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ON THE DENSITY OF THE RANGE FOR  
SOME SUPERQUADRATIC OPERATORS

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ABSTRACT

We study the existence of periodic solutions for a class of Hamiltonian systems of the form

$$\dot{z} - JH_z(t, z) = f(t) .$$

By using the Leray-Schauder Theorem to solve a modified problem and passing to a limit, we show  $\dot{z} - JH_z(t, z)$  has dense range in  $L^2$ . We also obtain similar density results for second order Hamiltonian operators.

AMS(MOS) Subject Classifications: 34C25, 35L10

Key Words: Hamiltonian systems, nonlinear hyperbolic equations, dense ranges,  
periodic solutions

Work Unit Number 1 (Applied Analysis)

### SIGNIFICANCE AND EXPLANATION

There has been a lot of recent work on showing certain classes of Hamiltonian systems of ordinary differential equations have multiple time periodic solutions. This paper addresses the simpler question of when do such operators have dense range in  $L^2$ . Several results of this type are proved both for general and for second order Hamiltonian systems.

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Keywords: periodic solutions, operator (mathematics);  
nonlinear hyperbolic equations, Leray-Schauder theorem,  
ordinary differential equations.  
ON THE DENSITY OF THE RANGE FOR  
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§1. Introduction

During the past few years there has been a considerable amount of research on the existence of multiple solutions of "superlinear" differential equations. By superlinear we mean the equations possess a nonlinearity which grows more rapidly than linearly at infinity. This research has been done in the setting of boundary value problems for semilinear elliptic partial differential equations, and periodicity problems for Hamiltonian systems of ordinary differential equations and semilinear wave equations. (See e.g. [1-7]) For example, Rabinowitz proved [2]

Theorem 1.1 Let  $H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$  and satisfy: there is an  $r > 0$  and  $\mu > 2$  such that  $0 < \mu H(z) < H_z(z) \cdot z$

for all  $|z| > r$ . Then for all  $T, R > 0$ , the Hamiltonian system

$$\dot{z} = JH_z(z) \tag{1.2}$$

possesses a T-periodic solution  $z(t)$  with  $\max_{t \in [0, T]} |z(t)| > R$ .

The existence of multiple periodic solutions of (1.2) is based on a variational formulation of (1.2) which is invariant under a group of symmetries. A natural question to ask is what happens if a perturbation is made which destroys the symmetries. This has been done by Bahri and Berestycki [6] for (1.2) who proved

Theorem 1.3 Let H satisfy conditions

- <1>  $H \in C^2(\mathbb{R}^{2n}, \mathbb{R})$
- <2>  $0 < \mu H(z) < H_z(z) \cdot z$  for all  $z \in \mathbb{R}^{2n}, |z| > R, \mu > 2$ .
- <3>  $a|z|^{p+1} - b < H(z) < a_1|z|^{q+1} + b_1$  with  $1 < p < q < 2p + 1$   
where  $a, a_1 > 0, b, b_1 > 0$  and  $R > 0$  are constants.

Let  $T > 0$  be given and  $f \in C^1(\mathbb{R}, \mathbb{R}^{2n})$  be a given  $T$ -periodic function. Then

$$\dot{z} - JH_z(z) = f(t) \quad (1.4)$$

possesses infinitely many distinct  $T$ -periodic solutions  $\{z_k\}_{k \in \mathbb{N}}$ . Moreover

$$\|z_k\|_{L^\infty} \rightarrow +\infty \text{ as } k \rightarrow +\infty.$$

We will prove another kind of existence result for (1.4), namely

Theorem 1.5 Let  $H$  satisfy

$$(H1) \quad H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$$

$$(H2) \quad \lim_{|z| \rightarrow +\infty} \frac{H_z(z) \cdot z}{|z|^2} = +\infty.$$

Then for any  $T > 0$ , there exists a dense set  $D$  in the space  $E$  of  $T$ -periodic functions in  $L^2([0, T], \mathbb{R}^{2n})$  such that for any  $f \in D$ , (1.4) possesses a  $T$ -periodic solution. (i.e. the range of the operator  $I \frac{d}{dt} - JH_z(\cdot)$  is dense in  $E$ ).

Note that we require much milder conditions on  $H$  than Theorem 1.3, but we also get a weaker existence statement. However Theorem 1.5 suggests that a stronger result than Theorem 1.3 may be true.

The proof of Theorem 1.5 is fairly elementary and was motivated by an analogous kind of result for a semilinear wave equations that was recently proved by Tanaka [9]. A more complicated density theorem was proved by Bahri in an elliptic setting [8]. For other density results, we refer the readers to the results of Hofer [11] and Willem [12].

In §2 we will prove Theorem 1.5 as well as some other results when  $H = H(t, z)$ . In §3 we will prove some parallel results for second order Hamiltonian systems. Finally in §4 we will make a slight extension to Tanaka's work on semilinear wave equations.

## §2. First Order Superquadratic Hamiltonian Operators

Consider the perturbed Hamiltonian system

$$\dot{z} - JH_z(z) = f(t) \quad (2.1)$$

where  $z \in \mathbb{R}^{2n}$ ,  $\dot{z} \equiv \frac{dz}{dt}$ ,  $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ .  $I$  is the identity matrix on  $\mathbb{R}^n$  (or on  $\mathbb{R}^{2n}$  later).  $H : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ , and  $H_z(z)$  is its gradient. Let  $z \cdot y$  denote the usual inner product in  $\mathbb{R}^{2n}$  and  $|z|^2 = z \cdot z$ . For  $T > 0$ , let  $E$  denote all the  $T$ -periodic functions in  $L^2([0, T], \mathbb{R}^{2n})$ .

For our first result, we shall prove Theorem 1.5. Without loss of generality we may assume  $T = 2\pi$  in the following discussion.

Consider (2.1) in the space  $W = (W^{1,2}(S^1))^n$  with norm

$$\|z\|_W = \left( \int_0^{2\pi} (|\dot{z}|^2 + |z|^2) dt \right)^{1/2}.$$

Let  $(z, y)_E = \int_0^{2\pi} z \cdot y dt$ ,  $\|z\|_E = (z, z)_E^{1/2}$ .

To prove Theorem 1.5, we first study the modified problem

$$\dot{z} - \epsilon z + \epsilon J(\dot{z} - \epsilon z) - \lambda JH_z(z) = \lambda f(t) \quad (2.2)$$

where  $\lambda \in [0, 1]$ ,  $\epsilon \in (0, 1]$ . We shall establish an estimate for any solution of (2.2) in  $W$  independently of  $\lambda \in [0, 1]$ . By using the Leray-Schauder theorem, this shows that for each  $\epsilon \in (0, 1]$  the problem

$$\dot{z} - \epsilon z + \epsilon J(\dot{z} - \epsilon z) - JH_z(z) = f(t) \quad (2.3)$$

possesses a solution  $z_\epsilon$  in  $W$ . Then by showing that  $\epsilon \|z_\epsilon\|_W \rightarrow 0$  as  $\epsilon \rightarrow 0$ , we will obtain Theorem 1.5. This sort of approach was used by Tanaka in his study of semilinear wave equations [9].

To begin, we need the following estimate

Lemma 2.4 If  $H$  satisfies (H1) and

(H3) there is a constant  $C > 0$  such that

$$H_z(z) \cdot z > -C \quad \text{for all } z \in \mathbb{R}^{2n}.$$

Then if  $z$  is a solution of (2.2) in  $W$  we have

$$\|z\|_W < \frac{\sqrt{2}}{\epsilon} \|f\|_E + \frac{2}{\epsilon} \sqrt{\pi C} \quad (2.5)$$

Remark 2.6 (H2) implies (H3).

Proof. Notice that  $JJ = -I$ ,  $J^* = -J$ ,  $Jz \cdot z = 0$  for any  $z \in \mathbb{R}^{2n}$ .

Multiply (2.2) by  $J(\dot{z} - \epsilon z)$  and integrate over  $[0, 2\pi]$ . Since  $(z, \dot{z})_E = 0$  and  $(H_z(z), \dot{z})_E = 0$ , we get

$$\epsilon \|\dot{z}\|_E^2 + \epsilon^3 \|z\|_E^2 + \lambda \epsilon (H_z(z), z)_E = \lambda (f, J\dot{z})_E - \lambda \epsilon (f, Jz)_E. \quad (2.7)$$

By (H3) we get

$$\begin{aligned} \epsilon \|\dot{z}\|_E^2 + \epsilon^3 \|z\|_E^2 - 2\pi C \lambda \epsilon &< \lambda \|f\|_E \|\dot{z}\|_E + \lambda \epsilon \|f\|_E \|z\|_E \\ &< \frac{\epsilon^3}{2} (\|\dot{z}\|_E^2 + \|z\|_E^2) + \frac{1}{\epsilon^3} \|f\|_E^2 \end{aligned}$$

so

$$\|z\|_W^2 < \frac{2}{\epsilon} \|f\|_E^2 + \frac{4\pi C}{\epsilon^2}$$

and this gives (2.5). □

Now we let  $I - \epsilon J$  act on (2.2) and get

$$(1 + \epsilon^2)\dot{z} - \epsilon(1 + \epsilon^2)z - \lambda [JH_z(z) + \epsilon H_z(z)] = \lambda [f - \epsilon Jf]$$

i.e.

$$\dot{z} - \epsilon z = \frac{\lambda}{1 + \epsilon^2} [JH_z(z) + \epsilon H_z(z) + f - \epsilon Jf]. \quad (2.8)$$

Since  $I - \epsilon J$  is invertible, (2.8) is equivalent to (2.2). We will solve (2.8)

in  $W$ . In order to do so, consider the corresponding linear problem

$$\dot{z} - \epsilon z = g(t). \quad (2.9)$$

We have

Lemma 2.10 For any  $g \in E$ , (2.9) possesses a unique solution  $z \in W$ , which satisfies

$$\|z\|_W < \frac{2}{\epsilon} \|g\|_E. \quad (2.11)$$

Proof Suppose  $z$  is a solution of (2.9). Multiply (2.9) by  $\dot{z} - \frac{1}{\epsilon} z$ , and then integrate over  $[0, 2\pi]$ . We get

$$\| \dot{z} \|_{\mathbb{E}}^2 + \| z \|_{\mathbb{E}}^2 = (g, \dot{z})_{\mathbb{E}} - \frac{1}{\epsilon} (g, z)_{\mathbb{E}} < \frac{2}{\epsilon} \| g \|_{\mathbb{E}} \| z \|_{\mathbb{W}}$$

which gives (2.11).

Observe that (2.11) implies the uniqueness of the solution of (2.9).

For any  $g \in \mathbb{E}$ , expand  $g$  in Fourier series

$$g = a_0 + \sum_{j=1}^{\infty} (a_j \sin jt + b_j \cos jt) \quad a_0, a_j, b_j \in \mathbb{R}^{2n}$$

By (2.11), it is easy to find that

$$z = -\frac{a_0}{\epsilon} + \sum_{j=1}^{\infty} \frac{1}{(j^2 + \epsilon^2)} [(jb_j - \epsilon a_j) \sin(jt) - (ja_j + \epsilon b_j) \cos(jt)]$$

is the solution of (2.9). □

Denote the solution operator of (2.9) by  $S_{\epsilon}$ . Then  $S_{\epsilon}$  is continuous from  $\mathbb{E}$  to  $\mathbb{W}$  by (2.11). For  $f \in \mathbb{E}$ , define

$$T_{\epsilon}(y) = \frac{1}{1+\epsilon^2} [JH_z(y) + \epsilon H_z(y) + f - \epsilon Jf] \quad \text{for all } y \in \mathbb{W}.$$

Since  $H \in C^1(\mathbb{E}^{2n}, \mathbb{R})$ , by the Rellich theorem,  $T_{\epsilon}$  is compact from  $\mathbb{W}$  to  $\mathbb{E}$ . Let

$K_{\epsilon} = S_{\epsilon} \circ T_{\epsilon}$ . Then  $K_{\epsilon}$  is a compact operator from  $\mathbb{W}$  to itself. By Lemma 2.4, for any  $z \in \mathbb{W}$ ,  $\lambda \in [0, 1]$  if  $z = \lambda K_{\epsilon} z$  then

$$\| z \|_{\mathbb{W}} < \text{const.} (\epsilon, \| f \|_{\mathbb{E}}).$$

Therefore by the Leray-Schauder theorem (Theorem 10.3 [10]),  $K_{\epsilon}$  has a fixed point

$z \in \mathbb{W}$  and we have proved

Proposition 2.12 If  $H$  satisfies (H1) and (H3), then for any  $\epsilon \in (0, 1]$  and any  $f \in \mathbb{E}$ , (2.3) possesses a solution  $z \in \mathbb{W}$ .

In order to prove Theorem 1.5 we need the following estimates.

Lemma 2.13 If  $H$  satisfies (H1) and (H2),  $f \in W$ , and  $\epsilon \in (0,1)$ , then any solution  $z_\epsilon$  of (2.3) satisfies

$$\lim_{\epsilon \rightarrow 0} \epsilon \|z_\epsilon\|_W = 0$$

Proof. For convenience we omit the subindex  $\epsilon$  of  $z_\epsilon$ . From (2.7) with  $\lambda = 1$  we have

$$\epsilon \|z\|_E^2 + \epsilon^3 \|z\|_E^2 + \epsilon (H_z(z), z)_E = - (f, Jz)_E - \epsilon (f, Jz)_E \quad (2.14)$$

By (H2), for any  $K > 0$ , there is a constant  $C_K > 0$  such that

$$H_z(z) \cdot z > K |z|^2 - C_K \quad \text{for all } z \in \mathbb{R}^{2n} \quad (2.15)$$

Therefore (2.14) gives us

$$\epsilon K \|z\|_E^2 - 2\pi \epsilon C_K < (\|f\|_E + \|f\|_E) \|z\|_E < \frac{\epsilon K}{2} \|z\|_E^2 + \frac{1}{\epsilon K} \|f\|_W^2$$

so

$$\|z\|_E^2 < \frac{1}{K} \|f\|_W^2 + \frac{2\pi C_K}{K} \epsilon^2$$

and

$$\lim_{\epsilon \rightarrow 0} \|z\|_E^2 < \frac{1}{K} \|f\|_W^2.$$

Letting  $K \rightarrow \infty$ , we get

$$\lim_{\epsilon \rightarrow 0} \epsilon \|z\|_E = 0. \quad (2.16)$$

By (2.14) and (2.15) with  $K = 1$  we get

$$\epsilon \|z\|_E^2 - 2\pi \epsilon C_1 < \|z\|_E (\|f\|_E + \|f\|_E)$$

so

$$\|z\|_E^2 < 2\epsilon \|z\|_E \|f\|_W + 2\pi C_1 \epsilon^2$$

and by (2.16) this gives

$$\lim_{\epsilon \rightarrow 0} \epsilon \|z\|_E = 0$$

This completes the proof of Lemma 2.13.  $\square$

The proof of Theorem 1.5

Given  $f \in W$ , by Proposition 2.12, we have that for any  $\epsilon \in (0,1)$ , the problem

$$\dot{z} - JH_z(z) = \epsilon z - \epsilon J(\dot{z} - \epsilon z) + f$$

has solution  $z_\epsilon \in W$ . Lemma 2.13 shows that  $\epsilon z_\epsilon - \epsilon J(\dot{z}_\epsilon - \epsilon z_\epsilon) + f \rightarrow f$  in  $E$  as  $\epsilon \rightarrow 0$ . So this shows that any  $f \in W$  belongs to the closure in  $E$  of the range of the operator  $I \frac{d}{dt} - JH_z(\cdot)$ . Since  $W$  is dense in  $E$ , this completes the proof of Theorem 1.5.  $\square$

Next we will study some generalized versions of Theorem 1.5. Their proofs are close to that of Theorem 1.5 and therefore we will not carry out all details.

Consider

$$\dot{z} - JH_z(t,z) = f(t) \quad (2.17)$$

We shall prove

Theorem 2.18 Given  $T > 0$ . Let  $H$  satisfy

(H4)  $H \in C^1(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R})$

(H5)  $H(t+T, z) = H(t, z)$  for all  $(t, z) \in \mathbb{R} \times \mathbb{R}^{2n}$ .

(H6) There is a constant  $\mu > 2$  such that

$$\lim_{|z| \rightarrow +\infty} \frac{H_z(t, z) \cdot z}{|z|^\mu} = +\infty \quad \text{for all } t \in \mathbb{R}.$$

(H7) There are constants  $a, b > 0$ , such that

$$H_t(t, z) > -a|z|^{\mu/2} - b \quad \text{for all } (t, z) \in \mathbb{R} \times \mathbb{R}^{2n}.$$

Then there exists a dense set  $D \subset E$  such that for any  $f \in D$ , (2.17) possesses a  $T$ -periodic solution (i.e. the range of the operator  $I \frac{d}{dt} - JH_z(t, \cdot)$  is dense in  $E$ ).

Remark 2.19 To illustrate the conditions (H6) and (H7), we may consider the case

$$H(t, z) = H_0(z) + h(t)H_1(z)$$

where  $H_0, H_1$  and  $h$  are  $C^1$ ,  $h$  is  $T$ -periodic,  $H_0$  satisfies (H6), and  $H_1$  satisfies (H7). For example

$$H(t, z) = |z|^4 \log(|z|^2 + 1) + (\sin t) |z|^2$$

To prove this theorem, we consider the modified problem

$$\dot{z} - \epsilon z + \epsilon J(\dot{z} - \epsilon z) - \lambda J H_z(t, z) = \lambda f(t) \quad (2.20)$$

where  $\lambda \in [0, 1]$ ,  $\epsilon \in (0, 1]$ .

We have the following estimates

**Lemma 2.21** Suppose  $H$  satisfies (H4), (H5), (H6) and

(H7) There exist constants  $a, b > 0$  such that

$$H_t(t, z) > -a|z|^\mu - b \quad \text{for all } (t, z) \in \mathbb{R} \times \mathbb{R}^{2n}$$

If  $z$  is a solution of (2.20) then

$$\|z\|_W < C$$

where  $C = C(\epsilon, \|f\|_E, H)$

**Proof.** Multiply (2.20) by  $J(\dot{z} - \epsilon z)$  and integrate over  $[0, 2\pi]$ . Then we get

$$\epsilon \|\dot{z}\|_E^2 + \epsilon^3 \|z\|_E^2 + \lambda [\epsilon (H_z(t, z), z)_E + \int_0^{2\pi} H_t(t, z) dt] = \lambda (f, J\dot{z})_E - \lambda \epsilon (f, Jz)_E \quad (2.22)$$

By (H6) for any  $K > 0$  there is a constant  $C_K > 0$  such that

$$H_z(t, z) \cdot z > K|z|^\mu - C_K \quad \text{for all } (t, z) \in \mathbb{R} \times \mathbb{R}^{2n}$$

so

$$\begin{aligned} \epsilon \|\dot{z}\|_E^2 + \epsilon^3 \|z\|_E^2 + \lambda [\epsilon K \|z\|_L^\mu - \epsilon 2\pi C_K - a \|z\|_L^\mu - 2\pi b] \\ < \lambda (f, J\dot{z}) - \lambda \epsilon (f, Jz) \end{aligned} \quad (2.23)$$

Taking  $K = \frac{a}{\epsilon}$ , we get

$$\begin{aligned} \epsilon \|\dot{z}\|_E^2 + \epsilon^3 \|z\|_E^2 < 2\pi C_K + 2\pi b + \|f\|_E (\|\dot{z}\|_E + \|z\|_E) \\ < 2\pi(C_K + b) + \frac{\epsilon^3}{2} \|z\|_W^2 + \frac{1}{\epsilon^3} \|f\|_E^2 \end{aligned}$$

Therefore

$$\|z\|_W^2 < \frac{4\pi}{\epsilon^3} (C_K + b) + \frac{2}{\epsilon^6} \|f\|_E^2 \quad \text{where } K = \frac{a}{\epsilon}$$

This completes the proof.  $\square$

Lemma 2.24 Let  $H$  satisfy (H4), (H5), (H6) and

(H9) There are constants  $a, b > 0$ ,  $0 < p < \mu - 1$  such that

$$H_\varepsilon(t, z) > -a|z|^p - b \quad \text{for all } (t, z) \in \mathbb{R} \times \mathbb{R}^{2n}$$

Given  $f \in W$ , if  $z_\varepsilon$  is a solution of the problem

$$\dot{z} - \varepsilon z + \varepsilon J(\dot{z} - \varepsilon z) - JH_\varepsilon(t, z) = f(t) \quad (2.25)$$

corresponding to  $\varepsilon \in (0, 1]$ , we have

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \|z_\varepsilon\|_E = 0 \quad (2.26)$$

Proof. For convenience, we omit the subindex  $\varepsilon$  of  $z_\varepsilon$ .

From (2.22) with  $\lambda = 1$ , (H6) and (H9), we get

$$\varepsilon \|\dot{z}\|_E^2 + \varepsilon K \|z\|_{L^\mu}^\mu - 2\pi \varepsilon C_K - a \int_0^{2\pi} |z|^p dt - 2\pi b \langle f, J\dot{z} \rangle + \varepsilon \|f\|_E \|z\|_E$$

Since

$$|z|^p < \frac{\varepsilon K}{2a} |z|^\mu + \left(\frac{\varepsilon K}{2a}\right)^{-\frac{p}{\mu-p}}$$

$$\varepsilon \|\dot{z}\|_E^2 + \frac{\varepsilon K}{2} \|z\|_{L^\mu}^\mu < 2\pi(\varepsilon C_K + b) + \left(\frac{\varepsilon K}{2a}\right)^{-\frac{p}{\mu-p}} + \langle f, J\dot{z} \rangle + \varepsilon \|f\|_E \|z\|_E \quad (2.27)$$

From now on we shall use  $A_1, A_2$ , etc., to denote constants independent of  $\varepsilon$  and  $K$ .

Since

$$\langle f, J\dot{z} \rangle < \|f\|_E \|\dot{z}\|_E < \frac{\varepsilon}{2} \|\dot{z}\|_E^2 + \frac{A_1}{\varepsilon}$$

and

$$\varepsilon \|f\|_E \|z\|_E < \varepsilon A_2 \|z\|_{L^\mu} < \frac{\varepsilon K}{4} \|z\|_{L^\mu}^\mu + A_3 \varepsilon K^{-\frac{1}{\mu-1}}$$

Substituting into (2.27) we get

$$\frac{\varepsilon}{2} \|\dot{z}\|_E^2 + \frac{\varepsilon K}{4} \|z\|_{L^\mu}^\mu < A_4 (\varepsilon C_K + 1 + \varepsilon^{-1} + \varepsilon K^{-\frac{1}{\mu-1}} + (\varepsilon K)^{-\frac{p}{\mu-2}})$$

so

$$\|z\|_E^\mu < A_5 \|z\|_L^\mu < \frac{A_6}{K} (\epsilon^\mu C_K + \epsilon^{\mu-1} + \epsilon^{\mu-2} + \epsilon^\mu K^{-\frac{1}{\mu-1}} + \epsilon^{\frac{\mu(\mu-1-p)}{\mu-p}} K^{-\frac{p}{\mu-p}})$$

Since  $\mu > 2$  and  $p < \mu - 1$ , we get

$$\lim_{\epsilon \rightarrow 0} \|z\|_E^\mu < \frac{A_6}{K} + A_6 K^{-\frac{\mu}{\mu-p}} \rightarrow 0 \quad \text{as } K \rightarrow \infty$$

This proves (2.26) □

Lemma 2.28 Let  $H$  satisfy (H4) (H5) and (H7) and  $f \in W$ . Then if  $z_\epsilon$  is a solution of (2.25) corresponding to  $\epsilon \in (0, 1]$ , we have

$$\lim_{\epsilon \rightarrow 0} \epsilon \|z_\epsilon\|_W = 0 \quad (2.29)$$

Proof. Since (H7) implies (H9) (with  $p = \frac{\mu}{2}$ ), by Lemma 2.24 we only need to estimate

$\|z_\epsilon\|_E^2$ . From (2.27), since  $f \in W$ , we get

$$\epsilon \|z_\epsilon\|_E^2 < A_1 (\epsilon C_K + 1 + (\epsilon K)^{-\frac{p}{\mu-p}}) + \|f\|_E \|z_\epsilon\|_E + \|f\|_E \|z_\epsilon\|_E$$

or

$$\|z_\epsilon\|_E^2 < A_1 (\epsilon^2 C_K + \epsilon + \epsilon^{\frac{\mu-2p}{\mu-p}} K^{-\frac{p}{\mu-p}}) + 2\|f\|_W \|z_\epsilon\|_E$$

Therefore

$$\lim_{\epsilon \rightarrow 0} \|z_\epsilon\|_E^2 < A_1 K^{-\frac{p}{\mu-p}} \rightarrow 0 \quad \text{as } K \rightarrow \infty$$

here since without loss of generality we assume  $p > 0$ . This completes the proof of (2.29). □

The proof of Theorem 2.18

(H7) implies (H8) and (H9). Arguing as in Theorem 1.5 by using Lemma 2.21, we get a solution  $z_\epsilon$  of (2.25) corresponding to each  $\epsilon \in (0, 1]$ . So

Lemma 2.24 Let H satisfy (H4), (H5), (H6) and

(H9) There are constants  $a, b > 0$ ,  $0 < p < \mu - 1$  such that

$$H_\varepsilon(t, z) > -a|z|^p - b \quad \text{for all } (t, z) \in \mathbb{R} \times \mathbb{R}^{2n}$$

Given  $f \in W$ , if  $z_\varepsilon$  is a solution of the problem

$$\dot{z} - \varepsilon z + \varepsilon J(\dot{z} - \varepsilon z) - JH_\varepsilon(t, z) = f(t) \quad (2.25)$$

corresponding to  $\varepsilon \in (0, 1]$ , we have

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \|z_\varepsilon\|_E = 0 \quad (2.26)$$

Proof. For convenience, we omit the subindex  $\varepsilon$  of  $z_\varepsilon$ .

From (2.22) with  $\lambda = 1$ , (H6) and (H9), we get

$$\varepsilon \|z\|_E^2 + \varepsilon K \|z\|_{L^\mu}^\mu - 2\varepsilon c_K - a \int_0^{2\pi} |z|^p dt - 2\pi b < (f, Jz) + \varepsilon \|f\|_E \|z\|_E$$

Since

$$|z|^p < \frac{\varepsilon K}{2a} |z|^\mu + \left(\frac{\varepsilon K}{2a}\right)^{-\frac{p}{\mu-p}}$$

$$\varepsilon \|z\|_E^2 + \frac{\varepsilon K}{2} \|z\|_{L^\mu}^\mu < 2\pi(\varepsilon c_K + b) + \left(\frac{\varepsilon K}{2a}\right)^{-\frac{p}{\mu-p}} + (f, Jz) + \varepsilon \|f\|_E \|z\|_E \quad (2.27)$$

From now on we shall use  $A_1, A_2$ , etc., to denote constants independent of  $\varepsilon$  and  $K$ .

Since

$$(f, Jz) < \|f\|_E \|z\|_E < \frac{\varepsilon}{2} \|z\|_E^2 + \frac{A_1}{\varepsilon}$$

and

$$\varepsilon \|f\|_E \|z\|_E < \varepsilon A_2 \|z\|_{L^\mu} < \frac{\varepsilon K}{4} \|z\|_{L^\mu}^\mu + A_3 \varepsilon K^{-\frac{1}{\mu-1}}$$

Substituting into (2.27) we get

$$\frac{\varepsilon}{2} \|z\|_E^2 + \frac{\varepsilon K}{4} \|z\|_{L^\mu}^\mu < A_4 (\varepsilon c_K + 1 + \varepsilon^{-1} + \varepsilon K^{-\frac{1}{\mu-1}} + (\varepsilon K)^{-\frac{p}{\mu-2}})$$

so

$$\|z\|_E^\mu < A_5 \|z\|_{L^\mu}^\mu < \frac{A_6}{K} (\epsilon^\mu C_K + \epsilon^{\mu-1} + \epsilon^{\mu-2} + \epsilon^\mu K^{-\frac{1}{\mu-1}} + \epsilon^{\frac{\mu(\mu-1-p)}{\mu-p}} K^{-\frac{p}{\mu-p}})$$

Since  $\mu > 2$  and  $p < \mu - 1$ , we get

$$\lim_{\epsilon \rightarrow 0} \|z\|_E^\mu < \frac{A_6}{K} + A_6 K^{-\frac{\mu}{\mu-p}} \rightarrow 0 \quad \text{as } K \rightarrow \infty$$

This proves (2.26) □

Lemma 2.28 Let  $H$  satisfy (H4) (H5) and (H7) and  $f \in W$ . Then if  $z_\epsilon$  is a solution of (2.25) corresponding to  $\epsilon \in (0, 1]$ , we have

$$\lim_{\epsilon \rightarrow 0} \epsilon \|z_\epsilon\|_W = 0 \quad (2.29)$$

Proof. Since (H7) implies (H9) (with  $p = \frac{\mu}{2}$ ), by Lemma 2.24 we only need to estimate

$\epsilon \|z_\epsilon\|_E$ . From (2.27), since  $f \in W$ , we get

$$\epsilon \|z_\epsilon\|_E^2 < A_1 (\epsilon C_K + 1 + (\epsilon K)^{-\frac{p}{\mu-p}}) + \|f\|_E \|z_\epsilon\|_E + \|f\|_E \|z_\epsilon\|_E$$

or

$$\epsilon \|z_\epsilon\|_E^2 < A_1 (\epsilon^2 C_K + \epsilon + \epsilon^{\frac{\mu-2p}{\mu-p}} K^{-\frac{p}{\mu-p}}) + 2\|f\|_W \epsilon \|z_\epsilon\|_E$$

Therefore

$$\lim_{\epsilon \rightarrow 0} \epsilon \|z_\epsilon\|_E^2 < A_1 K^{-\frac{p}{\mu-p}} \rightarrow 0 \quad \text{as } K \rightarrow \infty$$

here since without loss of generality we assume  $p > 0$ . This completes the proof of

(2.29). □

The proof of Theorem 2.18

(H7) implies (H8) and (H9). Arguing as in Theorem 1.5 by using Lemma 2.21, we get a solution  $z_\epsilon$  of (2.25) corresponding to each  $\epsilon \in (0, 1]$ . So

$$\dot{z}_\epsilon - JH_z(t, z_\epsilon) = \epsilon z_\epsilon - \epsilon J(\dot{z}_\epsilon - \epsilon z_\epsilon) + f(t)$$

By Lemma 2.28,  $\epsilon z_\epsilon - \epsilon J(\dot{z}_\epsilon - \epsilon z_\epsilon) + f \rightarrow f$  in  $W$  as  $\epsilon \rightarrow 0$ . This completes the proof of Theorem 2.18.  $\square$

Let  $W^{-1}$  be the completion of  $E$  with respect to the  $W^{-1,2}([0, T])$  norm. Next we show how (H7) can be weakened at the expense of density in  $W^{-1}$ . We have

**Theorem 2.30** Given  $T > 0$ . Let  $H$  satisfy (H4), (H5), (H6), and (H9). Then there exists a dense set  $D \subset W^{-1}$  such that for any  $f \in D$ , (2.17) possesses a  $T$ -periodic solution. (i.e., the range of the operator  $I \frac{d}{dt} - JH_z(t, \cdot)$  is dense in  $W^{-1}$ ).

**Proof** Since (H9) implies (H8), Lemma 2.21 is true. Arguing as in discussion 1.5, we get a solution  $z_\epsilon$  of (2.25) corresponding to each  $\epsilon \in (0, 1)$ . So

$$-(z_\epsilon, \dot{\varphi})_E - (JH_z(t, z_\epsilon), \varphi)_E = (f + \epsilon z_\epsilon - \epsilon^2 Jz_\epsilon, \varphi)_E - \epsilon (Jz_\epsilon, \dot{\varphi})_E$$

for any  $\varphi \in W$ .

By Lemma 2.24, for  $f \in E \cap W^{-1}$

$$(f + \epsilon z_\epsilon - \epsilon^2 Jz_\epsilon, \varphi)_E - \epsilon (Jz_\epsilon, \dot{\varphi})_E \rightarrow (f, \varphi)_E \quad \text{as } \epsilon \rightarrow 0.$$

Since  $E$  is dense in  $W^{-1}$ , this completes the proof.  $\square$

The following result where  $\mu > 3$  in (H6) allows us to weaken (H7).

**Theorem 2.31** Given  $T > 0$ . Let  $H$  satisfy (H4), (H5) and

(H10) There is a constant  $\mu > 3$  such that

$$\lim_{|z| \rightarrow +\infty} \frac{H_z(t, z) \cdot z}{|z|^\mu} = +\infty \quad \text{for all } t \in \mathbb{R}$$

and (H8). Then the conclusion of Theorem 2.30 is still true.

We need the following estimate

**Lemma 2.32** Let  $H$  satisfy (H4), (H5), (H10), and (H8). Then the conclusion of the Lemma 2.24 is true.

**Proof** Multiplying (2.20) by  $J(\dot{z} - z)$ , we get

$$\begin{aligned} & \varepsilon \dot{z}^2_E + \varepsilon^2 |z|_E^2 + (H_z(t, z), z)_E + \int_0^{2\pi} H_t(t, z) dt \\ & = (f, J\dot{z})_E - (f, Jz)_E + \varepsilon (z, J\dot{z})_E + (z, Jz)_E \end{aligned}$$

By (H10), for any  $K > 0$  there is a constant  $C_K > 0$  such that

$$H_z(t, z) \cdot z > K|z|^\mu - C_K \quad \text{for all } (t, z) \in \mathbb{R} \times \mathbb{R}^{2n}$$

Hence by (H8) we get

$$\varepsilon \dot{z}^2_E + K|z|_E^\mu - 2\varepsilon C_K - a|z|_E^\mu - 2\pi b < |f|_E \dot{z}^2_E + |f|_E |z|_E + 2|z|_E \dot{z}^2_E \quad (2.33)$$

Observe that

$$|f|_E \dot{z}^2_E < \frac{\varepsilon}{4} \dot{z}^2_E + \frac{1}{\varepsilon} \Lambda_1,$$

$$|f|_E |z|_E < \Lambda_2 |z|_E^\mu < |z|_E^\mu + \Lambda_3,$$

and

$$2|z|_E \dot{z}^2_E < \frac{\varepsilon}{4} \dot{z}^2_E + \frac{4}{\varepsilon} |z|_E^2 < \frac{\varepsilon}{4} \dot{z}^2_E + |z|_E^\mu + \Lambda_4 \varepsilon^{-\frac{\mu}{\mu-2}}$$

Substituting into (2.33) we get

$$(K-a-2)|z|_E^\mu < \Lambda_5 (C_K + 1 + \varepsilon^{-1} + \varepsilon^{-\frac{\mu}{\mu-2}})$$

and

$$\begin{aligned} |z|_E^\mu & < \Lambda_6 |z|_E^\mu < \frac{\Lambda_7}{(K-a-2)} (\varepsilon^\mu (C_K + 1) + \varepsilon^{\mu-1} + \varepsilon^{\frac{\mu(\mu-3)}{\mu-2}}) \\ & < \frac{\Lambda_7}{(K-a-2)} (\varepsilon^\mu (C_K + 1) + 2) \end{aligned}$$

Therefore

$$\lim_{\varepsilon \rightarrow 0} |z|_E^\mu < \frac{2\Lambda_7}{(K-a-2)} \rightarrow 0 \quad \text{as } K \rightarrow +\infty.$$

This completes the proof.  $\square$

Now combining Lemma 2.21 and Lemma 2.32, we get Theorem 2.31.

### §3. Second Order Superquadratic Hamiltonian Operators

In this section, results similar to those of §2 will be proved for the second order Hamiltonian system

$$\ddot{q} + V_q(q) = f(t) \quad (3.1)$$

where  $q : \mathbb{R} \rightarrow \mathbb{R}^n$ ,  $\ddot{q} = \frac{d^2}{dt^2} q$ .  $V_q(\cdot)$  is the gradient of  $V : \mathbb{R}^n \rightarrow \mathbb{R}$ . Let  $E$  be all the  $T$ -periodic functions in  $L^2([0, T], \mathbb{R}^n)$ . We shall prove

Theorem 3.2 Let  $V$  satisfy

$$(V1) \quad V \in C^1(\mathbb{R}^n, \mathbb{R})$$

$$(V2) \quad \lim_{|q| \rightarrow +\infty} \frac{V_q(q) \cdot q}{|q|^2} = +\infty$$

Then for any  $T > 0$ , there exists a dense set  $D \subset E$  such that for any  $f \in D$ , (3.1) possesses a  $T$ -periodic solution (i.e. the range of the operator  $I \frac{d^2}{dt^2} + V_q(\cdot)$  is dense in  $E$ ).

Remark 3.3 (3.1) is a special case of (2.1) with  $H(q, x) = \frac{1}{2} |x|^2 + V(q)$ , which satisfies (V2) but not (H2). (Here  $(q, x) \in \mathbb{R}^{2n}$ ).

Since the discussion of the second order Hamiltonian system closely parallels that of the first order case and is simpler, we will be rather sketchy in our exposition here.

In order to prove the Theorem 3.2, we consider a modified problem

$$\ddot{q} + \varepsilon \dot{q} + \varepsilon q + \lambda V_q(q) = \lambda f(t) \quad (3.4)$$

in the space  $W^2 \equiv (W^{2,2}(S^1))^n$ , where  $\varepsilon \in (0, 1]$ ,  $\lambda \in [0, 1]$ ,  $T = 2\pi$ .

We need the following estimate

Lemma 3.5 If  $V$  satisfies (V1) and

$$(V3) \quad V_q(q) \cdot q > -M \quad \text{for some } M > 0 \text{ and all } q \in \mathbb{R}^n.$$

Then the solution  $q$  of (3.4) satisfies

$$\|q\|_{W^2} < C$$

where  $C = C(\varepsilon, \|f\|_E, V)$

Proof Multiplying (3.4) by  $\dot{q}$  and integrating over  $[0, 2\pi]$ , gives

$$\| \dot{q} \|_{\mathbb{E}}^2 < \frac{1}{\epsilon} \| f \|_{\mathbb{E}} \quad (3.6)$$

Multiplying (3.4) by  $q$  and integrating over  $[0, 2\pi]$ , we get (by (3.6))

$$\| q \|_{\mathbb{E}}^2 < \frac{4}{\epsilon^3} \| f \|_{\mathbb{E}}^2 + \frac{4\pi M}{\epsilon} \quad (3.7)$$

Therefore, there is a constant  $M_1 = M_1(\epsilon, \| f \|_{\mathbb{E}}, V) > 0$  such that

$$\| q \|_{L^\infty} < \Lambda_1 \| q \|_{\mathbb{E}} < M_1$$

$$\text{Let } M_2 = \sup_{q \in (-M_1, M_1)^n} |V_q(q)|$$

From the equation (3.4), we get

$$\begin{aligned} \| \ddot{q} \|_{\mathbb{E}}^2 &< 4(\epsilon^2 \| \dot{q} \|_{\mathbb{E}}^2 + \epsilon^2 \| q \|_{\mathbb{E}}^2 + |V_q(q)|^2 + \| f \|_{\mathbb{E}}^2) \\ &< 4\left(\left(\frac{5}{\epsilon} + 1\right) \| f \|_{\mathbb{E}}^2 + 4\pi M \epsilon + 2\pi M_2^2\right). \end{aligned} \quad (3.8)$$

Now (3.6), (3.7) and (3.8) give the Lemma.  $\square$

Consider the linear problem

$$\ddot{q} + \epsilon \dot{q} + \epsilon q = g(t). \quad (3.9)$$

We have

Lemma 3.10 Given  $\epsilon \in (0, 1)$ , for any  $g \in \mathbb{E}$ , (3.9) possesses a unique solution  $q$  in  $W^2$  which satisfies

$$\| q \|_{W^2} < \left(1 + \frac{6}{\epsilon^2}\right)^{1/2} \| g \|_{\mathbb{E}}$$

Proof. As in Lemma 2.10.  $\square$

Using Lemma 3.5 and 3.10, as in §2, we get

Proposition 3.11 If  $V$  satisfies (V1) and (V8), then for any  $\epsilon \in (0, 1]$  any  $f \in \mathbb{E}$

$$\ddot{q} + \epsilon \dot{q} + \epsilon q + V_q(q) = f(t) \quad (3.12)$$

possesses a solution in  $W^2$ .

In order to prove Theorem 3.2, we have

Lemma 3.13 Let  $V$  satisfy (V1) and (V2),  $f \in W^1 = (W^{1,2}(S^1))^n$ . If  $q_\epsilon$  is a solution of (3.12) corresponding to  $\epsilon \in (0,1]$ , then

$$\lim_{\epsilon \rightarrow 0} \epsilon \|q_\epsilon\|_{W^1} = 0$$

Proof. As in Lemma 2.13 □

The above Lemma completes the proof of the Theorem 3.2.

As in section 2, there are analogous results for the problem

$$\ddot{q} + V_q(t,q) = f(t) \tag{3.14}$$

In particular we have the following two theorems

Theorem 3.15 Given  $T > 0$ , let  $V$  satisfy

(V4)  $V \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$

(V5)  $V(t+T, q) = V(t, q)$  for all  $(t, q) \in \mathbb{R} \times \mathbb{R}^n$

(V6) There is a constant  $\mu > 2$  such that

$$\lim_{|q| \rightarrow +\infty} \frac{V_q(t,q) \cdot q}{|q|^\mu} = +\infty \quad \text{for all } t \in \mathbb{R}$$

(V7) There are constants  $a, b > 0$  such that

$$V_t(t,q) < a|q|^{\mu/2} + b \quad \text{for all } (t,q) \in \mathbb{R} \times \mathbb{R}^n.$$

Then there exists a dense set  $D \subset E$  such that for any  $f \in D$ , (3.14) possesses a  $T$ -periodic solution, (i.e. the range of the operator  $I \frac{d^2}{dt^2} + V_q(t, \cdot)$  is dense in  $E$ .)

Let  $W^{-1}$  be the completion of  $E$  with respect to the  $W^{-1,2}([0,T])$  norm. Then we have

Theorem 3.16 Given  $T > 0$ , let  $V$  satisfy (V4), (V5), (V6),

and (V8) There are constants  $a, b > 0$ , such that

$$V_t(t,q) < a|q|^{\mu-1} + b \quad \text{for all } (t,q) \in \mathbb{R} \times \mathbb{R}^n.$$

Then there exists a dense set  $D \subset W^{-1}$  such that for any  $f \in D$ , (3.14) possesses a T-periodic solution (i.e. the range of the operator  $I \frac{d^2}{dt^2} + V_q(t, \cdot)$  is dense in  $W^{-1}$ ).

The proofs of the above two theorems follow in the same fashion as those already done and shall be omitted.

#### §4. Superlinear Hyperbolic Operators

We conclude this paper with some remarks about the hyperbolic case.

Let  $C^\infty$  be the real vector space of arbitrarily often continuously differentiable functions on  $(0, \pi) \times \mathbb{R}$  which are T-periodic in  $t \in \mathbb{R}$  and satisfy  $u(0, t) = u(\pi, t) = 0$  for all  $t \in \mathbb{R}$ . Denote by  $E$  the completion of  $C^\infty$  with respect to the norm

$$\|u\|_E = (u, u)_E^{1/2} \quad \text{where} \quad (u, v)_E = \int_0^T \int_0^\pi u(x, t) \cdot v(x, t) dx dt .$$

Let  $W^{-1}$  be the completion of  $E$  with respect to the  $W^{-1,2}((0, \pi) \times (0, T))$  norm.

Call  $u \in E$  a weak solution of the problem

$$u_{tt} - u_{xx} + u_t = f(x, t) \quad (x, t) \in (0, \pi) \times \mathbb{R} \quad (4.1)$$

$$u(0, t) = u(\pi, t) = 0 \quad t \in \mathbb{R} \quad (4.2)$$

$$u(x, t + T) = u(x, t) \quad (x, t) \in (0, \pi) \times \mathbb{R}$$

for  $f \in E$ , if

$$(u, \phi_{tt} - \phi_{xx} - \phi_t)_E = (f, \phi)_E \quad \text{for all } \phi \in C^\infty .$$

In [9], Tanaka studied the problem

$$u_{tt} - u_{xx} + g(u) = f(x, t) , \quad (x, t) \in (0, \pi) \times \mathbb{R} \quad (4.3)$$

with boundary and periodicity condition (4.2).

He proved

Theorem 4.4 Given  $T > 0$  let  $g$  satisfy

$$\langle 1 \rangle \quad g \in C(\mathbb{R}, \mathbb{R})$$

$$\langle 2 \rangle \quad G(s) < C(1 + sg(s)) \quad \text{for some } C > 0 \quad \text{and all } s \in \mathbb{R}$$

$$\text{where } G(s) = \int_0^s g(t) dt$$

$$\langle 3 \rangle \quad \lim_{|s| \rightarrow +\infty} \frac{g(s)}{s} = +\infty.$$

Then for all  $f(x,t)$  in a dense subset  $D \subset E$ , (4.3) (4.2) possesses a weak solution.

By using estimates similar to those in §2, together with Tanaka's techniques, one can easily extend his result to the following problem

$$u_{tt} - u_{xx} + g(x,t,u) = f(x,t) \quad (x,t) \in (0,\pi) \times \mathbb{R} \quad (4.5)$$

with the boundary and periodicity condition (4.2).

We have

Theorem 4.6 Given  $T > 0$ , let  $g$  satisfy

$$(G1) \quad g \in C((0,\pi) \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$$

$$(G2) \quad g(x,t+T,s) = g(x,t,s) \quad \text{for all } (x,t,s) \in (0,\pi) \times \mathbb{R} \times \mathbb{R}$$

(G3) There is a constant  $C > 0$  such that

$$G(x,t,s) < C(1 + g(x,t,s)s) \quad \text{for all } (x,t,s) \in (0,\pi) \times \mathbb{R} \times \mathbb{R}$$

$$\text{where } G(x,t,s) = \int_0^s g(x,t,\tau) d\tau$$

(G4) There is a constant  $\mu > 2$  such that

$$\lim_{|s| \rightarrow +\infty} \frac{g(x,t,s)s}{|s|^\mu} = +\infty \quad \text{for all } (x,t) \in (0,\pi) \times \mathbb{R}$$

(G5) There are constants  $a, b > 0$  such that

$$|g_t(x,t,s)| < a|s|^{\frac{\mu-2}{2}} + b \quad \text{for all } (x,t,s) \in (0,\pi) \times \mathbb{R} \times \mathbb{R}$$

Then there exists a dense set  $D \subset E$  such that for any  $f \in D$ , (4.5) (4.2) possesses a  $T$ -periodic weak solution.

Theorem 4.7 Given  $T > 0$ , let  $g$  satisfy (G1) (G2) (G3) (G4) and

(G6) There are constants  $a, b > 0$ , such that

$$|g_t(x,t,s)| < a|s|^{\mu-2} + b \quad \text{for all } (x,t,s) \in (0,\pi) \times \mathbb{R} \times \mathbb{R}$$

Then there exists a dense set  $D \subset W^{-1}$  such that for any  $f \in D$ , (4.5) (4.2) possesses a  $T$ -periodic weak solution.

Remark 4.8 Tanaka considered the problem

$$\begin{aligned}u_{tt} - \Delta u + g(x,t,u) &= f(x,t) & (x,t) \in \Omega \times \mathbb{R} \\u(x,t) &= 0 & (x,t) \in \partial\Omega \times \mathbb{R} \\u(x,t+T) &= u(x,t) & (x,t) \in \Omega \times \mathbb{R}\end{aligned}$$

where  $\Omega \subset \mathbb{R}^n$  is a bounded domain with smooth boundary and  $g$  was independent of  $x$  and  $t$ . In a similar fashion if  $g$  satisfies (G1) - (G5) (or (G6)) and

$$(G7) \quad |g(x,t,s)| < M(1 + |s|^p), \text{ for all } (x,t,s) \in \Omega \times \mathbb{R} \times \mathbb{R}$$

with constants  $M > 0$ ,  $p > 0$  if  $n = 2$ ,  $0 < p < \frac{n}{n-2}$  if  $n > 2$ , then results similar to Theorems 4.5 and 4.7 are still true. Here (G7) gives us the compactness of the composition operator  $\tilde{g} : W^{1,2} \rightarrow L^{\frac{2n}{n-2}} \rightarrow E$ , where  $\tilde{g}(u) = g(\cdot, \cdot, u(\cdot, \cdot))$ . We shall omit the details of the proofs.

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