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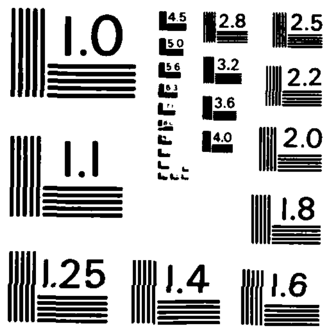
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CENTER FOR STOCHASTIC PROCESSES

Department of Statistics
University of North Carolina
Chapel Hill, North Carolina

REAL INVERSION FORMULAS FOR LAPLACE AND STIELTJES TRANSFORMS

by

Jozef L. Teugels

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REAL INVERSION FORMULAS FOR LAPLACE AND STIELTJES TRANSFORMS

by

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Abstract

We provide probabilistic proofs for a number of real inversion formulas for the Laplace and for the Stieltjes transform.

Keywords: Inversion formulas, Laplace transform, Stieltjes transform

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1. Introduction

Assume μ to be a bounded measure on $\mathbb{R}_+$. Let f be a kernel changing μ into the transformed function

$$g(t) := \int_0^\infty f(t,u)d\mu(u). \tag{1}$$

In the literature dealing with operational calculus the recovery of μ from g is mostly accomplished by complex inversion formulas. One knows that for a variety of integral transforms (i.e. Laplace, Stieltjes), real inversion formulas are available [6,10,12]. These formulas can also be obtained as applications of classical results from probability theory such as the weak law of large numbers and the central limit theorem.

To abbreviate the writing we put for $0 < y_1 < y_2 < \infty$

$$\mu\{y_1;y_2\} = \tfrac{1}{2}\mu\{y_1\} + \mu(y_1,y_2) + \tfrac{1}{2}\mu\{y_2\} \tag{2}$$

2. Inversions using lattice variables

Let X be a non-negative lattice variable with mean u > 0 and finite variance $\sigma^2 > 0$. Take $X_1, X_2, \ldots$ a sample from X. Define the sum

$$S_n(u) = X_1 + X_2 + \ldots + X_n \, , \qquad n \geq 1$$

and put

$$d_n(u,t) = P\{S_n(u) \leq [nt]\}.$$

By the weak law of large numbers we know that $S_n(u)/n \xrightarrow{P} u$ while $[nt]/n \to t$. Hence $d_n(u,t) \to 0$ if $t < u$, while $d_n(u,t) \to 1$ if $t > u$. For $u = t$ note that $\{[nu] - nu\}/\sqrt{n} \to 0$ while

$$d_n(u,u) = P\left\{\frac{S_n(u) - nu}{\sigma\sqrt{n}} \leq \frac{[nu] - nu}{\sigma\sqrt{n}}\right\} \to \frac{1}{2}$$

by the central limit theorem and the uniform convergence to the normal law [7, p. 139]. Hence

$$d_n(u,t) \to \begin{cases} 0 & t < u \\ \frac{1}{2} & t = u \\ 1 & t > u \, . \end{cases}$$

By Lebesgue's theorem and (2) for $0 < y_1 < y_2 < \infty$

$$\mu\{y_1;y_2\} = \lim_{n\to\infty} \int_0^\infty \{d_n(u,y_2) - d_n(u,y_1)\} d\mu(u) \tag{3}$$

is a potential inversion formula.

Moreover by a local limit theorem [4, p. 225] or [1, p. 233]

(4) $$h_n(u,t) := \sigma\sqrt{2\pi n}\; P\{S_n(u) = [nt]\} \to \begin{cases} 0 & t \neq u \\ 1 & t = u . \end{cases}$$

By proper approximation (4) will suggest a function of the form

$$e_n(v) = ch_n(v,1) \to \begin{cases} 0 & v \neq 1 \\ 1 & v = 1 \end{cases}$$

for a constant c; this implies an inversion formula for the point mass at y, i.e.

(5) $$\mu\{y\} = \lim_{n\to\infty} \int_0^\infty e_n\left(\frac{u}{y}\right) d\mu(y).$$

Before turning to examples let us introduce the abbreviation $(0 < y_1 < y_2 < \infty)$

(6) $$A_n = \{m\colon [ny_1] + 1 \le m \le [ny_2]\}.$$

Example 2.1. Poisson distribution and Laplace transform.

Put for the Laplace transform of μ,

(7) $$g(t) := \int_0^\infty e^{-tu} d\mu(u).$$

Then for $m \in \mathbb{N}$

(8) $$(-1)^n g^{(n)}(t) = \int_0^\infty e^{-tu} u^n d\mu(u).$$

Let X be Poisson with mean u so that

$$d_n(u,t) = \sum_{m=0}^{[nt]} e^{-nu} \frac{(nu)^m}{m!} .$$

By (2) and (3) we get

$$\mu\{y_1;y_2\} = \lim_{n\to\infty} \sum_{m\in A_n} \int_0^\infty e^{-nu} \frac{(nu)^m}{m!} d\mu(u)$$

$$= \lim_{n\to\infty} \sum_{m\in A_n} \frac{(-nu)^m}{m!} g^{(m)}(n).$$

This inversion formula can be found in [12, p. 295]. It can be fruitfully used in giving a probabilistic proof of Bernstein's theorem on completely monotone functions; see [2, p. 191].

For the point mass we get from (4)

$$h_n(u,t) = \sqrt{2\pi nu}\; e^{-nu} \frac{(nu)^{[nt]}}{[nt]!} .$$

Stirling's formula easily yields $c = 1$ and

$$e_n(v) = \exp - n\{v - 1 + \log v\}.$$

Collecting results we get

Theorem 2.1. Let μ be a bounded measure and $g(t)$ its Laplace transform. Then for $0 < y_1 < y_2 < \infty$

$$\mu\{y_1;y_2\} = \lim_{n\to\infty} \sum_{m\in A_n} \frac{(-nu)^m}{m!} g^{(m)}(n);$$

and for $0 < y < \infty$

$$\mu\{y\} = \lim_{n\to\infty}(-\frac{e}{y})^n g^{(n)}(\frac{n}{y}).$$

The latter formula occurs also in [12, p. 298].

Example 2.2. Pascal distribution and Stieltjes transform.

In the sequel we also deal with the (generalized) Stieltjes transform.

$$\text{(9)} \qquad \phi_p(t) := \int_0^\infty (u + t)^{-p} d\mu(u)$$

where $p > 0$.

In evaluating successive derivatives of ϕ_p the next lemma will be useful.

Lemma 2.2. For $a, c \in \mathbf{N}$, $-p \notin \mathbf{N}_0$, $u > 0$, $t > 0$

$$\text{(10)} \quad D^a\{t^b D^c(u + t)^{-p}\} = \frac{\Gamma(p + a + c)}{\Gamma(p)}(-1)^{a+c} t^b (u + t)^{-p-a-c} F(-a,-b,-p-a-c+1;\frac{u+t}{t});$$

in particular

$$\text{(11)} \quad D^a\{t^{p+a+c-1} D^c(u + t)^{-p}\} = \frac{\Gamma(p + a + c)}{\Gamma(p)}(-1)^c t^{p+c-1} u^a (u + t)^{-p-a-c}.$$

Proof. Apply Leibniz's rule for differentiation of products; and then, for $\alpha \in \mathbf{N}$

$$F(-\alpha,-\beta,-\gamma;x) = \sum_{m=0}^{\alpha} \frac{\binom{\alpha}{m}\binom{\beta}{m}}{\binom{\gamma}{m}}(-x)^m.$$

To prove (11), note that $F(-\alpha,\gamma,\gamma;x) = (1 - x)^\alpha$ if $\alpha \in \mathbf{N}$ by [5, p. 1040]. □

Let X be geometric with mean u. Then

$$d_n(u,t) = \sum_{m=0}^{[nt]} \binom{n+m-1}{m} p^m (1-p)^n$$

where $u = EX = \frac{1-p}{p}$ or $p = (u+1)^{-1}$. For the point mass we note that

$$h_n(u,t) = \sigma\sqrt{2\pi n}\binom{n+m-1}{m}\frac{u^m}{(1+u)^{n+m}}$$

where $m = [nt]$. Stirling's formula again applies to yield

$$e_n(v) = \left(\frac{2\sqrt{v}}{1+v}\right)^{2n} . \tag{12}$$

Theorem 2.3. Let μ be a bounded measure and $\phi_1(t)$ its (ordinary) Stieltjes transform. Then for $0 < y_1 < y_2 < \infty$

$$\mu\{y_1;y_2\} = \lim_{n\to\infty} \frac{1}{\Gamma(n)} \sum_{m\in A_n} \frac{1}{m!} D^n\{t^{n+m-1}(-D)^{m-1}\phi_1(t)\}_{t=1}$$

$$= \lim_{n\to\infty} \frac{1}{\Gamma(n)} \sum_{m\in A_n} \frac{(-1)^{m-1}}{m!} D^{n+m-1}\{t^n\phi_1(t)\}_{t=1};$$

for $0 < y < \infty$,

$$\mu\{y\} = \lim_{n\to\infty} \frac{2^{2n}y}{\Gamma(2n)} D^n\{y^{2n-1}(-D)^{n-1}\phi_1(y)\}.$$

Proof. From (3) and the current choice of d_n we find

$$\mu\{y_1;y_2\} = \lim_{n\to\infty} \sum_{m\in A_n} \binom{n+m-1}{m} \int_0^\infty \frac{u^n}{(u+1)^{m+n}} d\mu(u).$$

Put $a = n$, $b = n + m - 1$, $c = m - 1$, $p = 1$ in (11) to find the first formula. To get the second, put $a = n + m - 1$, $b = n$, $c = 0$, $p = 1$ in (10) and recall that $F(a,b,c;x) = F(b,a,c;x)$.

For the point mass we get from (5) and (12)

$$\mu\{y\} = \lim_{n\to\infty}(4y)^n \int_0^\infty \frac{u^n}{(u+y)^{2n}}\, d\mu(u).$$

So we take $a = n$, $b = 2n - 1$, $c = n - 1$ and $p = 1$ in (11). □

Example 2.3. Binomial distribution and Stieltjes transform.

Let X be Bernoulli on 0 and 1 so that

$$P\{X = k\} = \begin{cases} 1 - v & k = 0 \\ v & k = 1. \end{cases}$$

Then

$$P\{S_n(v) \le [nt]\} = \sum_{\ell=0}^{[nt]} \binom{n}{\ell} v^\ell (1 - v)^{n-\ell}$$

suggests a potential inversion. However we should adapt it so that $v \in \mathbb{R}_+$ rather than $v \in [0.1]$. So define

$$d_n(u,t) = \sum_{\ell=0}^{[n/(1+t)]} \binom{n}{\ell} u^{n-\ell}(1 + u)^{-n}.$$

Theorem 2.4. Let μ be a bounded measure and $\phi_1(t)$ its (ordinary) Stieltjes transform. Then for $0 < y_1 < y_2 < \infty$

$$\mu\{y_1;y_2\} = \lim_{n\to\infty} \sum_{\ell\in B_m} \frac{1}{(n-1)!}\binom{n}{\ell} D^{n-\ell}\{t^{n-1}(-D)^{\ell-1}\phi_1(t)\}_{t=1}$$

$$= \lim_{n\to\infty} \sum_{\ell\in B_m} \frac{1}{(n-1)!}\binom{n}{\ell} D^{n-1}\{t^{n-\ell}\phi_1(t)\}_{t=1},$$

where $B_m = \{\ell : [\frac{n}{y_2+1} + 1] \le \ell \le [\frac{n}{y_1+1}]\}$.

The proof is basically the same as that of the previous theorem.

3. Inversions using also densities

Let $X_1, X_2, \ldots$ be a sequence of independent non-negative random variables with common distribution F, mean 1 and finite variance. Let also $Y_1, Y_2, \ldots$ be such a sequence but with distribution G, and independent of the sequence $\{X_i\}$.

Define $S_n = X_1 + \ldots + X_n$ and $T_n = Y_1 + \ldots + Y_n$ and look at

$$d_n(u) = P\{S_n/T_n \leq u\};$$

by combining a weak law and the central limit theorem, one easily finds that

$$d_n(u) = \int_0^\infty F^{(n)}(ut)dG^{(n)}(t) \to \begin{cases} 0 & u < 1 \\ \frac{1}{2} & u = 1 \\ 1 & u > 1 \end{cases}$$

where $F^{(n)}(G^{(n)})$ is the n-fold convolution of F(G) with itself. The particular case where $Y_i = 1$ is important and yields $d_n(u) = F^{(n)}(nu)$.

The potential inversion formula reads now

$$\mu\{y_1;y_2\} = \lim_{n\to\infty} \int_0^\infty \{d_n(\frac{u}{y_1}) - d_n(\frac{u}{y_2})\}d\mu(u) \tag{13}$$

while a point mass formula can be derived by using

$$e_n(v) = \frac{c}{\sqrt{n}} v d_n'(v)$$

if d_n is absolutely continuous.

Example 3.1. Exponential distribution and Laplace transform.

Here $Y_i = 1$ and X_i has an exponential density with mean 1. Hence

$$(14) \qquad d_n(u) = \frac{1}{\Gamma(n)} \int_0^{nu} e^{-v} v^{n-1} dv$$

and so we have

Theorem 3.1. Let μ be a bounded measure and $g(t)$ its Laplace transform. Then for $0 < y_1 < y_2 < \infty$

$$(15) \qquad \mu\{y_1;y_2\} = \lim_{n\to\infty} \int_{y_1}^{y_2} \frac{(-n)^n}{\Gamma(n)} \frac{g^{(n)}(\frac{n}{t})}{t^{n+1}} dt.$$

Proof. From (13) and (14) we see that

$$\mu\{y_1;y_2\} = \lim_{n\to\infty} \int_0^{\infty} d\mu(u) \int_{nuy_2^{-1}}^{nuy_1^{-1}} \frac{1}{\Gamma(n)} e^{-v} v^{n-1} dv$$

$$= \lim_{n\to\infty} \int_{y_1}^{y_2} \frac{(-n)^n}{\Gamma(n)} \frac{dt}{t^{n+1}} \int_0^{\infty} e^{-u\frac{n}{t}} (-u)^n d\mu(u)$$

by Fubini's theorem. □

The point mass formula coincides with that of th. 2.1.

The appealing form of (15) suggests that if the sequence

$$k_n(t) := \frac{(-n)^n}{\Gamma(n)} \frac{g^{(n)}(\frac{n}{t})}{t^{n+1}}$$

converges a.e. on $\mathbb{R}_+$ and is bounded by an L_1-function then μ is absolutely continuous and its derivative is given by the inversion

$$(16) \qquad \frac{d\mu}{dt} = \lim_{n\to\infty} \frac{(-n)^n}{\Gamma(n)} \frac{g^{(n)}(\frac{n}{t})}{t^{n+1}} .$$

The inversion formula (16) is well known [6,12]; its probabilistic proof due to Feller [3] has been the inspiring source for the current paper. Formula (16) has been used by Jagerman [9] to perform numerical inversion of the Laplace transform. For other applications see a paper by Vinogradov [13].

Example 3.2. Exponential distributions and Stieltjes transform.

Let $F(x) = G(x) = 1 - e^{-x}$; then $d_n(u)$ turns out to be a beta distribution on $\mathbb{R}_+$, i.e.

$$d_n(u) = \frac{\Gamma(2n)}{\Gamma^2(n)} \int_0^u \frac{v^{n-1}}{(1+v)^{2n}} dv.$$

Theorem 3.2. Let μ be a bounded measure and $\phi_1(t)$ be its (ordinary) Stieltjes transform. Then for $0 < y_1 < y_2 < \infty$

$$\mu\{y_1;y_2\} = \lim_{n\to\infty} \frac{1}{\Gamma^2(n)} \int_{y_1}^{y_2} D^n\{t^{2n-1}(-D)^{n-1}\phi_1(t)\}dt$$

$$= \lim_{n\to\infty} \frac{(-1)^{n-1}}{\Gamma^2(n)} \int_{y_1}^{y_2} t^{n-1}D^{2n-1}\{t^n\phi_1(t)\}dt;$$

for $0 < y < \infty$

$$\mu\{y\} = \lim_{n\to\infty} \frac{(-4y)^n}{\Gamma(2n)} D^{2n-1}\{y^n\phi_1(y)\}.$$

Proof. We easily get

$$\mu\{y_1;y_2\} = \lim_{n\to\infty} \frac{\Gamma(2n)}{\Gamma^2(n)} \int_{y_1}^{y_2} t^{n-1}dt \int_0^\infty \frac{u^n}{(u+t)^{2n}} d\mu(u).$$

We get the two formulas by applying lemma 2.2 with the two choices

$a = n,\ b = 2n - 1,\ c = n - 1,\ p = 1$ in (11)

$a = 2n - 1,\ b = n,\ c = 0,\ p = 1$ in (10)

as in the proof of th. 2.3.

For the point mass we get e_n as in (12); the resulting formula comes from the second choice above. ❑

For the above formulas see [12, p. 350]. For applications, see [8].

The last example shows how an almost trivial probabilistic argument widens the applicability of the previous examples.

Example 3.3. **Exponential distributions and generalized Stieltjes transforms.**

Return to the previous example but look at S_{n+b} and T_{n+a+p} instead of S_n and T_n where $a,b \in \mathbb{Z}$ are fixed. Then

$$\lim_{n\to\infty} P\{\frac{S_{n+b}}{T_{n+a+p}} \le u\} = \lim_{n\to\infty} P\{\frac{S_n}{T_n} \le u\}.$$

Hence we have another candidate for d_n, i.e.

$$d_n(u) = \frac{\Gamma(2n+a+b+p)}{\Gamma(n+a+p)\Gamma(n+b)} \int_0^u \frac{v^{n+b-1}}{(1+v)^{2n+a+b+p}}\, dv.$$

By the usual procedure we obtain also

$$e_n(v) = v^{\frac{b-a-p}{2}} \{\frac{2\sqrt{v}}{1+v}\}^{2n+a+b+p}.$$

If we now go through the same routine as in example 3.2 with a replaced by n + b and c by n + a we obtain the following results of E.R. Love and A. Byrne [10,11]. We note that we have to restrict p to $\mathbb{N}_0$ in view of the probabilistic set up.

Theorem 3.3. Let μ be a bounded measure ; let $\phi_p(t)$ be its generalized Stieltjes transform. Let $a, b \in \mathbb{Z}$. Then for $0 < y_1 < y_2 < \infty$

$$\mu\{y_1;y_2\} = \lim_{n\to\infty} \frac{\Gamma(p)}{\Gamma(n+a+p)\Gamma(n+b)} \int_{y_1}^{y_2} D^{n+b}\{t^{2n+a+b+p-1}(-D)^{n+a}\phi_p(t)\}\,dt;$$

for $0 < y < \infty$

$$\mu\{y\} = \lim_{n\to\infty} \frac{2^{2n+a+b+p}\Gamma(p)}{\Gamma(2n+a+b+p)}\, yD^{n+b}\{y^{2n+a+b+p-1}(-D)^{n+a}\phi_p(y)\}.$$

Note that by using different random variables one can get a variety of real inversion formulas for the same Stieltjes transform ϕ_1. Needless to point out how th. 3.2 follows from the last result.

4. Remarks.

1. The main drawback of the method used in this paper is that the relevant integral transform has to be recovered from a potential inversion formula.

2. The main virtue of the paper lies in singling out the relevance of the asymptotic behavior of the sequence d_n for $n \to \infty$; probabilistic arguments often provide this behavior immediately.

3. There exist additive counterparts for the formula (13). For if

$$d_n(v) \to \begin{cases} 0 & v < 0 \\ \frac{1}{2} & v = 0 \\ 1 & v > 0 \end{cases}$$

where $v \in \mathbb{R}$, then a potential inversion formula for a transform on $\mathbb{R}$ is

$$\mu\{y_1;y_2\} = \lim_{n\to\infty} \int_{\mathbb{R}} \{d_n(u - y_1) - d_n(u - y_2)\} d\mu(u).$$

The best known example is $d_n(v) = \frac{1}{\pi} \int_{-\infty}^{nv} \frac{\sin u}{u} du$ leading to the inversion formula for the Fourier transform

$$\mu\{y_1;y_2\} = \lim_{n\to\infty} \int_{y_1}^{y_2} dx \{\frac{1}{2\pi} \int_{-n}^{n} e^{-izx} \phi(z) dz\}$$

where $\phi(z) = \int_{\mathbb{R}} e^{izu} d\mu(u)$.

Another example is $d_n(v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\sqrt{n}v} e^{-\frac{1}{2}x^2} dx$ leading to a real inversion formula for the Gauss-Weierstrasz transform [6]

$$w_\lambda(t) = \sqrt{\frac{t}{2\pi}} \int_{-\infty}^{\infty} \exp - \frac{\lambda}{2}(u - t)^2 d\mu(u).$$

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