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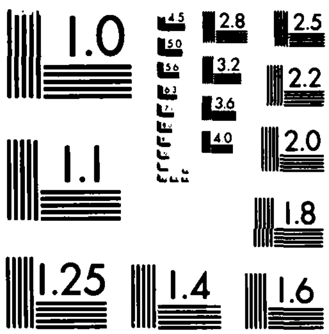

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DISCRETE-TIME CONVERSION FOR SIMULATING SEMI-MARKOV PROCESSES

Bennett L. Fox and Peter W. Glynn

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

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Bennett L. Fox[1,3] and Peter W. Glynn[2,4]

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ABSTRACT

To simulate long-run averages of time integrals of a recurrent semi-Markov process efficiently, convert to discrete-time by simulating only an imbedded chain and computing the conditional expectations of everything else needed given the sequence of states visited. This reduces asymptotic variance and eliminates generating holding-time variates. In this setting, uniformizing continuous-time Markov chains is not worthwhile. ~~We~~ The authors generalize beyond semi-Markov processes and cut ties to regenerative simulation methodology.

AMS (MOS) Subject Classifications: 65C05, 60K15

Key Words: Simulation, conditional Monte Carlo approach, semi-Markov process

Work Unit Number 5 - Optimization and Large Scale Systems

[1] Département d'informatique et de recherche opérationnelle, Université de Montréal.

[2] Mathematics Research Center and Department of Industrial Engineering, University of Wisconsin-Madison.

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SIGNIFICANCE AND EXPLANATION

A broad class of stochastic systems which are studied in operations research may be modelled as semi-Markov processes. Frequently, one is interested in obtaining an estimate, via simulation, for the steady-state average of such a process. In this paper, we offer new insights on an easily implemented procedure, which can substantially improve the accuracy of such an estimate. The basic idea involves passing from the continuous-time semi-Markov process to an appropriate discrete-time sequence, by conditioning out holding time variables.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

DISCRETE-TIME CONVERSION FOR SIMULATING SEMI-MARKOV PROCESSES

Bennett L. Fox[1,3] and Peter W. Glynn[2,4]

1. Introduction

Let X be a positive recurrent semi-Markov processes and f be a real-valued function. We want to estimate the time average

$$r = \lim_{T\to\infty} \left(1/T\right) E \int_0^T f\left(X(t)\right) dt \tag{1}$$

assuming it exists. Section 2.1 starts with a standard regenerative simulation approach and then converts to discrete time by conditioning on the sequence $Y = \left(Y_0, Y_1, \ldots\right)$ of states visited in an imbedded Markov chain. This reduces variance and also, typically, the work to simulate. Hordijk, Iglehart, and Schassberger [6] adopt the same approach for the special case of continuous-time Markov chains, except that they prove variance reduction by explicit calculation without mentioning the general principle that computing conditional expectations reduces variance. Peter Lewis informed us that he too was aware that this conditional Monte Carlo approach would streamline the proofs in [6]. Fox and Glynn [5] obtain an analog of these results for certain <u>finite-horizon</u> semi-Markov processes. Uniformization pays in [5] but not here as shown in [6] and, with less effort, in section 3.

Section 2.2 considers more general processes and cuts ties to the regenerative approach.

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[1] Département d'informatique et de recherche opérationnelle, Université de Montréal.

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2. Discrete-time conversion reduces variance

In section 2.1 we sketch how to generalize results of [6] to regenerative semi-Markov processes and at the same time indicate how to simplify proofs. Section 2.2 gives a formal proof and further generalizes the class of processes considered in [6]; more importantly, it allows nonregenerative approaches.

In section 2, we limit the discussion to a comparison of asymptotic variances. To estimate these variances and construct confidence intervals, see for example Bratley, Fox, and Schrage [2], chapter 3. Standardized time series can also be used, as in Schruben [7].

2.1 Regenerative framework

Let i be a (convenient) recurrent state and let S_j be the time at which state i is visited for the j-th time. We assume that state i is visited at time 0 and set $S_0 = 0$. Put

$$\tau_j = S_{j+1} - S_j, \tag{2}$$

the length of the j-th regeneration cycle. Following for example the development in Bratley, Fox, and Schrage [2], section 3.7, define

$$V_j = \int_{(S_j, S_{j+1})} f(X(t))\,dt \tag{3}$$

$$\bar{V}_n = (1/n) \sum_{j=0}^{n-1} V_j \tag{4}$$

$$\bar{\tau}_n = (1/n) \sum_{j=0}^{n-1} \tau_j \tag{5}$$

$$D_k = V_k - r\tau_k \tag{6}$$

$$\hat{r}_n = \bar{V}_n / \bar{\tau}_n \tag{7}$$

$$\tilde{\tau}_n = E[\bar{\tau}_n \mid Y] \tag{8}$$

$$\tilde{r}_n = E[\bar{V}_n \mid Y]/\tilde{\tau}_n \tag{9}$$

We reach state Y_j at time T_j. Let

$$\alpha_k = T_{k+1} - T_k \tag{10}$$

$$P(\alpha_k \in dt \mid Y) = F(Y_k, Y_{k+1}, dt) \tag{11}$$

$$\mu(x,y) = \int_0^\infty zF(x,y,dz). \tag{12}$$

As jump N_n ends, we visit state i for the n-th time. This gives

$$\tilde{\tau}_n = (1/n) \sum_{j=0}^{N_n-1} \mu(Y_j, Y_{j+1}) \tag{13}$$

$$E[\bar{V}_n \mid Y] = (1/n) \sum_{j=0}^{N_n-1} f(Y_j)\mu(Y_j, Y_{j+1}), \tag{14}$$

so we can compute $\tilde{r}_n$ in (9). Computing and accessing the expected transition times $\mu(Y_j, Y_{j+1})$ and the transition probabilities are similar tasks. Fox and Glynn [5] and references cited there discuss the latter. When $\mu(Y_j, Y_{j+1})$ depends only on Y_j, the former job is normally easy.

Since X is a semi-Markov process (by assumption), the α_k's are conditionally independent given Y. So Y regenerates implies X regenerates. Assuming certain mild moment conditions [inequality (24) will do] and mimicking standard arguments, we get

$$\sqrt{n}(\hat{r}_n - r) \Rightarrow (\delta/E\tau_1)\ N(0,1) \tag{15}$$

$$\sqrt{n}(\tilde{r}_n - r) \Rightarrow (\sigma/E\tilde{\tau}_1)\ N(0,1) \tag{16}$$

where

$$\delta^2 = \text{Var } D_k \tag{17}$$

$$\sigma^2 = \text{Var}\left(E\left[D_k \mid Y\right]\right). \tag{18}$$

Using the standard variance-reducing property of conditional expectation, we get $\sigma < \delta$. Clearly, $E\tau_1 = E\tilde{\tau}_1$, so $\tilde{r}_n$ has smaller variance than $\hat{r}_n$.

2.2 Generalization and formal proof

Now we neither assume that Y is a Markov chain nor that the α_k's are conditionally independent given Y. The only structural assumptions relating Y and X are (11) and (19). With these provisos, define Y_k, α_k, and $\mu(x,y)$ as before. Let I be an indicator, $T_0 = 0$, $T_k \uparrow \infty$, and

$$X(t) = \sum_{k=0}^{\infty} Y_k \, I\left(T_k < t < T_{k+1}\right). \tag{19}$$

Thus, X is more general than a semi-Markov process.

The obvious estimator is

$$\hat{R}_n = \left(1/T_n\right) \int_0^{T_n} f\left(X(s)\right) ds \tag{20}$$

Typically, the expected work to generate $\hat{R}_n$ or $\hat{r}_n$ grows at rate n whether n indexes "transitions" as in this section or cycles as in section 2.1. So we compare efficiencies of various estimators with respect to n.

Roughly speaking, to convert to discrete time we compute conditional expectations given Y. The alternative estimator

$$\tilde{R}_n = \sum_{k=0}^{n-1} f\left(Y_k\right)\mu\left(Y_k, Y_{k+1}\right) \Big/ \sum_{k=0}^{n-1} \mu\left(Y_k, Y_{k+1}\right) \tag{21}$$

is certainly plausible. The only difference between $\hat{R}_n$ defined by (20) and $\hat{r}_n$ defined by (7) lies in what n indexes; likewise, for $\tilde{R}_n$ and $\tilde{r}_n$. We show that

$\tilde{R}_n$ beats $\hat{R}_n$ in a precise sense, provided that $\tilde{R}_n$ is no harder to compute than $\hat{R}_n$.

We assume that X satisfies

(A) $T_n/n \Rightarrow \beta > 0$ (β constant)

(B) there are finite constants r_1 and σ_1 such that

$$(1/\sqrt{n}) \int_0^{T_n} f(X(s)-r_1)ds \Rightarrow \sigma_1 \; N(0,1)$$

(C) there are finite constants r_2 and σ_2 such that

$$(1/\sqrt{n}) \sum_{k=0}^{n-1} (f(Y_k)-r_2)\mu(Y_k,Y_{k+1}) \Rightarrow \sigma_2 \; N(0,1)$$

(D) the sequences $\{n^{-1} \sum_{k=0}^{n-1} f(Y_k)\alpha_k : n > 1\}$,

$$\{T_n/n : n > 1\}, \text{ and } \{n^{-1}[\int_0^{T_n} (f(X(s))-r)ds]^2 : n > 1\}$$

are uniformly integrable.

Section 2.3 contains examples.

Not surprisingly, we have

Proposition. $r_1 = r_2$.

Proof. From (A) and (B), we get $\frac{1}{n}\int_0^{T_n} f(X(s))ds \Rightarrow r_1\beta$. Equivalently, $\frac{1}{n}\sum_{k=0}^{n-1} f(Y_k)\alpha_k \Rightarrow r_1\beta$. From (D), we also get weak convergence to $r_1\alpha$ in the function space L_1. From p. 306 of Chung [4], we therefore get

$$\frac{1}{n}\sum_{k=0}^{n-1} E(f(Y_k)\alpha_k \mid Y) \Rightarrow r_1\beta;$$

i.e. $$\frac{1}{n}\sum_{k=0}^{n-1} f(Y_k)\mu(Y_k,Y_{k+1}) \Rightarrow r_1\beta.$$

The left side is the numerator of $\tilde{R}_n$. Likewise, for the denominator of $\tilde{R}_n$ we get

$$\frac{1}{n}\sum_{k=0}^{n-1} \mu(Y_k, Y_{k+1}) \Rightarrow \beta .$$

From (C), $\tilde{R}_n \Rightarrow r_2$. Comparing these results proves that $r_1 = r_2$. ∎

We are now ready to compare $\hat{R}_n$ and $\tilde{R}_n$. From (A) and (B):

$$\sqrt{n}\,(\hat{R}_n - r_1) \Rightarrow (\sigma_1/\beta)\ N(0,1); \tag{22}$$

from (A) and (C):

$$\sqrt{n}\,(\tilde{R}_n - r_2) \Rightarrow (\sigma_2/\beta)\ N(0,1). \tag{23}$$

That confidence intervals with fixed coverage based on $\tilde{R}_n$ are asymptotically shorter than those based on $\hat{R}_n$ now follows from our main

<u>Theorem</u>. $\sigma_2^2 < \sigma_1^2$.

<u>Proof</u>. By Jensen's inequality for conditional expectations (see p. 302 of Chung [4]),

$$\begin{aligned} n^{-1}E\Big(\int_0^{T_n}(f(X(s))-r)\,ds\Big)^2 &> n^{-1}E\Big[\Big(\int_0^{T_n}(f(X(s))-r)\,ds \mid Y\Big)^2\Big] \\ &= n^{-1}E\Big[\sum_{j=0}^{n-1}(f(Y_j)-r)\mu(Y_j,Y_{j+1})\Big]^2. \end{aligned}$$

By (D) the extreme left side converges to σ_1^2. We show that the right side converges to σ_2^2, which requires a uniform integrability argument.

Let

$$A_{nk} = \{n^{-1}\Big[\sum_{j=0}^{n-1}(f(Y_j)-r)\mu(Y_j,Y_{j+1})\Big]^2 > k\}.$$

Since A_{nk} is Y-measurable, apply Jensen's inequality again to get

$$E\{I_{A_{nk}}(\frac{1}{n}\sum_{j=0}^{n-1}(f(Y_j)-r)\mu(Y_j,Y_{j+1}))^2\} \le E[(I_{A_{nk}}(\frac{1}{n}\int_0^{T_n} f(X(s)-r)\,ds))^2].$$

Use (C) to see that $\lim_{k\to\infty}\sup_n P(A_{nk}) \to 0$ and then theorem 4.5.3 of Chung [4] to see that the term in brackets goes to 0 uniformly in n as $k \to \infty$. This proves uniform integrability of the term in braces. ∎

2.3 Examples

1. Let X be a semi-Markov process for which Y is regenerative: there exists s such that $P[Y_n = s$ infinitely often$] = 1$. Set $T = \inf\{n \ge 1: Y_n = s\}$ and assume that $Y_0 = s$ and

$$E(\sum_{k=0}^{T-1}(|f(Y_k)| + 1)\alpha_k)^4 < \infty. \tag{24}$$

Then (A)-(D) hold; for uniform integrability, see Chung [3].

2. Let X be a semi-Markov process for which Y is a stationary ψ-mixing process. If Y is ψ-mixing, then $\{f(Y_n)\alpha_n\}$ is ψ-mixing with mixing coefficients doubled since

$$|P\{f(Y_0)\alpha_0 \in A,\ f(Y_n)\alpha_n \in B\} - P\{f(Y_0)\alpha_0 \in A\}\cdot P\{f(Y_n)\alpha_n \in B\}|$$

$$= |E\{P\{f(Y_0)\alpha_0 \in A \mid Y\}\cdot P\{f(Y_n)\alpha_n \in B \mid Y\}\} - P\{f(Y_0)\alpha_0 \in A\}\cdot P\{f(Y_n)\alpha_n \in B\}$$

$$= |Eg_1(Y_0,Y_1)\cdot g_2(Y_n,Y_{n+1}) - Eg_1(Y_0,Y_1)\cdot Eg_2(Y_n,Y_{n+1})|$$

$$(\text{where } g_1(x,y) = P\{f(Y_0)\alpha_0 \in A \mid Y_0 = x,\ Y_1 = y\}$$

$$g_2(x,y) = P\{f(Y_n)\alpha_n \in B \mid Y_n = x,\ Y_{n+1} = y\})$$

$$\le 2\psi_n Eg_1(Y_0,Y_1) = 2\psi_n\cdot P\{f(Y_0)\alpha_0 \in A\},$$

the inequality following from (20.28) of Billingsley [1]. Suppose that

$$\sum_{n=0}^{\infty} \psi_n^{1/2} < \infty$$

and $$E[(f(Y_0)+1)\alpha_0]^2 < \infty .$$

Then, (A) - (D) hold (see theorem 20.1 of Billingsley [1]).

3. Uniformization loses

For the special case of continuous-time Markov chains with conservative generator Q and jump rates having least upper bound $\lambda < \infty$, we can uniformize. This corresponds to the equal holding-time method of [6]. The method of section 2.1, without uniformization, corresponds to the constant holding-time method of [6]. With uniformization, the imbedded discrete-time Markov chain W has null jumps from a state to itself. Thus, to generate any fixed number of regeneration cycles with the equal holding-time method requires more jumps, hence more work. It also gives more variance, as shown in [6].

We now give a simpler proof. Delete all null jumps from W to get Y, giving $\sigma(Y) \subseteq \sigma(W)$ where $\sigma(H)$ is the σ-field generated by H. By proposition G.1 in [5] for example, we get higher variance by conditioning on the latter. Thus

$$\mathrm{Var}\left(E[D_k \mid Y]\right) \leq \mathrm{Var}\left(E[D_k \mid W]\right). \tag{25}$$

Nevertheless, what is the best way to uniformize? To get a legitimate representation, we must choose the (Poisson) clock intensity $\theta \geq \lambda$ in the uniformized process. Choosing $\theta = \lambda$ stochastically minimizes the number of jumps in W to simulate a fixed number of regeneration cycles, hence stochastically minimizes work. Hordijk, Iglehart, and Schassberger [6] show by explicit calculation that choosing $\theta = \lambda$ also minimizes σ^2 in (18). An outline of an easier proof follows. Subscript W and X to indicate clock intensity. Put X_λ and X_θ on the same probability space using three synchronized streams of common random numbers. Stream 1 generates the common non-null jump subsequences of W_λ and W_θ. Stream 2 generates the null jumps so that each null-jump sequence in W_λ is at most as long as the corresponding null-jump sequence in W_θ. Stream 3 generates clock chimes so that chime j of the θ-clock sounds before or with chime j of the λ-clock. Appendix B of [5], in a more general setting, spells out the details. Thus

$$\sigma\left(W_\lambda\right) \subseteq \sigma\left(W_\theta\right) \tag{26}$$

for all $\theta > \lambda$. Again using the fact that conditioning on less reduces variance,

$$\operatorname{Var}\left(E\left[D_k \mid \sigma\left(W_\lambda\right)\right]\right) < \operatorname{Var}\left(E\left[D_k \mid \sigma\left(W_\theta\right)\right]\right), \tag{27}$$

illustrating the power of the conditional Monte Carlo approach.

In the setting of section 2.2 specialized to continuous-time Markov chains, similar arguments give counterparts to (25) and (27). Since uniformization increases both variance and work, don't use it there either.

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