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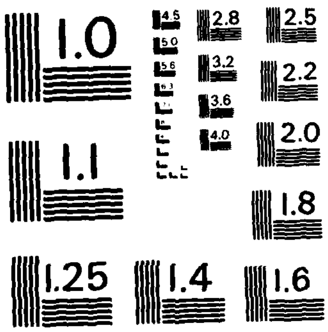

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AN OPTIMAL DESIGN PROBLEM FOR SUBMERGED BODIES

by

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DEPARTMENT OF MATHEMATICAL SCIENCES

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INTRODUCTION:

When a body, floating on the surface of an infinite, ideal, inviscid, irrotational fluid is subjected to a periodic vertical displacement, a wave pattern is created in the fluid and the problem of determining this pattern from a knowledge of the body geometry and applied forces is well known in fluid mechanics. In problems with both partially and fully submerged objects, quantities of physical interest are not only the wave patterns which may be derived from the velocity potential but also functionals of the potential such as added mass and damping factors which measure the distribution of energy in the fluid, e.g. Weyhausen and Laitone [26, p. 567]. These factors are, of course, dependent on the body geometry and the present paper is devoted to showing how these quantities may be optimized over restricted classes of body geometry.

Specifically we will study the problem of the optimal design of a floating body, totally submerged in a fluid of finite depth. In the terminology of optimal control, this is a problem of optimization of geometrical elements (see Lions [17]).

In his classic paper [13], John formulated this problem of a partially immersed heaving body as a boundary value problem for the velocity potential which satisfies Laplace's equation with given Neumann data on the submerged portion of the boundary, a linearized free surface condition on the mean free surface, a homogeneous Neumann condition on the bottom of the fluid container, and a radiation condition. Under certain restrictions on the body shape, in particular that it be convex, smooth, and have normal intersection with the free surface, he proved the existence of a unique solution and formulated an integral equation for the velocity potential evaluated on the body surface. Kleinman has shown in [15] that the geometrical conditions of John may be somewhat relaxed: corners are allowable as are non-normal intersections with the free surface, and convexity is not necessary. However the essential restriction that vertical rays from the free surface intersect the body at most once remains. Relaxation of this restriction has been reported by Ursell [25] for the two dimensional case. In [15] it is also shown that the integral equation, suitably modified possesses unique solutions i.e. methods are provided for showing how one may eliminate the problem of the existence of so-called irregular frequencies. We should point out here that such irregular frequencies are an artifact of the integral equation method; they are particular values of the parameter appearing in the free surface boundary condition for which the integral equation has multiple solutions. However, the original boundary value problem has at most one solution and so we see that an irregular frequency does not correspond to a physical pathology. Moreover, such irregular frequencies are dependent on the shape of the body. As we will be looking at a family of such bodies it will be essential to our analysis that we use an integral equation formulation that is free of such irregular frequencies and one of our first tasks will be to establish such a boundary integral equation for our problem.

When the body is completely submerged, John's uniqueness proof no longer applies, however, Maz'ja [19] has provided a proof for a class of bodies subject again to certain geometric restrictions. In this case the boundary value problem may once more be reformulated as a uniquely solvable integral equation. A virtue of the integral equation formulation of problems for partial differential equations in unbounded regions exterior to a bounded obstacle lies in the apparent advantage that numerical algorithms for bounded domains (the domains of the integral operators) have over algorithms for unbounded domains.

Problems involving optimization with respect to the domain have been treated by a number of authors, among whom we mention Cea and his co-workers [5], [6], Chenais [7] and Pironneau [21], [23]. This last author has published a book [24] on this general topic. But the great majority of the papers in this area treat problems in bounded domains. In this paper, we deal with a problem whose natural setting is an unbounded domain and we will find the boundary integral equation method particularly convenient. With respect to optimization problems in exterior domains, using boundary integral equations, we refer to the recent treatment of the three dimensional inverse acoustic scattering problem by Angell, Colton, and Kirsch [1] and the subsequent numerical treatment of an inverse two dimensional problem by Kirsch [14].

As in [1], our approach to the optimization problem will depend on the reformulation of the original boundary value problem (which here includes not only boundary conditions given on the bounded surface of the body, but also on the bottom and on the free surface which are of infinite extent), as a uniquely solvable integral equation defined on the boundary of the body. The first section of this paper is devoted to this question. While the use of an appropriate radiation condition and Green's theorem will lead to an integral equation, it is important to note that the equivalence of this integral equation formulation and the original boundary value problem depends on a uniqueness theorem for the boundary value problem. This is already a non-trivial problem which is, as yet, not fully understood. As mentioned earlier, for the case of a totally submerged body, Maz'ja [19] has given a proof of uniqueness for a restricted class of bodies. The recent and interesting paper of A. Hulme [21] discusses the result of Maz'ja and most effectively describes the geometric meaning of the result. Since it is essential to have a uniqueness result for the original boundary value problem, we will confine ourselves to this class of bodies; our results can, however, be extended immediately to any situation in which uniqueness obtains.

The second section of the paper is then devoted to the formulation and proof of the existence of a solution, optimal in the sense of being a body with minimal (or maximal) added mass or damping, in any compact set of admissible surfaces. We point out here that this problem is highly non-linear: the cost function involves the solution of the boundary value problem relative to the unknown boundary.

Since the class of all admissible surfaces is convex we study necessary optimality conditions in section 3. There, we compute the Gateaux derivative of the cost functional using, basically, the arguments introduced by Hadamard [11]. The problem of computing the derivative of a "domain functional" appears in a number of works including [2], [5], [6], and [7]. The techniques used here allow us to deal with a much more general situation than that, for example, of Kirsch [14] in so far as we work in three dimensions and make no a priori analyticity assumptions on the parametric representation of the surface. Moreover we do not use techniques of complex variables e.g. conformal mapping arguments or reflection principles.

As one would expect, since we are dealing with a constrained optimization problem, the necessary conditions take the form of a variational inequality and we conclude section 3 with a brief discussion of a numerical procedure for finding solutions of this inequality. It is our intention to present concrete numerical results elsewhere.

1. The Basic Boundary Value Problem

In this discussion we will consider the following geometry: A Cartesian coordinate system is centered at a point on the linearized free surface and oriented so that the free surface coincides with the (x,z)-plane $(y=0)$ and the fluid extends from $y=-h$ to $y=0$. The submerged body will be a simply connected subset of $\mathbb{R}^3$ lying in a strip $\mathbb{R}^2\times[-h,-\epsilon_0]$ whose boundary Γ is a bounded Lyapunov surface of index 1. We will place certain additional restrictions on Γ and these will be indicated as they become necessary. We denote the interior of the body by D^-, and the points of the strip $\mathbb{R}^2\times[-h,0]$ and exterior to the body by D^+. The condition that the surface be Lyapunov of index 1 guarantees, among other things, that there exists a Lipschitz continuous normal $\hat{n}$ at all points of Γ. We emphasize that $\hat{n}$ is oriented so that it points into D^+. Points will be denoted by $p=(x_p,y_p,z_p)$ with cylindrical coordinates $p=(\rho_p,\theta_p,y_p)$ and the subscripts will be omitted if there is no danger of confusion.

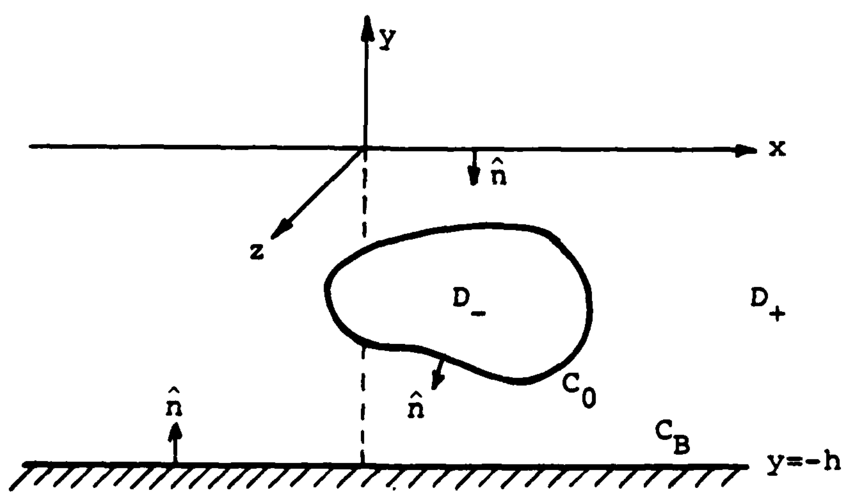

Figure 1

We concern ourselves with the following boundary value problem:

$$
(1.1)\quad \begin{cases} \text{(a)} & \Delta\phi = 0 \quad \text{in } D^{+} \\ \text{(b)} & \dfrac{\partial\phi}{\partial n} + k\phi = 0 \quad \text{on } y=0 \\ \text{(c)} & \dfrac{\partial\phi}{\partial n} = 0 \quad \text{on } y=-h \\ \text{(d)} & \dfrac{\partial\phi}{\partial n} = g \quad \text{on } \Gamma \end{cases}
$$

together with a radiation condition

$$
(1.1)\quad \text{(e)}\quad \frac{\partial\phi}{\partial\rho} - ik_0\phi = o(\rho^{-\frac{1}{2}})
$$

where $g \in C(\Gamma)$ and k_0 is the root with largest real part of the transcendental equation

$$
(1.2)\quad k_n \sinh k_n h = k \cosh k_n h \ .
$$

Recall that $\hat{n}$ always points into D^{+}.

In [19], Maz'ja introduced a restricted class of boundaries for which this boundary value problem has at most one solution. We formulate that theorem here as follows:

<u>Theorem 1.1</u>: Let V be the vector field in $\mathbb{R}^3$ defined by

$$
V = \left[\frac{\rho(y^2-\rho^2)}{\rho^2+y^2}\right]\tilde{\rho} - \left[\frac{2\rho^2 y}{\rho^2+y^2}\right]\tilde{y} \ .
$$

Then the homogeneous boundary value problem (1) with g=0 has only the trivial solution provided

$$
(1.3)\quad V\cdot\hat{n} \geq 0 \quad \text{on } \Gamma \ .
$$

A discussion of this result and its geometric significance may be found in A. Hulme [12]. We will refer to the class of all such surfaces as the Maz'ja class.

Following John [13] we introduce the Green's function for this problem which is normalized to have the form

$$(1.4)\quad \gamma(p,q) = -\frac{1}{2\pi}\frac{1}{|p-q|} + R(p,q)$$

where the function R has bounded derivatives with respect to q for points $p\in\Gamma$ (see [13; p. 96]) and γ satisfies conditions 1.1b, c and e. Using this Green's function to define single and double layer potentials, the usual jump conditions can be established as in the potential-theoretic case since the singular behavior of γ and $\frac{\partial}{\partial n_q}\gamma$ is determined by the first term in (1.4). For convenience, we record these results here:

$$(1.5)\quad \lim_{p\to\Gamma^\pm} \frac{\partial}{\partial n_p}\int_\Gamma u(q)\gamma(p,q)\,d\Gamma_q = \pm u(p) + \int_\Gamma u(p)\frac{\partial\gamma(p,q)}{\partial n_p}\,d\Gamma_q\,,$$

$$(1.6)\quad \lim_{p\to\Gamma^\pm} \int_\Gamma u(q)\frac{\partial}{\partial n_q}\gamma(p,q)\,d\Gamma_q = \mp u(p) + \int_\Gamma u(q)\frac{\partial\gamma(p,q)}{\partial n_q}\,d\Gamma_q$$

where $p\to\Gamma^\pm$ means p approaches Γ from $D^\pm$.

Moreover, if u is a solution of the boundary value problem (1.1) then one may use Green's Theorem to draw the familiar relation

$$(1.7)\quad \int_\Gamma \left[\gamma(p,q)\frac{\partial u(q)}{\partial n_q} - \frac{\partial}{\partial n_q}(\gamma(p,q))u(q)\right]d\Gamma_q = \begin{cases} 2u(p), & p\in D^+ \\ u(p), & p\in\Gamma \\ 0, & p\in D^- \end{cases}$$

If one then uses the boundary condition (1.1d) we have,

$$(1.8) \quad \int_\Gamma \gamma(p,q)g(q)d\Gamma_q - \int_\Gamma u(q) \frac{\partial}{\partial n_q} [\gamma(p,q)]d\Gamma_q = u(p)$$

or, in operator notation

$$(1.9) \quad (I+\bar{K}^*)u = \int_\Gamma \gamma(p,q)g(q)d\Gamma_q$$

where $\bar{K}^*$ is the boundary integral operator with kernel $\frac{\partial\gamma}{\partial n_q}$. We pause to remark that, given a solution, ω, of this integral equation we may represent the solution of the boundary value problem, according to the relation (1.7) by

$$(1.10) \quad u(p) = \tfrac{1}{2}\int_\Gamma \gamma(p,q)g(q)d\Gamma_q - \tfrac{1}{2}\int_\Gamma \omega(q) \frac{\partial\gamma(p,q)}{\partial n} d\Gamma_q, \; p\in D^+$$

and, again using the jump relations one sees easily that

$$(1.11) \quad u\big|_\Gamma = \omega ,$$

which is a direct relationship between the solution of the boundary integral equation and the boundary values taken on by the solution. Such a direct relation does not obtain when one uses a layer approach in which one assumes that the solution u has a representation as a single layer,

$$u(p) = \int_\Gamma \omega(q)\gamma(p,q)d\Gamma_q ,$$

and then the boundary condition and jump relations are employed to obtain an integral equation for ω.

Our first task is to discuss the unique solvability of (1.9).

<u>Theorem 1.2</u>: Let Γ be Lyapunov of index 1 and belong to the Maz'ja class. Let $g \in C(\Gamma)$. Then the integral equation (1.9) has a unique solution.

<u>Proof</u>: Since $\bar{K}^*$ is an integral operator with a weak singularity on the smooth surface Γ it is compact (e.g. [8]) hence the Fredholm alternative applies. Thus the integral equation in question either has a unique solution in $L^2(\Gamma)$ (or in fact in $C(\Gamma)$) or the homogeneous adjoint equation has non-trivial solutions in which case (1.9) has solutions if and only if the function $\int_\Gamma \gamma g d\Gamma$ is orthogonal to all solutions of the homogeneous adjoint equation

$$(1.12) \quad (I+\bar{K})\overset{\Omega}{\omega} = 0 ,$$

where $\bar{K}$ is the boundary integral operator with kernel $\frac{\partial \bar{\gamma}}{\partial n_p}$. We first prove that any solution to (1.12) is orthogonal to $\int_\Gamma \gamma g d\Gamma$ for any $g \in L^2(\Gamma)$.

Assume that $\overset{\Omega}{\omega}$ is a solution of (1.12) and define

$$\psi(p) := \int_\Gamma \bar{\gamma}(p,q)\overset{\Omega}{\omega}(q)d\Gamma_q , \quad p \in D^+ .$$

Certainly $\overset{\Omega}{\omega}$, as a solution of (1.12) is continuous on Γ and the jump condition (1.5) yields

$$\frac{\partial \psi}{\partial n} = \overset{\Omega}{\omega} + \bar{K}\overset{\Omega}{\omega} = 0 \quad \text{on } \Gamma .$$

Hence the function ψ is a solution of the exterior homogeneous submerged floating body problem and so, by

Theorem 1.1, $\psi \equiv 0$ in $\bar{D}^+$. Therefore,

$$0 = \int_\Gamma g(q)\bar{\psi}(q)\,d\Gamma_q = \int_\Gamma \int_\Gamma \gamma(q,p)\mathring{\omega}(p)\,d\Gamma_p g(q)\,d\Gamma_q$$

and because the Green's function is symmetric we obtain

$$\int_\Gamma \int_\Gamma \gamma(p,q)g(q)\,d\Gamma_q \mathring{\omega}(p)\,d\Gamma_p = 0$$

which is the required orthogonality condition. Thus the equation (1.9) always has at least one solution.

To see that (1.9) has at most one solution, we assume that $\mathring{\bar{\omega}}^*$ is a non-trivial solution of $(I+\bar{K}^*)\mathring{\bar{\omega}}^*=0$. Define v in D_- by

$$v(p) = \int_\Gamma \frac{\partial\gamma(p,q)}{\partial n_q}\mathring{\bar{\omega}}^*(q)\,d\Gamma_q, \quad p\in D^- .$$

Then, again using the jump relations we have

$$v(p) = (I+\bar{K}^*)\mathring{\bar{\omega}}^*(p) = 0$$

and so v is a solution of the homogeneous interior Dirichlet problem for the Laplacian. Since there are no non-trivial solutions of this problem, v must vanish in D^- and so $\frac{\partial v}{\partial n^-}(p)=0$ for $p\in\Gamma$

Now define v in D^+ by

$$v(p) = \int_\Gamma \frac{\partial\gamma(p,q)}{\partial n_q}\mathring{\bar{\omega}}^*(q)\,d\Gamma_q, \quad p\in D^+ .$$

Since $\mathring{\bar{\omega}}^*$ is continuous on Γ and Γ is Lyapunov of index 1, $\frac{\partial v}{\partial n^+}(p)$ exists and is equal to $\frac{\partial v}{\partial n^-}(p)=0$ by a Theorem of Lyapunov (Günter [9 ;p. 297]). So again by Maz'ja's uniqueness theorem, v must vanish in D^+ and so, by the jump conditions we have

$$-\mathring{\bar{\omega}}^*+\bar{K}^*\mathring{\bar{\omega}}^* = 0 \; .$$

But $\mathring{\bar{\omega}}^*+\bar{K}\mathring{\bar{\omega}}^*=0$ as well, hence $\mathring{\bar{\omega}}^*=0$ which is a contradiction and the theorem is established.

2. The Optimization Problem

Let $\Gamma_0=\{p\in\mathbb{R}^3 \,\big|\, |p|=1\}$ denote the surface of the unit ball in $\mathbb{R}^3$ and let $C^{1,1}(\Gamma_0)$ denote the space of continuously differentiable functions whose first derivatives satisfy a Lipschitz condition and which is equipped with the usual Hölder norm $||\cdot||_{1,1}$ (see e.g. [8]). We will assume that we are given a family of surfaces which can be described by $C^{1,1}$ parameterizations:

$$(2.1)\quad \Gamma(f) = \{p\in\mathbb{R}^3 | p=f(\tilde{p})\tilde{p}+p_0,\ \tilde{p} = \frac{p-p_0}{|p-p_0|}\}$$

where $f:\Gamma_0\to\mathbb{R}^3$ is an element of $C^{1,1}(\Gamma_0)$ and $p_0\in\mathbb{R}^2\times(-h,-\epsilon_0)$. Let a and b be two positive constants and define the subset $F_{a,b}\subset C^{1,1}(\Gamma_0)$ by

$$(2.2)\quad F_{a,b} = \{f\in C^{1,1}(\Gamma_0) \mid ||f||_{1,1}\le b,\ f(\tilde{p})\tilde{p}+p_0\in\mathbb{R}^2\times(-h,-\epsilon_0) \text{ and } f(\tilde{p})\ge a,\ \ \tilde{p}\in\Gamma_0\}\ .$$

Definition 2.1: A surface S in $\mathbb{R}^3$ will be called admissible provided S can be described by a parameterization $f\in F_{a,b}$ and S is contained in the Maz'ja class (c.f. Theorem 1.1). Note that since each admissible surface is completely determined by the function f, we will henceforth simply refer to "the surface f" although, when convenient, we will use the notation $\Gamma(f)$. Clearly, each admissible surface describes a surface bounding a bounded region which contains a ball of radius a/2 and center p_0 in its interior.

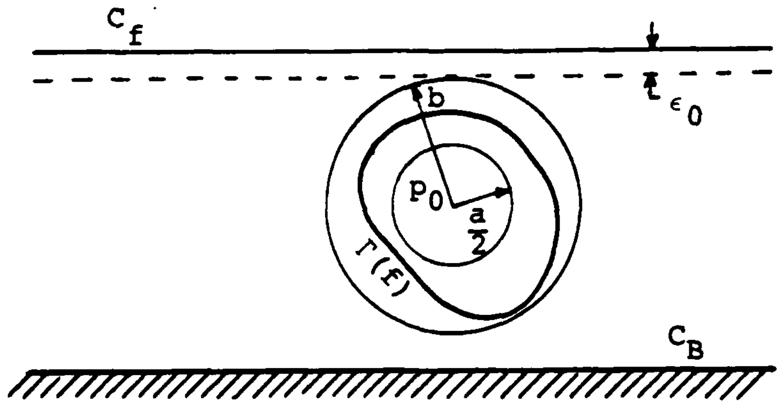

Figure 2

We will, when necessary, denote the region in $\mathbb{R}^2\times(-h,0)$ exterior to an admissible surface f by D_f^+ and that interior to the surface by D_f^-.

Let U_{ad} be a compact subset of admissible surfaces. For example, since the imbedding $C^2(\Gamma_0)\to C^{1,1}(\Gamma_0)$ is compact, we may choose U_{ad} to be the subset of admissible functions in $C^2(\Gamma_0)$. This particular choice leads to a non-linear optimization problem over a closed convex set. The convexity will be advantageous for subsequent numerical considerations.

We now wish to consider a <u>family</u> of boundary value problems of the type discussed in section 1 which may be considered as indexed by U_{ad}:

$$
(2.3)\begin{cases}
\text{(a)} & \Delta\phi(p) = 0, \quad p\in D_f^+ , \\
\text{(b)} & \dfrac{\partial\phi}{\partial n} + k\phi = 0 \quad \text{on } y=0 , \\
\text{(c)} & \dfrac{\partial\phi}{\partial n} = 0 \quad \text{on } y=-h , \\
\text{(d)} & \dfrac{\partial\phi}{\partial n} = g \quad \text{on } \Gamma(f) , \\
\text{(e)} & \dfrac{\partial\phi}{\partial\rho} - ik_0\phi = o(\rho^{-\frac{1}{2}}) .
\end{cases}
$$

Note that, because we are considering a family of boundary value problems, the data g in (2.3d) must be defined throughout the domain formed by the union of all admissible surfaces. This is indeed the case for heaving motion where $g=-\hat{n}\cdot\hat{y}$.

With this understanding, each choice of surface $f\in U_{ad}$ gives rise, according to Theorem 1.1, to a potential $\phi=\phi(p;f)$, $p\in D_f^+$. The class of optimization problems that we discuss below then have the form: Let $J:C(\Gamma_0)\to\mathbb{R}$ be continuous and define a map $J[\cdot]:U_{ad}\to\mathbb{R}$ by

$$(2.4)\quad J[f]:=J(\tilde{\phi}(\cdot;f)),\quad f\in U_{ad}$$

where

$$(2.5)\quad \tilde{\phi}(\tilde{p};f):=\phi(f(\tilde{p})\tilde{p}+p_0;f),\quad \tilde{p}\in\Gamma_0 .$$

The optimization problem is then to find $f_0\in U_{ad}$ such that

$$(2.6)\quad J[f_0]\le J[f],\quad \text{for all } f\in U_{ad}$$

or

$$(2.7)\quad J[f_0]\ge J[f],\quad \text{for all } f\in U_{ad} .$$

We will confine our discussion to the problem of minimization. This is sufficiently general since the problem of maximizing a functional J may always be replaced by that of minimizing -J.

Specific forms of the functional J of (2.4) may be chosen to reflect desirable design criteria. For example, as mentioned in the introduction one may choose J to represent the added mass of the hull. In this case, the problem of interest is that of minimizing the functional J in order to reduce the hydrodynamic force on the ship hull, a goal which is obviously of great importance to ship design. Indeed, it is well known (see [26;pp. 563-567]) that the added mass of a particular hull may be represented by

$$M_a = \mathrm{Re} \int_{\Gamma(f)} \phi(p) \frac{\partial \phi}{\partial n} d\Gamma .$$

This form, in light of the boundary condition (2.3d), leads to the functional

$$(2.8) \quad J[f] = \mathrm{Re} \int_{\Gamma_0} \tilde{\phi}(\tilde{p};f) g(f(\tilde{p})\tilde{p}+p_0) J_f(\tilde{p}) d\Gamma_{\tilde{p}} ,$$

where J_f is the Jacobian of the transformation $p=f(\tilde{p})\tilde{p}+p_0$. As we will see below, this form indeed defines a continuous functional on U_{ad}.

Returning to the more general problem of minimizing the functional (2.4), our first task is to show that the functions $\tilde{\phi}(\cdot;f)$ related to the family of boundary value problems (2.3) via the equation (2.5) depend continuously on the choice of surface f, i.e. that the mapping $f \mapsto \tilde{\phi}(\cdot;f)$ is continuous as a map from U_{ad} into $C(\Gamma_0)$. Recall that the solution of the boundary value problem (2.3) is given in D_f^+ (see also equation (1.10)) by

$$(2.9) \quad \phi(p) = \tfrac{1}{2}\int_{\Gamma(f)} \gamma(p,q) g(q) d\Gamma_q - \tfrac{1}{2}\int_{\Gamma(f)} \frac{\partial}{\partial n_q} \gamma(p,q) u(q) d\Gamma_q ,$$

where u is a solution of the integral equation (1.9) on $\Gamma(f)$. Thus $\phi(p;f)\big|_{\Gamma(f)}=u(p;f)$ and we may write the integral equation (1.9), suppressing the dependence of ϕ on f as

$$(2.10)\quad \phi(p)+\int_{\Gamma(f)} \frac{\partial}{\partial n_q}\gamma(p,q)\phi(q)d\Gamma_q = \int_{\Gamma(f)} \gamma(p,q)g(q)d\Gamma_q .$$

Theorem 1.2 guarantees that this equation possesses one and only one solution for each given $f\in U_{ad}$ and $g\in C(\Gamma(f))$.

We now wish to use the integral equation (2.10) to study the mapping $f\mapsto\tilde{\phi}(\cdot;f)$ as a map from U_{ad} into $C(\Gamma_0)$. In particular we wish to show that this map is continuous and, consequently, a functional J of the form given by the relation (2.4) will assume its minimum value on U_{ad}. To this end, we consider the integral equation

$$(2.11)\quad \phi(f(\tilde{p})\tilde{p}+p_0)+\int_{\Gamma_0} [\frac{\partial}{\partial n_q}\gamma(f(\tilde{p})\tilde{p}+p_0,f(\tilde{q})\tilde{q}+p_0)]\phi(f(\tilde{q})\tilde{q}+p_0)J_f(\tilde{q})d\Gamma_{\tilde{q}}$$
$$= \int_{\Gamma_0} \gamma(f(\tilde{p})\tilde{p}+p_0,f(\tilde{q})\tilde{q}+p_0)g(f(\tilde{q})\tilde{q}+p_0)J_f(\tilde{q})d\Gamma_{\tilde{q}}$$

obtained from the integral equation (2.10) by means of the change of variables $p=f(\tilde{p})\tilde{p}+p_0$. Note that, under our hypotheses, the Jacobians, J_f, of the transformations, are uniformly bounded on Γ_0 with respect to $f\in U_{ad}$ since U_{ad} is a bounded set with respect to the $C^{1,1}$-norm. Indeed, writing $q=f(\theta_q,\phi_q)\tilde{q}+p_0$, $\tilde{q}=(\sin\theta_q \cos\phi_q, \sin\theta_q \sin\phi_q, \cos\theta_q)$, then, for any given $f\in U_{ad}$, the Jacobian has the form

$$(2.12)\quad J_f(\theta_q,\phi_q)=f(\theta_q,\phi_q)[f(\theta_q,\phi_q)^2\ \sin^2\theta_q+(\frac{\partial}{\partial\phi}\ f(\theta_q,\phi_q))^2 +(\frac{\partial}{\partial\theta}\ f(\theta_q,\phi_q))^2\sin^2\theta_q]^{\frac{1}{2}}.$$

Since each term of the second factor on the right is bounded by $||f||^2_{1,1}$ and $|f(\theta_q,\phi_q)|\leq||f||_{1,1}$ we see that

$$(2.13)\quad \sup_{\tilde{q}\in\Gamma_0} |J_f(q)| \leq \sqrt{3}\ ||f||^2_{1,1} \leq b^2\sqrt{3}\ .$$

Introducing the functions $\tilde{\phi}(\tilde{p}):=\phi(f(\tilde{p})\tilde{p}+p_0)$, $g_f(\tilde{p})=g(f(\tilde{p})\tilde{p}+p_0)$, $a_f(\tilde{p},\tilde{q})=[\frac{\partial}{\partial n_q}\ \gamma(f(\tilde{p})\tilde{p}+p_0,\ f(\tilde{q})\tilde{q}+p_0)]J_f(\tilde{q})$ and $b_f(\tilde{p},\tilde{q})=[\gamma(f(\tilde{p})\tilde{p}+p_0;f(\tilde{q})\tilde{q}+p_0)]J_f(\tilde{q})$, then equation (2.11) assumes the convenient form

$$(2.14)\quad \tilde{\phi}(\tilde{p})+\int_{\Gamma_0} a_f(\tilde{p},\tilde{q})\tilde{\phi}(\tilde{q})\,d\Gamma_{\tilde{q}} = \int_{\Gamma_0} b_f(\tilde{p},\tilde{q})g_f(\tilde{q})\,d\Gamma_{\tilde{q}}\ .$$

Let $\mathcal{B}(C(\Gamma_0))$ denote the class of bounded linear operators on $C(\Gamma_0)$. We define two families of operators $\{A_f\}_{f\in U_{ad}}$ and $\{B_f\}_{f\in U_{ad}}$ in $\mathcal{B}(C(\Gamma_0))$ by the relations

$$(2.15)\quad A_f(\psi)(\tilde{p}) = \int_{\Gamma_0} a_f(\tilde{p},\tilde{q})\psi(\tilde{q})\,d\Gamma_{\tilde{q}},\quad \tilde{p}\in\Gamma_0$$

and

$$(2.16)\quad B_f(\psi)(\tilde{p}) = \int_{\Gamma_0} b_f(\tilde{p},\tilde{q})\psi(\tilde{q})\,d\Gamma_{\tilde{q}},\quad \tilde{p}\in\Gamma_0\ .$$

We will establish the continuity of the map $f\mapsto\tilde{\phi}$ by showing that the kernels a_f and b_f are weakly singular for all $f\in U_{ad}$ and that the operators A_f and B_f depend continuously on f i.e. that the maps $f\mapsto A_f$ and $f\mapsto B_f$ of U_{ad} into $\mathcal{B}(C(\Gamma_0))$

are continuous when $\mathcal{B}(C(\Gamma_0))$ is equipped with the uniform operator topology. To this end, we will need two crucial estimates which are given in the following:

Lemma 2.1: For any δ, $0<\delta<\frac{1}{2}$, the kernels a_f and b_f are weakly singular and satisfy the inequalities:

$$(2.17)\quad |a_f(\tilde{p},\tilde{q})-a_g(\tilde{p},\tilde{q})| \leq ||f-g||^{\delta}_{1,1} \frac{c_1}{|\tilde{p}-\tilde{q}|^{1+2\delta}}$$

and

$$(2.18)\quad |b_f(\tilde{p},\tilde{q})-b_g(\tilde{p},\tilde{q})| \leq ||f-g||^{\delta}_{1,1} \frac{c_2}{|\tilde{p}-\tilde{q}|^{1+\delta}}$$

where c_1 and c_2 are positive constants. The proof is quite technical and we relegate it to an appendix. An estimate similar to (2.17) was used in [1]. The present estimates are used in the proof of the first part of the next theorem.

Theorem 2.1: Let $\mathcal{B}(C(\Gamma_0))$ denote the space of bounded linear operators on $C(\Gamma_0)$ equipped with the uniform operator topology and assume that the map $f \mapsto g_f$ from $C^{1,1}(\Gamma_0)$ into $C(\Gamma_0)$ is Hölder continuous. Then the mappings $f \mapsto A_f$ and $f \mapsto B_f$ of $C^{1,1}(\Gamma_0)$ into $\mathcal{B}(C(\Gamma_0))$ are Hölder continuous. Moreover, since the set U_{ad} is compact, the map $f \mapsto \tilde{\phi}(\cdot;f)$ of U_{ad} into $C(\Gamma_0)$, where $\tilde{\phi}(\cdot;f)$ is the unique solution of the boundary integral equation (2.14), is Hölder continuous.

Proof: Consider, first, the mapping $f \mapsto B_f$. Using the estimate (2.18) we have

$$|B_f(\psi)(\tilde{p})-B_g(\psi)(\tilde{p})| \le \int_{\Gamma_0} \left|b_f(\tilde{p},\tilde{q})-b_g(\tilde{p},\tilde{q})\right| |\psi(\tilde{q})| d\Gamma_{\tilde{q}}$$

$$\le ||\psi||_\infty ||f-g||_{1,1}^{\delta} \quad c_2\int_{\Gamma_0} |\tilde{p}-\tilde{q}|^{-1-\delta} d\Gamma_{\tilde{q}}$$

This establishes the Hölder continuity of $f \mapsto B_f$. It then follows from the assumed continuity of the map $f \mapsto g_f$ that the map $f \mapsto G_f$ from U_{ad} into $C(\Gamma_0)$, where G_f is defined by

$$(2.19) \quad G_f(\tilde{p}) := B_f(g_f)(\tilde{p}) = \int_{\Gamma_0} b_f(\tilde{p},\tilde{q})g_f(\tilde{q})d\Gamma_{\tilde{q}} \ ,$$

is continuous. We may now rewrite the integral equation (2.14), using the definition (2.19), as

$$(2.20) \quad \tilde{\phi}(\tilde{p}) + \int_{\Gamma_0} a_f(\tilde{p},\tilde{q})\tilde{\phi}(\tilde{q})d\Gamma_{\tilde{q}} = G_f(\tilde{p}), \quad \tilde{p}\in\Gamma_0 \ .$$

This equation is of exactly the same form as that treated in Theorem 2.2 of [1]. The remainder of the proof follows, almost verbatim, the proof in [1] and we do not repeat the arguments here.

The continuity of the mappings $f \mapsto \tilde{\phi}(\cdot;f)$ and $f \mapsto G_f$ lead immediately to the following corollary which establishes the existence of solutions of our optimization problem.

Corollary 2.1: Under the hypotheses of Theorem 2.1, the functional $J[\cdot]$ defined by the equation (2.8) is continuous as a map from U_{ad} into $\mathbb{R}$ and consequently takes on its absolute minimum on the set U_{ad}.

There is, of course, no guarantee that the solution of our minimization problem is unique. We can, however, formulate a continuous dependence result for the solution set of the minimization problem. To this end, we consider the functional as a continuous map on the product space $C(\Gamma_0)\times C(\Gamma_0)$. For example, we may think of the added mass functional defined in equation (2.8) as a functional depending on the given data g as well as on f.

For a given functional $J(f,g)$ we set

$$(2.21)\quad i_g = \inf\{J(f,g)\,|\,f\epsilon U_{ad}\}$$

and define the set-valued mapping

$$(2.22)\quad \Phi(g) = \{f\epsilon U_{ad}\,|\,J(f,g)=i_g\}\ .$$

Note that, by Theorem 1.2, for all $g\epsilon C(\Gamma_0)$, $\Phi(g)\neq\emptyset$. In fact, if $g\epsilon V\subset C(\Gamma_0)$, V closed, then the graph of this set-valued mapping is closed, as is shown in the following theorem.

Theorem 2.2: Let $V\subset C(\Gamma_0)$ be closed and $J:U_{ad}\times V\to\mathbb{R}$ be continuous. Then

(a) the real-valued function $g\mapsto i_g$ is a continuous map from V into $\mathbb{R}$, and

(b) the set-valued function Φ, defined by the relation (2.21), has a closed graph.

Proof: Suppose that $\{g_n\}$ and $\{f_n\}$ are sequences in V and U_{ad} respectively such that $g_n\to g$ in $C(\Gamma_0)$ and $f_n\to f$ in $C^{1,1}(\Gamma_0)$, and $f_n\epsilon\Phi(g_n)$. Since U_{ad} is compact V is closed, $f\epsilon U_{ad}$, and $\Phi(g)$ is defined. We wish to show, first, that $i_{g_n}\to i$, and second, that $f\epsilon\Phi(g)$.

Now, let $f_0 \in \Phi(g)$. Then, the continuity of J implies that $i_{g_n} \to J(f,g)$. Moreover, by definition of i_{g_n}, $J(f_0,g_n) \geq i_{g_n}$ for all n. Again by continuity of J, $J(f_0,g_n) \to J(f_0,g) = i_g$. Hence $i_g \leq J(f,g) = \lim_{n\to\infty} i_{g_n} \leq \lim_{n\to\infty} J(f_0,g_n) = i_g$. Therefore $J(f,g) = i_g = \lim_{n\to\infty} i_{g_n}$ and $f \in \Phi(g)$.

Corollary 2.2: The optimal solutions of the minimization problem for the functional (2.8) depend continuously on the boundary data g in the sense of Theorem 2.2.

3. Optimality Conditions

We turn now to the question of finding necessary conditions for the constrained optimization problem. Specifically, we will treat the problem of minimizing the functional (2.8) representing added mass, under the additional assumption that the subset of admissible surfaces, U_{ad}, is convex as well as compact. Such an additional requirement is not overly restrictive since, as we remarked in section 2, the class of all admissible surfaces is convex in $C^{1,1}(\Gamma_0)$. If, as suggested there, we take U_{ad} to be the class of all admissible C^2-surfaces, U_{ad} will be convex.

As is well known, the basic necessary condition in this context takes the form of a variational inequality. For convenience, we recall here some basic definitions.

Definition 3.1: Let X be a vector space, Y a normal linear space and A a mapping defined on a subset $D \subset X$ with range in Y. Let $x_0 \varepsilon D$ and $\phi \varepsilon X$. If the limit

$$(3.1) \quad A'(x_0;\phi) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}[A(x_0+\epsilon\phi)-A(x_0)]$$

exists, then A is said to have a Gateaux differential $A'(x_0;\phi)$ at x_0 in the direction ϕ. If $A'(x_0;\phi)$ exists for all $\phi \varepsilon X$, then A is said to be Gateaux differentiable at x_0. If $A'(x_0;\phi)$ exists for all $\phi \varepsilon X$ and if the map $\phi \mapsto A'(x_0;\phi)$ is linear and continuous then this map is called the Gateaux derivative of A at the point x_0. In this case we will denote this map by $A'(x_0)$. Thus $A'(x_0)$ is a linear continuous map from X to Y.

Remark: We note that it makes sense to consider the limit (3.1) only if $x_0+\epsilon\phi\varepsilon D$ for all ϵ sufficiently small.

The basic theorem, stated here in a form slightly weaker than usual (compare e.g. [4, p.119]) is the following:

Theorem 3.1: Suppose X is a vector space, $K\subset X$ is a convex subset, and $A:K\to\mathbb{R}$. If $x_0\varepsilon K$ is such that $A(x_0)\le A(x)$ for all $x\varepsilon K$ and if $A'(x_0;x-x_0)$ exists for all $x\varepsilon K$ then

$$(3.2)\quad A'(x_0;x-x_0)\ \ge 0 \quad \text{for all } x\varepsilon K\ .$$

Proof: Since x_0 minimizes A on K and since K is convex, $x_0+\epsilon(x-x_0)\varepsilon K$ for all $\epsilon\varepsilon[0,1]$ and all $x\varepsilon K$ and

$$A(x_0+\epsilon(x-x_0))\ \ge\ A(x_0)\ .$$

Hence

$$\frac{A(x_0+\epsilon(x-x_0))-A(x_0)}{\epsilon}\ \ge\ 0$$

for all $\epsilon\varepsilon(0,1]$ and all $x\varepsilon K$. Since, by hypothesis, $A'(x_0;x-x_0)$ exists for all $x\varepsilon K$,

$$(3.3)\quad A'(x_0;x-x_0)\ =\ \lim_{\epsilon\to 0^+}\ \frac{A(x_0+\epsilon(x-x_0)-A(x_0)}{\epsilon}\ \ge\ 0\ .$$

Remark: The slight difference in the form of the statement of Theorem 3.1 from that usually found in the literature (e.g. [4] loc.cit.) lies in our assumption that $A'(x_0;h)$ exists only for those $h=(x-x_0)$, $x\varepsilon K$. This is crucial for our considerations since the functional we treat is only defined on a convex set $(C^{1,1}(\Gamma_0)$ functions representing surfaces in the Maz'ja class) and cannot be interpreted outside of this convex set. Thus it is impossible for us to consider arbitrary $C^{1,1}$-perturbations of the form $x_0+\epsilon h$.

The inequality (3.2) is the basic variational inequality which expresses a necessary condition for the problem of minimizing A over K. We remark that, if A is convex, then the condition (3.2) is also sufficient for a (unique) minimum. We will not consider this question in what follows.

The application of this result to our concrete situation $(x=C^{1,1}(\Gamma_0), K=U_{ad})$ will be possible provided we can show that the limit in (3.3) exists for the functional J defined in (2.8). If, indeed, this limit exists for all $f_1 \varepsilon U_{ad}$ i.e. if, for every $f_1 \varepsilon U_{ad}$,

$$J'[f_1;f-f_1]:=\lim_{\epsilon\to 0^+} \frac{J[f_1+\epsilon(f-f_1)]-J[f_1]}{\epsilon}$$

exists for all $f \varepsilon U_{ad}$, then solutions of the corresponding variational inequality will be candidates for optimal solutions. Notice that, in light of the existence theorem of section 2, there is no question concerning the existence of at least one solution of the variational inequality

(3.4) $J'[f_1;f-f_1] \geq 0$, for all $f \varepsilon U_{ad}$,

since this inequality will necessarily be satisfied at any optimal solution in U_{ad}. What is needed here is, first, the calculation of $J'[f_1;f-f_1]$ and, second, the description of an appropriate approximation scheme for the solution of the resulting variational inequality (3.4).

With regard to the differentiability of J we remark that a number of authors, see e.g. [10], [3], [9], [24], and [4] have discussed similar questions with regard to what are generally referred to as "domain functions". Many such discussions (see e.g. [9], [14]) consider two dimensional problems and rely heavily on conformal mapping arguments. Here, we return to Hadamard's technique (see [3]) to establish the form of J'. This derivation depends on a technical lemma which we state below and whose proof, which is somewhat involved, we will relegate an appendix. Recall that the functional J depends on the surface f through the solution of the boundary value problem (2.3). Actually we consider a special class of boundary value problems, namely those for which the given values of the normal derivative on Γ are in fact the values of the normal derivative of a potential function which is defined throughout $\mathbb{R}^2 \times [h,0]$. That is, the boundary condition (1.1d) is replaced by

$$\frac{\partial u}{\partial n} = -\frac{\partial \hat{u}}{\partial n} \quad \text{on } \Gamma$$

where $\nabla^2 \hat{u} = 0$ in $\mathbb{R}^2 \times (-h,0)$. Of course $\hat{u}$ will not satisfy the boundary and radiation conditions appropriate to the problem. In this sense $\hat{u}$ plays the role of the incident field in scattering problems. In this vein we define

$$u^T = u + \hat{u}$$

which behaves as a total field and satisfies the condition

$$\frac{\partial u^T}{\partial n} = 0 \quad \text{on } \Gamma \ . \tag{3.5}$$

Lemma 3.1: Let $\hat{u}$ be harmonic and let u_1 and u_2 be solutions of the boundary value problem (2.3) corresponding to the admissible surfaces $\Gamma_1=\Gamma(f_1)$ and $\Gamma_2=\Gamma(f_2)$ with D_i^- and D_i^+ denoting the interior and exterior respectively of Γ_i, $i=1,2$. Let $u_i^T=u_i+\hat{u}$, $i=1,2$. If we define $\Delta\hat{J}$ by

$$(3.6)\quad \Delta\hat{J} := \int_{\Gamma_1} u_1 \frac{\partial u_1}{\partial n}\, d\Gamma_1 - \int_{\Gamma_2} u_2 \frac{\partial u_2}{\partial n}\, d\Gamma_2$$

then

$$\Delta\hat{J} = \int_{\Gamma_2} \hat{u}\frac{\partial \hat{u}}{\partial n}\, d\Gamma_1 - \int_{\Gamma_1} \hat{u}\frac{\partial \hat{u}}{\partial n} + \int_{D_1^-\setminus(D_1^-\cap D_2^-)} \nabla u_1^T\cdot\nabla u_2^T dV - \int_{D_1^-\setminus(D_1^-\cap D_2^-)} \nabla u_1^T\cdot\nabla u_1^T dV$$

or

$$(3.7)\quad \Delta\hat{J} = \int_{D_2^-\setminus(D_1^-\cap D_2^-)} [|\nabla\hat{u}|^2+(\nabla u_1^T\cdot\nabla u_2^T)]dV - \int_{D_1^-\setminus(D_1^-\cap D_2^-)} [|\nabla\hat{u}|^2+(\nabla u_1^T\cdot\nabla u_1^T)]dV.$$

The proof of this lemma is given in Appendix B.

We may now use this form for $\Delta\hat{J}$ in order to compute $J'[f_1;f-f_1]$ for $f,f_1\varepsilon U_{ad}$.

Theorem 3.2: Let $f_1\varepsilon U_{ad}$ and consider the functional

$$(3.8)\quad J[f] = \mathrm{Re}\int_{\Gamma(f)} \phi(p;f)g(p)d\Gamma_p,\quad f\varepsilon U_{ad},$$

where $\phi(p,f)$ is a solution of the boundary integral equation (1.9) appropriate to the boundary $\Gamma(f)$. Then, for all $f\varepsilon U_{ad}$, the Gateaux differential $J'(f_1;f-f_1)$ exists and has the form

$$(3.9)\quad J'(f_1;f-f_1) = \mathrm{Re}\int_{\Gamma_0} (f(\tilde{p})-f_1(\tilde{p}))(f_1(\tilde{p}))^2\{[\nabla\hat{u}(f(\tilde{p})\tilde{p}+p_0)]^2 + [\hat{n}\times\nabla u^T(f(\tilde{p})\tilde{p}+p_0)]^2\}d\Gamma_{\tilde{p}}\ .$$

Proof: For ease of notation we write $v=f-f_1$ and let $\Gamma_\epsilon=\Gamma(f_1+\epsilon v)$, $0\le\epsilon\le 1$. Then

$$(3.10)\quad \frac{1}{\epsilon}[J(f_1+\epsilon v)-J(f_1)] = \frac{1}{\epsilon}\,\mathrm{Re}\left\{\int_{D_\epsilon^-\setminus(D_1^-\cap D_\epsilon^-)}[|\nabla\hat{u}|^2+(\nabla u_1^T\cdot\nabla u_\epsilon^T)]dV - \int_{D_1^-\setminus(D_1^-\cap D_\epsilon^-)}[|\nabla\hat{u}|^2+(\nabla u_1^T\cdot\nabla u_\epsilon^T)]dV\right\}$$

$$= \frac{1}{\epsilon}\,\mathrm{Re}[I_1-I_2]\ .$$

Consider the first of the integrals in (3.10). Then, iterating the integral we may write, using spherical coordinates,

$$(3.11)\quad I_1:=\int_{D_\epsilon^-\setminus(D_1^-\cap D_\epsilon^-)}|\nabla\hat{u}|^2+(\nabla u_1^T\cdot\nabla u_\epsilon^T)dV$$

$$= \int_{A_1}\left\{\int_{R(\theta,\phi)}[|\nabla\hat{u}|^2+(\nabla u_1^T\cdot\nabla u_\epsilon^T)]r^2dr\right\}d\sigma$$

where A_1 is a measurable subset of Γ_0, $R(\theta,\phi)$ is an interval of r-values and $d\sigma=\sin\theta d\theta d\phi$.

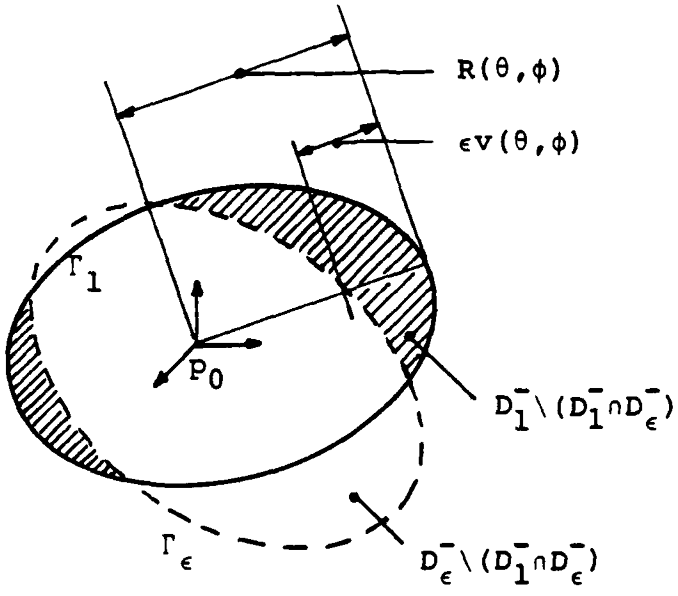

Figure 3

Using the mean value theorem equation (3.11) becomes

$$I_1 = \int_{A_1}\left\{\int_{f_1(\theta,\phi)}^{f_1(\theta,\phi)+\epsilon v(\theta,\phi)} [|\nabla\hat{u}|^2+(\nabla u_1^T\cdot\nabla u_\epsilon^T)]r^2dr\right\}d\sigma$$

$$= \epsilon\int_{A_1} v(\theta,\phi)[f_1(\theta,\phi)+\lambda(\epsilon,\theta,\phi)v(\theta,\phi)]^2[|\nabla\hat{u}|^2+(\nabla u_1^T\cdot\nabla u_\epsilon^T)]\Big|_{r=f_1+\lambda h} d\sigma$$

where $0\le\lambda\le\epsilon v$. Similarly, we arrive at the same expression for I_2 recognizing, in that case, that $f_1(\theta,\phi)+\epsilon v(\theta,\phi)\le f_1(\theta,\phi)$. Note that $A_1\cup A_2=\Gamma_0$, that $\lambda=0(\epsilon)$ and that, according to Theorem 2.1, the solutions of the integral equations depend continuously in the $C^{1,1}$ norm on the functions f. Hence, decomposing ∇u_1^T and ∇u_ϵ^T into normal and tangential directions and recalling that $\frac{\partial u^T}{\partial n} = 0$, we have

$$\frac{1}{\epsilon}[J(f_1+\epsilon v)-J(f_1)] = \mathrm{Re}\int_{\Gamma_0} v(f_1+\epsilon v)^2[(\nabla\hat{u})^2+\nabla u^T\cdot\nabla u_\epsilon^T]\Big|_{r=f_1+\lambda v} d\Gamma_0$$

$$\to \mathrm{Re}\int_{\Gamma_0} vf_1^2[(\nabla\hat{u})^2+(\hat{n}\times\nabla u^T)^2]d\Gamma_0, \text{ as } \epsilon\to 0$$

which completes the proof.

Note that the differential is linear and continuous with respect to $v=f-f_1$ and that the variational inequality takes the concrete form

$$(3.12)\quad \mathrm{Re}\int_{\Gamma_0}(f(\tilde{p})-f_1(\tilde{p}))[f_1(\tilde{p})]^2\{[\nabla\hat{u}(f(\tilde{p})\tilde{p}+p_0)]^2 +[\hat{n}\times\nabla u^T(f(\tilde{p})\tilde{p}+p_0)]^2\}d\Gamma_{\tilde{p}} \ge 0 .$$

Having the explicit form of the Gateaux differential we may indicate here, very briefly, the questions which must be addressed in introducing a concrete numerical procedure for the approximation of solutions of the variational inequality (3.12). Systematic discussions of the general problem may be found, e.g., in the book of Cea [4] or in the review paper of Oden and Kikuchi [21]. We repeat that here there is no question of the existence of a solution of (3.12) in light of Theorem 2.1. To find an approximate solution of (3.12), let $\{S_h\}_{0<h<1}$ be a dense sequence of finite dimensional subspaces of $C^{1,1}(\Gamma_0)$ and let $U_{ad}^h = U_{ad} \cap S_h$. Here as usual, h denotes an appropriate index, normally the mesh parameter. The sequence $\{U_{ad}^h\}_{0<h<1}$ is a family of subsets of $C^{1,1}(\Gamma_0)$ approximating in some sense the constraint set U_{ad}. The approximation of (3.12) will then involve seeking a function $f_{1h} \in U_{ad}^h$ such that

$$(3.13) \quad J'[f_{1h}; f_{1h} - f_h] \geq 0, \quad \text{for all } f_h \in U_{ad}^h .$$

Note that, since U_{ad} is compact, $U_{ad} \cap S_h$ is a compact subset of a finite dimensional space. Again we may appeal to continuity of J on U_{ad} and hence of the restriction of J to $U_{ad} \cap S_h$ to establish the existence of solutions of these approximating inequalities.

Since U_{ad}^h is a subset of the finite-dimensional space S_h, one may express f_{1h} and f_h in the form

$$(3.14) \quad f_{1h} = \sum_{i=1}^{N} \alpha_i \phi_i, \quad f_h = \sum_{i=1}^{N} \beta_i \phi_i ,$$

if $\{\phi_i\}_{i=1}^N$ denotes the collection of basis functions spanning S_h, where α_i and β_i are components of f_1 and f with respect to the basis functions, respectively. By substituting (3.14) into (3.13), and using the linearity of $J'(f;v)$ in v, we then reduce the variational inequality problem to an <u>optimization problem</u> in $\mathbb{R}^N$: Find $\{\alpha_i\}\epsilon K_h$ such that

$$(3.15)\quad \sum_{i=1}^{N}(\alpha_i-\beta_i)J'(\sum_{j=1}^{N}\alpha_j\phi_j)\cdot\phi_i \geq 0, \text{ for all } \{\beta_i\}\epsilon K_h$$

where

$$K_h := \left\{\{\beta_i\}\epsilon\mathbb{R}^N \,\middle|\, f_h = \sum_{i=1}^{N}\beta_i\phi_i\epsilon U_{ad}^h\subset S_h\right\}.$$

Then several standard algorithms in use in the theory of constrained optimization problems may be directly applicable to our study. Four major methods of this type are available. These are the classical method of successive approximations (or fixed point methods), pointwise relaxation methods, penalty methods and Lagrange multiplier methods. However, it is understood that the success of any of the aforementioned schemes, and the questions of convergence of the solutions of the optimization problems (3.15) to a solution of the variational inequality (3.12) will depend heavily on detailed analysis of how the approximate constraint sets K_h, and the subspace S_h are chosen, as well as on establishing Gårding-type estimates for the operator J'. We shall pursue these questions in a separate communication.

APPENDIX A:

In Lemma 2.1 we introduced estimates in order to prove that the operators A_f and B_f, defined by equations (2.14) and (2.15), are continuous on $C(\Gamma_0)$ and that the mappings $f \mapsto A_f$ and $f \mapsto B_f$ from U_{ad} into $\mathcal{B}(C(\Gamma_0))$ are continuous. Here, we outline the calculations needed to establish those estimates.

We begin by remarking that, since the set U_{ad} is bounded in the $C^{1,1}$-norm, the elements of U_{ad} satisfy a uniform Lipschitz condition. Hence, writing $p_f = f(\tilde{p})\tilde{p} + p_0$ and $q_f = f(\tilde{q})\tilde{q} + p_0$, where we affix the suffix on p_f and q_f to accentuate the dependence on f,

$$\text{(A1)} \quad |p_f - q_f| \le |f(\tilde{p})|(\tilde{p}-\tilde{q})| + |f(\tilde{p}) - f(\tilde{q})|\tilde{q} \le M|\tilde{p}-\tilde{q}|$$

where M is independent of f (recall that $|f(\tilde{p})| \le b$ for $\tilde{p} \in \Gamma_0$).

Moreover, we can obtain a lower estimate on $|\tilde{p}-\tilde{q}|$ as follows if we recall that for all $f \in U_{ad}$, $f(\tilde{p}) \ge a$ pointwise on Γ_0:

$$\begin{aligned} |p_f - q_f| &\ge |[f(\tilde{p})(\tilde{p}-\tilde{q})] - [(f(\tilde{q}) - f(\tilde{p}))\tilde{q}]| \\ &\ge \Big| |f(\tilde{p})||\tilde{p}-\tilde{q}| - |f(\tilde{q}) - f(\tilde{p})| \Big| \end{aligned}$$

from which we obtain

$$\text{(A2)} \quad |p_f - q_f| + |f(\tilde{q}) - f(\tilde{p})| \ge |f(\tilde{p})||\tilde{p}-\tilde{q}| \; .$$

But since $f(\tilde{p}) = |f(\tilde{p})|$, we have

$$|f(\tilde{q}) - f(\tilde{p})| = \Big| |f(\tilde{q})| - |f(\tilde{p})| \Big| = \Big| |q_f - p_0| - |p_f - p_0| \Big| \le |p_f - q_f|$$

which, when combined with (A2) and the pointwise lower bound on f, yields

(A3) $\frac{a}{2}|\tilde{p}-\tilde{q}| \le |p_f-q_f|$.

Turning now to an estimate for the kernels a_f, we recall (see the relation (1.4)) that the normal derivative of γ can be written in the form

(A4) $$\frac{\partial}{\partial n_q}[\gamma(p_f,q_f)] = -\frac{n(q_f)\cdot(p_f-q_f)}{2\pi|p_f-q_f|^3} + Q(p_f,q_f)$$

where Q is bounded. The unit normal to the surface has the form

$$n(q) = \tilde{n}(q)/J_f(\theta_q,\phi_q)$$

where (θ_q,ϕ_q) are the spherical angles associated with the point $q\epsilon\Gamma(f)$ and $\tilde{n}(q)$ is the normal vector to the surface $\Gamma(f)$ whose components are computed in the usual way from the parametric equations for the surface in spherical coordinates. Explicitly, if $q=f(\theta_q,\phi_q)\tilde{q}+p_0$ and $\tilde{q}=(\sin\theta_q\ \cos\phi_q,\ \sin\theta_q\ \sin\phi_q,\ \cos\theta_q)$ then

$$\tilde{n}(q) = \frac{\partial q}{\partial\theta_q}\times\frac{\partial q}{\partial\phi_q} = -f\,\frac{\partial f}{\partial\theta_q}\sin\theta_q\tilde{\theta}_q - f\,\frac{\partial f}{\partial\phi_q}\tilde{\phi}_q + f^2\sin\theta_q\,\tilde{q}$$

where

$$\tilde{\theta}_q = \frac{\partial\tilde{q}}{\partial\theta_q} \quad\text{and}\quad \tilde{\phi}_q = (1/\sin\theta_q)\,\frac{\partial\tilde{q}}{\partial\phi_q}\ ,$$

and

$$J_f(\theta_q,\phi_q) = f[f^2\sin^2\theta_q+(\frac{\partial f}{\partial\phi_q})^2+(\frac{\partial f}{\partial\theta_q})^2\sin^2\theta_q]^{\frac{1}{2}}\ .$$

Since the surface is described by a function $f \in C^{1,1}(\Gamma_0)$, the usual Lyapunov-type estimates yield

$$\text{(A5)} \quad |n(q_f)\cdot(p_f-q_f)| \le \alpha |p_f-q_f|^2$$

hence, with (A4),

$$\left|\frac{\partial}{\partial n_q}\gamma(p_f,q_f)\right| \le \frac{c}{|p_f-q_f|} + |Q(p_f,q_f)| \ .$$

With the inequality (A3) and the boundedness of Q, this last estimate becomes

$$\left|\frac{\partial}{\partial n_q}\gamma(p_f,q_f)\right| \le \frac{\tilde{c}}{|\tilde{p}-\tilde{q}|}$$

and, since the Jacobian $J_f(\tilde{q})$ is also bounded (see the estimate (2.13)), we have

$$|a_f(\tilde{p},\tilde{q})| = \left|\frac{\partial}{\partial n_q}\gamma(p_f,q_f)J_f(\tilde{q})\right| < \frac{c_0}{|\tilde{p}-\tilde{q}|} \ .$$

Thus the kernels a_f are weakly singular.

It is much easier to see that the kernels b_f are weakly singular and we omit the simple estimates.

In order to establish the first estimate (2.17) of Lemma 2.1 we use the following estimate from [1 ;p. 52]:

$$\text{(A6)} \quad |n_f(q_f)\cdot(q_f-p_f)-n_g(q_g)\cdot(q_g-p_g)| \le \beta |\tilde{p}-\tilde{q}|^{2(1-\delta)} ||f-g||^{\delta}_{1,1}, \quad 0<\delta<\tfrac{1}{2},$$

where $n_f(q_f)$ is the unit normal to the surface $\Gamma(f)$ at the point $q_f \in \Gamma(f)$ and $n_g(q_g)$ is similarly defined on the surface $\Gamma(g)$.

Now, using (A1), (A3), (A5), and (A6) we have

$$\Delta := \left| \frac{n_f(q_f)\cdot(q_f-p_f)}{|q_f-p_f|^3} - \frac{n_g(q_g)\cdot(q_g-p_g)}{|q_g-p_g|^3} \right|$$

$$= \left| \frac{n_f(q_f)\cdot(q_f-p_f)-n_g(q_g)\cdot(q_g-p_g)}{|q_f-p_f|^3} + n_g(q_g)\cdot(q_g-p_g) \cdot \left(\frac{1}{|q_f-p_f|^3} - \frac{1}{|q_g-p_g|^3}\right) \right|$$

$$\leq \hat{c}|\tilde{p}-\tilde{q}|^{-1-2\delta}||f-g||^{\delta}_{1,1}+\hat{\hat{c}}|\tilde{p}-\tilde{q}|^2 \cdot \left| \frac{[|q_g-p_g|^3-|q_f-p_f|^3]^{1-\delta}[|q_g-p_g|^3-|q_f-p_f|^3]^{\delta}}{|\tilde{p}-\tilde{q}|^6} \right|$$

or, since both $|q_g-p_g|$ and $|q_f-p_f|$ satisfy (A1),

(A7) $$\Delta \leq \hat{c}|\tilde{p}-\tilde{q}|^{-1-2\delta}||f-g||^{\delta}_{1,1}+\hat{d}|\tilde{p}-\tilde{q}|^{-4}|\tilde{p}-\tilde{q}|^{3-3\delta}\left||q_g-p_g|^3-|q_f-p_f|^3\right|^{\delta}.$$

The last factor in the second term may be estimated as follows where we write $\Delta f=q_f-p_f=f(\tilde{q})\tilde{q}-f(\tilde{p})\tilde{p}$, and similarly for g:

$$\left||g(\tilde{q})\tilde{q}-g(\tilde{p})\tilde{p}|^3-|f(\tilde{q})\tilde{q}-g(\tilde{p})\tilde{p}|^3\right| = \left||\Delta g|-|\Delta f|\right| \left||\Delta g|^2+|\Delta g||\Delta f|+|\Delta f|^2\right|$$

$$= A\cdot B \ .$$

Then

(A8) $$A \leq |\Delta g-\Delta f| \leq |g(\tilde{q})-f(\tilde{q})|+|g(\tilde{p})-f(\tilde{p})| \leq 2||f-g||_{1,1}$$

while each term in B is of the form $|f(\tilde{p})\tilde{p}-f(\tilde{q})\tilde{q}|<c|\tilde{p}-\tilde{q}|$ and so $B\leq 3c|\tilde{p}-\tilde{q}|^2$.

Combining this last result with (A7) then yields

$$\Delta \leq [c_1|\tilde{p}-\tilde{q}|^{-1-2\delta}+c_2|\tilde{p}-\tilde{a}|^{-(1+\delta)}]||f-g||^{\delta}_{1,1}$$

which establishes the estimate (2.17).

In order to establish the last estimate (2.18) we write

$$\text{(A9)}\quad |b_f(\tilde{p},\tilde{q})-b_g(\tilde{p},\tilde{q})| = |\gamma(p_f,q_f)J_f(\tilde{q})-\gamma(p_g,q_g)J_g(\tilde{q})|$$

$$\leq |\gamma(p_f,q_f)-\gamma(p_g,q_g)||J_f(\tilde{q})|+|\gamma(p_g,q_g)||J_f(\tilde{q})-J_g(\tilde{q})|.$$

Since, according to the estimate (2.13), the Jacobian is bounded we may estimate the first term on the right hand side of the inequality (A9), by

$$B_1:=|\gamma(p_f,q_f)-\gamma(p_g,q_g)||J_f(\tilde{q})| \leq \beta\left|\frac{1}{|p_f-q_f|}-\frac{1}{|p_g-q_g|}\right|$$

since R is bounded. Then

$$B_1\leq\beta\frac{||p_g-q_g|-|p_f-q_f||}{|p_f-q_f||p_g-q_g|} \leq \beta\left|\frac{[|p_g-q_g|-|p_f-q_f|]^{1-\delta}[|p_g-q_g|-|p_f-q_f|]^{\delta}}{a^2|\tilde{p}-\tilde{q}|^2}\right|$$

$$\leq \frac{2\beta M}{a^2}\frac{1}{|\tilde{p}-\tilde{q}|^{1+\delta}}[|g(\tilde{q})-f(\tilde{q})|+|g(\tilde{p})-f(\tilde{p})|] \quad |\tilde{p}-\tilde{q}|^{-(1+\delta)}||f-g||_{1,1}^{\delta}$$

or, as in (A8)

$$\text{(A10)}\quad B_1 \leq \tilde{\beta}|\tilde{p}-\tilde{q}|^{-(1+\delta)}||f-g||_{1,1}^{\delta}.$$

For the second term on the right of (A9) we have the inequalities

$$B_2:=|\gamma(p_g,q_g)||J_f(\tilde{q})-J_g(\tilde{q})| \leq \frac{\tilde{\tilde{\beta}}}{|\tilde{p}-\tilde{q}|}|J_f(\tilde{q})-J_g(\tilde{q})|.$$

Examining the second factor and using the explicit form for the Jacobian given in (2.12), we have

$$
\begin{aligned}
|J_f-J_g| \le\ & |f-g|\,[f^2\sin^2\theta+f_\phi^2+f_\theta^2\sin^2\theta]^{\frac{1}{2}}+|g|\,\Big|[f^2\sin^2\theta+f_\phi^2+f_\theta^2\sin^2\theta]^{\frac{1}{2}} \\
& \qquad -[g^2\sin^2\theta+f_\phi^2+f_\theta^2\sin^2\theta]^{\frac{1}{2}}\Big| \\
\le\ & \rho_1||f-g||_{1,1}+\rho_2\,\frac{[f^2\sin^2\theta+f_\phi^2+f_\theta^2\sin^2\theta]-[g^2\sin^2\theta+g_\phi^2+g_\theta^2\sin^2\theta]}{[f^2\sin^2\theta+f_\phi^2+f_\theta^2\sin^2\theta]^{\frac{1}{2}}+[g^2\sin^2\theta+g_\phi^2+g_\theta^2\sin^2\theta]^{\frac{1}{2}}} \\
\le\ & \rho_1||f-g||_{1,1}+\rho_2\left\{\frac{|f^2-g^2|\,|\sin\theta|^2}{(|f|+|g|)\,|\sin\theta|}+\frac{|f_\phi^2-g_\phi^2|}{|f_\phi|+|g_\phi|}\right. \\
& \qquad \left.+\frac{|f_\theta^2-g_\theta^2|\,|\sin\phi|^2}{(|f_\theta|+|g_\theta|)\,|\sin\phi|}\right\} \\
\le\ & \rho_1||f-g||_{1,1}+\rho_2\left\{\frac{(|f|+|g|)\,|f-g|}{|f|+|g|}+\frac{(|f_\phi|+|g_\phi|)\,|f_\phi-g_\phi|}{|f_\phi|+|g_\phi|}\right. \\
& \qquad \left.+\frac{(|f_\theta|+|g_\theta|)\,|f_\theta-g_\theta|}{|f_\theta|+|g_\theta|}\right\} \\
\le\ & (\rho_1+3\rho_2)\,||f-g||_{1,1}\ .
\end{aligned}
$$

Hence $B_2 \le \dfrac{c_4}{|\tilde{p}-\tilde{q}|}\,||f-g||_{1,1}^{\delta}$ and this, combined with the estimate (A10), yields

$$
\text{(A11)}\quad |b_f(\tilde{p},\tilde{q})-b_g(\tilde{p},\tilde{q})| \le \left(\frac{c_3}{|\tilde{p}-\tilde{q}|^{1+\delta}}+\frac{c_4}{|\tilde{p}-\tilde{q}|}\right)||f-g||_{1,1}^{\delta}
$$

from which Equation (2.18) follows.

APPENDIX B:

This appendix is devoted to the proof of Lemma 3.1. The functional $J(\Gamma)=\int_\Gamma u\frac{\partial u}{\partial n}\, d\Gamma$ where u is the solution of the boundary value problem (2.3) corresponding to the particular surface is often referred to as a domain functional. As we recorded in the introduction, the consideration of optimization problems involving such functionals goes back to Hadamard [11]. Other references have been cited in the introduction. Here we return to the strategy of Hadamard in order to establish our result.

Proof of Lemma 3.1:

We begin by recalling some notation: $\hat{u}$ is an harmonic function in $\mathbb{R}^2\times(-h,0)$ while, for i=1,2, Γ_i is a bounded surface of class $C^{1,1}$ and u_i is the corresponding solution of the boundary value problem (2.3), i.e.,

(B1) $\Delta u_i = 0$ in D_i^+ ,

(B2) $\frac{\partial u_i}{\partial n} + ku_i = 0$ on $y=0$,

(B3) $\frac{\partial u_i}{\partial n} = 0$ on $y=-h$,

(B4) $\frac{\partial u_i}{\partial n} = -\frac{\partial \hat{u}}{\partial n}$ on Γ_i ,

and

(B5) $\frac{\partial u_i}{\partial r} - ik_0u_i = 0(r^{-\frac{1}{2}})$.

We divide our discussion into two cases:

<u>Case I</u>: Γ_1 lies entirely inside Γ_2.

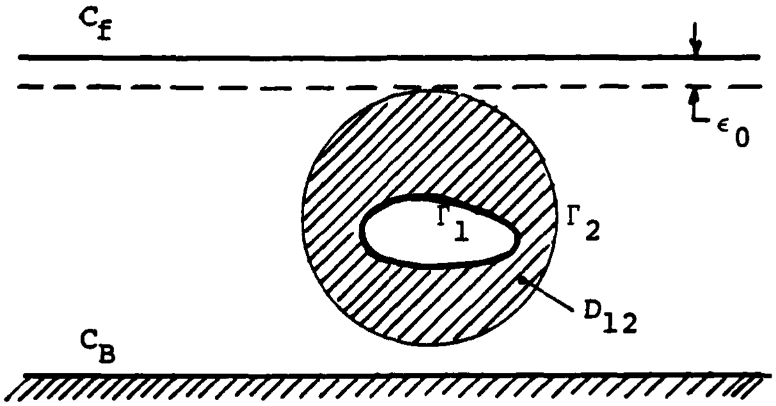

Figure 4

In this case, we denote the shell determined by Γ_1 and Γ_2 by D_{12}. Notice that, initially, u_2 is defined only on and exterior to Γ_2. However, we may extend u_2 inside Γ_2 as a C^1 function although this extended function will not, in general, satisfy Laplace's equation interior to Γ_2. For details, the reader is referred to Theorem 16.IV of Miranda [10]. We will assume, throughout the remainder of this proof, that all functions are so extended, in accordance with Miranda's method, as to make the expressions which we will encounter meaningful. Writing $u_i^T=u_i+\hat{u}$, $i=1,2$, and recognizing that $\partial u_1^T/\partial n=0$ on Γ_1 and that $\Delta u_1=0$ exterior to Γ_1 and, in particular in D_{12}, we have, with the divergence theorem,

$$\int_{D_{12}} \nabla u_1^T \cdot \nabla u_2^T dV = \int_{D_{12}} \nabla \cdot (u_1^T \nabla u_2^T) dV = \int_{\Gamma_2} u_2^T \frac{\partial u_1^T}{\partial n} d\Gamma$$

$$= \int_{\Gamma_2} \hat{u} \frac{\partial \hat{u}}{\partial n} d\Gamma + \int_{\Gamma_2} u_2 \frac{\partial \hat{u}}{\partial n} d\Gamma + \int_{\Gamma_2} \hat{u} \frac{\partial u_1}{\partial n} d\Gamma + \int_{\Gamma_2} u_2 \frac{\partial u_1}{\partial n} .$$

Using the boundary condition (B4),

$$\text{(B6)}\quad \int_{D_{12}} \nabla u_1^T \cdot \nabla u_2^T dV = \int_{\Gamma_2} \hat{u}\, \frac{\partial \hat{u}}{\partial n}\, d\Gamma - \int_{\Gamma_2} u_2\, \frac{\partial u_2}{\partial n}\, d\Gamma + \int_{\Gamma_2} \hat{u}\, \frac{\partial u_1}{\partial n} + \int_{\Gamma_2} u_2\, \frac{\partial u_1}{\partial n}\ .$$

But

$$\text{(B7)}\quad \int_{\Gamma_2} (\hat{u}\, \frac{\partial u_1}{\partial n} - u_1\, \frac{\partial \hat{u}}{\partial n}) d\Gamma - \int_{\Gamma_1} (\hat{u}\, \frac{\partial u_1}{\partial n} - u_1\, \frac{\partial \hat{u}}{\partial n}) d\Gamma = 0$$

since $\hat{u}$ and u_1 are both harmonic in D_{12}. (Note carefully that we always take the normal to Γ_i as the exterior normal. This accounts for the choice of signs in (B7).) We may now use (B7) to replace the third integral on the right hand side of (B6). Hence

$$\text{(B8)}\quad \int_{D_{12}} \nabla u_1^T \cdot \nabla u_2^T dV = \int_{\Gamma_2} \hat{u}\, \frac{\partial \hat{u}}{\partial n}\, d\Gamma - \int_{\Gamma_2} u_2\, \frac{\partial u_2}{\partial n}\, d\Gamma$$

$$+ \int_{\Gamma_2} u_i\, \frac{\partial \hat{u}}{\partial n}\, d\Gamma + \int_{\Gamma_1} \hat{u}\, \frac{\partial u_1}{\partial n}\, d\Gamma - \int_{\Gamma_1} u_1\, \frac{\partial \hat{u}}{\partial n}\, d\Gamma + \int_{\Gamma_2} u_2\, \frac{\partial u_1}{\partial n}\, d\Gamma\ .$$

But

$$\frac{\partial u_i}{\partial n} = -\frac{\partial \hat{u}}{\partial n} \text{ on } \Gamma_i,\ i=1,2, \quad \text{and} \quad \int_{\Gamma_2} (u_2\, \frac{\partial u_1}{\partial n} - u_1\, \frac{\partial u_2}{\partial n}) d\Gamma = 0\ .$$

Indeed this last relation follows since both u_1 and u_2 are harmonic outside Γ_2 and we have, therefore,

$$\int_{\Gamma_2} (u_2\, \frac{\partial u_1}{\partial n} - u_1\, \frac{\partial u_2}{\partial n}) d\Gamma = -\int_{C_f} (u_2\, \frac{\partial u_1}{\partial n} - u_1\, \frac{\partial u_2}{\partial n}) dC_f$$

$$+ \lim_{R\to\infty} \int_{\partial D_R} (u_2\, \frac{\partial u_1}{\partial n} - u_1\, \frac{\partial u_2}{\partial n}) ds$$

where ∂D_R is the surface of the cylinder D_R of radius R. Noting that u_1 and u_2 satisfy the free surface condition (B2), the integral over the free surface is simply

$$\int_{C_f} \left(u_2 \frac{\partial u_1}{\partial n} - u_1 \frac{\partial u_2}{\partial n}\right) dC_f = \int_{C_f} [u_2(-ku_1) - u_1(-ku_2)]\, dC_f = 0 \;.$$

Moreover, the radiation condition insures that the second term on the right is zero. For detailed calculations, see Kleinman [15;pp. 7-9]. Therefore the relation (B8) reduces to

$$\text{(B9)} \quad \int_{D_{12}} \nabla u_1^T \cdot \nabla u_2^T dV = \int_{\Gamma_1} u_1 \frac{\partial u_1}{\partial n}\, d\Gamma - \int_{\Gamma_2} u_2 \frac{\partial u_2}{\partial n}\, d\Gamma + \int_{\Gamma_2} \hat{u} \frac{\partial \hat{u}}{\partial n}\, d\Gamma - \int_{\Gamma_1} \hat{u} \frac{\partial \hat{u}}{\partial n}$$

or, rearranging terms,

$$\text{(B10)} \quad \Delta\hat{J} = \int_{\Gamma_1} u_1 \frac{\partial u_1}{\partial n}\, d\Gamma - \int_{\Gamma_2} u_2 \frac{\partial u_2}{\partial n}\, d\Gamma = \int_{D_{12}} \nabla u_1^T \cdot \nabla u_2^T dV + \int_{D_{12}} |\nabla \hat{u}|^2 dV \;.$$

This is exactly the conclusion of Lemma 3.1 for Case I since, in this case, $D_2^- \setminus (D_1^- \cap D_2^-) = D_{12}$ and $D_1^- \setminus (D_1^- \cap D_2^-) = \emptyset$.

<u>Case II</u>: The surfaces Γ_1 and Γ_2 intersect.

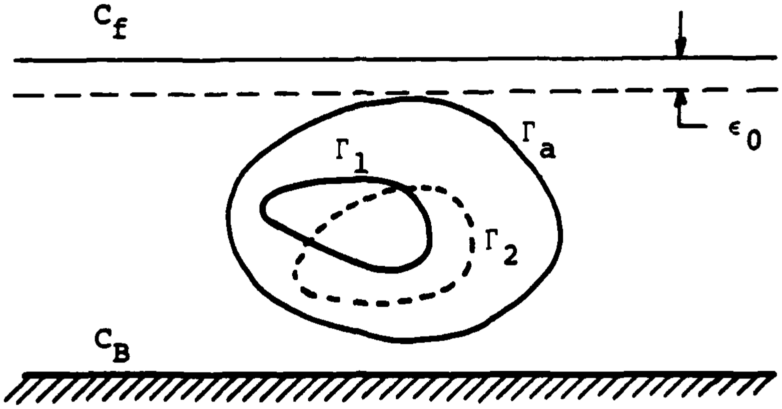

Figure 5

Here, we use the device of Hadamard to extend the result of Case I (see e.g. Bergman and Schiffer [3]). Thus, we introduce an auxiliary surface Γ_0 containing both Γ_1 and Γ_2 in its interior and apply the results of the previous case. With self-explanatory notations, we may write

$$\text{(B11)} \quad \int_{\Gamma_1} u_1 \frac{\partial u_1}{\partial n} d\Gamma - \int_{\Gamma_a} u_a \frac{\partial u_a}{\partial n} d\Gamma = \int_{D_{1a}} \nabla u_1^T \cdot \nabla u_a^T dV + \int_{\Gamma_a} \hat{u} \frac{\partial \hat{u}}{\partial n} d\Gamma - \int_{\Gamma_1} \hat{u} \frac{\partial \hat{u}}{\partial n} d\Gamma$$

and

$$\text{(B12)} \quad \int_{\Gamma_2} u_2 \frac{\partial u_2}{\partial n} d\Gamma - \int_{\Gamma_a} u_a \frac{\partial u_a}{\partial n} d\Gamma = \int_{D_{2a}} \nabla u_2^T \cdot \nabla u_a^T dV + \int_{\Gamma_a} \hat{u} \frac{\partial \hat{u}}{\partial n} d\Gamma - \int_{\Gamma_2} \hat{u} \frac{\partial \hat{u}}{\partial n} d\Gamma \ .$$

Subtracting equation (B12) from (B11) we obtain

$$\text{(B13)} \quad \int_{\Gamma_1} u_1 \frac{\partial u_1}{\partial n} d\Gamma + \int_{\Gamma_2} u_2 \frac{\partial u_2}{\partial n} d\Gamma = \int_{D_{1a}} \nabla u_1^T \cdot \nabla u_a^T dV - \int_{D_{2a}} \nabla u_a^T \nabla u_a^T dV$$
$$+ \int_{\Gamma_2} \hat{u} \frac{\partial \hat{u}}{\partial n} d\Gamma - \int_{\Gamma_1} \hat{u} \frac{\partial \hat{u}}{\partial n} d\Gamma \ .$$

Notice that the right hand side is apparently dependent on the surface Γ_a and the solution u_a while the left hand side is not. To see that this dependence is illusory, we examine the first two integral terms on the right hand side of (B13). By appropriate addition and subtraction we have

$$\text{(B14)} \int_{D_{1a}} \nabla u_1^T \cdot \nabla u_a^T dV - \int_{D_{2a}} \nabla u_2^T \cdot \nabla u_a^T dV = \left[\int_{D_{1a}} \nabla u_1^T \cdot (\nabla u_a^T - \nabla u_2^T)\, dV + \int_{D_{1a}} \nabla u_1^T \cdot \nabla u_2^T dV \right]$$

$$- \left[\int_{D_{2a}} \nabla u_2^T \cdot (\nabla u_a^T - \nabla u_1^T)\, dV + \int_{D_{1a}} (\nabla u_2^T \cdot \nabla u_1^T)\, dV \right]$$

$$= \int_{D_{1a}} \nabla u_1^T \cdot (\nabla u_a^T - \nabla u_2^T)\, dV - \int_{D_{2a}} \nabla u_2^T \cdot (\nabla u_a^T - \nabla u_1^T)\, dV$$

$$+ \int_{D_2^- \setminus (D_1^- \cap D_2^-)} \nabla u_1^T \cdot \nabla u_2^T dV - \int_{D_1^- \setminus (D_1^- \cap D_2^-)} \nabla u_1^T \cdot \nabla u_2^T dV \,.$$

But

$$\int_{D_{1a}} \nabla u_1^T \cdot (\nabla u_a^T - \nabla u_2^T)\, dV - \int_{D_{2a}} \nabla u_2^T \cdot (\nabla u_a^T - \nabla u_1^T)\, dV$$

$$= \int_{\Gamma_a} (u_a^T - u_2^T) \frac{\partial u_1^T}{\partial n}\, d\Gamma - \int_{\Gamma_a} (u_a^T - u_1^T) \frac{\partial u_2^T}{\partial n}\, d\Gamma$$

since $\frac{\partial u_1^T}{\partial n} = 0$ on Γ_1 and $\frac{\partial u_2^T}{\partial n} = 0$ on Γ_2. Also $\frac{\partial u_a^T}{\partial n} = 0$ on Γ_a and thus, after rearranging terms, the relation (B14) becomes

$$\text{(B15)} \int_{D_{1a}} \nabla u_1^T \cdot \nabla u_a^T dV - \int_{D_2} \nabla u_2^T \cdot \nabla u_a^T dV$$

$$= \int_{\Gamma_a} \left[u_a^T \frac{\partial}{\partial n}(u_1^T - u_2^T) - (u_1^T - u_2^T) \frac{\partial u_a^T}{\partial n} + u_1^T \frac{\partial u_2^T}{\partial n} - u_2^T \frac{\partial u_1^T}{\partial n} \right] d\Gamma$$

$$+ \int_{D_2^- \setminus (D_1^- \cap D_2^-)} \nabla u_1^T \cdot \nabla u_2^T dV - \int_{D_1^- \setminus (D_1^- \cap D_2^-)} \nabla u_1^T \cdot \nabla u_2^T dV \,.$$

Now, writing $u_i^T = u_i + \hat{u}$, $i=a,1,2$, we have

$$\text{(B16)}\quad \int_{\Gamma_a}\left[u_a^T \frac{\partial}{\partial n}(u_1-u_2)-(u_1-u_2)\frac{\partial u_a^T}{\partial n} + u_1^T \frac{\partial u_2^T}{\partial n} - u_2^T \frac{\partial u_1^T}{\partial n}\right]d\Gamma$$

$$= \int_{\Gamma_a}\Bigg[\hat{u} \frac{\partial}{\partial n}(u_1-u_2)-(u_1-u_2)\frac{\partial \hat{u}}{\partial n} + u_a \frac{\partial}{\partial n}(u_1-u_2)$$

$$- (u_1-u_2)\frac{\partial u_a}{\partial n} + u_1^T \frac{\partial u_2^T}{\partial n} - u_2^T \frac{\partial u_1^T}{\partial n}\Bigg]d\Gamma\ .$$

An argument completely analogous to that given above in Case I allows us to conclude that

$$\int_{\Gamma_a}\left[u_a \frac{\partial}{\partial n}(u_1-u_2)-(u_1-u_2) \frac{\partial u_a}{\partial n}\right]d\Gamma = 0$$

and so, as can be easily checked by writing out all terms, the right hand side of (B16) reduces to

$$\int_{\Gamma_a} (u_1 \frac{\partial u_2}{\partial n} - u_2 \frac{\partial u_1}{\partial n})d\Gamma = 0\ ,$$

again because u_1 and u_2 both satisfy the free surface and radiation conditions.

Combining this last result with equation (B13) and (B15) we have

$$\text{(B17)}\quad \int_{\Gamma_1} u_1 \frac{\partial u_1}{\partial n}\, d\Gamma-\int_{\Gamma_2} u_2 \frac{\partial u_2}{\partial n}\, d\Gamma = \int_{D_2^-\setminus(D_1^-\cap D_2^-)} \nabla u_1^T\cdot\nabla u_2^T dV$$

$$- \int_{D_1^-\setminus(D_1^-\cap D_2^-)} \nabla u_1^T\cdot\nabla u_2^T dV+\int_{\Gamma_2} \hat{u} \frac{\partial \hat{u}}{\partial n}\, d\Gamma-\int_{\Gamma_1} \hat{u} \frac{\partial \hat{u}}{\partial n}\, d\Gamma$$

which is the desired result.

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