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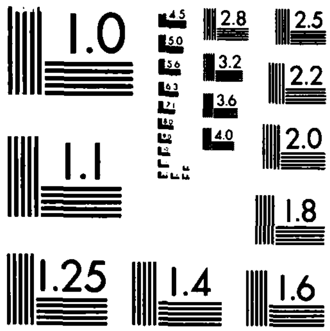

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Let $X = [X_{ij}: i, j=1, 2, \ldots]$ be an infinite dimensional random matrix, T_p be a $p \times p$ nonnegative definite random matrix independent of X, for $p=1,2,\ldots$ Suppose $\frac{1}{p} \operatorname{tr} T_p^k \to H_k$ a.s. as $p\to\infty$ for $k=1,2,\ldots$, and $\Sigma H_{2k}^{-1/2k} = \infty$. Then the spectral distribution of $A_p = X_p X_p' T_p$ where $X_p = [X_{ij}: i=1,\ldots,p;\ j=1,\ldots m]$ tends to a nonrandom limit distribution as $p\to\infty$ but $p/n \to y > 0$, under the mild conditions that X's are i.i.d. and $EX_{11}^2 < \infty$.

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LIMITING SPECTRAL DISTRIBUTION FOR A CLASS
OF RANDOM MATRICES

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LIMITING SPECTRAL DISTRIBUTION FOR A CLASS OF RANDOM MATRICES

Y. Q. Yin

ABSTRACT

Let $\mathcal{X} = \{X_{ij} : i, j = 1, 2, \ldots\}$ be an infinite dimensional random matrix, T_p be a $p \times p$ nonnegative definite random matrix independent of $\mathcal{X}$, for $p = 1, 2, \ldots$. Suppose $\frac{1}{p} \operatorname{tr} T_p^k \to H_k$ a.s. as $p \to \infty$ for $k = 1, 2, \ldots$, and $\sum H_{2k}^{-1/2k} = \infty$. Then the spectral distribution of

$$A_p = \frac{1}{n} X_p X_p^T T_p$$

where $X_p = [X_{ij} : i = 1, \ldots, p;\ j = 1, \ldots, n]$, tends to a nonrandom limit distribution as $p \to \infty$, $n \to \infty$ but $p/n \to y > 0$, under the mild conditions that X_{ij}'s are i.i.d. and $EX_{11}^2 < \infty$.

1. INTRODUCTION

The spectra for random matrices of the form $\frac{1}{n} X_p X_p' T_p$ are important in many fields. Many results are available for the special case where $T_p = I_p$ = identity, (e.g., see Grenander-Silverstein (1977), Wachter (1978), Jonsson (1983), and Yin and Krishnaiah (1983)).

In Yin and Krishnaiah (1984), the case when T_p is an arbitrary positive definite matrix was investigated for the first time. In that paper, it was assumed that the entries of $X_p = [x_{ij} : i = 1,\ldots,p; j = 1,\ldots,n]$ are i.i.d. and normal. A new combinatorial technique was developed in a previous ~~that~~ paper to prove the existence of a ~~the~~ limiting spectral distribution.

That ~~The above~~ work can be generalized in two directions. First, we can generalize to the case when X_p has isotropic columns. This work was done in Yin and Krishnaiah (1983) and Bai, Yin, and Krishnaiah (1984). In the second direction, we can prove the result by assuming that X_p has i.i.d. entries with minimum moment requirements. This paper ~~The present work~~ is devoted to this goal. In this paper, we have succeeded to prove the existence of limiting spectral distribution by assuming only that the second moment exists. The keys to reach this goal are (1) truncation technique and (2) sophisticated combinatorial techniques. The two-stage truncation method works in proving the main result. To prove the main result, we have to generalize the notion of Q-graph to a new kind of graphs —— M-graphs. Some properties of M-graphs are developed here.

In this paper, we have succeeded to prove the existence of the limiting spectral distribution in the sense of "a.s." convergence.

2. PRELIMINARIES

Let A be a $p \times p$ matrix with p real eigenvalues $\ell_1 \leq \ell_2 \leq \ldots \leq \ell_p$. We define a distribution function by

$$F^A(x) = \frac{1}{p} |\{ i : \ell_i \leq x\}|,$$

where $|\{\ldots\}|$ denotes the number of elements in the set $\{\ldots\}$. In the sequel, $F^A(x)$ will be referred to as the spectral distribution of the matrix A.

In this paper, we are interested in proving the convergence of the spectral distributions $\{F^{A_p}(\cdot)\}$ of a sequence $\{A_p\}$ of random matrices to a nonrandom distribution $F(\cdot)$. Here A_p is of the form

$$A_p = \frac{1}{n} X_p X_p' T_p$$

and is defined on a common probability space $(\Omega, \mathcal{F}, P)$. The definitions of X_p and T_p and basic hypothesis are given below.

(A) $\mathcal{X} = \{X_{ij} : i, j = 1, 2, \ldots\}$ is an infinite random matrix of i.i.d. entries. $EX_{11}^2 < \infty$. $X_p = [X_{ij} : 1 \leq i \leq p;\ 1 \leq j \leq n]$ is a submatrix of $\mathcal{X}$; here $n = n(p) \to \infty$ and $p/n \to y$ for some finite number $y > 0$.

(B) For each integer $p \geq 1$, $T_p = [t_{ij} : 1 \leq i, j \leq p]$ is a $p \times p$ nonnegative definite random matrix and T_p is independent of $\mathcal{X}$. Here $t_{ij} = t_{ij}(p)$ may depend on p.

(C) There exists a sequence $(H_1, H_2, \ldots)$ of positive numbers such that

$$\sum H_{2k}^{-1/2k} = +\infty, \text{ and} \tag{2.1}$$

(D_1) $\frac{1}{p} \operatorname{tr} T_p^k \to H_k$ as $p \to \infty$, in pr., for any $k \geq 1$, or (2.2)

(D_2) $\frac{1}{p} \operatorname{tr} T_p^k \to H_k$ as $p \to \infty$, a.s., for any $k \geq 1$. (2.3)

Theorem 2.1 Suppose the conditions (A), (B), (C), (D_1) (or (D_2)) are true. Then the spectral distributions of $A_p = \frac{1}{n} X_p X_p' T_p$ converge to a nonrandom distribution function in pr. (a.s.).

Remark According to the Strong Representation Theorem (Bai-Liang (1984)), if (D_1) holds, we can reconstruct a sequence of random matrices $\{\tilde{T}_p\}$ such that

(1) $\tilde{T}_p$ and T_p have a common distribution, for each p.

(2) $\mathrm{tr}\tilde{T}_p^k \to H_k$, a.s. for each k, as $p \to \infty$.

(3) $\{X_{ij}, i, j = 1, 2, \ldots\}$ is independent of $\tilde{T}_p$, for each p.

Thus, to prove Theorem 2.1, we need only to prove the a.s. part under the conditions (A), (B), (C), and (D_2).

3. SOME RESULTS IN GRAPH THEORY

At first, we generalize the notion of Q-graph introduced in Yin and Krishnaiah (1984).

Let V and E be two finite sets, called the vertex set and edge set, respectively. The numbers of elements of V and E are denoted by v and k, respectively. A multigraph is a single-valued mapping $\Gamma : E \to V \times V$. The multigraph will be denoted by (V,E,Γ) or simply by Γ.

If $e \in E$, $\Gamma(e) = (B_1, B_2)$, $B_1 \in V$, $B_2 \in V$, we say that B_1 and B_2 are the ends of e. We do not distinguish (B_1, B_2) and (B_2, B_1). Note that for two vertices B_1, B_2 in V, there may be several edges e in E such that $\Gamma(e) = (B_1, B_2)$. Given B V, let n_1 be the number of edges with different ends and one of the ends is B and let n_2 be the number of edges whose ends both are B. Then the number $n_1 + 2n_2$ is called the degree of B and it is denoted by deg(B).

<u>Definition</u> A multigraph (V,E,Γ) is called an M-graph, if

(1) for each $B \in V$, deg (B) ≥ 2,

(2) there is a partition $W = \{C_1,\ldots,C_w\}$ of the vertex set V, the elements of W are called classes, such that

(3) for each class, the sum of degrees of vertices in it is even, and

(4) Γ is W-connected.

In condition (4), "W-connectedness" is defined as follows. For each pair of classes C_a and C_b, there are classes $C_a = C_{a_0}, C_{a_1}, C_{a_2},\ldots,C_{a_d} = C_b$ such that C_{a_i} and $C_{a_{i+1}}$ $(i = 0, 1,\ldots,d - 1)$ are directly connected, i.e. there is an edge $e \in E$ such that one end of e is in C_{a_i} and the other in $C_{a_{i+1}}$.

An M-graph is denoted by (V,E,Γ,W), or simply by (Γ,W), or more simply by Γ if there is no confusion. The number of classes in W will be denoted by w.

In an M-graph (V,E,Γ,W), we denote by v_i the number of vertices with degree i. Evidently, $v_1 = 0$ and

$$v_1 + v_2 + \cdots + v_{2k} = v, \tag{3.1}$$

$$v_1 + 2v_2 + \cdots + 2kv_{2k} = 2k. \tag{3.2}$$

Recall that v and k are the cardinal numbers of V and E, respectively.

A sequence of vertices $\{B_{b_1}, B_{b_2}, \ldots, B_{b_c}\}$ is called a chain if any two neighboring vertices B_{b_i}, $B_{b_{i+1}}$ $(i = 1, 2, \ldots, c-1)$ are two ends of an edge and $B_{b_2}, \ldots, B_{b_{c-1}}$ are of degree 2.

A chain $\{B_{b_1}, \ldots, B_{b_c}\}$ is called singular, if $\deg B_{b_1} > 2$ and $\deg B_{b_c} > 2$.

A chain $B_{b_1}, B_{b_2}, \ldots, B_{b_c}$ is called a free cycle if $B_{b_1} = B_{b_c}$ and $\deg B_{b_1} = 2$, too. A one-vertex free cycle is called a loop. If the ends of an edge belong to a chain, we say that this edge is an edge of the chain.

<u>Lemma 3.1</u> In an M-graph (Γ,W), the number of singular chains equals $(k - v_2)$.

<u>Proof</u> Obvious.

<u>Lemma 3.2</u> In an M-graph (Γ,W), if each vertex has degree 2, then Γ is a collection of free cycles, and the number of free cycles $f \le k - w + 1$.

<u>Proof</u> For a proof, see Yin and Krishnaiah (1984).

<u>Lemma 3.3</u> In an M-graph (Γ,W), if $v_2 < k$, then

$$f \le \frac{1}{2}(k + v_2) - w.$$

Recall that f, w and v_2 are the numbers of free cycles, classes and vertices of degree 2, respectively.

<u>Proof</u> We apply induction on f.

Suppose $f = 0$. Let m be the number of classes which contain only one vertex with degree 2 and no other vertex. Since Γ is an M-graph, for each $B \in V$, $\deg B \geq 2$, by definition condition (3) and (3.2), we have

$$2m + 4(w - m) \leq 2k.$$

Note $m \leq v_2$, we get

$$w \leq \frac{1}{2}(k + v_2)$$

i.e.

$$0 = f \leq \frac{1}{2}(k + v_2) - w.$$

Suppose there are $f + 1$ free cycles in our graph. Let Z be a free cycle with its vertex set V_Z and edge set E_Z, and let W_Z be the set of all those classes in W which contain vertices of Z only.

Now delete Z from our M-graph. Consider the residue graph with its vertex set $V' = V \setminus V_Z$, edge set $E' = E \setminus E_Z$, $\Gamma' = \Gamma|_{E'}$ = the restriction of Γ on E', $\Gamma' : E' \to V' \times V'$. Let $W' = \{C \cap V' : C \in W \setminus W_Z\}$. Thus, W' is a partition of V' into disjoint classes. Since $v_2 < k$, i.e. there is at least one vertex in V whose degree is greater than two, so when we delete Z, this vertex is not deleted, hence V', E' and W' are not empty. The new graph (V', E', Γ') is not necessarily W'-connected. But W' can be split into disjoint subsets : $W'_1, \ldots, W'_d$, say, such that if V'_i is the set of vertices of V' which belong to some class of W'_i, $E'_i = \{e \in E' : \Gamma(e) \subset V'_i\}$, $\Gamma'_i = \Gamma'|E'_i$, then $(V'_i, E'_i, \Gamma'_i, W'_i)$ are M-graphs, and classes from different W'_i are not directly connected through edges in E'. Since $v_2 < k$, there is at least one graph $(V'_i, E'_i, \Gamma'_i, W'_i)$ with $v'_{2i} < |E'_i|$, where v'_{2i} is the number of vertices with degree 2 of the graph Γ'_i. Without loss of generality, we assume that

$$|E'_i| > v'_{2i}, \text{ for } i = 1, 2, \ldots, c,\ c \geq 1,$$

and

$$|E_i'| = v_{2i}', \text{ for } i = c+1,\dots,d.$$

By induction hypothesis, if f_i is the number of free cycles of Γ_i',

$$f_i \leq \frac{1}{2}(|E_i'| + v_{2i}') - |W_i'|, \; i = 1, 2, \dots, c,$$

and by Lemma 3.2,

$$f_i \leq |E_i'| - |W_i'| + 1 = \frac{1}{2}(|E_i'| + v_{2i}') - |W_i'| + 1, \; i = c+1,\dots,d.$$

Summing up with respect to i, we get

$$|W'| \leq \frac{1}{2}(|E'| + v_2') - f + (d - c)$$

here v_2' is the number of vertices with degree 2 in V'. We show that $|W_Z| \leq |E_Z| - d$. In fact,

$$W_Z = \{C \in W : C \setminus V_Z = \phi\} = W^* \setminus W^{**}$$

where

$$W^* = \{C \in W : C \cap V_Z \neq \phi\},$$

$$W^{**} = \{C \in W : C \cap V_Z \neq \phi, \; C \cap V' \neq \phi\}.$$

But, classes are disjoint and nonempty, so

$$|W^*| \leq |V_Z| \leq |E_Z|.$$

On the other hand, for each $i = 1,\dots,d$, there is a class $C \in W$ such that C contains a vertex of V_i' and a vertex in V_Z. In fact, given any class C' of W with $C' \cap V_i' \neq \phi$, let C_0 be an arbitrary class of W_Z. Since the whole graph is class-connected, there are classes $C_0, C_1, \dots, C_g = C'$ of W such that $C_{j-1} \neq C_j$ are directly connected for $j = 1,\dots,g$. Let C_j be the first class in the above sequence which contains a vertex of V_i'.

Suppose $e \in E$ connects C_{j-1} and C_j, i.e. $\Gamma(e) \cap C_{j-1} \neq \Phi$ and $\Gamma(e) \cap C_j \neq \Phi$. If $\Gamma(e) \cap C_j$ is a vertex of V_i', $\Gamma(e) \cap C_{j-1}$ must also be a vertex of V_i', contradicting the minimality of j. But C_j cannot contain any vertex of V_ℓ', for $\ell \neq i$, so $\Gamma(e) \cap C_j$ is a vertex of V_Z. Thus, C_j is a class with the required property : it contains a vertex of V_Z and a vertex of V_i'. But W_i', $i = 1,\ldots,d$, are disjoint, so

$$|W^{**}| \geq d.$$

Therefore

$$|W_Z| \leq |E_Z| - d.$$

Also, it is evident that

$$|E| = |E'| + |E_Z|, \quad |W| = |W'| + |W_Z|,$$

and

$$v_2 = v_2' + |E_Z|.$$

Therefore,

$$w \leq |W'| + |E_Z| - d \leq \frac{1}{2}(|E'| + v_2') - f + (d - c) + |E_Z| - d$$

$$= \frac{1}{2}(|E| + v_2) - f - c$$

$$\leq \frac{1}{2}(k + v_2) - f - 1,$$

which completes the proof of Lemma 3.3 by induction.

The following lemmas are useful in the proof of Theorem 2.1. Some of them are well-known and are quoted below without proof except Lemma 3.4.

<u>Lemma 3.4</u> If in the sum

$$S = \sum f_1(a_1)\ldots f_c(a_c)g_1(b_1, b_2)\ldots g_d(b_{2d-1}, b_{2d})$$

each index occurs at least two times, $b_{2i-1} \ddagger b_{2i}$, $i = 1, 2,\ldots,d$ and the indices run over $\{1, 2,\ldots,p\}$, then

$$S^2 \le \sum_{i=1}^{p} f_1^2(i) \cdots \sum_{i=1}^{p} f_c^2(i) \sum_{i,j=1}^{p} g_1^2(i,j) \cdots \sum_{i,j=1}^{p} g_d^2(i,j).$$

Here indices $a \equiv b$ means that a and b always take the same value.

Proof We will prove this lemma by induction on $c + d$ and by using Schwartz's Inequality.

Now, let $c + d > 2$, we shall discuss the following two cases.

Case 1 $c = 0$. We have

$$S^2 \le \sum_{b_1 b_2} g_1^2(b_1, b_2) \sum_{b_1 b_2} \left(\sum g_2(b_3, b_4) \cdots g_d(b_{2d-1}, b_{2d}) \right)^2.$$

If for some $i > 1$, $\{b_{2i-1}, b_{2i}\} \equiv \{b_1, b_2\}$ then g_i can be taken out of the bracket. If for some $i > 1$, $\{b_{2i-1}, b_{2i}\} \cap \{b_1, b_2\}$ has only one element, for fixed b_1 and b_2, g_i can be regarded as a new f function. In any case, the product under the inner summation has less factors than that in S and has the required form. By induction we can get the conclusion of the lemma.

Case 2 $c \ge 1$. We have

$$S^2 \le \sum_{a_1=1}^{p} f_1^2(a_1) \sum_{a_1=1}^{p} \left(\sum f_2(a_2) \cdots f_c(a_c) g_1(b_1, b_2) \cdots g_d(b_{2d-1}, b_{2d}) \right)^2.$$

If for some $i > 1$, $a_i \equiv a_1$, then $f_i(a_i)$ can be taken out of the bracket. If for some $j \ge 1$, $b_j \equiv a_1$, for fixed a_1, the $\left[\frac{j+1}{2}\right]$-th g function can be regarded as a new f function. In any case, the product under the inner summation has less factors than that in S and has the required form. By induction, we have proved the lemma.

Lemma 3.5 (Ky Fan) Let A and B be two $p \times n$ matrices and F, G be the spectral distributions of AA' and BB', respectively. Then

$$\|F - G\| = \sup_x |F(x) - G(x)| \le \frac{1}{p}\,\mathrm{rank}(A - B).$$

Lemma 3.6 (J. von Neumann(1937)) Let A and B be two $p \times n$ matrices and $\{\lambda_i\}\{k_i\}$ be the eigenvalues of AA' and BB' respectively. Then

$$|\mathrm{tr}AB'| \le \sum_{i=1}^{p} \sqrt{\lambda_i k_i}$$

Here $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_p \ge 0$ and $k_1 \ge k_2 \ge \dots \ge k_p$.

From Lemma 3.6 it follows immediately that

$$\sum_{i=1}^{p} (\sqrt{\lambda_i} - \sqrt{k_i})^2 = \sum_{i=1}^{p} (\lambda_i + k_i) - 2\sum_{i=1}^{p} \sqrt{\lambda_i k_i}$$

$$\le \mathrm{tr}AA' + \mathrm{tr}BB' - 2\mathrm{tr}AB' = \mathrm{tr}(A - B)(A - B)'. \qquad (3.3)$$

Definition (Dudley). Let G and F be two probability measures. Define D-metric of G and F to be

$$D(F.G) = \sum_{i=1}^{\infty} \left| \int f_i dF - \int f_i dG \right| 2^{-i}.$$

where $\{f_i\}$ is a sequence of functions which is uniformly dense in the set of functions from R^1 to $[0,1]$ satisfying $|f(x) - f(y)| \le |x - y|$ for any x and y. It is well known that the topology deduced by D-metric in the space of all one-dimensional distributions is the same topology of weak convergence, (see Dudley (1966)).

If F and G are spectral distributions of AA' and BB' respectively, then from (3.3) we get

$$D^2(F.G) \le \left(\frac{1}{p} \sum_{i=1}^{p} |\lambda_i - k_i|\right)^2 \le \frac{1}{p}\sum_{i=1}^{p}(\sqrt{\lambda_i} + \sqrt{k_i})^2 \, \frac{1}{p}\sum_{i=1}^{p}(\sqrt{\lambda_i} - \sqrt{k_i})^2$$

$$\le \frac{2}{p^2}\,\mathrm{tr}\,(AA' + BB')\,\mathrm{tr}(A - B)(A - B)'. \qquad (3.4)$$

Lemma 3.7 (Hoeffding (1963)). Let ξ_n be a binomial random variable with parameters n and η. Then for any $\varepsilon > 0$,

$$P(|\frac{1}{n}\xi_n - \eta| \geq \varepsilon) \leq 2\exp\{-n\ \ \varepsilon^2/(2\eta + \varepsilon)\}.$$

Lemma 3.8 Let $B = \{B_1,\ldots,B_v\}$ be a partition of the set $\{1, 2,\ldots,2\ell\}$. Then consider the sum

$$S = \sum t_{i_2 i_3} t_{i_4 i_5} \cdots t_{i_{2\ell} i_1}$$

where $i_1, i_2\ldots,i_{2\ell}$ run over the set $\{1, 2,\ldots,p\}$ but subject to the condition that if a, a' belong to the same set B_b for some b, then $i_a \equiv i_{a'}$. Consider a multigraph (V, E, Γ), where $V = B$, $E = \{e_1,\ldots,e_\ell\}$, and $\Gamma(e_i) = \{B(2i), B(2i + 1)\}$, $(B(a) \stackrel{d}{=} B_b$ iff $a \in B_b)$. If (V, E, Γ) is an M-graph with some partition $W = \{C_1,\ldots,C_w\}$ of vertices, such that $\deg(B_i) > 2$ for at least one i, then

$$S = O(p^{\ell-w}), \qquad \text{a.s.}.$$

Proof Summing up with respect to all free indices (i.e. indices which occur just twice in $t_{i_2 i_3} t_{i_4 i_5} \cdots t_{i_{2\ell} i_1}$, or equally, those indices i_a, for which $a \in B_b$ for some b with $|B_b| = 2$). Then we have

$$S = \operatorname{tr} T_p^{n_1} \cdots \operatorname{tr} T_p^{n_f} \sum t^{(a_1)}_{m_1 m_1} \cdots t^{(a_r)}_{m_r m_r} t^{(b_1)}_{g_1 h_1} \cdots t^{(b_s)}_{g_s h_s}.$$

Here $n_1,\ldots,n_f$ are the lengths of all the f free cycles, $a_1,\ldots,a_r$ are the lengths of singular chains the two ends of which are identical, and $b_1,\ldots b_s$ are the lengths of the singular chains their ends are not equal, $g_1 \not\equiv h_1,\ldots,g_s \not\equiv h_s$, and $T_p^a = (t_{ij}^{(a)})$. By definition of a singular chain, each of the indices $m_1,\ldots,m_r$, $g_1,\ldots,g_s$, $h_1,\ldots,h_s$ occurs at least three times. By Lemma 3.4,

$$S^2 \leq (\operatorname{tr} T_p^{n_1})^2 \cdots (\operatorname{tr} T_p^{n_f})^2 \operatorname{tr} T_p^{2a_1} \cdots \operatorname{tr} T_p^{2a_r} \operatorname{tr} T_p^{2b_1} \cdots \operatorname{tr} T_p^{2b_s}.$$

Since $\operatorname{tr} T_p^c = O(p)$ for any $c \geq 1$,

$$S = O\left(p^{f+(1/2)(r+s)}\right), \qquad \text{a.s.}.$$

By Lemma 3.3, and Lemma 3.1, if v_2 is the number of B_b with $|B_b| = 2$,

$$f \leq \frac{1}{2}(\ell + v_2) - w,$$

$$r + s = (\ell - v_2)$$

Therefore,

$$S = O\left(p^{(1/2)(\ell+v_2)+(1/2)(\ell-v_2)-w}\right)$$
$$= O(p^{\ell-w}).$$

4. PROOF OF THEOREM 2.1

Let $V_p = T_p^{1/2}$, $F_p = F^{A_p}$, $A_p = \frac{1}{n} X_p X_p' T_p$. Define $\hat{X}_{ij} = X_{ij} I_{[|X_{ij}|<\frac{1}{2}n]}$ and $\hat{X}_p = [\hat{X}_{ij};\ i = 1,\dots,p;\ j = 1,\dots,n]$. Let $\hat{F}_p(x)$ be the spectral distribution of $\frac{1}{n} \hat{X}_p \hat{X}_p' T_p$. According to Ky Fan Inequality, we have

$$\|F_p - \hat{F}_p\| \le \frac{1}{p} \operatorname{rank}[V_p(X_p - \hat{X}_p)] \le \frac{1}{p} \operatorname{rank}(X_p - \hat{X}_p)$$
$$\le \frac{1}{p} \left|\{(i,j) : |X_{ij}| \ge \tfrac{1}{2}n;\ i \le p,\ j \le n\}\right| \triangleq \frac{1}{p} \xi_n,$$

where $\triangleq$ means "denoted by." Write

$$\eta = P(|X_{11}| \ge \tfrac{1}{2}\sqrt{n}) = o\left(\frac{1}{n}\right).$$

From Hoeffding's Inequality, we get for p large enough

$$P(\| F_p - \hat{F}_p\| \ge \varepsilon) \le P\left(\frac{1}{np} \xi_n \ge \frac{\varepsilon}{n}\right)$$
$$\leqq 2\exp\left\{-np\left(\frac{\varepsilon}{n}\right)^2 \Big/ \left(2\eta + \frac{\varepsilon}{n}\right)\right\}$$
$$\leqq 2\exp\{-p\ \varepsilon/2\}, \text{ for } \forall\ \varepsilon > 0.$$

Thus,

$$\|F_p - \hat{F}_p\| \to 0 \qquad \text{a.s.}. \tag{4.2}$$

Let

$$\tilde{X}_{ij} = \hat{X}_{ij} - E\hat{X}_{ij}$$

and $\tilde{X}_p = (\tilde{X}_{ij}) = \hat{X}_p - E\hat{X}_p$. Denote by $\tilde{F}_p$ the spectral distribution of $\frac{1}{n} \tilde{X}_p \tilde{X}_p' T_p$. Again using Ky Fan's Inequality, we get

$$\| \hat{F}_p - \tilde{F}_p\| \le \frac{1}{p} \to 0. \tag{4.3}$$

Therefore, to prove $\{F_p\}$ has a limiting spectral distribution, we need only to prove that $\{\tilde{F}_p\}$ has a limiting spectral distribution.

Now define

$$\bar{X}_{ij} = \tilde{X}_{ij} I_{[|X_{ij}|<\frac{1}{2}\log n]} - E\tilde{X}_{ij} I_{[|X_{ij}|<\frac{1}{2}\log n]}$$

and let

$$\bar{X}_p = (\bar{X}_{ij}),\ \bar{A}_p = \frac{1}{n} \bar{X}_p \bar{X}_p' T_p$$

and $\bar{F}_p$ be the spectral distribution of $\bar{A}_p$. From (3.4) we get

$$D^2(\tilde{F}_p, \bar{F}_p) \leq 2\, \Delta_1 \Delta_2 \tag{4.4}$$

where

$$\Delta_1 = \frac{1}{np} \sum_{i=1}^{p} \sum_{j=1}^{n} \left[\left(\sum_{k=1}^{p} v_{ik} \tilde{X}_{kj}\right)^2 + \left(\sum_{k=1}^{p} v_{ik} \bar{X}_{kj}\right)^2\right]$$

$$\Delta_2 = \frac{1}{np} \sum_{i=1}^{p} \sum_{j=1}^{n} \left(\sum_{k=1}^{p} v_{ik} (\tilde{X}_{kj} - \bar{X}_{kj})\right)^2.$$

We shall prove

$$\Delta_1 \to 2H_1, \text{ a.s.} \tag{4.5}$$

$$\Delta_2 \to 0, \quad \text{a.s..} \tag{4.6}$$

If $E_T(\cdots) = E(\cdots|T)^*$ stands for the conditional expectation given $T = \{T_1, T_2, \ldots\}$, by independence and $E(\tilde{X}_{kj} - \bar{X}_{kj}) = 0$, we have

$$E_T\Delta_2 = \frac{1}{p} \sum_{i=1}^{p} \sum_{k=1}^{p} v_{ik}^2 E(\tilde{X}_{11} - \bar{X}_{11})^2$$

$$= \frac{1}{p} \operatorname{tr} T_p E(\tilde{X}_{11} - \bar{X}_{11})^2. \tag{4.7}$$

Recalling the definitions of $\tilde{X}_{11}$ and $\tilde{X}_{11}$, we have

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$^*E_T$ can be defined precisely as follows. Given Borel function $f(T,X)$, if μ is the probability distribution of X on the value space $\mathbb{R}$ of X, then

$$E_T f = \int_{\mathbb{R}} f(T,X)\mu(dX).$$

$$E(\tilde{X}_{11} - \bar{X}_{11})^2 = \mathrm{var}(\tilde{X}_{11} I_{[|X_{11}|\geq\frac{1}{2}\log n]})$$

$$\leq E\tilde{X}_{11}^2 I_{[|X_{11}|\geq\frac{1}{2}\log n]}$$

$$\leq 2E\hat{X}_{11}^2 I_{[|X_{11}|\geq\frac{1}{2}\log n]} + 2(E\hat{X}_{11})^2 P(|X_{11}| \geq \frac{1}{2}\log n)$$

$$\leq 2EX_{11}^2 I_{[|X_{11}|\geq\frac{1}{2}\log n]} + 2EX_{11}^2 P(|X_{11}| \geq \frac{1}{2}\log n)$$

$$\to 0. \qquad (p \to \infty). \tag{4.8}$$

From (4.7) and (4.8), we have

$$E_T\Delta_2 \to 0 \qquad (p \to \infty), \text{ for almost all } T. \tag{4.9}$$

Also, we have

$$E_T(\Delta_2 - E_T\Delta_2)^2 \leq \frac{1}{np^2}\Bigg[\sum_{k=1}^{p}\left(\sum_{i=1}^{p} v_{ik}^2\right)^2 E(\tilde{X}_{11} - \bar{X}_{11})^4$$

$$+ 4\sum_{\substack{1\leq k_1,k_2\leq p\\ k_1\neq k_2}}\left(\sum_{i=1}^{p} v_{ik_1}v_{ik_2}\right)^2 \left(E(\tilde{X}_{11} - \bar{X}_{11})^2\right)^2\Bigg]$$

$$\leq \frac{1}{np}\left\{\frac{1}{p}\operatorname{tr} T_p^2 E(\tilde{X}_{11} - \bar{X}_{11})^4 + \frac{4}{p}\operatorname{tr} T_p^2\left(E(\tilde{X}_{11} - \bar{X}_{11})^2\right)^2\right\}$$

$$= \frac{1}{np} E(\tilde{X}_{11} - \bar{X}_{11})^4 (H_2 + 0(1)) + 0(p^{-2}), \text{ for almost all } T.$$

Recalling the definitions of $\tilde{X}_{11}$ and $\bar{X}_{11}$ and $\frac{p}{n} \to y > 0$, we can prove that

$$E(\tilde{X}_{11} - \bar{X}_{11})^4 \leq 16E\tilde{X}_{11}^4 = (16)^2 EX_{11}^4 I_{[|X_{11}|<\frac{1}{2}\sqrt{n}]}.$$

Hence

$$\sum_{p=1}^{\infty} E_T(\Delta_2 - E_T\,\Delta_2)^2 < \infty, \text{ for almost all } T. \tag{4.10}$$

Here we have used the fact that $\sum_{p=1}^{\infty} \frac{1}{p^2} EX_{11}^4 I_{[|X_{11}| \leq K\sqrt{p}]} < \infty$, for any $K > 0$ fixed.

(4.9) and (4.10) ensure that

$$\Delta_2 \to 0 \quad \text{for almost all } X, \text{ for almost all } T. \tag{4.11}$$

By Fubini's Theorem, we have proved (4.6). By the same approach, we can prove (4.5). Hence, from (4.4) it follows that

$$D_2(\tilde{F}_p, \overline{F}_p) \to 0 \quad \text{a.s.}. \tag{4.12}$$

From (4.2), (4.3), (4.12), to prove $\{F_p\}$ has a limit spectral distribution, we need only to prove $\{\overline{F}_p\}$ has a limit spectral distribution. For simplicity, we drop the symbols imposed on X_{ij} and F_p, and we assume that

$$\left.\begin{array}{ll} (1) & |X_{ij}| < \frac{1}{2}\log n \\ (2) & EX_{ij} = 0 \\ (3) & EX_{ij}^2 \leq 1 \text{ and } EX_{ij}^2 \to 1 \quad (p \to \infty). \end{array}\right\} \tag{4.13}$$

Note that X_{ij} depends on p.

Now, we shall first prove that

$$\begin{aligned} E_T \int x^k dF_p(x) &= \frac{1}{pn^k} E_T \operatorname{tr}(X_p X_p' T_p)^k \to \\ &\to E_k = \sum_{w=1}^{k} y^{k-w} \sum \frac{k!}{n_1!\ldots n_w! w!} H_1^{n_1} H_2^{n_2} \ldots H_w^{n_w}, \\ &p \to \infty \quad \text{as } p \to \infty, \text{ for almost all } T, \end{aligned} \tag{4.14}$$

where the inner summation is carried out over all nonnegative integer solutions to the equations

$$\left.\begin{aligned} n_1 + n_2 + \cdots + n_w &= k - w + 1, \\ n_1 + 2n_2 + \cdots + wn_w &= k. \end{aligned}\right\} \tag{4.15}$$

For a given integer $k \geq 1$, let

$$R_p = \int x^k dF^{A_p}(x) = \frac{1}{p} \operatorname{tr}(A_p^k) = \frac{1}{pn^k} \operatorname{tr}(X_p X_p' T_p)^k$$

$$= \frac{1}{pn^k} \sum X_{i_1 j_1} X_{i_2 j_1} X_{i_3 j_2} X_{i_4 j_2} \cdots X_{i_{2k-1} j_k} X_{i_{2k} j_k} t_{i_2 i_3} t_{i_4 i_5} \cdots t_{i_{2k} i_1}.$$

Here the summation is taken in such a way that $i_1, i_2, \ldots, i_{2k}$ run over the set $\{1, 2, \ldots, p\}$ and $j_1, \ldots, j_k$ run over the set $\{1, 2, \ldots, n\}$.

We have

$$E_T R_p = \frac{1}{pn^k} \sum t_{i_2 i_3} t_{i_4 i_5} \cdots t_{i_{2k} i_1} E \prod_{q=1}^{k} (X_{i_{2q-1} j_q} X_{i_{2q} j_q})$$

$$= \frac{1}{pn^k} \sum_A \sum_{r_1, \ldots, r_w}' \sum_{(i)} t_{i_2 i_3} t_{i_4 i_5} \cdots t_{i_{2k} i_1} \prod_{\nu=1}^{w} E \prod_{q \varepsilon A} (X_{i_{2q-1} r_\nu} X_{i_{2q} r_\nu}).$$

Here

$\sum_A$ means the summation over all partitions $A = \{A_1, \ldots, A_w\}$ of the set $\{1, \ldots, k\}$,

$\sum'_{r_1, \ldots, r_w}$ means the summation for the indices $r_1, \ldots, r_w$ running over $\{1, 2, \ldots, n\}$ but being kept distinct from each other;

$\sum_{(i)}$ means the summation for $i_1, i_2, \ldots, i_{2k}$ running over the set $\{1, 2, \ldots, p\}$.

But, by i.i.d., we have

$$E_T R_p = \frac{1}{pn^k} \sum_A n(n-1)\ldots(n-w+1) \sum_{(i)} t_{i_2 i_3} t_{i_4 i_5} \cdots t_{i_{2k} i_1}$$

$$\prod_{\nu=1}^{w} E \prod_{q \varepsilon A_\nu} (X_{i_{2q-1},1} X_{i_{2q},1}).$$

Now let W be the partition of $\{1, 2, \ldots, 2k\}$ induced by A, i.e. $W = \{A_1^*, A_2^*, \ldots, A_w^*\}$, where $A_i^* = \bigcup_{s \varepsilon A_i} \{2s-1, 2s\}$.

Let $V = \{B_1,\ldots,B_v\}$ be any partition of $\{1, 2,\ldots,2k\}$. We say that V is a refinement of W, if each B_b is a subset of some A_a^*. We have

$$E_T R_p = \frac{1}{pn^k} \sum_A n(n-1)\ldots(n-w+1) \sum_{V\leq W} \sum_{(i)|V,W} t_{i_2 i_3} t_{i_4 i_5} \cdots t_{i_{2k} i_1}$$

$$\prod_{\nu=1}^{w} \prod_{B_b\subseteq A_\nu^*} E \prod_{s\in B_b} x_{i_s 1}.$$

Here

$\sum_{V\leq W}$ is the summation for all partitions V of $\{1, 2,\ldots,2k\}$ which are refinements of W;

$\sum_{(i)|V,W}$ means the summation for $i_1, i_2,\ldots,i_{2k}$ running over the set $\{1, 2,\ldots,p\}$ but subject to the condition that if b, b' are in the same A_a^* then $i_b = i_{b'}$ $\Leftrightarrow$ b, b' belong to the same V-class.

Thus, $E \prod_{s\in B_b} x_{i_s 1} = Ex_{11}^{|B_b|}$, and

$$E_T R_p = \sum_A \sum_{V\leq W} \frac{(n)_w}{pn^k} R(W,V)K(V),$$

where

$$(n)_w = n(n-1)\ldots(n-w+1),$$

$$R(W,V) = \sum_{(i)|V,W} t_{i_2 i_3} t_{i_4 i_5} \cdots t_{i_{2k} i_1}$$

and

$$K(V) = \prod_{b=1}^{v} Ex_{11}^{|B_b|}. \quad \text{(Note that } K(V) = 0 \text{ if } |B_b| = 1 \text{ for some } b.)$$

Let $A = \{A_1,\ldots,A_w\}$ be a partition of $\{1, 2,\ldots,k\}$, $W = \{A_1^*,\ldots,A_w^*\}$ where $A_a^* = \bigcup_{c\in A_a} \{2c-1, 2c\}$, $a = 1,\ldots,w$. Let $V = \{B_1,\ldots,B_v\}$ be any partition of $\{1, 2,\ldots,2k\}$ such that $V \leq W$ and $|B_b| \geq 2$ for all b. We define a graph (E,V,Γ,W) as follows. V is the vertex set, i.e. there are

v vertices $B_1,\ldots,B_v$. The edge set $E = \{e_1,\ldots,e_k\}$ contains k edges. The function $\Gamma : E \to V \times V$ is defined by $\Gamma(e_c) = \{B(2c), B(2c+1\}$, where $B(a) = B_b$ iff $a \in B_b$. But $2k + 1$ is regarded as 1.

It is easy to verify that $\Gamma(V,W)$ is an M-graph, if two vertices are defined to belong to the same class iff they are subsets of the same set A_a^*.

By Lemma 3.8, it is easy to see that if $|B_b| > 2$ for some b, then $R(W,V) = O(p^{k-w})$, so by the obvious inequality: $K(V) = O(\log^{2k} p)$,

$$\frac{(n)_w}{pn^k} R(W,V)K(V) = O(p^{-(k+1-w)} p^{k-w} (\log p)^{2k} = o(1), \text{ a.s. .}$$

So, we consider only those V for which each $|B_b| = 2$, and the number of cycles of $\Gamma(V,W)$ is just $k - w + 1$. Thus

$$E_T R_p = \sum_{A, V \leq W} \sum{}'' \frac{(n)_w}{pn^k} R(W,V)K(V) + o(1), \text{ a.s.}$$

where $\sum''$ means the summation over those $V = \{B_b : b = 1,\ldots,v\}$, for which $\nu_2 = k$ and $f = k - w + 1$.

By the same argument as in Yin and Krishnaiah (1984)

$$E_T R_p \to E_k = \sum_{w=1}^{k} y^{k-w} \sum_{\substack{n_1+\ldots+n_w = k-w+1 \\ n_1+2n_2+\ldots+wn_w = k}} \frac{k!}{n_1!\ldots n_w! w!} H_1^{n_1} H_2^{n_2} \ldots H_w^{n_w} \text{ for almost all T.}$$

And it is easy to verify that $\sum E_{2k}^{-1/2k} = +\infty$.

We will now prove that if $R_p = \int x^k dF_p(x)$, then

$$\sum_{p=1}^{\infty} E_T(R_p - E_T R_p)^2 < \infty \text{ for almost all T.}$$

We have

$$\mathrm{Var}_T(R_p) = E_T R_p^2 - E_T^2 R_p = \frac{1}{p^2 n^{2k}} \sum t_{i_2 i_3} t_{i_4 i_5} \ldots t_{i_{2k} i_1} t_{i_{2k+2} i_{2k+3}} \ldots t_{i_{4k} i_{2k+1}}$$
$$\left(E \prod_{q=1}^{2k} (X_{i_{2q-1} j_q} X_{i_{2q} j_q}) - E \prod_{q=1}^{k} (X_{i_{2q-1} j_q} X_{i_{2q} j_q}) E \prod_{q=k+1}^{2k} (X_{i_{2q-1} j_q} X_{i_{2q} j_q}) \right).$$

Here, $i_1,\ldots,i_{4k}$ run over the set $\{1, 2,\ldots,p\}$ and $j_1,\ldots,j_{2k}$ run over the set $\{1, 2, \ldots, n\}$.

Let $S_1 = \{1, 2, \ldots, k\}$, $S_2 = \{k+1, \ldots, 2k\}$. If D is any set of numbers, D* will denote the set $\bigcup_{x \in D} \{2x-1, 2x\}$.

We ~~have~~

$$\mathrm{Var}_T R_p = \frac{1}{p^2 n^{2k}} \sum_A \sum_{V \leq A^*} \sum'_{r_1,\ldots,r_w} \sum_{(i)|A^*,V} t_{i_2 i_3} t_{i_4 i_5} \cdots t_{i_{2k} i_1} t_{i_{2k+2} i_{2k+3}} \cdots t_{i_{4k} i_{2k+1}} \cdot$$

$$\cdot \left(\prod_{a=1}^{w} E \prod_{q \in A_a^*} X_{i_q r_a} - \prod_{a=1}^{w} \left[E \prod_{q \in A_a^* \cap S_1^*} x_{i_q r_a} \, E \prod_{q \in A_a^* \cap S_2^*} x_{i_q r_a} \right] \right)$$

$$= \frac{1}{p^2 n^{2k}} \sum_A \sum_{V \leq A^*} \sum'_{r_1,\ldots,r_w} \sum_{(i)|A^*,V} t_{i_2 i_3} \cdots t_{i_{2k} i_1} t_{i_{2k+2} i_{2k+3}} \cdots t_{i_{4k} i_{2k+1}}$$

$$\left(\prod_{b=1}^{v} E x_{11}^{|B_b|} - \prod_{b=1}^{v} \left[E x_{11}^{|B_b \cap S_1^*|} \, E x_{11}^{|B_b \cap S_2^*|} \right] \right)$$

Here

$\sum_A$ means summation over all possible partitions $A = \{A_1, \ldots, A_w\}$ of $\{1, 2, \ldots, 2k\}$,

$\sum_{V \leq A^*}$ means summation over all possible partitions V of $\{1, 2, \ldots, 4k\}$ which is a refinement of $A^* = \{A_1^*, \ldots, A_w^*\}$.

$\sum'_{r_1,\ldots,r_w}$ means the summation for $r_1, \ldots, r_w$ running over the set $\{1, 2, \ldots, n\}$, but being kept different from each other;

$\sum_{(i)|A^*,V}$ means the summation for $i_1, \ldots, i_{4k}$ running over the set $\{1, 2, \ldots, p\}$ but if c, c' belong to the same class A_a^* then $i_c = i_{c'}$, iff c, c' belong to a same class B_b.

So, if

$$K(V) = \prod_{b=1}^{v} Ex_{11}^{|B_b|} - \prod_{b=1}^{v} Ex_{11}^{|B_b \; S_1^*|} Ex_{11}^{|B_b \; S_2^*|},$$

$$R(A^*,V) = \sum_{(i)|A^*,V} t_{i_2 i_3} t_{i_4 i_5} \cdots t_{i_{2k} i_1} t_{i_{2k+2} i_{2k+3}} \cdots t_{i_{4k} i_{2k+1}}$$

$$(n)_w = n(n-1)\ldots(n-w+1),$$

then

$$\mathrm{Var}_T R_p = \sum_A \sum_{V \leq A^*} \frac{(n)_w}{p^2 n^{2k}} R(A^*,V)K(V).$$

We may suppose that

1. $|B_b| \geq 2$ for all b,
2. at least for one b, $|B_b \cap S_1^*| \cdot |B_b \cap S_2^*| \neq 0$.

For, otherwise $K(V) = 0$.

Now for each pair of partitions $A = \{A_1,\ldots,A_w\}$ of $\{1, 2,\ldots,2k\}$ and $V = \{B_1,\ldots,B_v\}$ of $\{1, 2,\ldots,4k\}$ such that $V \leq A^* = \{A_1^*,\ldots,A_w^*\}$. We construct a graph $G(A^*,V)$ as follows:

(1) the vertex set is V,

(2) there are 2k edges $e_1,\ldots,e_{2k}$; $E = \{e_1,\ldots,e_{2k}\}$

(3) $\Gamma : E \to V \times V$ is defined as follows:

$$\Gamma(e_1) = \{B(2), B(3)\}, \Gamma(e_2) = \{B(4), B(5)\},\ldots,\Gamma(e_k) = \{B(2k), B(1)\}$$
$$\Gamma(e_{k+1}) = \{B(2k+2), B(2k+3)\},\ldots,\Gamma(e_{2k}) = \{B(4k), B(2k+1)\}.$$

Here $B(a) = B_b$ iff $a \in B_b$.

(4) Classes are $C_a = \{B_b : B_b \subset A_a^*\}$, $a = 1,\ldots,w$.

It is easy to verify that $G(A^*,V)$ is an M-graph. It is class-connected owing to 2.

Now we show that $R(A^*,V) = O(p^{2k-w})$.

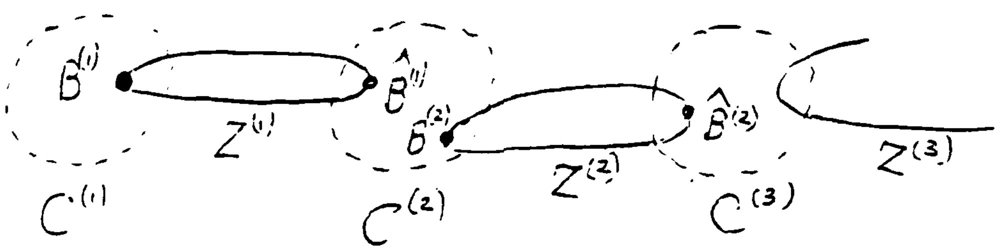
<u>Case 1</u> there exists b such that $|B_b| > 2$. Then by Lemma 3.8, we have $R(A^*,V) = O(p^{2k-w})$.

Case 2 $|B_b| = 2$ for all b. Consider the b, for which $|B_b \cap S_1^*| \cdot |B_b \cap S_2^*| \neq 0$. Such a vertex will be called mixed. It is evident that any class has even number of mixed vertex.

Now our graph $G(A^*,B)$ is a Q-graph. We are going to show that it is not maximal, i.e. the number of cycles $\leq 2k - w$.

Suppose it is maximal. Then, we know that any free cycle cannot meet a class at more than one vertex.

Let $B^{(1)}$ be a mixed vertex in the class $C^{(1)}$. Let $Z^{(1)}$ be the free cycle containing $B^{(1)}$. $Z^{(1)}$ must contain another mixed vertex $\hat{B}^{(1)}$, $\hat{B}^{(1)} \overline{\in} C^{(1)}$. Suppose $\hat{B}^{(1)} \in$ class $C^{(2)}$.



$C^{(2)}$ contains another mixed vertex $B^{(2)}$. $B^{(2)}$ belongs to a cycle $Z^{(2)}$. $Z^{(2)} \neq Z^{(1)}$. $Z^{(2)}$ has another mixed vertex $\hat{B}^{(2)}$. $\hat{B}^{(2)}$ class $C^{(3)}$, $C^{(3)} \neq C^{(2)}$. $C^{(3)}$ has another mixed vertex $B^{(3)}$, $B^{(3)}$ belongs to cycle $Z^{(3)}$. $Z^{(3)} \neq Z^{(2)}$.

Because there are only finitely many cycles and classes. We may suppose $Z^{(1)},\ldots,Z^{(a)}$ are different and $Z^{(a+1)} = Z^{(1)}$. But for maximal Q-graph, such situation can't occur.

So $G(A^*,B)$ is not a maximal Q-graph and

$$R(A^*,V) = O(p^{2k-w})$$

Thus,

$$\mathrm{Var}_T R_p = \sum_A \sum_{V \leq A^*} \frac{(n)_w}{p^2 n^{2k}} O(p^{2k-w}) K(V)$$

$$= O\left(\frac{(\log p)^{2k}}{p^2}\right). \qquad \text{a.s.}$$

and from this it is easy to deduce that

$$\sum_{p=1}^{\infty} E(R_p - E_T R_p)^2 < \infty, \text{ for almost all } T.$$

So, the conclusion of Theorem 2.1 follows.

REFERENCES

[1] BAI, Z. D., YIN, Y. Q. and KRISHNAIAH, P. R. (1984). On limiting spectral distribution of product of two random matrices when the underlying distribution is isotropic. Technical Report No. 84-48, Center for Multivariate Analysis, University of Pittsburgh.

[2] DUDLEY, R. M. (1966). Convergence of Baire measures. Studia Math., 27, 251-268.

[3] KY FAN (1951). Maximum properties and inequalities for the eigenvalues of completely continuous operators. Proc. Nat. Acad. Sci. USA, 37, 760-766.

[4] GRENANDER, V. and SILVERSTEIN, J. (1977). Spectral analysis of networks with random topologies. SIAM Journal, Appl. Math., 32(2), 499-519.

[5] HOEFFDING, W. (1963). Probability inequalities for sums of bounded random variables. JASA, March, 13-30.

[6] VON NEUMANN, J. (1937). Some matrix-inequalities and metrization of matric space. Tomsk Univ. Rev. 1, 286-300.

[7] WACHTER, K. W. (1978). The strong limits of random matrix spectra for sample matrices of independent elements. Ann. Prob., 6, 1-18.

[8] YIN, Y. Q. and KRISHNAIAH, P. R. (1984). A limit theorem for the eigenvalues of product of two random matrices. J. Mult. Anal., 13, 489-507.

[9] YIN, Y. Q. and KRISHNAIAH, P. R. (1983). Limit theorems for the eigenvalues of the sample covariance matrix when the underlying distribution is isotropic. Technical Report No. 84-14, Center for Multivariate Analysis, University of Pittsburgh.

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