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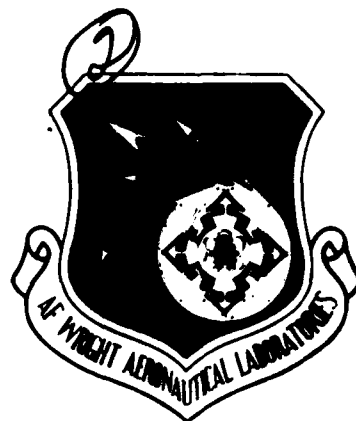
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COMPUTER-AIDED DESIGN OF ROBUST DECENTRALIZED CONTROLLERS

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19. ABSTRACT (Continue on reverse if necessary and identify by block number)
Significant progress was made on all of the tasks proposed for the Phase I (feasibility) effort. In addition to the 4 tasks that was originally proposed, a new subtask on simultaneously stabilizing controllers was included, and a model of a flexible structure was used for which two types of fault tolerant controllers were designed. The concept of fixed modes was extended to that of a decentralized fixed determinant and significant new results for characterizing fixed determinants were obtained. A characterization of all decentralized controllers was obtained based on which conditions for the existence of robust decentralized controllers were derived, in case of a two channel system. Finally, an iterative solution (of nonlinear programming type) was proposed for optimal decentralized filtering of a two-channel system. In this way a firm foundation in Phase I was laid based on which a state-of-the-art approach to computer-aided design of robust decentralized control problems can be carried out. *Additional keywords: numerical methods and procedures; flight dynamics, trajectories.*

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FOREWORD

This report describes the work performed by Scientific, Systems, Inc.
from September 1984 to May 1985 under Air Force Award No. F33615-84-C3619.
The work represents the first of a three-phase project aimed at the deve-
lopment of state-of-the-art analytical and computational tools for
computer-aided robust decentralized control. The objective of the Phase I
effort was to demonstrate the feasibility of the approach. Dr. Siva Banda
was the project manager for the Air Force, Wright Aeronautical
Laboratories, Flight Dynamics Laboratory.

Dr. Raman Mehra was the Project Supervisor. Dr. Hamid Razavi was the
principal investigator. Professor M. Vidasagar of the University of
Waterloo was the consultant. Special thanks goes to Ms. Alina Bernat for
her excellent supervision of the documentation.

TABLE OF CONTENTS

1. INTRODUCTION

This document comprises the final report on the Air Force Contract no. F33615-84-C-3619 "Computer-Aided Design of Robust Decentralized Controllers," based on the Stable Factorization approach. For the Phase I (feasibility) effort we proposed "to make a start on the design of robust decentralized controllers by tackling the following problems:"

1) numerical computation of optimally robust controllers

2) decentralized stabilization of an n-channel system

3) robust decentralized control of a two-channel system

4) optimal decentralized filtering for a two-channel system

In addition to the Phase I proposed tasks, in this report we include a new subtask (1.(b)) on simultaneous stabilization. Also a model of a flexible structure is presented for which two types of "robust" controllers are computed.

We are happy to report that substantial progress was obtained in all of the four original tasks as well as the added substask 1.(a). Our preliminary computational experience with the design of the above-mentioned controllers has been very good.

As a result of successful completion of Phase I effort, a firm foundation has been laid for the Phase II undertaking.

2. TASK 1

NUMERICAL COMPUTATION OF OPTIMALLY ROBUST CONTROLLERS

The overall goal of this task was to devise numerical techniques for obtaining optimally robust controllers. The specific objective of this task was to obtain controllers which are robust against the class of plant perturbations introduced in Vidyasagar and Kimura (1984) known as the "stable factor perturbations". Recall that this class , denoted as S (N_0 , D_0 , r), consists of all plants P of the form P (s) = N (s) [D (s)]$^{-1}$ for which $|| [N' - N'_0 \quad D' - D'_0] || < |r(jw)|$ for all w, where r is a proper and stable rational function and prime denotes transposition. Thus the class of stable factor perturbations consists of all plants whose stable factors are 'close' to the stable factors of the nominal plant P_0. The function r(jw) plays the role of an "uncertainty profile", as a function of frequency.

The advantage of stable factor perturbations over for example multiplicative or additive perturbations is that the former does not have the often restrictive requirement on the perturbed and unperturbed plants that they have the same number of poles in the unstable region. Furthermore, stable factor perturbation results do not require that the perturbed plant poles or zeros on the jw-axis be fixed in their location. In order to illustrate the significance of these more relaxed conditions, a model of a flexible structure will be discussed, and its relevance to stable factor uncertainty will be described.

In addition to the tasks spelled out in the proposed Phase I effort,
we will present a second type of robustness result. (Furthermore the
robustness results will be applied to stabilize a flexible structure.)
This second type of robustness relates to the concept of simultaneous sta-
bilization introduced in Vidyasagar and Viswanadham (1982). Recall that
the simultaneous stabilization problem is as follows: Given the nominal
plant P_o and a number of contingent plant conditions P_1 , $---$, P_r ,
determine a single controller, when one exists, such that it stabilizes not
only the nominal plant but also the other contingent plants P_1 , $----$, P_r .
(For details see Razavi, Mehra, and Vidyasagar (1984) and the above paper).

Before we present the techniques for solving the above two types of
robustness problems we describe a simple model we shall use to obtain spe-
cific controller designs. Consider a 4-disk vibrating system indicated in
Figure 1.
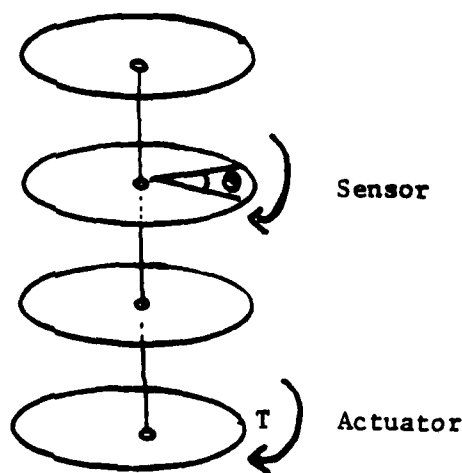
Figure 1. 4-disk system.

The disks are connected by torsion springs. Let K_i denote the spring constant for the spring between i-th and (i+1)-th disk; let J_i denote the inertia for i-th disk. The equations of motion for the system are:

$$\begin{bmatrix} \ddot{\Theta}_1 \\ \ddot{\Theta}_2 \\ \ddot{\Theta}_3 \\ \ddot{\Theta}_4 \end{bmatrix} + \begin{bmatrix} \dfrac{K_1}{J_1} & -\dfrac{K_1}{J_1} & 0 & 0 \\ -\dfrac{K_1}{J_2} & \dfrac{(K_1 + K_2)}{J_2} & -\dfrac{K_2}{J_2} & 0 \\ 0 & -\dfrac{K2}{J_3} & \dfrac{(K2 + K_3)}{J_3} & -\dfrac{K_3}{J_3} \\ 0 & 0 & -\dfrac{K_3}{J_4} & \dfrac{K_3}{J_4} \end{bmatrix} \begin{bmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \\ \Theta_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dfrac{T_3}{J_3} \\ 0 \end{bmatrix} \quad (S)$$

where Θ_i denotes the angle of deflection of the i-th disk from normal position. Each dot indicates differentiation with respect to time. The right hand side indicates the scaled torques applied (in this case to the 3-rd disk).

The motivation for this model is that by applying actuator inputs (torques) u_i at disk i one wishes to control the outputs Θ_j, for i equal to or different from j. The transfer function from each input u_i to each output Θ_j can be easily calculated from the dynamic system (S). For the purpose of illustration we assume that the spring constants K_i are all equal to unity. The nominal plant is further assumed to have $J_i = 1$, i=1, 2, 3, 4 (e.g. equal inertia). Note that due to the fact that there is no

damping in the system, the transfer function from any input to any output

will have all of its poles and zeros located on the jw-axis.

For example the transfer function from u_3 to Θ_4 will be of the form

$$P(s) = \frac{b(s)}{a(s)}$$

$$= \frac{s^2 + \beta_1^2}{s^2 \prod_{i=1}^{4} (s^2 + \alpha_i^2)}$$

Where β_1 and α_i are real positive scalars, corresponding to a zero at $z_1 = j\beta_1$ and poles $p_i = j\alpha_i$, $i = 1, 2, 3, 4$. The rigid body modes correspond to the poles $s^2 = 0$.

The preceding model will be used in the application of two robust controller design techniques based on the robustness notions that we discussed earlier: (a) robust controllers for 'discrete' or contingent plant perturbations (e.g. fixed controllers that simultaneously stabilize a nominal plant P_1 together with contingent plants $P_2, \_ \_ \_ \_, Pr.$). Here we consider the one input – two output case; furthermore let r=2, i.e. one contingency, (b) optimally robust controllers for stable factor perturbations. Sections 1.(a) and 1.(b) describe the computational techniques together with numerical results.

Subtask 1.(a) Simultaneous Stabilization of Two Vibrating Systems

In this section, we present an example of the simultaneous stabiliza-
tion technique introduced in Vidyasagar and Viswanadham (1982). The
technique is applied to a four-disk vibrating system described earlier.
The computation of the actual controller using state- space versions of the
stable factorization approach is carried out using the methods from the
forthcoming thesis of Minto (1985).

A control input is applied to the second disk from the top, and the two
outputs are the angular positions of the first and fourth disks respec-
tively. Two different situations are considered. In the first, all four
disks are identical. In the second, the first three disks are identical
and the last disk has a moment of inertia which is one quarter of those of
the rest. In the nominal case the system is denoted by P_1 (s) and equals
the following: (The particular pole zero locations correspond to the
experimental set-up described in Cannon and Rosenthal (1984))

Case 1 (All disks identical)

$$P_1 \ (s) = \frac{1}{a_1 \ (s)} \begin{bmatrix} b_{11} \ (s) \\ b_{12} \ (s) \end{bmatrix}$$

(1.1)

where

$$a_1(s) = s(s \pm j \ 23.81) \ (s \pm j \ 18.22)(s \pm j \ 9.863)$$

(1.2a)

$$b_{11}(s) = (s \pm j \ 20.85) \ (s \pm j \ 7.96)$$

(1.2b)

$$b_{12}(s) = (s \pm j \ 12.886)$$

Where the symbol $(s \pm \alpha j)$ denotes $(s + \alpha j)(s - \alpha j)$

Case 2 (First 3 disks are identical and moment of inertia of the last disk is one quarter of each of the rest)

$$P_2(s) = \frac{1}{a_2(s)} \begin{bmatrix} b_{21}(s) \\ b_{22}(s) \end{bmatrix}, \tag{1.3}$$

where

$$a_2(s) = s(s \pm j\ 29.34)(s \pm j\ 21.48)(s \pm j\ 11.938) \tag{1.4a}$$

$$b_{21}(s) = b_{22}(s) \tag{1.4b}$$

$$b_{22}(s) = (s \pm j\ 25.95) \tag{1.4c}$$

Note that we have assumed a unity gain since non-unity gains can be easily absorbed into the controller. The objective is to find a single controller C that stabilizes both P1 and P2. In the present instance both systems are entirely undamped, i.e. all poles and zeros are purely imaginary. If we stabilize the two systems by moving the poles into the left half-plane, it is possible to come up with designs that yield closed-loop systems which are nominally stable, but have poles with very low damping. To rule out this possibility, we define the stability region to be the region {s:Re s <-1}. Thus, in order for a system to be stable in this more restrictive sense, its poles must have real parts less than -1.

Mathematically, the easiest way to handle this requirement is to transform variables by replacing s by s-1. This gives two modified systems

$Q_1(s)$ and $Q_2(s)$ whose poles and zeros all lie along the shifted vertical axis Re $s=1$. If we now simultanteously stabilize Q_1 and Q_2 by a controller $C_{mod}(s)$, then the controller $C(s)=C_{mod}(s+1)$ moves the poles of P_1 and P_2 into the region Re $s<-1$.

Accordingly, define

$$Q_1(s)= \frac{1}{a_1(s-1)} \begin{bmatrix} b_{11}(s-1) \\ b_{12}(s-1) \end{bmatrix}, \quad Q_2(s)= \frac{1}{a_2(s-1)} \begin{bmatrix} b_{21}(s-1) \\ b_{22}(s-1) \end{bmatrix}, \quad (1.5)$$

where all functions are as in (1.2) and (1.4). Because the systems have only one input, it is easy to construct right-coprime factorizations for Q_1 and Q_2. Define, for example

$$f_1(s)=(s+1)(s+2)(s+3)(s+4)(s+5)(s+6)(s+7), \quad (1.6a)$$

$$f_2(s)=(s+1.5)(s+2.5)(s+3.5)(s+4.5)(s+5.5)(s+6.5)(s+7.5) \quad (1.6b)$$

$$\begin{bmatrix} d_1(s) \\ n_{11}(s) \\ n_{12}(s) \end{bmatrix} = \frac{1}{f_1(s)} \begin{bmatrix} a_1(s-1) \\ b_{11}(s-1) \\ b_{12}(s-1) \end{bmatrix}, \quad \begin{bmatrix} d_2(s) \\ n_{21}(s) \\ n_{22}(s) \end{bmatrix} = \frac{1}{f_2(s)} \begin{bmatrix} a_2(s-1) \\ b_{21}(s-1) \\ b_{22}(s-1) \end{bmatrix} \quad (1.7)$$

Then$\left(\begin{bmatrix} n_{i1} \\ n_{i2} \end{bmatrix}, d_i \right)$ is an r.c.p. of Q_i for $i=1,2$. It is easy to construct

state-space realizations for the matrices in (1.7). Expand the various

polynomials as follows:

$$f_i(s) = s^7 + \sum_{k=0}^{} \phi_{ik}s^k \tag{1.8a}$$

$$a_i(s) = s^7 + \sum_{k=o}^{6} \alpha_{ik}s^k \tag{1.8b}$$

$$b_{ij}(s) = \sum_{k=o}^{6} \beta_{ijk}s^k \tag{1.8c}$$

Now let A_i be the companion matrix

$$A_i = \begin{bmatrix} -\phi_{10} & -\phi_{11} & \cdots & -\phi_{15} & -\phi_{16} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ & & & & \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \in R^{7 \times 7}, \tag{1.9a}$$

and define

$$B_i = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \end{bmatrix}' \in R^{7 \times 1}, \tag{1.9b}$$

$$C_i = \begin{bmatrix} a_{10} - \alpha_{10} & \cdots & a_{16} - \alpha_{16} \\ b_{110} & \cdots & b_{116} \\ b_{120} & \cdots & b_{126} \end{bmatrix} \in R^{3 \times 7}, \tag{1.9c}$$

Then the quadruple (A_i, B_i, C_i, E_i) is a state-space realization of the "stacked" 3x1 system $[d_i \quad n_{11} \quad n_{12}]'$. Here E has been used in place of the

usual output feedthrough symbol D to avoid confusion with 'denominator'
term in coprime factors.

Next, consider the 3x2 system

$$\begin{bmatrix} d_1 & d_2 \\ n_{11} & n_{21} \\ n_{12} & n_{22} \end{bmatrix} =: T \tag{1.10}$$

A state-space realization of T(s) can now be readily constructed, as
follows:

$$A_t = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \epsilon R^{14 \times 14}, \qquad B_t = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \epsilon R^{14 \times 2} \tag{1.11a}$$

$$C_t = [c_1 \quad c_2] \; \epsilon R^{3 \times 14}, \qquad E_t = [E_1 \quad E_2] \; \epsilon R^{3 \times 2} \tag{1.11b}$$

Now we summarize the procedure of Vidyasagar (1985), Vidyasagar et al.
(1985) for simultaneously stabilizing the two plants Q_1 and Q_2 . Select
units u_1 and u_2 in S, and try to find a matrix $R(s) \epsilon S^{1 \times 3}$ such that

$$R(s) \; T(s) = [u_1(s) \quad u_2(s)]. \tag{1.12}$$

If such an R can be found, then R can be partitioned as $[d_c \; n_{c1} \; n_{c2}]$,
and then

$$C(s) = \frac{1}{d_c(s)} [n_{c1}(s) \; n_{c2}(s)] \tag{1.13}$$

is a controller that solves the simultaneous stabilization problem. The
argument in Vidyasagar(1985), Vidyasagar et al. (1985) is that, since
T has more rows than columns, generically it has a stable left inverse, say
Z. Then we can choose the units u_1 and u_2 arbitrarily, and set $R=[u_1 \ u_2]Z$.

However, in the present instance, T does _not_ have a stable left
inverse, since the matrix $T(\infty)=E_t$ only has rank 1. In this case, we
proceed as follows (Note that the procedure is from Minto (1985)):
Let $t_1(s)$, $t_2(s)$ denote the two columns of $T(s)$, and partition B_t, E_t as

$$B_t \ = \ [b_1 \quad b_2], \qquad E_t \ = \ [e_1 \quad e_2]. \qquad (1.14)$$

(Hereafter we drop the subscript "t" on all matrices.) Then it is easy to
see that the quadruple (A, b_i, c, e_i) is a state-space realization of t_i.
Also, $e_1 = e_2$. Now, we want to solve

$$R \ [t_1 \quad t_2] \ = \ [u_1 \quad u_2]. \qquad (1.15)$$

This is equivalent to

$$R \ [t_1 \quad t_2-t_1] \ = \ [u_1 \quad u_2-u_1]. \qquad (1.16)$$

Now (A, b_2-b_1, c, e_2-e_1) is a state-space realization of t_2-t_1. Since
$e_2-e_1=0$, $t_2(s)-t_1(s)$ is strictly proper. Hence, (1.16) implies that u_2
$-u_1$ is also strictly proper. In other words, the two units u_1 and u_2 can
not be chosen independently, but must satisfy $u_2(\infty)=u_1(\infty)$. Now (1.16) is
equivalent to

$$R \: [t_1 \quad (s + \gamma) \quad (t_2 - t_1 \:)] = [u_1 \quad (s + \gamma) \: (u_2 - u_1 \:)], \qquad (1.17)$$

where γ is any positive number. Since $(A, b_2 - b_1, c, 0)$ is a state-space realization of $t_2 - t_1$, one can show without much difficulty that a state-space realization of $(s+\gamma)(t_2(s) - t_1(s))$ is given by $(A, (A+\gamma I)(b_2 - b_1), c, c(b_2 - b_1))$. If the vector $c(b_2 - b_1)$ is linearly independent of e_1, then the matrix $[t_1 \quad (s+\gamma)(t_2 - t_1)]$ has a left inverse, and (1.17) can be solved for R. Otherwise the procedure has to be repeated. The algorithm can be stated as follows:

Step 1: Set i=0

Step 2: Is e_2 linearly independent of e_1 ? If so, go to step 5; if not, go to step 3

Step 3: Since e_2 is linearly dependent on e_1, there exists a constant v_i such that $e_2 = v_i \: e_1$. Replace b_2 by $b_2 \: - v_i \: b_1$, e_2 by $e_2 - v_i \: e_1 = 0.$

Step 4: Replace b_2 by $(A+\gamma I) \: b_2$, e_2 by $c \: b_2$. Increment i and go to step 2.

Step5: Stop

At this stage, one gets an equation of the following form

$$R \: (s) \: [t_1 \: (s) \quad \bar{t}_2 \: (s) \:] = [u_2 \: (s) \quad \bar{u}_2 \: (s) \:], \qquad (1.18)$$

- 12 -

where, letting k denote the total number of iterations, we have

$$\bar{u}_2 (s) = u_2 (s) - \sum_{i=0}^{k} v_i (s + \gamma)^i u_i (s) \qquad (1.19)$$

Also, the termination criterion ensures that the matrix $[t_1 (s) \ \bar{t}_2 (s)]$ has a stable left inverse. Hence, once suitable units u_1 and u_2 are chosen, the matrix R (s) can be found as

$$R (s) = [u_1 (s) \ \bar{u}_2 (s)] \ \bar{T}^l (s), \qquad (1.20)$$

where $\bar{T}^l (s)$ is a stable left inverse of $[t_1 (s) \ \bar{t}_2 (s)]$.

To complete the discussion, it only remains to show how to choose the units u_1 and u_2 . Now any function f in s can be expanded in a power series of the form

$$f (s) = \sum_{i=0}^{\infty} f_i (s + \gamma) ^{-i} . \qquad (1.21)$$

Now, the power series of the units u_1 and u_2 must satisfy certain relationships to ensure that various quantities are strictly proper. For instance, if we simply choose $u_1 (s) = 1$, then the first k terms in the power series of $u_2 (s)$ must be $v_0 + v_2 (s + \gamma) ^{-1} + \ldots + v_{k-1} (s + \gamma)^{-(k+1)}$, where the constants v_i are generated in step 3 of the algorithm. Thus any unit u_2 can be chosen, so that the above condition is satisfied.

In the problem at hand, we chose $\gamma = 8$, and it turned out that the algorithm had to be run _five_ times, and the constants were

$$v_0 = 1, \ v_1 = -3.5, \ v_2 = 461.58, \ v_3 = 6769.9, \ v_4 = -75847 \qquad (1.22)$$

If we perform the bilinear transformation $z = (s-8) / (s+8)$, which maps $s=\infty$ into $z=1$, we conclude that the power series (in terms of z) for u_2 must look like

$$u_2 = 3.0798 + 3.7164 \ z - 0.18256 \ z^2 - 2.9765 \ z^3 - 1.1573 \ z^4 + \dots \quad (1.23)$$

Fortunately, it turns out that if we add just one term, namely $10z^5$, then u_2 is a unit, i.e. its zeros are in the open left half of the s-plane.

The procedure leads to a nineteenth order controller, whose state-space realization is given in Appendix B. The twenty-six closed-loop poles with plants P_1 and P_2 are given below:

With Plant P_1 and controller

-1

-1.517

-1.5

-1.333

-1.25

-1.20

-1.167

-1.143

-1.118

-1.126

-1.155$\pm$j0.1236

-1.006$\pm$j0.077

-1.002$\pm$j0.0048

-1.002$\pm$j0.0044

-1.011$\pm$j0.004

-1.012$\pm$j0.006

-1.013$\pm$j0.006

-1.013$\pm$j0.003

With Plant P_2 and controller

-2.481

-1.667

-1.517

-1.4

-1.286

-1.222

-1.043

-1.095

-1.018

-1.017

-1.015

-1.013

-1.016+j0.124

-1.006+j0.077

-1.002+j0.048

-1.002+j0.044

-1.003+j0.039

-1.137+j0.037

-1.105+j0.025

It can be noted that all closed-loop poles have damping ratios very close to 1.

In summary, in an actual controller design problem one would apply order reduction techniques to replace the nineteenth order controller by something smaller. One would also scale the A,B,C,E matrices of the controller in such a way that the numbers do not have a very large dynamic range.

Subtask 1(b): <u>Numerical Computation of Optimally Robust Controller</u>

Recall that in this task the class of perturbations that we are interested in are those of 'stable factor'; this class was denoted by $S(N_0, D_0, r)$. The following necessary and sufficient result was derived by Vidyasagar and Kimura (1984). A Controller C that stabilizes the nominal plant $P_0 = N_0 D_0^{-1}$ also stabilizes all plants in the class $S(N_0, D_0, r)$ if and only if $\|[X \; Y]r\| < 1$, where $C = -Y^{-1}X$ and X and Y are the solutions to the Bezout identity. Stated more generally, given the class $S(N_0, D_0, r)$ of stable factor perturbations, there exists a controller which robustly sta-

bilizes all plants in $S(N_0, D_0, r)$ if and only if $\|[Y \ X]' + [-\tilde{N} \ \tilde{D}]'R\| < 1$

for some stable rational matrix R, where $(\tilde{N}, \tilde{D})$ denote the right coprime

factorization of P. Given an uncertainty profile $r \epsilon S$, consider λr for all

real values of the parameter λ. An optimally robust controller is that

which stabilizes the plants in the set $S(N_0, D_0, \lambda r)$ with the largest value

of λ.

With this review, our aim in this subtask is to compute the following

minimization problem:

$$\text{Min } \|F - GRH\| \qquad (1.24)$$
$$R \epsilon H^\infty$$

where $G \epsilon R_\infty^{n \times 1}$, $\quad H \epsilon R_\infty^{k \times m} \quad$ with $\quad n > 1, \quad k < m$

(other cases are simpler and follow from this problem.) To see that the

above minimization solves the robust controller problem described above

simply make the associations:

$$F \ =: \ [Y \ X]' \qquad (1.25)$$

$$G \ =: \ [-\tilde{N} \ \tilde{D}]' \qquad (1.26)$$

$$H \ = \ I \qquad (1.27)$$

The robust controller is obtained from the optimal R, via Youla's
parameterization.

We will call the above minimization problem the "tall" plant minimiza-

tion problem as the free variable R appears on the right of G, and G is

"tall". The proposed procedure for solving (1.24) turned out to be dif-

ficult to implement – as the proposed minimax problem became intractable.

Instead we present the following approach. (Throughout we assume that G and H have full column or row rank as appropriate; otherwise the procedure can be slightly modified.) The procedure will be summarized without proof as an algorithm. The main idea is that by a series of manipulations we will transform the tall plant problem (1.24) to a square plant problem (1.28) which then can be readily solved.

Step 1:

$$\text{Compute} \qquad G = G_1\, G_0, \; (G_0 \text{ square})$$

$$H = H_0{}'\, H_1{}', \; (H_0 \text{ square})$$

where the subscripts 1 and 0 stand for inner and outer, respectively

$$\text{Let} \qquad Q = G_0\, R\, H_0$$

Step 2:   Compute complementary inner for G_1

i)   Perform row permutations if necessary to get

$$G_1 = \left[\begin{array}{c} D \\ \hline N \end{array}\right] \quad \text{with } |D| \neq 0$$

ii)   Let $P(Z) = -[D'(Z^{-1})]^{-1}\, N(Z^{-1})$

Compute r.c.f. of P

$P(Z) = U(Z)\, V^{-1}(Z)$

$U \in R^{m \times (n-m)}$, $V \in R^{(n-m) \times (n-m)}$

Let $W = \begin{bmatrix} U \\ V \end{bmatrix} \in R^{n \times (n-m)}$

Compute inner outer factorization of W:

$$W = W_i \, W_0$$

$$W_0 \in R^{(n-m) \times (n-m)} \quad , \quad W_i \in R^{n \times (n-m)}$$

W_i is the complementary inner for G_i:

$$\tau = [G_i \vdots W_i]$$

Where $\tau \in R^{n \times n}$ is inner

Step 3:

Compute $\quad \varphi * F =: \begin{bmatrix} A & B \\ \hline C & D \end{bmatrix}$

(Partition $\psi * F$ such that Q and A are additively compatible. Note that $\varphi * =: \Phi'(-s)$.)

Step 4:

Let $\mu = \max \left\{ \left| C \ D \right|_\infty, \left| \begin{matrix} B \\ D \end{matrix} \right|_\infty \right\}$

Step 5:

Choose $\quad \gamma > \mu$

Let $\quad K = [C \ D]$

Step 6:

Compute $\quad L = (\gamma^2 I - K* \ K)^{-1/2}$

(Notation: If $A = V^T V$, then $V =: A^{1/2}$)

Compute $\qquad M = (I - BLL^{*}B^{*})^{-1/2}$

If $\qquad \| B\, L \|_{\infty} > 1$ increase γ and go to Step 6, otherwise go

to Step 7

Step 7:

Compute $\qquad J^{*} = \text{Min } \| MAL - S \|_{\infty}$

$$S \in M(H_{\infty})$$

i) Let $\qquad A_2 = MAL; \quad A_2 \in M(L_{\infty})$

ii) Compute l.c.f. of A_2,

$$A_2 = (T(z))^{-1}\, U(z)$$

iii) Compute inner outer factorization of T:

$$T = T_1\, T_0; \text{ where } T_0 \text{ is a unit}$$

Then

$$A_2 = (T_1(z))^{-1}\, V(z)$$

Where

$$V(z) = T_0^{-1}\, U$$

Step 8:

Let $\qquad b = |T_1|$

Then $\qquad \| MAL - S \|_{\infty} = \| T_1^{adj}\, V - b\, S \|_{\infty}$

where $\qquad x^{adj}$ indicates adjoint of x.

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Thus the minimization problem (1.24) becomes

$$J^* = \text{Min} \quad \| T_i^{adj} V - b \, S \|_\infty \qquad (1.28)$$

$$S \in M(H_\infty)$$

The right hand side of (1.28) is a 'square' plant minimization problem which can be solved using methods of Section 7.4 of Vidyasagar (1985)

Step 9:

If $J^* < 1$ decrease γ and go to Step 6

If $J^* > 1$ increase γ and go to Step 6

If $J^* = 1$ go to Step 10

Step 10:

Set $\qquad Q = M^{-1} \, S \, L^{-1}$

then, $\qquad R^* = G_0^{-1} \, Q \, H^{-1}{}_0$

is the optimum R in (1.24).

We conclude this section with a controller design which achieves stable factor robustness against pole/zero movements on the jw-axis of the four disk vibrating (non-minimum phase) system. The controller order turns out to be smaller (than that of the previous controller) but some of the closed loop poles are not as well damped.

In the interest of brevity, only the final results are summarized here. Nominal plant transfer function from 3rd to 4th disk:

$$P_0(s) = k \frac{s^2 + 195}{s^2(s^2+156)(s^2+361)(s^2+576)}$$

One such robustly stabilizing controller c(s) for $P_0(s)$ is:

$$c(s) = \frac{(s-132.44)(s+27.14)(s+90.26)(s+3.6)(s+0.99)(s^2-17.5\,s+398.5)}{(s+190.35)(s+37.70)(s+0.489\pm30.935j)(s+11.61\pm20.89j)(s+8.86\pm5.307j)}$$

The value of the gain k is 827,000 (Cannon and Rosenthal (1984)). The closed loop nominal system poles are

100x

$$
\begin{aligned}
&-1.9036 + 0.0000i \\
&-0.0063 + 0.3096i \\
&-0.0063 - 0.3096i \\
&-0.0036 + 0.2427i \\
&-0.1183 + 0.2120i \\
&-0.0036 - 0.2427i \\
&-0.1183 - 0.2120i \\
&-0.0040 + 0.1772i \\
&-0.0040 - 0.1772i \\
&-0.0017 + 0.1235i \\
&-0.1001 + 0.0603i \\
&-0.0017 - 0.1235i \\
&-0.1001 - 0.0603i \\
&-0.0370 + 0.0000i \\
&-0.0181 + 0.0000i \\
&-0.0073 + 0.0000i
\end{aligned}
$$

The variations in inertia cause the zero and the poles to vary from their nominal values. Figure 2 indicates the range for the zero and one pole variation. The nominal value for the zero is 13.96 and for the pole is 12.48. The acceptable range for pole zero variations is shown by the darkened lines. Other pole variations of about the same percentage variation range turn out to be stabilized by the fixed controller.
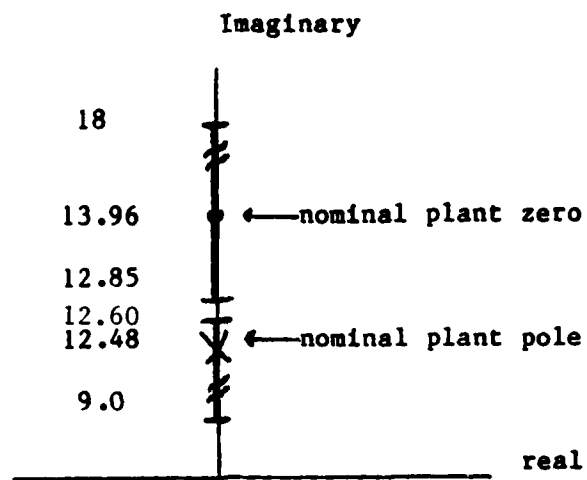
Figure 2. Pole-zero variation range

3. TASK 2

DECENTRALIZED STABILIZATION OF AN n-CHANNEL SYSTEM

Recall that task 2 consisted of finding a parameterization of all decentralized controllers that stabilize a given plant.

Suppose a plant P has dimensions ℓxm, so that it has m inputs and ℓ outputs. Partition the inputs and outputs into k disjoint subsets each, where k is the number of <u>channels</u>. Without loss of generality, renumber the inputs and outputs in such a way that the first channel consists of the first m_1 inputs and ℓ_1 outputs, the second channel of the next m_2 inputs and ℓ_2 outputs, and so on. Then we have integers m_1 ,..., m_k , and ℓ_1 ,..., ℓ_k such that $\sum_{i=1}^{k} m_i = m$ and $\sum_{i=1}^{k} \ell_i = \ell$. This partitioning of inputs and outputs is called an <u>information structure</u>. With these definitions, a controller C is said to be <u>decentralized</u> if it is block-diagonal of the form

$$C = \begin{bmatrix} c_1 & & & 0 \\ & c_2 & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \cdot \\ 0 & & & & c_k \end{bmatrix} , \qquad (2.1)$$

where c_i has dimensions $m_i \times \ell_i$. The objectives of this task are (i) to derive conditions under which a given plant can be stabilized by a decentralized controller, and to find an expression for all controllers that stabilize it.

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We now summarize some results from Vidyasagar (1985) concerning stabilization. Given a plant p and a controller c, we form the composite transfer matrix

$$H\ (P,\ C) = \begin{bmatrix} (I+PC)^{-1} & -P(I+CP)^{-1} \\ C(I+PC)^{-1} & (I+CP)^{-1} \end{bmatrix} , \qquad (2.2)$$

and say that <u>C stabilizes P</u> if the above transfer matrix is stable. Let S denote the set of scalar stable transfer functions. Thus in the contineous-time case S consists of proper rational functions of the Laplacian variable s whose poles all lie in the open left half-plane. In the discrete-time case, S consists of proper rational functions of the unit advance operator z whose poles all lie inside the closed unit disk. Finally, let M (S) denote the set of matrices, of whatever order, whose elements all lie in S. Then, given any P, one can factor it in the form $N_p\ D_p^{-1} = \overline{D}_p^{-1}\ \overline{N}_p$, where N_p, D_p $\overline{N}_p$, $\overline{D}_p$ all belong to M (S), and moreover satisfy the so-called <u>Bezout identities</u>

$$X_p\ N_p + Y_p\ D_p = I, \quad \overline{N}_p\ \overline{X}_p + \overline{D}_p\ \overline{Y}_p = I, \qquad (2.3)$$

where X_p, Y_p, $\overline{X}_p$, $\overline{Y}_p$, also belong to M(S). We refer to the pair $(N_p,\ D_p)$ as a <u>right-coprime factorization</u> (r.c.f.) of P, and to the pair $(\overline{D}_p,\ \overline{N}_p)$ as a <u>left-coprime factorization</u> (l.c.f.) of P. Similar remarks and symbols apply to C.

A function f in S is called a <u>unit</u> if its reciprocal 1/f also belongs to S. A square matrix U in M (S) is called <u>unimodular</u> if its inverse U^{-1} also belongs to M (s). It is easy to show that U is unimodular if and only if its determinant (denoted by $|U|$) is a unit.

The following theorem is well-known, and forms the basis of this section; see (Vidyasagar 1985).

Theorem 2.1. Given P, C, let (N_p, D_p), (N_c, D_c) be r.c.f.'s of P and C respectively, and let $(\overline{D}_p, \overline{N}_p)$, $(\overline{D}_c, \overline{N}_c)$ be l.c.f.'s of P and C respectively. Then the following sattements are equivalent:

(i) C stabilizes P.

(ii) The matrix $\Delta = \overline{D}_c D_p + \overline{N}_c N_p$ is unimodular.

(iii) The matrix $\overline{\Delta} = \overline{D}_p D_c + \overline{N}_p N_c$ is unimodular.

Now we are in a position to present some new results. A key concept in decentralized stabilization is that of fixed modes (Davison and Wang (1973), Vidyasagar and Viswanadham (1983). In the present report, we extend this concept by defining a decentralized fixed determinant. Given a plant p and a controller c, it can be shown that the quantities $|\Delta|$ and $|\overline{\Delta}|$ (where Δ, $\overline{\Delta}$ are defined in theorem 2.1) differ only by a unit, and that each is a characteristic determinant of the closed-loop transfer matrix $H(p,c)$; see (Vidyasagar 1985.). The significance of this is that the unstable poles of $H(p,c)$ are precisely the RHP zeros of its characteristis determinant. Now the (decentralized) fixed determinant ϕ is defined as follows: Let $\phi = \phi(p,c)$ denote the quantity $|\Delta|$, and let R denote the set of constant decentralized feedback gain matrices. Then f, the fixed determinant, is given by

$$f = \underset{K \in R}{\text{g.c.d.}} \quad \phi(p,K) \tag{2.3}$$

In other words, f is the greatest common divisor of all characteristric determinants of all closed-loop systems obtained by applying a block-

diagonal constant output feedback to the given plant P. Obviously the RHP
zeros of P are precisely the RHP poles of P that remain fixed under all
such feedback gains. The next two theorems give some new and powerful
results characterizing f. (All proofs are given in Appendix A.)

Theorem 2.2. Let (N,D) be an r.c.f. of P, $(\overline{D}, \overline{N})$ an l.c.f. of P. Then
each of the following constructions gives an explicit formula for f: (i)
partition N and D by rows, as follows: D^1 consists of the first m_1 rows,
of D, D2 of the next m_2 rows, and so on until D^k consists of the last m_k
rows. Thus

$$
D = \begin{bmatrix} D^1 \\ D^2 \\ \cdot \\ \cdot \\ \cdot \\ D^k \end{bmatrix} , \tag{2.4}
$$

where D^1 has m_i rows. Similarly partition N as

$$
N = \begin{bmatrix} N^1 \\ N^2 \\ \cdot \\ \cdot \\ \cdot \\ N^k \end{bmatrix} \tag{2.5}
$$

where N^1 has l_i rows. Now define

$$
F^1 = \begin{bmatrix} D^1 \\ N^1 \end{bmatrix} , \qquad F = \begin{bmatrix} F^1 \\ F^2 \\ \cdot \\ \cdot \\ \cdot \\ F^k \end{bmatrix} , \tag{2.6}
$$

and note that F has dimensions $(m+\ell)$ xm. Then f is the g.c.d. of all mxm
minors of F consisting of exactly m_i rows from F_i, for each i.

(ii) partition $\overline{D}$, $\overline{N}$ as

$$D = [\overline{D}_1 \ \overline{D}_2 \ \dots \ \overline{D}_k], \quad N = [\overline{N}_1 \ \overline{N}_2 \ \dots \ \overline{N}_k], \tag{2.7}$$

where $\overline{D}_1$ has ℓ_i columns, $\overline{N}_i$ has m_i columns. Define

$$\overline{F}_i = [\overline{D}_i \ \overline{N}_i], \quad \overline{F} = [\overline{F}_1 \ \overline{F}_2 \ \dots \overline{F}_k], \tag{2.8}$$

and note that $\overline{F}$ has dimensions $\ell \times [\ell+m]$. Then f is the g.c.d. of all $\ell \times \ell$ minors of $\overline{F}$ consisting of exactly ℓ_i columns from $\overline{F}_i$.

Theorem 2.2 gives a very simple and direct characterization of fixed modes, or more generally fixed determinents. Theorem 2.3 below shows that an RHP pole of P that can not be moved by constant (decentralized) output feedback also remains fixed under dynamic (decentralized) output compensation. Strictly speaking this result is not new; however, the proof given here is very transparent.

Theorem 2.3. Let DC denote the set of all decentralized controllers

and define
$$g = \underset{C \ \epsilon \ DC}{\text{g.c.d.}} \quad \phi(P,C) \tag{2.9}$$
Then g = f.

Next, we present a decomposition principle that is useful in finding all decentralized stabilizing controllers. This result should be compared with (Zames 1981), (Callier and Desoer 1982) and (Vidyasagar 1985).

Theorem 2.4. Given a plant P, let DS (P) denote the set of all decentralized stabilizing controllers that stabilize P. Let C be a specific decentralized stabilizing controller for P, and define $P_1=P$

$(I+CP)-1$. Then $C+DS$ (P_1) is a subset of DS (P), and the two sets are equal if and only if C is stable.

The theorem should be interpreted as follows: Given a plant P, suppose one finds first a <u>particular</u> C that stabilizes P. Then, whenever C_1 stabilizes the resulting plant P_1 = $P(I+CP)^{-1}$, the controller $C+C_1$ stabilizes the original plant P. If C is stable, then every stabilizing controller for P can be expressed as the sum of C and a stabilizing controller for P_1. If C is unstable, then there exist stabilizing controllers for P that can not be decomposed in this fashion. The point of the theorem is that the set DS (P_1) is in general easier to determine than the set DS (P), since the first plant is stable. Thus, by using Theorem 2.4, one can reduce the problem of finding all decentralized stabilizing controllers for a <u>general</u> plant to one of finding all decentralized stabilizing controllers for a <u>stable</u> plant, under certain conditions. Specifically, if the given plant can be stabilized by a stable decentralized controller, such a reduction is always possible.

Youla et al. (1974) call a plant <u>strongly stabilizable</u> if it can be stabilized by a stable controller. Theorem 2.4 brings out the importance of the concept of <u>decentralized strong stabilizability</u>, which is the ability to be stabilized by a decentralized stable controller. While necessary and sufficient conditions are available for strong stabilizability, so far only necessary conditions are available for decentralized strong stabilizability.

At this point, it becomes necessary to choose between stating all results in their most general form or in their clearest form, we opt

to state all subsequent results in their clearest form by restricting

to the case where the plant P is square of dimension mxm, and there are

m channels. Thus a decentralized controller in this context is a diagonal

controller. It must be emphasized that all of the results below carry to a

general plant with a general information structure, but at the expense of

more cumbersome notation.

Let (N,D) be an r.c.p. of the plant P. Then both N and D are square

and have dimension mxm, and the matrices D^i, N^i in (2.4) -(2.5) are just

the rows of D and of N. Hence we choose to denote them by lower case

letters d^i, n^i. The matrix F in (2.6) is 2mxm and consits of an

interlearning of the rows of D and N. The minors of F whose g.c.d, equals

the fixed determinant f are those of the form

$$
\begin{bmatrix}
v^1 \\
v^2 \\
\cdot \\
\cdot \\
\cdot \\
v^m
\end{bmatrix}
, \tag{2.10}
$$

where v^i equals either d^i or n^i. Clearly there are 2^m of these. An easy way

to index these minors is to use binary notation. Let $\alpha \in \{0,1\}^m$, i. e. α

is a binary number with m bits. Let $v^i = d^i$ if $\alpha_i = 0$ and n^i if $\alpha_i = 1$.

Then, as α varies over $\{0,1\}^m$, the corresponding minor v_α defined in (2.10)

traces out all the minors whose g.c.d. equals the fixed determinant f.

Theorem 2.5. Let $\underline{0}$ denote the binary zero with m bits, In order for

P to be stabilizable by a stable diagonal controller, a necessary condition

is the following. Define

$$a = v_{\underline{0}} = |D|, \quad b = \text{g.c.d.} \quad v_\alpha \qquad (2.11)$$
$$\alpha \neq \underline{0}$$

Then a and b must satisfy the parity interlacing property; that is, between every pair of real RHP zeros of b, there must be an even number of zeros of a, or equivalently, a must have the same sign at all real RHP zeros of b.

The condition of Theorem 2.5 should be compared to the condition for strong stabilizability by a (not necessarily decentralized) controller, as given in Youla et al (1974), Vidyasagar and Viswanadham (1982): Let n denote the g.c.d. of all elements of N; then a and n satisfy the parity interlacing property. The latter condition is necessary and sufficient, while the former (in Theorem 2.5) is only necessary.

Next, we turn to the problem of finding all decentralized stabilizing controllers. If the plant is strongly decentrally stabilizable, then the results below can be simplified using Theorem 2.4. The following lemma is very useful for this purpose.

<u>Lemma 2.6.</u> Let g_{ij} $1 \leqslant i \leqslant \ell$, $0 \leqslant j \leqslant m$ be given elements of S, and suppose

that for some integer $j \geqslant 1$ the matrix

$$\begin{bmatrix} g_{10} & g_{1j} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ g_{\ell 0} & g_{\ell j} \end{bmatrix} \qquad (2.12)$$

has rank 2. Define

$$\alpha = \text{g.c.d.} \quad \{g_{ij}\} \qquad (2.13)$$
$$1 \leqslant i \leqslant \ell, \ 0 \leqslant j \leqslant m$$

under these conditions, for almost all functions a_j, $1 \leqslant j \leqslant m$ belong to S,

we have

$$\text{g.c.d.} \quad \left\{ g_{io} + \sum_{j=1}^{m} g_{ij} a_j \right\} = \alpha \qquad (2.14)$$
$$1 \leqslant i \leqslant \ell$$

Now we can tackle the problem of finding all decentralized stabilized controllers using the results of Corfmat and Morse (1976), together with Lemma 2.6. We first give a detailed treatment of a two-input, two-outputs systems to bring out the basic ideas clearly, and then move on to the general case of an m-input, m-output system.

consider a system

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \qquad (2.15)$$

which we wish to stabilize by a diagonal controller

$$C = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \qquad (2.16)$$

Let $(\tilde{D}, \tilde{N})$ be an l.c.f. of P, and partition the matrix $[\tilde{D} \ \tilde{N}]$ as

$$[\tilde{D} \ \tilde{N}] = [d_1 \ d_2 \ n_1 \ n_2] \qquad (2.17)$$

Since C is diagonal, it ras an r.c.p. of the form

$$\begin{bmatrix} D_c \\ N_c \end{bmatrix} = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \\ \beta_1 & 0 \\ 0 & \beta_2 \end{bmatrix} ; \qquad (2.18)$$

just let (β_i, α_i) be a coprime factorization of c_i, for $i=1,2$. In order for C to stabilize P, by theorem 2.1 a necessary and sufficient condition is that the quantity

$$u = |\tilde{D}D_c + \tilde{N}N_c| \qquad (2.19)$$

be a unit of S. However, from (2.17) and (2.18),

$$\tilde{D}D_c + \tilde{N}N_c = [\tilde{D} \quad \tilde{N}] \begin{bmatrix} D_c \\ N_c \end{bmatrix} = \begin{bmatrix} \alpha_1 \, d_1 + \beta_1 \, n_1 & \alpha_2 \, d_2 + \beta_2 \, n_2 \end{bmatrix} \quad (2.20)$$

Now for convenience we introduce the "wedge product" notation. Given two 2x1 vectors v_1 and v_2, the scalar $v_1 \wedge v_2$ equals the determinant of the 2x2 matrix $[v_1 \, v_2]$. It is easy to see that wedge product is a multilinear function. Now

$$u = (\alpha_1 \, d_1 + \beta_1 \, n_1) \wedge (\alpha_2 \, d_2 + \beta_2 \, n_2)$$
$$= \alpha_1 \, \alpha_2 \, (d_1 \wedge d_2) + \alpha_1 \, \beta_2 \, (d_1 \wedge n_2) + \beta_1 \, \alpha_2 \, (n_1 \wedge d_2) + p_1 \, p_2 \, (n_1 \wedge n_2)$$
$$(2.21)$$

clearly, unless

$$\text{g.c.d.} \ \{d_1 \wedge d_2, \ d_1 \wedge n_2, \ n_1 \wedge d_2, \ n_1 \wedge n_2\} = 1, \qquad (2.22)$$

u can never be a unit. It can be routinely verified that the above g.c.d. is precisely the fixed determinant defined in Theorem 2.2.

Suppose (2.22) is true. Then the problem of choosing a decentralized stabilizing controller becomes one of selecting α_1, α_2, β_1, β_2, such that u is a unit. For this purpose, note that

$$u = \alpha_1 \left[\alpha_2 (d_1 \wedge d_2) + \beta_2 (d_1 \wedge n_2)\right] + \beta_1 \left[\alpha_2 (n_1 \wedge d_2)\right.$$
$$\left. + \beta_2 (n_1 \wedge n_2)\right] \tag{2.23}$$

Hence, if

$$\text{g.c.d.}\left\{ \alpha_2 (d_1 \wedge d_2) + \beta_2 (d_1 \wedge n_2), \ \alpha_2 (n_1 \wedge n_2) + \beta_2 (n_1 \wedge n_2)\right\} = 1, \tag{2.24}$$

then it is possible to choose α_1, β_1 such that u is a unit; the converse is also true. Thus the question becomes: When is (2.24) true? This is where Lemma 2.6 is useful. It states that if (2.22) holds, then (2.24) also holds for almost all α_2, β_2, provided

$$\begin{bmatrix} d_1 \wedge d_2 & d_1 \wedge n_2 \\ n_1 \wedge d_2 & n_1 \wedge n_2 \end{bmatrix} =: g \neq 0$$

However, routine calculations show that

$$g = | D | \ P12 \ P21 \tag{2.26}$$

Hence the rank condition holds provided P12 $\neq$ 0 and P21 $\neq$ 0. This is the strong connectedness condition of comfort and Morse (1976).

Thus the strategy for finding all decentralized stabilizing controllers is as follows: Pick α_2, β_2 at random. Then for almost all choices of α_2,β_2 the condition (2.24) will hold. Next, select x,y such that

$$x \left[\alpha_2 (d_1 \wedge d_2) + \beta_2 (d_1 \wedge n_2)\right] + y \left[\alpha_2 (n_1 \wedge d_2) + \beta_2 (n_1 \wedge n_2)\right] = 1 \tag{2.27}$$

Then the set of all α_1, β_1 such that u=1 (and this is the only case that needs to be considered) is given by

$$\alpha_1 = x+r \left[\alpha_2 (n_1 \wedge d_2) + \beta_2 (n_1 \wedge n_2)\right],$$
$$\beta_1 = y-r \left[\alpha_2 (d_1 \wedge d_2) + \beta_2 (d_1 \wedge n_1)\right], \qquad (2.28)$$

as r varies over the set S. The formula (2.28) can be made cleaner by noting that $P=\tilde{D}_{-1} \tilde{N}$, and using the relationships between minors of p and minors of $\left[\begin{array}{cc} \tilde{D} & \tilde{N} \end{array}\right]$ (see Vidyasagar 1985, appendix B)). This gives

$$\alpha_1 = x + r \ |\tilde{D}| \ \left[\alpha_2 \ P_{11} + \beta_2 \ P_{12} \ \right], \qquad (2.29)$$
$$\beta_1 = y - r \ |\tilde{D}| \left[\alpha_2 + \beta_2 \ P_{22}\right].$$

The above stated discussion becomes a little simpler if the plant P is already stable. As shown by Theorem 2.4, it is possible under certain circumstances to restrict one's own attention to this case. To proceed further, we need the following result:

Theorem 2.7 (Youla et all 1976, Zames 1981) given a stable plant P, the set of all controllers that stabilize P is given by

$$s(P) = \left\{(I-RP)^{-1} \ R: \ R \epsilon M(s)\right\} \qquad (2.30)$$

Using this theorem, we can characterize all decentralized stabilizing controllers.

Theorem 2.8 Suppose P is a given 2x2 stable plant. Then the set of all diagonal C that stabilize P is given by

$$DS \ (P) = \left\{(I-RP)^{-1}R: \ R \epsilon M(s) \text{ and } r_{12} + P_{12} \left| R \right| = 0, \ r_{21} + p_{21} \left| R \right| = 0\right\} \qquad (2.31)$$

Thus any stable R substituted into the expression $(I-RP)_{-1}$ R yields a

controller that stabilizes P. However, in order for the resulting controller to be diagonal, R must satisfy the two additional conditions $r_{12} + P_{12} |R| = 0$, $r_{21} + P_{21} |R| = 0$. These conditions can be used as follows: pick any function $f \in S$, and set $r_{12} = -p_{12} f$, $r_{21} = -p_{21} f$, and select r_{11}, r_{12} arbitrarily except for the constraint $r_{11} r_{22} = f (1-p_{12} p_{21} f)$.

4. TASK 3

ROBUST DECENTRALIZED CONTROL OF A TWO-CHANNEL SYSTEM

An important practical problem is the following: Given a nominal plant description P_0, together with a "band of uncertainty" containing P_0, when does there exist a controller that stabilizes not only P_0 but also all plants within the band of uncertainty? This problem can be divided into two parts: (i) To determine necessary and sufficient conditions for robust stabilization and (ii) to determine whether there is a controller that satisfies these conditions. The first part was solved by Doyle and Stein (1981), Chen and Desoer (1982) and Vidyasagar and Kimura (1984), while the second part was solved by Vidyasagar and Kimura (1984). These results form the point of departure for Task 3, and are summarized below.

To make the theorem statements more compact, we introduce three classes of uncertainty. In each class, r is a given function belonging to the set S of stable transfer functions. Let P_0 be a given plant description, not necessarily stable, but free of jw-axis poles. Then the class $A(P_0, r)$ of <u>additive perturbations</u> consists of all plants P that have the same number of open RHP poles as P_0, and satisfy

$$|| P(jw) - P_0(jw) || < | r(jw) | \text{ for all } w. \tag{3.1}$$

The class $M(P_0, r)$ of <u>multiplicative perturbations</u> consists of all P of the form $(I+M)P_0$, such that P has the same number of open RHP poles as P_0, and in addition,

$$||M(jw)|| < |r (jw)| \text{ for all w.} \qquad (3.2)$$

Finally, set $(N_0\ D_0)$ be a particular r.c.f. of P_0. Then as discussed in Task 1, the class $S(P_0, r)$ of __stable factor perturbations__ consists of all plants P of the form $N(s)\ [D(s)]^{-1}$, where

$$\left\| \begin{array}{c} N\ (jw)\ -\ N_0\ (jw) \\ D\ (jw)\ -\ D_0\ (jw) \end{array} \right\| < |\ r(jw)\ | \quad \text{for all w.} \qquad (3.3)$$

In the case of stable factor perturbations, there are no restrictions on the relative numbers of RHP poles of P and of P_0.

__Theorem 3.1__ (Doyle and Stein 1981, Chen and Desoer 1982) A controller C that stabilizes P_0 also stabilizes all plants in the class $A\ (P_0,\ r)$ if and only if

$$||[C\ (I + P_0\ C)^{-1}]\ (jw)\ ||. \ |r\ (jw)\ | < 1 \text{ for all w.} \qquad (3.4)$$

C stabilizes all plants in the class $M\ (P_0,\ r)$ if and only if

$$||[P_0\ C\ (I + P_0\ C)^{-1}]\ (jw)\ ||. \ |r\ (jw)\ | < 1 \text{ for all w.} \qquad (3.5)$$

__Theorem 3.2__ (Vidyasagar and Kimura 1984). Suppose C stabilizes P_0, and choose an ℓ.c.p. (A, B) of C such that

$$A\ (s)\ D_0\ (s) + B\ (s)\ N_0\ (s) = I, \text{ for all S.} \qquad (3.6)$$

Then C stabilizes all plants in the class $S\ (p_0,\ r)$ if and only if

$$||([\ A\ \ B]\ r)\ (jw)\ || < 1 \text{ for all w.} \qquad (3.7)$$

__Theorem 3.3__ (Vidyasagar and Kimura 1984). Let (N_0, D_0), $(\overline{D}_0,\ \overline{N}_0)$ be a particular r.c.f. and a particular ℓ.c.f. of P_0, and choose $X, Y, \overline{X}, \overline{Y}$ in M(S) such that

$$X\ N_0 + Y\ D_0 = I, \quad \overline{N}_0\ \overline{X} + \overline{D}_0\ \overline{Y} = I. \qquad (3.8)$$

Then there exists a controller C that stabilizes all plants in the class A (P_0, r) if and only if there exists an ReM (S) such that

$$|| [(\overline{X}\ \overline{D} + D R \overline{D})\ r]\ (jw)\ || < 1 \text{ for all w.} \qquad (3.9)$$

There exists a controller C that stabilizes all plants in the class $M(P_0, r)$ if and only if there exists an ReM (S) such that

$$|| [(N\ X + NRD)\ r\ ]\ (jw)\ || < 1 \text{ for all w.} \qquad (3.10)$$

There exists a controller C that stabilizes all plants in the class $S(P_0, r)$ if and only if there exists an ReM (S) such that

$$|| r\ ([\ Y \quad X] + R\ [-N_0 \quad D_0\ ])\ (jw)\ || < 1 \text{ for all w.} \qquad (3.11)$$

Now we come to Task 3. The objective of this task is to determine conditions for the existence of robust <u>decentralized</u> controllers, in the special case of a two-channel system. We restrict attention to the case where P_0 is stable. In this case the problem is one of determining whether or not there exists a decentralized controller C that satisfies (3.4), (3.5) or (3.6), as appropriate. It turns out that this can be translated into a restriction on R using Theorem 2.8.

<u>Theorem 3.4</u> Suppose P_0 is 2x2 and decentrally strongly stabilizable. Let C_1 be a diagonal controller that stabilizies P_0, and let $P_1 = P_0$ $(I + C_1\ P_0\ )^{-1}$. Let qϵS be given. Then there exists a decentralized controller that stabilizes all plants in the class A $(P_0, q\ )$ if and only if there exists a matrix R S^{2x2} such that

$$r_{12} = -(P_1)_{12}\ |R|, \quad r_{21} = -\ (P_1)_{21}\ |R|, \qquad (3.12)$$

$$||q C_1\ (I - P_1\ C_1\ ) + q\ (I - P_1\ C_1\ )\ ||_\infty < 1. \qquad (3.13)$$

There exists a decentralized controller that stabilizes all plants in the class $M(P_0, q)$ if and only if there exists a matrix $R \epsilon S^{2 \times 2}$ satisfying (3.12) such that

$$|| q \ P_1 \ C_1 + q \ P_1 \ R(I - P_1 \ C_1 \ ) \ ||_{\infty} < 1. \tag{3.14}$$

Finally, let (N_0, D_0) be an r.c.f. of P_0 and let U be the unique unimodular matrix in $S^{2 \times 2}$ such that

$$\begin{bmatrix} N_0 \\ D_0 \end{bmatrix} U = \begin{bmatrix} P_1 \\ I- C_1 \ P_1 \end{bmatrix} . \tag{3.15}$$

Then there exists a decentralized controller that stabilizes all plants in the class $S(P_0, r)$ if and only if there exists a matrix R $S^{2 \times 2}$ satisfying (3.12) such that

$$|| \ (\ U \ [\ I \ \ C_1 \ ] + R \ [\ - P_1 \ \ \ I- P_1 \ C_1 \ ]) \ q \ ||_{\infty} < 1. \tag{3.16}$$

5. TASK 4

OPTIMAL DECENTRALIZED FILTERING FOR A TWO-CHANNEL SYSTEM

In this task, we propose an iterative solution procedure based on Theorem 2.8. Suppose P is 2x2, W is a given 2x2 weighting matrix, and we want to minimize the cost function

$$J = || \ W \ (I + PC)^{-1} \ ||_\infty, \tag{4.1}$$

where the norm $||.||_\infty$ is defined by

$$||F||_\infty = \text{ess.} \sup_W ||F \ (jw)||, \tag{4.2}$$

and we want to find the optimal <u>decentralized</u> C. If P is decentrally strongly stabilizable, then Theorem 2.4 allows us to assume without loss of generality that P is stable. In this case, the set of all $(I+PC)^{-1}$ that result from a stabilizinbg controller is given by I-PR, as R varies over M(S). However, only certain choices of R lead to decentralized controllers, as shown in Theorem 2.8. Thus the problem is one of minimizing $||W(I-PR)||_\infty$, subject to the constraint that

$$r_{12} = - \ p_{12} \ |R|, \ r_{21} = - \ p_{21} \ |R|. \tag{4.3}$$

Let us denote R by γ; then (4.3) implies that

$$r_{12} = - \ p_{12} \ \gamma, \ r_{21} = -p_{21} \ \gamma \ , \ r_{11} \ r_{22} = \gamma \ (1 - p_{12} \ p_{21} \ \gamma \ ). \tag{4.4}$$

The cost function to be minimized is $||W(I-PR)||_\infty$, where

$$I-PR = \begin{bmatrix} 1 - (p_{11} \ r_{11} + p_{12} \ r_{21} \ ) & - (p_{11} \ r_{12} + p_{12} \ r_{22} \ ) \\ - (p_{21} \ r_{11} + p_{22} \ r_{21} \ ) & 1 - (p_{21} \ r_{12} + p_{22} \ r_{22} \ ) \end{bmatrix}$$

$$= \begin{bmatrix} -\gamma\, p_{12}\, p_{21} - p_{11}\, r_{11} & -p_{11}\,\gamma - p_{12}\, r_{22} \\ p_{21}\, r_{11} - p_{22}\, p_{21}\,\gamma & 1 - \gamma\, p_{12}\, p_{21} - p_{22}\, r_{22} \end{bmatrix} \qquad (4.5)$$

The norm of this quantity, which is an __affine__ function of the vector $[\gamma\ r_{11}\ r_{22}]$, is to be minimized subject to the constraint

$$r_{11}\, r_{22} - \gamma\,(1 - p_{12}\, p_{21}\,\gamma) = 0. \qquad (4.6)$$

This can be done using nonlinear programming.

CONCLUSIONS

With various degrees of detail and complexity we have obtained signifi-
cant results on <u>all</u> of the proposed tasks, <u>plus</u> an additional subtask
together with two robust controller designs for a flexible structure model.
More specifically, two numerical procedures were given for construction of
two different types of robust controllers. In addition, the concept of
fixed modes was extended to that of fixed determinant, which is important
for decentralized control. Furthermore, it was demonstrated that for a
stable system, then exists a class of robust decentralized controllers for
a 2x2 system. Finally, an iterative approach was proposed to solve optimal
decentralized filtering for a two channel system. It appears evident that
in this Phase I preliminary effort we have demonstrated the feasibility of
the approach.

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In this appendix we give the proofs of all results that are original to this proposal.

<u>Proof of Theorem 2.2</u> First, let K be a block-diagonal matrix. Then, (I,K) is an ℓ.c.f. of K, so that $\phi(P,K)=|I \ D+K \ N|=|D+KN|$. Now

$$|D+KN| = |D| \ |I+KP| \tag{A.1}$$

Expanding the determinant $|I+KP|$ using the well-known formula for the determinant of the sum of a diagonal matrix and an arbitrary matrix (see e.g. Vidyasagar 1985) gives

$$\phi \ (\ P, \ K \ ) = |D|. \ 1 + \Sigma \ \text{principal minors of KP} \tag{A.2}$$

Now the minors of KP can be expanded by the Binet-Cauchy formula (Vidyasagar 1985) This gives

$$(KP)_{I,I} = \Sigma \ K_{I,J} \ P_{J,I} \tag{A.3}$$

$$\phi \ (P,K) = |D| + \Sigma \ K_{I,J} \ P_{J,I} \ |D| \tag{A.4}$$

However, (Vidyasagar 1985) the quantity $P_{J,I} \ |D|$ can be expressed as a minor of F, and several minors $K_{I,J}$ are structurally equal to zero because of the block-diagonal nature of K. Thus the only minors that appear in (A.4) are the minors of F referred to in the statement of Theorem 2.2. If ζ denotes the g.c.d. of these minors, then each term in (A.4) is a multiple of ζ, whatever be K. Hence ζ divides $\phi(P,K)$ for all K, and finally divides f, which is the g.c.d. of all $\phi(P,K)$.

To complete the proof, we show that f divides ζ. First, let K=0.

Then, from (A.4), $\phi(P,K)=|D|$. Hence, from (2.3), f divides $|D|$. Next, let

K be a matrix with a single nonzero element, which equals one. Then the

summation in (A.4) reduces to a single minor of F, wherein one row of D is

replaced by a row of N. Thus f divides this minor. (This part of the

argument is very similar to that in Vidyasagar and Viswanadham (1983) and

is therefore only sketched.) Next we choose K to contain two one's and the

rest zero. This will show that f divides appropriate minors of F con-

taining two rows of N and the rest from D. Finally, f divides the g.c.d.

of all these minors, which is ζ.

Proof of Theorem 2.3   Clearly g divides f since g is g.c.d. of a larger

set. To prove that f divides g, write

$$\phi\,(P,C) = |I + PC|. \quad |D_C\,|. \quad |D.|$$

$$= |D_C|. \quad |D| + \Sigma \text{ minors of } P. \quad |D|. \text{ minors of } C. \quad |D_C\,|$$

$$= |D_C|. \quad |D| + \Sigma \text{ minors of } F. \text{ minors of } [\;D_C\;N_C\;] \qquad (A.5)$$

However, every function in (A.5) is a multiple of ζ=f; hence $\phi(P,C)$ is a

multiple of f. Finally, the g.c.d. of all such $\phi(P,C)$ is a multiple of f,

i.e. f divides g.

Proof of Theorem 2.4   In (Vidyasagar 1985, Theorem (5.3.10)), it is

shown that if C stabilizes P, and C_1 stabilizes P_1, then $C+C_1$ stabilizes

P. Now, if C and C_1 are both decentralized, so is $C+C_1$. This proves that

$C+DS(P_1)$ is a subset of DS(P). Next, suppose C is decentralized but

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unstable. Then, from the proof of the above-cited theorem, we know that, for sufficiently small α, the controller αC does not belong to $S(P_1)$ (and therefore certainly not to $DS(P_1)$), even though $(1+\alpha)C \epsilon S(P)$. Since C is decentralized, so is $(1+\alpha)C$. Thus, $(1+\alpha)C$ $DS(P)$ but αC $DS(P_1)$, i.e. $C+DS(P_1)$ is a strict subset of $DS(P)$.

Proof of Theorem 2.5 Suppose C is stable and diagonal. Then, letting C_i denote the ii-th element of C, we get

$$|D + CN| = | d_1 + c_1 \; n_1 \; \ldots \; d_m + c_m \; n_m |$$

$$= |D| + \underset{\alpha \neq o}{\Sigma} \text{ a minor of } c \cdot \upsilon_\alpha$$

$$= |D| + \text{ a multiple of g.c.d. } \underset{\alpha \neq o}{\upsilon_\alpha} \tag{A.6}$$

In order for this quantity to be a unit, $|D|$ and g.c.d. $\upsilon\alpha$ must satisfy the parity interlacing property.

Proof of Lemma 2.6 see Vidyasagar et al. (1985).

Proof of Theorem 2.8 Let $C=(I-RP)^{-1}R$. Then $C=Y^{-1}X$, where $Y=I-RP$, $X=R$. Let the subscript i denote the i-th column of a matrix. Then, by Cramer's rule,

$$c_{12} = 0_1 \; c_{12} = o \text{ if and only if } y_1 \wedge x_1 = 0 \;, y_2 \wedge x_2 = 0 . \tag{A.7}$$

Let ei denote the i-th elementary unit vector, i.e. the i-th column of the identity matrix. Then $y_1 = e_1 - Rp_1$, $y_2 = e_2 - Rp_2$. Similarly, $x_1 = r_1$, $x_2 = r_2$. Thus

- 47 -

$$y_1 \wedge x_1 = 0 \text{ if and only if } (c_1 - RP_1) \wedge r_1 = 0 \qquad (A.8)$$

But $Rp1 = r_1 \, p_{11} + r_2 \, p_{21}$, Hence

$$y_1 \wedge x_1 = 0 \text{ if and only if } c_1 \wedge r_1 - (r_1 \, p_{11} + r_2 \, p_{21}) \wedge r_1 = 0$$
$$\text{if and only if } c_1 \wedge r_1 - p_{21} \, r_2 \wedge r_1 = 0 \qquad (A.9)$$

since $r_1 \wedge r_1 = 0$. Now

$$e_1 \wedge r_1 = \begin{vmatrix} 1 & r_{11} \\ 0 & r_{21} \end{vmatrix} = r_{21}, \ r_2 \wedge r_1 = r_1 \wedge r_2 = -|R|. \qquad (A.10)$$

Hence (A.9) becomes

$$y_2 \wedge x_1 = 0 \text{ if and only if } r_{21} = -p_{21} \, |R|. \qquad (A.11)$$

Similarly,

$$y_2 \wedge X_2 = 0 \text{ if and only if } r_{12} = -p_{12} \, |R| \qquad (A.12)$$

<u>Proof of Theorem 3.4</u> As shown in (Vidyasagar 1985, p.   ), one can choose

$$\begin{bmatrix} Y & X \\ -N & D \end{bmatrix} = \begin{bmatrix} I & C_1 \\ P_1 & I - P_1 \, C_1 \end{bmatrix}, \ \begin{bmatrix} D & -X \\ N & Y \end{bmatrix} = \begin{bmatrix} I - C_1 \, P_1 & C_1 \\ P_1 & I \end{bmatrix} (A.13)$$

as a doubly coprime factorization of P_0. Moreover, $C + C_1$ stabilizes P_0 if and only if

$$C_1 = R \, (I - P_{1R})^{-1}, \ R \in S^{2 \times 2}. \qquad (A.14)$$

Since C is diagonal, $C+C_1$ is diagonal if and only if C_1 is. By Theorem 2.8, C_1 is diagonal if and only if R satisfies (3.12). This reads to (3.13) and (3.14).

To prove (3.16), note that, since $(N_0 \ D_0)$ and $(P_1, \ I-C_1P_1)$ are both r.c.f.'s of P_0, there must exist a unimodular U such that (3.15) holds. Suppose C stabilizes P, i.e. $C-C_1$ stabilizes P_1. Then $C-C_1$ must be of the form

$$C-C_1 = (I-RP_1)^{-1} \ R \tag{A.15}$$

and so C must be of the form

$$C = (I - RP_1)^{-1} [\ C_1 + R \ (I - P_1 \ C_1] \tag{A.16}$$

Now an ℓ.c.f. $(\tilde{D}_c, \ \tilde{N}_c)$ of C such that

$$D_c \ D_0 + N_c \ N_0 = I \tag{A.17}$$

is given by

$$[\ D_c \ N_c \ ] = \upsilon \ [\ I - RP_1 \quad C_1 + R \ (\ I - P_1 \ C_1 \ )]$$

$$= \upsilon \ (\ [\ I \quad C_1 \ ] + R \ [\ - P1 \quad I - P_1 \ C_1 \ ]). \tag{A.18}$$

Controller Equations for Simultaneous Stabilization
(Section 1.(a))

Controller dynamic equations:

$$\frac{d}{dt} x_c(t) = (AC)x_c(t)+(BC)u_c(t)$$

$$y_c(t) = (CC)x_cH) + (EC)u_c(t)$$

Controller input u_c: 2x1   vector

          output y_c: 1x1   scalar

          state x_c: 19x1   vector

Controller matrices:

                  AC: 19x19   matrix

                  BC: 19x2   matrix

                  CC: 1x19   vector

                  EC: 1x2   vector

AC ::

Columns 1 to 6

			6		
-2.8000D+01	-3.2200D+02	-6.6496D+03	1.1790D+04	-2.7795D+06	5.4821D+06
1.0000D+00	0.0000D+00	3.5108D+02	-1.4043D+03	2.5951D+05	-5.1622D+05
0.0000D+00	1.0000D+00	-1.0454D+01	4.1816D+01	-2.8870D+04	5.7657D+04
0.0000D+00	0.0000D+00	8.3230D-02	3.6671D+00	-5.9031D+02	1.1733D+03
0.0000D+00	0.0000D+00	-1.8140D-02	1.0726D+00	6.8529D+01	-1.3720D+02
0.0000D+00	0.0000D+00	-2.1341D-03	8.5365D-03	1.2906D+00	-5.9829D-01
0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	4.5069D-01	9.8613D-02
0.0000D+00	0.0000D+00	-1.9169D+01	7.6676D+01	-2.6618D+04	5.3083D+04
0.0000D+00	0.0000D+00	2.5559D+00	-1.0223D+01	3.8734D+03	-7.7263D+03
0.0000D+00	0.0000D+00	-4.5883D-02	1.8353D-01	-1.7955D+02	3.5874D+02
0.0000D+00	0.0000D+00	-8.3230D-02	3.3292D-01	-9.4544D+01	1.8842D+02
0.0000D+00	0.0000D+00	1.8140D-02	-7.2560D-02	2.6334D+01	-5.2522D+01
0.0000D+00	0.0000D+00	2.1341D-03	-8.5365D-03	1.0204D+00	-2.0237D+00
0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	-2.7659D-01	5.5319D-01
0.0000D+00	0.0000D+00	2.1341D-03	-8.5365D-03	1.0758D+00	-2.1345D+00
0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00
0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00
0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00
0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00
0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00
0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00

Columns 7 to 12

			12		
-1.9873D+08	0.0000D+00	0.0000D+00	-4.6896D+03	1.8759D+04	-2.7663D+06
-2.3634D-07	0.0000D+00	0.0000D+00	3.5108D+02	-1.4043D+03	2.5951D+05
-4.2356D+06	0.0000D+00	0.0000D+00	-1.0454D+01	4.1816D+01	-2.8870D+04
-4.7122D+04	0.0000D+00	0.0000D+00	-9.1677D-01	3.6671D+00	-5.9031D+02
1.2467D+04	0.0000D+00	0.0000D+00	-1.8140D-02	7.2560D-01	6.8529D+01
1.5840D+02	0.0000D+00	0.0000D+00	-2.1341D-03	8.5365D-03	2.7061D-01
7.5288D+01	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	4.5069D-01
-3.3700D+06	-3.1500D+01	-4.1125D+02	-2.8935D+03	-1.1496D+04	-5.3274D+04

5. 0350D+05	1. 0000D+00	0. 0000D+00	2. 5559D+00	-1. 0223D+01	3. 8734D
-2. 7417D+04	0. 0000D+00	1. 0000D+00	-4. 5883D-02	1. 8353D-01	-1. 7955D
-1. 1119D+04	0. 0000D+00	0. 0000D+00	9. 1677D-01	3. 3292D-01	-9. 4544D
3. 3802D+03	0. 0000D+00	0. 0000D+00	1. 8140D-02	9. 2744D-01	2. 6334D
5. 0600D+01	0. 0000D+00	0. 0000D+00	2. 1341D-03	-8. 5365D-03	2. 0204D
-4. 6204D+01	0. 0000D+00	0. 0000D+00	0. 0000D+00	0. 0000D+00	-2. 7659D
5. 9849D+01	0. 0000D+00	0. 0000D+00	2. 1341D-03	-8. 5365D-03	1. 0758D
0. 0000D+00	0. 0000D+00	0. 0000D+00	0. 0000D+00	0. 0000D+00	0. 0000D
0. 0000D+00	0. 0000D+00	0. 0000D+00	0 0000D+00	0. 0000D+00	0. 0000D
0. 0000D+00	0. 0000D+00	0. 0000D+00	0. 0000D+00	0. 0000D+00	0. 0000D
0. 0000D+00	0. 0000D+00	0. 0000D+00	0 0000D+00	0. 0000D+00	0. 0000D

Columns 13 to 18

5. 4951D+06	-4. 0296D+08	-8. 0000D+01	2. 5600D+04	-4. 0960D+05	3. 2768D
-5. 1622D+05	6 5510D+07	0. 0000D+00	0. 0000D+00	0. 0000D+00	0 0000D
5. 7657D+04	-1. 6209D+07	0. 0000D+00	0. 0000D+00	0. 0000D+00	0. 0000D
1. 1733D+03	-1 1215D+05	0. 0000D+00	0. 0000D+00	0. 0000D+00	0 0000D
-1. 3720D+02	5. 1874D+04	0. 0000D+00	0. 0000D+00	0. 0000D+00	0. 0000D
-5 9829D-01	8. 6164D+02	0 0000D+00	0 0000D+00	0. 0000D+00	0. 0000D
-9. 0139D-01	3. 0395D+02	0. 0000D+00	0. 0000D+00	0. 0000D+00	0. 0000D
2 0721D+04	-1. 1988D+07	0 0000D+00	0. 0000D+00	0. 0000D+00	0 0000D
-7. 7263D+03	1. 8150D+06	0. 0000D+00	0. 0000D+00	0. 0000D+00	0. 0000D
3. 5874D+02	-1. 0678D+05	0. 0000D+00	0. 0000D+00	0. 0000D+00	0. 0000D
1. 8842D+02	-3. 7800D+04	0. 0000D+00	0. 0000D+00	0. 0000D+00	0. 0000D
-5. 2522D+01	1. 2101D+04	0. 0000D+00	0. 0000D+00	0. 0000D+00	0. 0000D
-2. 0237D+00	2. 2510D+01	0. 0000D+00	0. 0000D+00	0. 0000D+00	0. 0000D
1 5532D+00	-1. 8653D+02	0. 0000D+00	0. 0000D+00	0. 0000D+00	0 0000D
-2. 1345D+00	5. 9849D+01	-8. 0000D+00	0. 0000D+00	0. 0000D+00	0. 0000D
0 0000D+00	0. 0000D+00	1. 0000D+00	-8. 0000D+00	0. 0000D+00	0 0000D
0. 0000D+00	0. 0000D+00	0. 0000D+00	1. 0000D+00	-8. 0000D+00	0. 0000D
0. 0000D+00	0. 0000D+00	0. 0000D+00	0. 0000D+00	1. 0000D+00	-8. 0000D
0. 0000D+00	0. 0000D+00	0. 0000D+00	0. 0000D+00	0. 0000D+00	1. 0000D

Columns 19 to 19

-1. 0486D+07
0. 0000D+00
0 0000D+00
0. 0000D+00
0. 0000D+00
0. 0000D+00
0. 0000D+00
0. 0000D+00
0. 0000D+00
0. 0000D+00
0 0000D+00
0. 0000D+00
0 0000D+00
0 0000D+00
0 0000D+00
0 0000D+00
0 0000D+00
0 0000D+00
-8 0000D+00

BC   =

```
      4. 6896D+03    4. 0235D+05
     -3. 5108D+02   -8. 2540D+04
      1. 0454D+01    2. 3600D+04
      9. 1677D-01    1. 2818D+02
      1. 8140D-02   -7. 7673D+01
      2. 1341D-03   -1. 3664D+00
      0. 0000D+00   -4. 5069D-01
      1. 9169D+01    1. 6756D+04
     -2. 5559D+00   -2. 5850D+03
      4. 5983D -02   1. 5642D+02
      8. 3230D -02   5. 2589D+01
     -1. 6140D-02   -1. 7190D+01
     -2. 1341D-03    5. 5366D-02
      0. 0000D+00    2. 7659D-01
     -2. 1341D-03    0. 0000D+00
      0. 0000D+00    0. 0000D+00
      0. 0000D+00    0. 0000D+00
      0. 0000D+00    0. 0000D+00
      0. 0000D+00    0. 0000D+00
```

CC   =

```
      Columns   1 to    6
     3 5000D+01   -6. 9516D+02    1. 5938D+04   -3. 1472D+05    5. 3675D+06   -2 8090D+07

      Columns   7 to   12
     2. 6991D+08    3. 8500D+01   -1. 0745D+03    1. 9195D+04   -6. 2458D+05    6. 3156D+06

      Columns  13 to   18
    -6 7300D+07    3. 0853D+08    8. 0000D+01   -2. 5600D+04    4. 0960D+05   -3. 2768D+06

      Columns  19 to   19
     1. 0486D+07
```

EC   =

```
     -8. 9617D+03    0. 0000D+00
```

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END

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