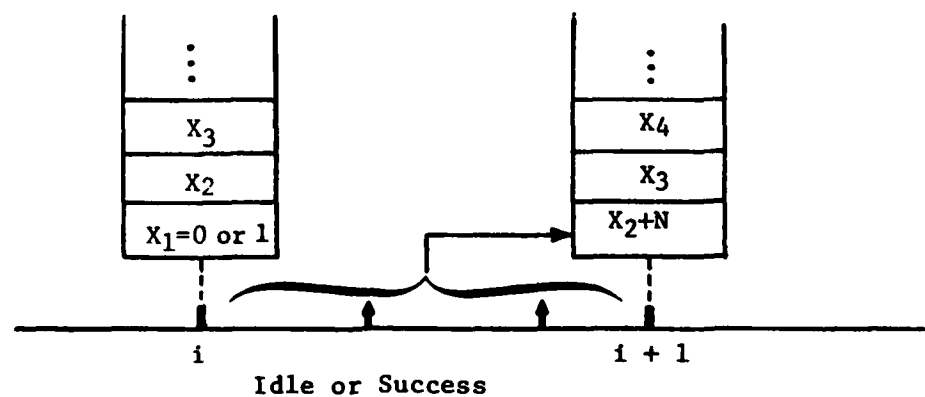
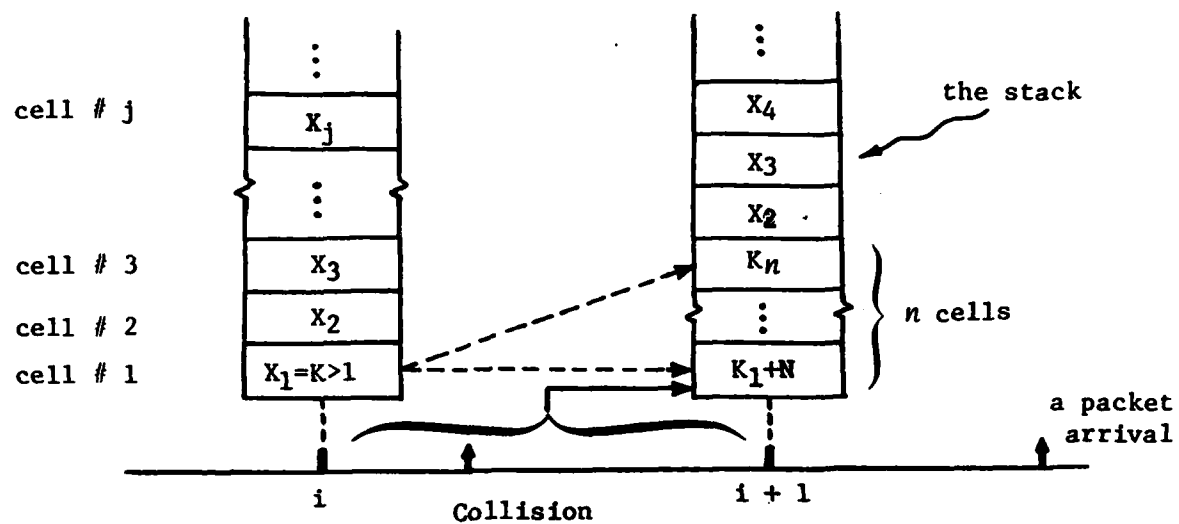


Figure 1

Illustration of the rules of the algorithm using the stack. X_j denotes the number of packet in cell #j, at the beginning of the i -th slot; N denotes the number of new arrivals during the i -th slot, and $K_1 + K_2 + \dots + K_n = K \geq 2$.

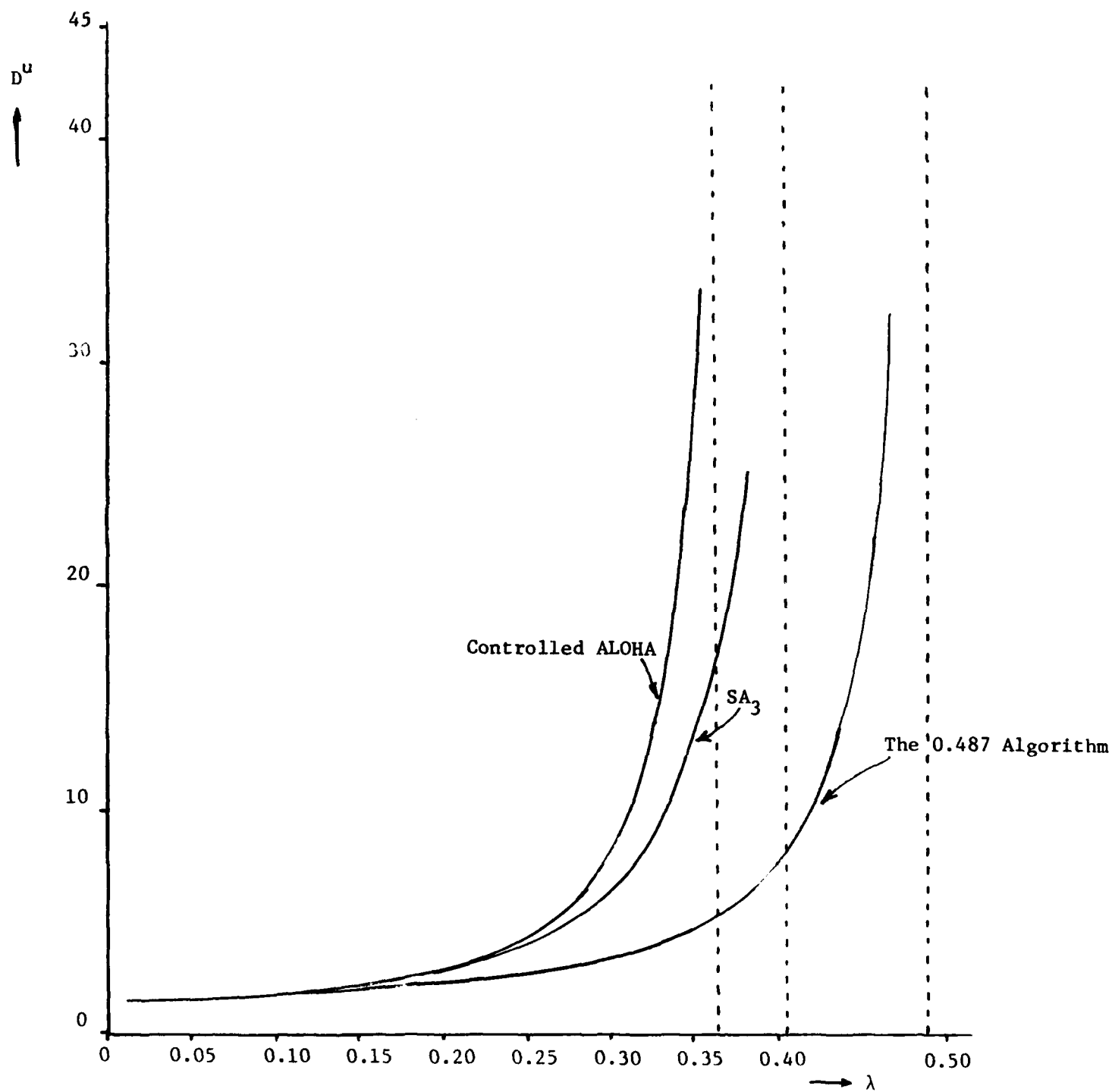


Figure 2

Upper Bounds on Delays--Comparison

APPENDIX A

We present, in a generalized format, some basic results regarding the approximate computation of solutions of infinite dimensionality linear systems [21]. Let A be a denumerable set of indices, and let E be the space of sequences $x = \{x(k)\}: A \rightarrow R$. Given a set $\{c_{ik}^L \in R, b_i^L \in R, i, k \in A\}$, let E^L be the subspace of E defined as follows: $E^L = \{x: \sum_{k \in A} |c_{ik}^L x(k)| < \infty \forall i \in A\}$. We define an operator

$L: E^L \rightarrow E$ as follows: $y(i) = L_i(x) = b_i^L + \sum_{k \in A} c_{ik}^L x(k) \quad i \in A, x \in E^L$. A point $s^L \in E^L$, such that,

$$s^L = L(s^L) \quad (A.1)$$

is called a fixed point of the operator L . (A.1) represents an infinite system of linear equations and a fixed point is a solution to this system. Given an operator L , we define its n -th power L^n as follows: $L^1(x_0) \stackrel{\Delta}{=} L(x_0)$, $L^{n+1}(x_0) \stackrel{\Delta}{=} L(L^n(x_0))$, $n=1,2,\dots$, provided that $x_0 \in E^L$, and $L^n(x_0) \in E^L$, for every $n \geq 1$. The sequence $\{x_n\} \stackrel{\Delta}{=} \{L^n(x_0)\}$, $n=1,2,\dots$ is called the power sequence of L , with initial point x_0 . A fixed point of L that is a pointwise limit of the power sequence of L , with initial point $x_0 = 0$, is called a principal fixed point of L , and is denoted by s_*^L . An operator $U: E^L \rightarrow E$ is called a majorant of L , iff,

$$\begin{aligned} |c_{ik}^L| &\leq c_{ik}^U & i, k \in A \\ |b_i^L| &\leq b_i^U & i \in A \end{aligned}$$

In this case, L is called a minorant of U . The notation $x < x'$, $x \leq x'$, $x, x' \in E$ means that $x(k) < x'(k)$, $(x(k) \leq x'(k))$, $k \in A$. A point $x \in E$ is called positive (nonnegative) iff, $0 < x$ ($0 \leq x$). By $|x|$ we denote the sequence defined by

$|x|(k) = |x(k)|$, $k \in A$. Theorems A1, A2 below are essentially theorems I, II § 2 of [21]. They relate the existence and uniqueness of a fixed point of L , to the existence of a fixed point of a majorant θ of L .

Theorem A.1 If θ is a majorant of L , and θ has a nonnegative fixed point S^θ , then both θ and L have principal fixed points $S_\star^\theta$, $S_\star^L$. Moreover, $0 \leq |S_\star^L| \leq S_\star^\theta \leq S^\theta$.

Theorem A.2 If θ is a majorant of L , and θ has a nonnegative fixed point S^θ , then the principal fixed point $S_\star^L$ of L is unique in the class $E_\star^\theta \subset E^L$, defined as follows.

$$E_\star^\theta = \{x \in E : \sup_{i \in A} \frac{|x(i)|}{s_\star^\theta(i)} < \infty\} \quad (1)$$

Furthermore, $S_\star^L$ is the pointwise limit of any power sequence of L , with initial point any point in $E_\star^\theta$.

Theorem A.3 below relates the existence and uniqueness of a fixed point of L , to the existence of a fixed point of a majorant θ of L , and it is a consequence of the theory of regular systems [21]. Its difference from theorems A1, A2, lies in the fact that, under the stated assumptions in it, we have, $S^\theta = S_\star^\theta$.

Theorem A.3 If θ is a majorant of L , and θ has a positive fixed point S^θ , such that,

$$\inf_{i \in A} \frac{b_1^\theta}{s^\theta(i)} > 0,$$

then $S_\star^\theta = S^\theta$. Therefore, theorem A.2 holds with $S_\star^\theta$ replaced by S^θ .

(1) We adopt the convention:

$$\frac{0}{0} = 1, \quad \frac{a}{0} = \infty, \quad a > 0$$

The following theorem relates the existence and uniqueness of a fixed point of some operator θ_2 , to the existence and uniqueness of such a point for another operator θ_1 , where the latter is not necessarily a majorant of the former.

Theorem A.4 Let θ_1, θ_2 , be two operators such that,

$$(a) \quad \theta_1 c_{ik} > \theta_2 c_{ik} \quad \forall i, k \in A, \quad \theta_1 b_i \in [0, \infty), \forall i \in A$$

(i) If θ_1 has a fixed point S^{θ_1} , and there exists a sequence $g: A \rightarrow \mathbb{R}$, such that,

$$(b) \quad g + S^{\theta_1} > 0$$

$$(c) \quad \sum_{k \in A} \theta_1 c_{ik} g(k) < \infty, \forall i \in A$$

$$(d) \quad \theta_2 b_i < (\theta_1 b_i + g(i) - \sum_{k \in A} \theta_1 c_{ik} g(k)) M, \quad \forall i \in A, \text{ for some } M > 0.$$

Then, θ_2 has a fixed point.

(ii) If (a), (b), (d) hold, for $g = 0$, then the solution of θ_2 is unique in the

class E^{θ_1} , where E^{θ_1} is as defined in Th. A.2.

(iii) If in addition to (a), (b), (d), we have that,

$$(e) \quad g + S^{\theta_1} > 0, \text{ and } \inf_{i \in A} \frac{\theta_1 b_i + g(i) - \sum_{k \in A} \theta_1 c_{ik} g(k)}{S^{\theta_1}(i) + g(i)} > 0,$$

then the fixed point S^{θ_2} of θ_2 is unique in the class $E_g^{\theta_1} \subset E^{\theta_2}$, defined as follows.

$$E_g^{\theta_1} = \{x \in E : \sup_{i \in A} \frac{|x(i)|}{S^{\theta_1}(i) + g(i)} < \infty\}$$

S^{θ_2} is the pointwise limit of any power sequence of θ_2 , with initial point in $E_g^{\theta_1}$.

Proof

Part (i): Let $Y^{0_1} = (S^{0_1} + g) M$. Since $S^{0_1} = O_1(S^{0_1})$, we have that,

$$Y^{0_1} / M - g = O_1(Y^{0_1} / M - g)$$

or

$$Y^{0_1}(i) = M(b_i^{0_1} + g(i) - \sum_{k \in A} c_{ik}^{0_1} g(k)) + \sum_{k \in A} c_{ik}^{0_1} Y^{0_1}(i) \quad (A.2)$$

From (A.2) and (b), we see that the operator O with parameters,

$$b_i^\theta = M(b_i^{0_1} + g(i) - \sum_{k \in A} c_{ik}^{0_1} g(k)), \quad i \in A$$

$$c_{ik}^\theta = c_{ik}^{0_1} \quad i, k \in A,$$

has a nonnegative fixed point $S^\theta = Y^{0_1}$. Because of (a) and (d), O is a majorant of O_2 . From theorem A.1, we conclude that O_2 has a fixed point.

Part (ii) : This follows from A.2, by observing that $S_\star^\theta = M S_\star^{0_1}$, and, therefore,

$$E_\star^\theta = E_\star^{0_1}.$$

Part (iii) : Under condition (e), theorem A.3 is applicable, and shows the uniqueness of the fixed point in $E_g^{0_1}$.

The following lemma is useful in identifying the class within which the fixed point of an operator is unique, in the case where the solution of the majorant is not exactly known.

Lemma A.1 if $S, F : A \rightarrow R$, and,

(a) S, F are nonnegative

$$(b) \sup_{i \in A} \frac{s(i)}{f(i)} < \infty$$

$$(c) \inf_{i \in A} \frac{s(i)}{f(i)} > 0,$$

then $\sup_{i \in A} \frac{|x(i)|}{f(i)} < \infty$, iff $\sup_{i \in A} \frac{|x(i)|}{s(i)} < \infty$, $x \in E$, i.e. the classes

$$E_S = \{x \in A: \sup_{i \in A} \frac{|x(i)|}{s(i)} < \infty\} \text{ and } E_F = \{x \in A: \sup_{i \in A} \frac{|x(i)|}{f(i)} < \infty\}, \text{ coincide.}$$

Proof For the "if" part let

$$\sup_{i \in A} \frac{|x(i)|}{s(i)} = A < \infty \rightarrow |x(i)| \leq A s(i), \quad i \in A \quad (A.3)$$

Because of (b), we have,

$$s(i) \leq B f(i), \quad i \in A, \quad B < \infty \quad (A.4)$$

From (A.3), (A.4), we conclude that, $|x(i)| \leq A B f(i)$, $i \in A$ or, $\sup_{i \in A} \frac{|x(i)|}{f(i)} \leq A B < \infty$

The proof of the "only if" part is similar.

The lemma below is used to establish the existence of a fixed point S^L of an operator L , as well as upper and lower bounds on S^L . Its proof, via induction, is straightforward.

Lemma A.2 Let L be an operator with nonnegative parameters i.e. :

$c_{ik}^L \geq 0 \quad i, k \in A, \quad b_i^k \geq 0, \quad i \in A$ If there exist points $Y^0, X^0 \in E^L$, such that,

- (a) $Y^0 \leq X^0$
- (b) $X^0 \geq L(X^0) \geq 0$
- (c) $Y^0 \leq L(Y^0)$,

then the power sequence of L , with initial points X^0 (Y^0) decreases (increases) monotonically and pointwise, to a fixed point S^L ($\bar{S}^L$). Furthermore, $Y^0 \leq \bar{S}^L \leq S^L \leq X^0$, and $S^L \geq 0$.

It is generally difficult to establish tight bounds on S^L , using the method exhibited by lemma A.2. The following theorem provides an alternative method for the computation of such bounds.

Theorem A.5 Let L be an operator with nonnegative parameters:

$$c_{ik}^L > 0, i, k \in A, \quad b_i^L > 0, i \in A.$$

Let S^L be a nonnegative fixed point of L , for which it is known that $L^L < S^L < U^L, L^L, S^L, U^L \in E^L$. Let $A_j \subset A$, A_j^c be the complement of A_j , and let ϕ_j, F_j, θ_j be the operators with parameters,

$$\phi_j = \theta_j = F_j = \begin{cases} c_{ik}^L, & i, k \in A_j \\ 0, & \text{Otherwise} \end{cases}$$

$$F_j = \begin{cases} b_i^L + \sum_{k \in A_j^c} c_{ik}^L s^L(k), & i \in A_j \\ 0, & \text{Otherwise} \end{cases}$$

$$\phi_j = \begin{cases} \rho_i + \sum_{k \in A_j^c} c_{ik}^L s^L(k), & \rho_i < b_i^L, i \in A_j \\ 0, & \text{Otherwise} \end{cases}$$

$$\theta_j = \begin{cases} \sigma_i + \sum_{k \in A_j^c} c_{ik}^L u^L(k), & \sigma_i > b_i^L, i \in A_j \\ 0, & \text{Otherwise} \end{cases}$$

Then, (a) F_j has a nonnegative fixed point $S_j^{F_j}$, such that,

$$S_j^{F_j}(i) = \begin{cases} s^L(i), & i \in A_j \\ 0, & \text{Otherwise} \end{cases}$$

(b) ϕ_j is a minorant of F_j , and its principal solution $S_j^{\phi_j}$ is such that,

$$0 < S_j^{\phi_j} < S_j^{F_j} < S_j^{F_j}$$

(c) θ_j is a majorant of F_j , and if $\sup_{i \in A_j} \frac{b_i}{L_j} < \infty$, then θ_j has a nonnegative fixed point $S_j^{\theta_j}$, such that,

$$0 < S_j^{F_j} < S_j^{\theta_j} < S_j^{\theta_j}$$

(d) If in addition to the previous conditions, also $S^L > 0$, and

$\inf_{i \in A_j} \frac{b_i^L}{S^L(i)} > 0$, then the operators Φ_j, F_j, Θ_j have respective

unique fixed points, $S^{\Phi_j}, S^{F_j}, S^{\Theta_j}$, in the class

$$E^{F_j} = \left\{ x \in E : \sup_{i \in E} \frac{|x(i)|}{F_j(i)} < \infty \right\}, \quad \text{and} \quad S^{\Phi_j} \leq S^{F_j} \leq S^{\Theta_j}$$

Remark If $\mathcal{D}_i$ is a finite set with $b_i^L > 0, \forall i \in \mathcal{D}_j$, conditions (c) and (d) are clearly satisfied. If in addition, $\rho_i = \sigma_i = b_i^L$, and $A_j \not\subset A$, then it can be

shown [21] that, $S_j^{F_j} \rightarrow S^L$, and $S_j^{\Theta_j} \rightarrow S^L$, pointwise.

The quantities of interest in the various random access algorithms are statistics of random variables, where many of those statistics are fixed points of some operator. Theorem A.6 is used to justify the latter statement and appeared in [15].

Theorem A.6 Let L be an operator with nonnegative parameters, that has a unique

nonnegative fixed point S^L in the class $E_g^L = \{x \in \mathcal{D} : \sup_{i \in \mathcal{D}} \frac{|x_i|}{g(i)} < \infty\}$.

Let $\{x_i^n\}, \{x_i\}, i \in \mathcal{D}, n \in \mathbb{N}$, be families of random variables, such that,

(a) $0 \leq x_i^n \leq x_i$, a.e. for every $i \in \mathcal{D}$

(b) $x_i^n \leq M_n g(i)$, a.e. for every $i \in \mathcal{D}, M_n < \infty$

(c) $f_n \leq L(f_n), f = L(f)$, where $f_n(i) = E\{x_i^n\}, f(i) = E\{x_i\}$

Then, f coincides with the unique fixed point S^L in E_g^L .

Proof

We observe that because of (b), then $f_n \in E_g^L$, and because of (c) and lemma A.2, then $f_n \leq S^L$. Since also f_n increases to f pointwise, because of (a) and the monotone convergence theorem, we conclude that $f \leq S^L$, and therefore $f \in E_g^L$.

The assertion now follows, from the fact that f is a fixed point of L .

APPENDIX B

In section 4.2, we saw that the computation of conditional expectations, $E\{X|d\}$, is required. In this appendix, we show that those conditional expectations can be computed with high accuracy. Let us define,

$E\{X|d, k\}$: The conditional expectation of the random variable X , given that the arrival interval contains k packets, and has length d .

Then,

$$E\{X|d\} = \sum_{k=0}^{\infty} E\{X|d, k\} e^{-\lambda d} \frac{(\lambda d)^k}{k!} \quad (B.1)$$

Using the rules of the algorithm, the quantities $E\{X|d, k\}$ can be computed recursively, as follows.

$$\begin{aligned} E\{\ell/d, k\} &= E\{\ell/1, k\}; \forall d \in F \\ E\{\ell/1, 0\} &= E\{\ell/1, 1\} = 1 \end{aligned} \quad (B.2)$$

$$E\{\ell/1, k\} = (1 + P_1^k + E\{\ell/1, k-1\}P_1^k + \sum_{i=2}^{k-1} E\{\ell/1, i\}P_1^k) / (1 - 2P_0^k); k \geq 2$$

$$\text{where } P_1^k = \binom{k}{1} 2^{-k}$$

$$\begin{aligned} E\{\delta/d, k\} &= d E\{\delta/1, k\}; \forall d \in F \\ E\{\delta/1, 0\} &= E\{\delta/1, k\} = 1 \end{aligned} \quad (B.3)$$

$$E\{\delta/1, k\} = (P_0^k + P_1^k + E\{\delta/1, k\}P_1^k + \sum_{i=2}^{k-1} E\{\delta/1, i\}P_1^k) / (2 \cdot (1 - P_0^k)); k \geq 2$$

$$\begin{aligned} E\{\ell^2/d, k\} &= E\{\ell^2/1, k\}; \forall d \in F \\ E\{\ell^2/1, 0\} &= E\{\ell^2/1, 1\} = 1 \\ E\{\ell^2/1, k\} &= (2E\{\ell/1, k\} + 2E\{\ell/k, k-1\} \cdot P_1^k + P_1^k - 1 + P_1^k E\{\ell^2/1, k-1\} + \sum_{i=2}^{k-1} E\{\ell^2/1, i\}P_1^k) / \\ &\quad (1 - 2P_0^k); k \geq 2 \end{aligned} \quad (B.4)$$

$$E\{\delta^2/d, k\} = d^2 E\{\delta^2/1, k\}; \forall d \in F$$

$$E\{\delta^2/1, 0\} = E\{\delta^2/1, 1\} = 1$$

$$\begin{aligned} E\{\delta^2/1, k\} &= (.25(P_1^k + P_0^k) + .5 P_0^k E\{\delta/1, k\} + .5 P_1^k E\{\delta/1, k-1\} + .25 E\{\delta^2/1, k-1\}P_1^k + \\ &\quad .25 \sum_{i=2}^{k-1} E\{\delta^2/1, i\}P_1^k) / (1 - .5P_0^k); k \geq 2 \end{aligned} \quad (B.5)$$

$$E\{\delta l/d, k\} = d E\{\delta l/1, k\} ; \forall d \in F$$

$$E\{\delta l/1, 0\} = E\{\delta l/1, 1\} = 1$$

$$E\{\delta l/1, k\} = (E\{\delta/1, k\} + .5P_1^k E\{l/1, k-1\} + .5P_0^k E\{l/1, k-1\} + .5P_1^k + .5E\{\delta/1, k-1\}P_1^k + .5P_1^k E\{\delta l/1, k-1\} + .5 \sum_{i=2}^{k-1} E\{\delta l/1, i\} P_1^k) / (1-P_0^k) ; k \geq 2 \quad (B.6)$$

$$E\{N/d, k\} = E\{N/1, k\} ; \forall d \in F$$

$$E\{N/1, 0\} = 0, E\{N/1, 1\} = 1$$

$$E\{N/1, k\} = P_1^k + P_1^k E\{N/1, k-1\} + \sum_{i=2}^{k-1} E\{N/1, i\} P_1^k ; k \geq 2 \quad (B.7)$$

$$E\{\omega/d, k\} = E\{\omega/1, k\} ; \forall d \in F$$

$$E\{\omega/1, 0\} = 0, E\{\omega/1, 1\} = 1 \quad (B.8)$$

$$E\{\omega/1, k\} = (P_1^k + E\{N/1, k\} + P_1^k E\{N/1, k-1\} + P_1^k E\{\omega/1, k-1\} + \sum_{i=2}^{k-1} E\{\omega/1, i\} P_1^k) / (1-2P_0^k) ; k \geq 2$$

$$E\{\psi/d, k\} = d E\{\psi/1, k\} ; \forall d \in F$$

$$E\{\psi/1, 0\} = 0, E\{\psi/1, 1\} = \frac{1}{2} \quad (B.9)$$

$$E\{\psi/1, k\} = (E\{N/1, k\}(1-P_0^k) - P_1^k E\{N/1, k-1\} + .5P_1^k + E\{\psi/1, k-1\}P_1^k + \sum_{i=2}^{k-1} E\{\psi/1, i\}P_1^k) / (2(1-P_0^k)) ; k \geq 2$$

From formulas (B.2)-(B.9), we see that a finite number, M, of terms from the infinite series (B.1), can be easily computed. Also, for large k values, and based on the recursive expressions, simple upper and lower bounds on $E\{X/d, k\}$ can be developed. Those bounds can be used to tightly bound the sum $\sum_{k=M+1}^{\infty} E\{X/d, k\} e^{-\lambda d} \frac{(\lambda d)^k}{k!}$

Remark It can be also proved that

$$E\{N/d\} = \lambda E\{\delta/d\}$$

$$E\{\psi/d\} = \lambda d E\{\delta/d\} - \lambda E\{\delta^2/d\}$$

END

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