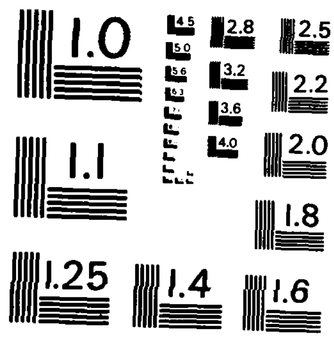
AD-A161 273 MOMENT AND GEOMETRIC PROBABILITY INEQUALITIES ARISING FROM ARRANGEMENT IN..(U) FLORIDA STATE UNIV TALLAHASSEE DEPT OF STATISTICS P J BOLAND ET AL. AUG 85 1/1

UNCLASSIFIED FSU-STATISTICS-M708 AFOSR-TR-85-0952 F/G 12/1 NL

END

FILMED

DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

MOMENT AND GEOMETRIC PROBABILITY INEQUALITIES ARISING FROM ARRANGEMENT INCREASING FUNCTIONS

by

Philip J. Boland[1], Frank Proschan[2], and Y. L. Tong[3]

FSU Statistics Report M708

AFOSR Technical Report No. 85-182

August, 1985

University College, Dublin
Department of Mathematics
Belfield, Dublin 4, IRELAND

and

The Florida State University
Department of Statistics
Tallahassee, Florida 32306-3033

and

Georgia Institute of Technology
School of Mathematics
Atlanta, Georgia 30332

[1]Research partially sponsored by AFOSR F49620-85-C-0007 and AFOSR 84-0113.

[2]Research sponsored by the Air Force Office of Scientific Research, AFSC, USAF, under Grant AFOSR F49620-85-C-0007.

[3]Research sponsored by the National Science Foundation under grant DMS-8502346.

AMS 1980 subject classification: Primary 60D05; secondary 62H10

Key Words and Phrases: Arrangement increasing, decreasing in transposition, exchangeable random vector, family of arrangement increasing densities, inequalities, moments, Laplace transforms, permutation.

A -

Moment and Geometric Probability Inequalities Arising from Arrangement Increasing Functions

by

Philip J. Boland, Frank Proschan, and Y. L. Tong

Abstract

A real valued function g of two vector arguments x and $y \in R^n$ is said to be arrangement increasing if it increases in value as the arrangement of components in x becomes increasingly similar to the arrangement of components in y. Hollander, Proschan and Sethuraman (1977) show that the composition of arrangement increasing functions is arrangement increasing. This result is used to generate some interesting probability inequalities of a geometric nature for exchangeable random vectors. Other geometric inequalities for families of arrangement increasing multivariate densities are also given, and some moment inequalities are obtained.

1. Introduction

Definition 1.1. For a given vector $\underline{x} = (x_1, \ldots, x_n) \in R^n$, we let $\underline{x}\uparrow = (x_{[n]}, \ldots, x_{[1]})$ and $\underline{x}\downarrow = (x_{[1]}, \ldots, x_{[n]})$ be respectively the vectors with the components of $\underline{x}$ arranged in increasing (decreasing) order. For any permutation π of $\{1, 2, \ldots, n\}$, we let $\underline{x}_\pi = (x_{\pi(1)}, \ldots, x_{\pi(n)})$.

For vectors $\underline{x}$, $\underline{y}$, $\underline{u}$, $\underline{v}$ in R^n, we write $(\underline{x}, \underline{y}) \overset{a}{=} (\underline{u}, \underline{v})$ if there exists a permutation π of $\{1, \ldots, n\}$ such that $\underline{x}_\pi = \underline{u}$ and $\underline{y}_\pi = \underline{v}$. We define $(\underline{x}, \underline{y}) \overset{a}{\le} (\underline{u}, \underline{v})$ if there exist a finite number of vectors $\underline{z}^1, \ldots, \underline{z}^K$ such that

(i) $(\underline{x}, \underline{y}) \overset{a}{=} (\underline{x}\uparrow, \underline{z}^1)$ and $(\underline{x}\uparrow, \underline{z}^K) \overset{a}{=} (\underline{u}, \underline{v})$ and

(ii) $\underline{z}^{i-1}$ can be obtained from $\underline{z}^i$ by an interchange of two components of $\underline{z}^i$, the first of which is less than the second.

Definition 1.2. A function g of two vector arguments $\underline{x}$ and $\underline{y} \in R^n$ for which $g(\underline{x}) \le g(\underline{y})$ when $\underline{x} \overset{a}{\le} \underline{y}$ is said to be arrangement increasing or AI by Marshall and Olkin (1979), and decreasing in transposition or DT by Hollander, Proschan and Sethuraman (1977). We shall use the terminology arrangement increasing (AI) in order to emphasize that such a function $g(\underline{x}, \underline{y})$ increases in value as the arrangement of components in $\underline{x}$ becomes increasingly similar to the arrangement of components in $\underline{y}$.

Hollander, Proschan and Sethuraman (1977) have shown that the class of arrangement increasing functions is closed under some basic operations. In particular the class of such functions is closed under mixtures, and the product of nonnegative AI functions is AI. If ϕ is an increasing function and $g(\underline{x}, \underline{y})$ is AI, then clearly $\phi(g(\underline{x}, \underline{y}))$ is AI. Perhaps the most powerful closure property they show however is that the composition of AI functions is AI.

Theorem 1.3. (Hollander, Proschan and Sethuraman). Let u be a measure defined on the Borel subsets of R^n such that for all Borel sets $A \subset R^n$ and all permutations π, $u(A) = u(A_\pi)$ where $A_\pi = \{\underline{y}: \underline{y} = \underline{x}_\pi = (x_{\pi(1)}, \ldots, x_{\pi(n)})$ for some $\underline{x} \in A\}$. If g_i is AI on $R^n \times R^n$ for $i = 1, 2$, then the composition g given by

$$g(\underline{a}, \underline{b}) = \int g_1(\underline{a}, \underline{x}) g_2(\underline{x}, \underline{b}) u(d\underline{x})$$

is AI, provided the integral exists.

We will illustrate some of the power of this result in Section 2 when we show that many geometric probabilities of random vectors $\underline{X}$ with exchangeable densities $f(\underline{x})$ are AI functions. In Section 3 we further illustrate this power with examples of geometric probability comparisons arising in families of AI multivariate densities. Some moment inequalities are also obtained.

2. Arrangement Increasing Functions from Exchangeable Random Vectors.

In this section we will assume $\underline{X} = (X_1, \ldots, X_n)$ is an exchangeable random vector (that is the distribution of $\underline{X}_\pi = (X_{\pi(1)}, \ldots, X_{\pi(n)})$ does not depend on the permutation π) with density or mass function $f(\underline{x})$. Many interesting geometric probability functions which are arrangement increasing as well as some moment inequalities may be generated by the use of the following corollaries.

Corollary 2.1. Let $\underline{X}$ be an exchangeable random vector with density or mass function $f(\underline{x})$. Let h^i be an AI function on $R^n \times R^n$ and $\phi_i = R \to R$ be nondecreasing for $i = 1, 2$. Then

$$\psi(\underline{a}, \underline{b}) = E_{\underline{X}}[\phi_1(h^1(\underline{a}, \underline{X}))\phi_2(h^2(\underline{X}, \underline{b}))] \text{ is AI in } (\underline{a}, \underline{b}) \in R^n \times R^n.$$

Proof: Let $g_1(\underline{a}, \underline{x}) = \phi_1(h^1(\underline{a}, \underline{x}))$ and $g_2(\underline{x}, \underline{b}) = \phi_2(h^2(\underline{x}, \underline{b}))$. Then g_1 and g_2 are AI (nondecreasing functions of AI functions are AI). As $u(d\underline{x}) = f(\underline{x})d\underline{x}$ is a permutation symmetric measure, we deduce from Theorem 1.3 that

$$\psi(\underline{a}, \underline{b}) = \int g_1(\underline{a}, \underline{x}) g_2(\underline{x}, \underline{b}) f(\underline{x}) d\underline{x}$$

is arrangement increasing in $\underline{a}$ and $\underline{b} \in R^n$.

Corollary 2.2. Let $\underline{X}$ be an exchangeable random vector with density or mass function $f(\underline{x})$, and h^i be an AI function on $R^n \times R^n$ for $i = 1, 2$. Then for any two constants c_1 and c_2,

$$P_{(\underline{a},\underline{b})}(\underline{X}) = \mathrm{Prob}(h^1(\underline{a}, \underline{X}) \geq c_1, h^2(\underline{X}, \underline{b}) \geq c_2)$$

is an arrangement increasing function of $\underline{a}$ and $\underline{b} \in R^n \times R^n$.

Proof: This follows from Corollary 2.1 by letting $\phi_i = I_{[c_i, +\infty]}$ (the characteristic function of the set $[c_i, +\infty)$) for $i = 1, 2$.

Example 2.3. The following are some elementary examples of AI functions which will be useful in illustrating Corollaries 2.1 and 2.2:

1. $h_1(\underline{u}, \underline{v}) = I_{\{u_i \leq v_i:\ i = 1, \ldots, n\}}$.
2. $h_2(\underline{u}, \underline{v}) = \sum_{i=1}^{n} u_i v_i$.
3. $h_3(\underline{u}, \underline{v}) = -\sum_{i=1}^{n} (u_i - v_i)^2$.
4. $h_4(\underline{u}, \underline{v}) = -\sum_{i=1}^{n} \frac{u_i^2}{v_i^2} \times I_{\{u_i > 0,\ v_i > 0:\ i = 1, \ldots, n\}}$.
5. $h_5(\underline{u}, \underline{v}) = \prod_{i=1}^{n} (u_i - v_i)^+$ where $(u_i - v_i)^+ = (u_i - v_i) I_{\{u_i > v_i\}}$.
6. $h_6(\underline{u}, \underline{v}) = -\max |u_i - v_i|$.
7. $h_7(\underline{u}, \underline{v}) = -\sum_{i=1}^{n} |u_i - v_i|$.

Example 2.4. Judicious selection of h^1 and h^2 from, say Example 2.3, can yield some useful AI geometric probability functions. We illustrate this with some examples.

1. The rectangular probability

$$\mathrm{Prob}(a_i \le X_i \le b_i: \; i = 1, \ldots, n) = \mathrm{Prob}(\underline{X} \in [\underline{a}, \underline{b}])$$

is an AI function of $\underline{a}$ and $\underline{b}$, as can be seen by using $h^1(\underline{a}, \underline{x}) = h_1(\underline{a}, \underline{x})$ and $h^2(\underline{x}, \underline{b}) = h_1(\underline{x}, \underline{b})$. (See also Boland (1985) for more on rectangular probabilities of this type).

2. $\mathrm{Prob}(\sum_{i=1}^{n} a_i X_i \ge c_1, \; \sum_{i=1}^{n} b_i X_i \ge c_2)$ is AI in $\underline{a}$ and $\underline{b}$, as can be seen by letting $h^1(\underline{a}, \underline{x}) = h_2(\underline{a}, \underline{x})$ and $h^2(\underline{x}, \underline{b}) = h_2(\underline{x}, \underline{b})$. For example, if $\underline{X} = (X_1, X_2)$ is exchangeable, then ▤ $= \mathrm{Prob}(4X_1 + X_2 \ge 4, \; 2X_1 + 4X_2 \ge 2)$ $\le \mathrm{Prob}(X_1 + 4X_2 \ge 4, \; 2X_1 + 4X_2 \ge 2) =$ ▥

(Insert Figure 2.1 here)

3. Of course any combination of two of the types of functions in Example 2.3 gives us a probability function which is AI in $\underline{a}$ and $\underline{b}$, such as

$$\mathrm{Prob}(\sum_{i=1}^{n} a_i X_i \ge c_1 \text{ and } \sum (X_i - b_i)^2 \le c_2)$$

or

$$\mathrm{Prob}(X_i \ge a_i \text{ for all } i = 1, \ldots, n \text{ and } \sum_{i=1}^{n} |X_i - b_i| \le c_2).$$

Remark 2.5. Let $\underline{X}$ be an exchangeable random vector with mass or density function $f(\underline{x})$. Let h^i be an AI function for $i = 1, 2$, and assume $(\underline{a}, \underline{b}), (\underline{a}', \underline{b}') \in R^n \times R^n$ are given, where $(\underline{a}, \underline{b}) \overset{a}{\le} (\underline{a}', \underline{b}')$.

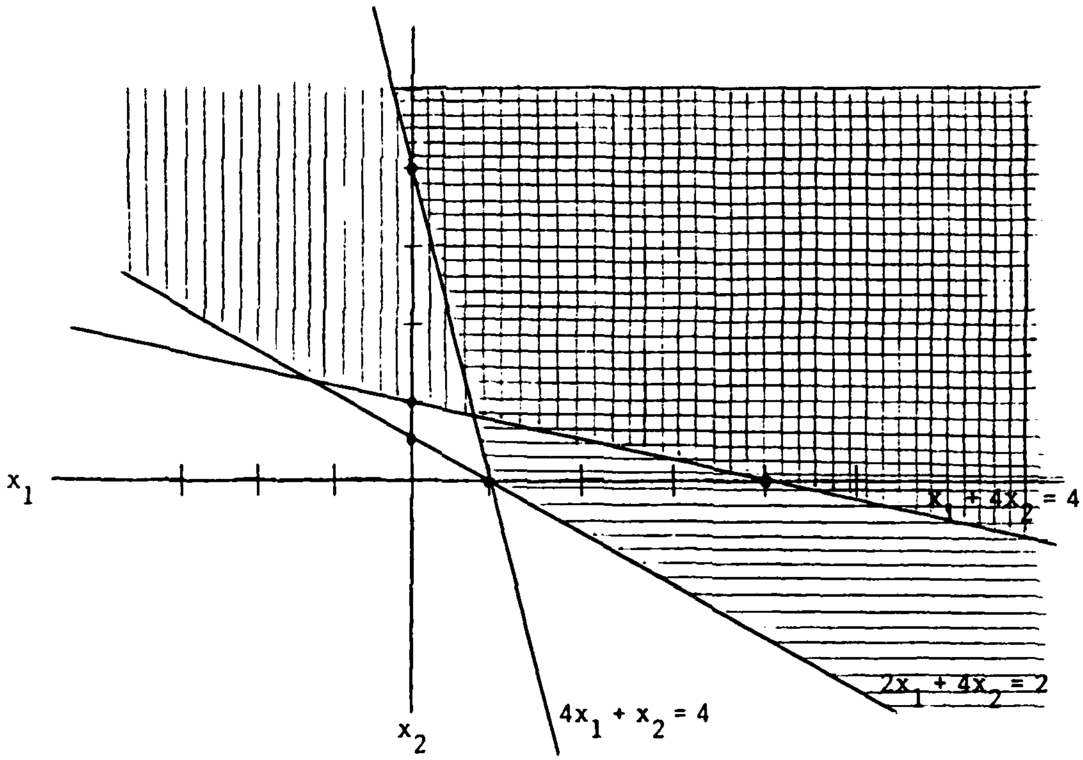


Figure 2.1 Regions generated by AI functions.

We use the notation $\underline{Y} = (Y_1, Y_2) \equiv (h^1(\underline{a}, \underline{X}), h^2(\underline{X}, \underline{b}))$ and $\underline{Y}' = (Y_1', Y_2') \equiv (h^1(\underline{a}', \underline{X}), h^2(\underline{X}, \underline{b}'))$. Because $\underline{X}$ is exchangeable, $\underline{Y}$ and $\underline{Y}'$ have the same marginals. In other words the distributions of the marginals of $\underline{Y}$ are unaffected by a permutation of the components of $\underline{a}$ or $\underline{b}$, although their joint distribution may be altered.

Corollary 2.2 says that (Y_1', Y_2') is in a sense <u>more positively quadrant dependent</u> than (Y_1, Y_2). See Lehmann (1966) and Barlow and Proschan (1981) for concepts of dependence.

We might say that the bivariate vector (U_1', U_2') is <u>more positively associated</u> than (U_1, U_2) if U_i' and U_i have the same distribution for $i = 1, 2$ and

$$\mathrm{cov}(\phi_1(U_1'), \phi_2(U_2')) \geq \mathrm{cov}(\phi_1(U_1), \phi_2(U_2))$$

for every pair ϕ_1, ϕ_2 of nondecreasing functions. (See Esary, Proschan and Walkup (1967) for more on association). Then Corollary 2.1 implies that (Y_1', Y_2') is more positively associated than (Y_1, Y_2) since for any nondecreasing ϕ_1 and ϕ_2,

$$\mathrm{cov}(\phi_1(Y_1'), \phi_2(Y_2')) - \mathrm{cov}(\phi_1(Y_1), \phi_2(Y_2)) = E(\phi_1(Y_1')\phi_2(Y_2')) - E(\phi_1(Y_1)\phi_2(Y_2)).$$

<u>Remark 2.6</u>. Corollary 2.1 yields moment inequalities when $\underline{X}$ is exchangeable taking values in $[0, +\infty)^n$. Suppose that $\underline{X}$ is of this type and let us use the notation of Remark 2.5 where however $(\underline{a}, \underline{b})$, $(\underline{a}', \underline{b}') \in [0, +\infty)^n \times [0, +\infty)^n$ and h^1 and h^2 take nonnegative values.

Let $\phi_i(y) = y^{m_i}$ for $y \geq 0$ and m_i a positive integer, $i = 1, 2$. Then Corollary 2.1 implies that when $(\underline{a}, \underline{b}) \overset{a}{\leq} (\underline{a}', \underline{b}')$,

$$E(Y_1'^{m_1} Y_2'^{m_2}) \geq E(Y_1^{m_1} Y_2^{m_2}) \quad \text{for all } m_i, m_2 \geq 1.$$

3. Arrangement Increasing Probabilities for AI families of Densities.

Many families of multivariate densities $\{f_{\underline{\lambda}}(\underline{x})\}$ have the property that the function $\phi(\underline{\lambda}, \underline{x}) = f_{\underline{\lambda}}(\underline{x})$ is arrangement increasing in the parameter $\underline{\lambda}$ and the outcome $\underline{x}$. The multinomial

$$\phi_1(\underline{\lambda}, \underline{x}) = N! \prod_{i=1}^{n} \frac{\lambda_i^{x_i}}{x_i!} \qquad 0 < \lambda_i, \ x_i = 0, 1, \ldots i = 1, \ldots, n \text{ and } N > 0$$

and the multivariate normal distribution with common variance and common covariance

$$\phi_2(\underline{\lambda}, \underline{x}) = (2\pi)^{-\frac{n}{2}} |\Sigma|^{-\frac{1}{2}} \ e^{-\frac{1}{2}(\underline{x}-\underline{\lambda})\Sigma^{-1}(\underline{x}-\underline{\lambda})'}$$

(where Σ is the positive definite covariance matrix with σ^2 on the diagonal and $\rho\sigma^2$ elsewhere and $\rho > \frac{-1}{n-1}$)

are but two such examples. For many other examples see Hollander, Proschan and Sethuraman (1977).

The following corollary of Theorem 1.3 enables one to construct many AI functions of a geometric type for AI families of densities.

Corollary 3.1. Let $\{f_{\underline{\lambda}}(\underline{x})\}$ be an AI family of probability densities (or mass functions). Suppose h is an AI function and that c is an arbitrary constant. Then

$$P_{\underline{\lambda}, \underline{a}}(\underline{X}) = \text{Prob}(h(\underline{a}, \underline{x}) \geq c: \ \underline{X} \text{ has density } f_{\lambda}(\underline{x}))$$

is an AI function of $\underline{\lambda}$ and $\underline{a}$.

Proof: This follows from Theorem 1.3 by letting $g_1(\underline{a}, \underline{x}) = I_{\{h(\underline{a},\underline{x}) \geq c\}}$, $g_2(\underline{x}, \underline{\lambda}) = f_{\underline{\lambda}}(\underline{x})$ and u either Lebesgue measure or an appropriate counting measure on a discrete set.

We now illustrate Corollary 3.1 with diverse examples.

Example 3.2. We assume that $\underline{X}$ is a multivariate random vector with density given by $f_{\underline{\lambda}}(\underline{x})$, and that the family $\{f_{\underline{\lambda}}(\underline{x})\}$ is an AI family of densities.

1. $F_{\underline{\lambda}}(\underline{a}) = \mathrm{Prob}_{\underline{\lambda}}(X_i \le a_i: \ i = 1, \ldots, n)$ and $\bar{F}_{\underline{\lambda}}(\underline{a}) = \mathrm{Prob}_{\lambda}(X_i > a_i: \ i = 1, \ldots, n)$ are AI in $\underline{\lambda}$ and $\underline{a}$. (See Hollander, Proschan and Sethuraman (1977)).

2. $\mathrm{Prob}_{\underline{\lambda}}(\sum_{i=1}^{n} a_i X_i \ge c)$ is AI in $\underline{\lambda}$ and $\underline{a}$. As a special case, it follows that if $X_i > 0$ for all i and $\underline{\lambda}$, then $\mathrm{Prob}_{\underline{\lambda}}(\sum_{i=1}^{n} \frac{X_i}{a_i} \le 1)$ is AI in $\underline{\lambda}$ and $\underline{a}$, where $\underline{a} \in (0, +\infty)^n$.

3. $\mathrm{Prob}_{\underline{\lambda}}(\sum_{i=1}^{n} (X_i - a_i)^2 \le c)$ is AI in $\underline{\lambda}$ and $\underline{a}$. Hence for a given $\underline{\lambda}$, the probability that $\underline{X}$ lies in a sphere of radius $\sqrt{c}$ with center $\underline{a} = (a_1, \ldots, a_n)$, increases as the order of the coordinates of $\underline{a}$ becomes more similar to the order of coordinates in $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n)$.

   Similarly if follows that

   $$\mathrm{Prob}_{\underline{\lambda}}(\sum_{i=1}^{n} |X_i - a_i| \le c) \text{ and } \mathrm{Prob}_{\underline{\lambda}}(|X_i - a_i| \le c: \ i = 1, \ldots, n)$$

   are both AI in $\underline{\lambda}$ and $\underline{a}$.

4. If $\underline{X} \in [0, +\infty)^n$ with probability 1 for all $\underline{\lambda}$, then $\mathrm{Prob}_{\underline{\lambda}}(\sum_{i=1}^{n} X_i^2/a_i^2 \le c)$ is AI in $\underline{\lambda}$ and $\underline{a}$ where $\underline{a} \in (0, +\infty)^n$.

5. $\text{Prob}_{\underline{\lambda}}\left(\prod_{i=1}^{n}(X_i - a_i)^+ \geq c\right)$ is AI in $\underline{\lambda}$ and $\underline{a}$. The boundary of the above region for the 2 dimensional case is a hyperbola, illustrated in Figure 3.1.

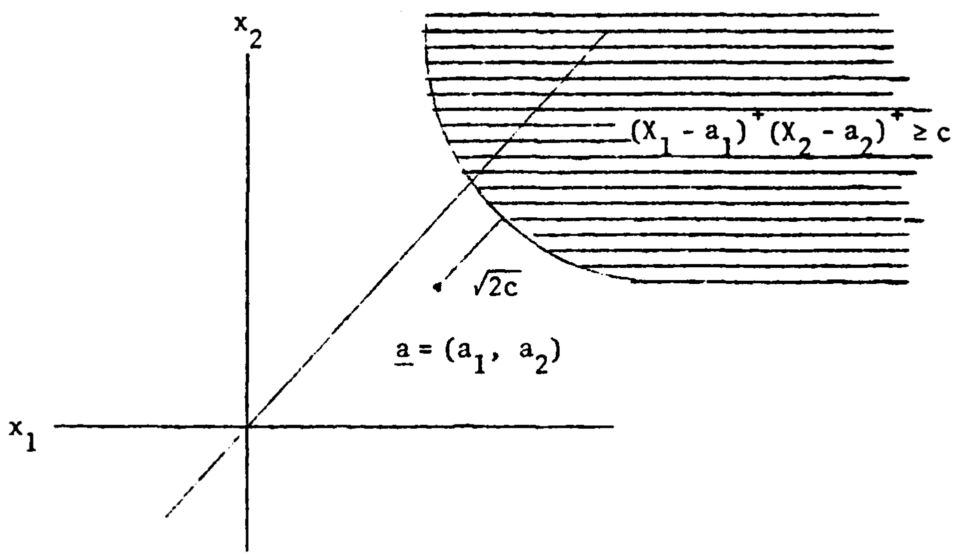


Figure 3.1 Hyperbola

<u>Example 3.3.</u> Suppose that $\{f_{\underline{\lambda}}(\underline{x})\}$ is an AI family of densities where each $f_{\underline{\lambda}}(\underline{x})$ has support in $[0, +\infty)^n$. Now $\prod_{i=1}^{n} X_i^{m_i}$ is an AI function $\underline{m}$ and $\underline{x}$, and hence a further application of Theorem 1.3 yields that $E_{\underline{\lambda}}\left(\prod_{i=1}^{n} X_i^{m_i}\right) = u_{\underline{\lambda}}^{m_1, \ldots, m_n}$ is an AI function of $\underline{\lambda}$ and $\underline{m}$.

Now $e^{\underline{x}\cdot\underline{t}} = e^{\sum_{i=1}^{n} x_i t_i}$ is also an AI function of $\underline{x}$ and $\underline{t}$. It is easy to see therefore that the multivariate Laplace transform

$$\text{MGF}_{\underline{X},\ \underline{\lambda}}(\underline{t}) \equiv M_{\underline{\lambda}}(\underline{t}) = \int e^{\underline{x}\cdot\underline{t}} f_{\underline{\lambda}}(\underline{x})\, d\underline{x}$$

is also an AI function in $\underline{\lambda}$ and $\underline{t}$.

References

Barlow, R. E. and Proschan, F. (1981). Statistical Theory of Reliability and Life Testing - Probability Models, To Begin With.

Boland, P. J. (1985). Integrating Schur Concave Densities on Rectangles, unpublished manuscript.

Esary, J. D., Proschan, F., and Walkup, D. W. (1967). Association of Random Variables, with Applications, Ann. Math. Statist. 38, 1466-1474.

Hollander, M., Proschan, F., and Sethuraman, J. (1977). Functions Decreasing in Transposition and Their Applications in Ranking Problems, Ann. Statist. 5, 722-733.

Lehmann, E. L. (1966). Some Concepts of Dependence, Ann. Math. Statist. 37, 1137-1153.

Marshall, A. and Olkin, I. (1979). Inequalities: Theory of Majorization and its Applications. Academic Press, New York.

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

AFOSR-TK- ~~AFOSR-85-102~~	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
TITLE (and Subtitle) MOMENT AND GEOMETRIC PROBABILITY INEQUALITIES ARISING FROM ARRANGEMENT INCREASING FUNCTIONS		5. TYPE OF REPORT & PERIOD COVERED Technical
		6. PERFORMING ORG. REPORT NUMBER
AUTHOR(s) Philip J. Boland, Frank Proschan, & Y.L. Tong		8. CONTRACT OR GRANT NUMBER(s) F49620-85-C-0007
PERFORMING ORGANIZATION NAME AND ADDRESS The Florida State University Department of Statistics Tallahassee, FL 32306-3033		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
CONTROLLING OFFICE NAME AND ADDRESS The U.S. Air Force Air Force Office of Scientific Research Bolling Air Force Base, DC 20332		12. REPORT DATE August, 1985
		13. NUMBER OF PAGES 10
MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report)
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

DISTRIBUTION STATEMENT (of this report)

distribution unlimited

7. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

SUPPLEMENTARY NOTES

KEY WORDS Arrangement increasing, decreasing in transposition, exchangeable random vector, family of arrangement increasing densities, inequalities, moments, Laplace transforms, permutation.

ABSTRACT (Continue on reverse side if necessary and identify by block number)

A real valued function g of two vector arguments $\underline{x}$ and $\underline{y} \in R^n$ is said to be arrangement increasing if it increases in value as the arrangement of components in $\underline{x}$ becomes increasingly similar to the arrangement of components in $\underline{y}$. Hollander, Proschan and Sethuraman (1977) show that the composition of arrangement increasing functions is arrangement increasing. This result is used to generate some interesting probability inequalities of a geometric nature for exchangeable random vectors. Other geometric inequalities for families of arrangement increasing multivariate densities are also given, and some moment inequalities are obtained.

END

FILMED

1-86

DTIC