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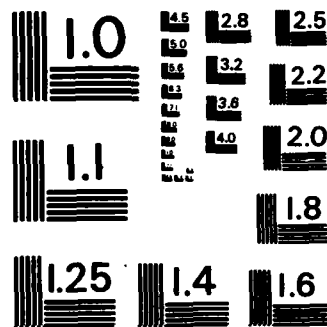
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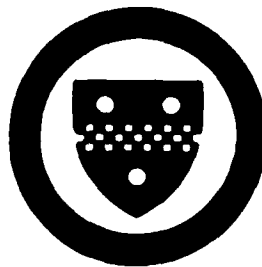
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AN INEQUALITY CONCERNING THE
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AND EMPIRICAL DISTRIBUTIONS*

Lin-Cheng Zhao
Center for Multivariate Analysis

Center for Multivariate Analysis

University of Pittsburgh



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August 1985

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1. INTRODUCTION

The result. Let $x_1, \dots, x_r$ be r points in R^d , and A be a class of Borel sets in R^d . Denote by $\Delta^A(x_1, \dots, x_r)$ the number of distinct sets in $\{\{x_1, \dots, x_r\} \cap A, A \in A\}$. Define

$$m^A(r) = \max_{x_1, \dots, x_r \in R^d} \Delta^A(x_1, \dots, x_r).$$

Vapnik and Chervonenkis (1971) showed that either $m^A(r) = 2^r$ for any positive integer r or $m^A(r) \leq r^s + 1$, where s is the smallest k such that $m^A(k) \neq 2^k$. A class of sets A for which the latter case holds will be called a V-C class with index s .

Suppose that μ is a probability measure on R^d . Let $X_1, X_2, \dots$ be a sequence of i.i.d. random vectors with common distribution μ , and μ_n be the empirical distribution of $X_1, \dots, X_n$. Denote a "distance" between μ_n and μ by

$$D_n(A, \mu) = \sup_{A \in A} |\mu_n(A) - \mu(A)|.$$

Throughout this paper we assume that $D_n(A, \mu)$, $\sup_{A \in A} |\mu_n(A) - \mu_{2n}(A)|$ and $\sup_{A \in A} \mu_n(A)$ are all random variables. We shall prove the following

Theorem 1. Let A be a V-C class with index s such that

$$\sup_{A \in A} \mu(A) \leq \delta \leq 1/8. \quad (1)$$

Then for any $\epsilon > 0$ we have

$$\begin{aligned} P\{D_n(A, \mu) > \epsilon\} &\leq 5(2n)^s \exp(-n\epsilon^2/(91\delta+4\epsilon)) \quad (2) \\ &\quad + 7(2n)^s \exp(-\delta n/68) \\ &\quad + 2^{2+s} n^{1+2s} \exp(-\delta n/8), \end{aligned}$$

provided $n \geq \max (12\sigma/\epsilon^2, 68(1+s)(\log 2)/\delta)$.

The proof of (2) is based on an important inequality proved by Devroye and Wagner (1980).

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2. HISTORICAL NOTES

A few remarks concerning this inequality are in order. In 1971, Vapnik and Chervonenkis proved that, for any $\epsilon > 0$

$$P\{D_n(A, \mu) > \epsilon\} \leq 4 \exp(-n\epsilon^2/8) E \Delta^A(x_1, \dots, x_{2n}). \quad (3)$$

This inequality is quite general since no restrictions such as (1) are imposed. In using this inequality, an estimate of $m^A(n)$ must be given, see, for example, Gaenssler and Stute (1979), Wenocur and Dudley (1981).

The weakness of (3) lies in the fact that, in many applications $\epsilon = \epsilon_n \rightarrow 0$ as $n \rightarrow \infty$. In this case $n\epsilon_n^2$ may not tend to ∞ or tend to ∞ very slowly. For this reason, the inequality proved by Devroye and Wagner (1980) is sometimes more useful. They proved that, if $\sup_A \mu(A) \leq \delta \leq \frac{1}{2}$, then for any $\epsilon > 0$

$$\begin{aligned} P\{D_n(A, \mu) > \epsilon\} &\leq 4m^A(2n) \exp(-n\epsilon^2/(64\delta+4\epsilon)) \\ &\quad + 2P\{\sup_A \mu_{2n}(A) > 2\delta\} \end{aligned} \quad (4)$$

for $n \geq 8\delta/\epsilon^2$. If we further have

$$\sup_{A \in \mathcal{A}} \sup_{x, y \in A} ||x - y|| \leq \rho < \infty$$

and

$$\sup_{x \in R^d} \mu(S(x, \rho)) \leq \delta \leq \frac{1}{2}, \quad (5)$$

here $||\cdot||$ is the L_2 or L_∞ norm in R^d , and $S(x, \rho)$ is the closed ball with radius ρ

centered at x , then

$$\begin{aligned} P\{D_n(A, \mu) > \epsilon\} &\leq 4m^A(2n) \exp(-n\epsilon^2/(64\delta+4\epsilon)) \\ &+ 4n \exp(-n\delta/10) \end{aligned} \quad (6)$$

for $n \geq \max(1/\delta, 8\delta/\epsilon^2)$.

This inequality is most useful when A is the class of balls with the same diameter (norm L_2 or L_∞). Otherwise δ may be much larger than $\sup_A \mu(A)$, and (6) gives no improvement over (3). Chen and Zhao (1984) made an essential improvement in the one-dimensional case:

Let A be a class of intervals in R^1 , satisfying $\sup_{I \in A} \mu(I) \leq \delta \leq 1$.

Then there exists positive absolute constants $C_0, C_1, \dots, C_4$ such that for any $\epsilon > 0$

$$\begin{aligned} P\{\sup_{I \in A} |\mu_n(I) - \mu(I)| > \epsilon\} \\ \leq C_1 \epsilon^{-1/\sqrt{\delta/n}} \exp(-C_2 n \epsilon^2 / \delta) + C_3 \exp(-C_4 n \epsilon), \end{aligned} \quad (7)$$

provided $n/\log n > C_0/\epsilon$.

The proof of (7) relies on a result concerning the strong approximation to Brownian bridge of the empirical process on R^1 . The argument fails in the general case $d > 1$. The inequality (2), to be proved in the next section, gives a satisfactory generalization to the case $d \geq 1$.

3. PROOF OF THEOREM 1

Set

$$\delta_j = 2^{2^{-1}} + 2^{-2} + \dots + 2^{-j} \delta, \quad j=1,2,\dots,r,$$

where r will be chosen later. Then

$$\delta < \delta_1 < \delta_2 < \dots < \delta_r < 2\delta \leq \frac{1}{4}$$

When $n \geq 12\delta/\epsilon^2$ we have $n \geq 8\delta_1/\epsilon^2$. From (4), the definition of V-C class and the fact that

$$\sup_A \mu(A) \leq \delta_1 \leq \frac{1}{4},$$

it follows that

$$\begin{aligned} P\{D_n(A, \mu) > \epsilon\} &\leq 4\{(2n)^s + 1\} \exp(-n\epsilon^2/(64\delta_1 + 4\epsilon)) \\ &\quad + 2P\{\sup_A \mu_{2n}(A) > 2\delta_1\} \\ &\leq 5(2n)^s \exp(-n\epsilon^2/(64\sqrt{2}\delta + 4\epsilon)) + 2P\{D_{2n}(A, \mu) > \delta_1\}, \end{aligned}$$

provided $n \geq 12\delta/\epsilon^2$.

When $\delta n \geq 68(1+s)\log 2$, we have $2^{j-1}n \geq 8\delta_j/\delta_{j-1}^2$ for $j = 2, 3, \dots, r$. As before, from (4) and $\sup_A \mu(A) \leq \delta_2 \leq \frac{1}{4}$, it follows that

$$\begin{aligned} P\{D_n(A, \mu) > \epsilon\} &\leq 5(2n)^s \exp(-n\epsilon^2/(91\delta + 4\epsilon)) \\ &\quad + (2 \cdot 5)(2 \cdot 2n)^s \exp(-2n\delta_1^2/(64\delta_2 + 4\delta_1)) \\ &\quad + 2^2 P\{D_{2^2 n}^2(A, \mu) > \delta_2\}, \end{aligned}$$

provided $n \geq \max(68(1+s)\log 2/\delta, 12\delta/\epsilon^2)$.

Using (4) and $\sup_A \mu(A) \leq \delta_j \leq \frac{1}{4}$ repeatedly, we obtain

$$\begin{aligned} P\{D_n(A, \mu) > \epsilon\} &\leq 5(2n)^s \exp(-n\epsilon^2/(91\delta+4\epsilon)) \\ &+ \sum_{j=1}^{r-1} 2^j \cdot 5(2^j, 2n)^s \exp(-2^j n \delta_j^2 / (68\delta_{j+1})) \\ &+ 2^r P\{D_{2^r n}(A, \mu) > \delta_r\} \stackrel{\Delta}{=} J_{1,n} + J_{2,n} + J_{3,n}, \end{aligned} \quad (8)$$

provided $n \geq \max(68(1+s)\log 2/\delta, 12\delta/\epsilon^2)$.

It is easy to see that

$$2^j \delta_j^2 / \delta_{j+1} \geq 2j\delta, \quad j=1, \dots, r-1. \quad (9)$$

Hence it follows from (8), (9) and $2^{1+s} \leq e^{\delta n/68}$ that

$$\begin{aligned} J_{2,n} &\leq 5(2n)^s \sum_{j=1}^{r-1} 2^{(1+s)j} \cdot \exp(-2^j n \delta_j^2 / (68\delta_{j+1})) \\ &\leq 5(2n)^s \sum_{j=1}^{\infty} (2^{1+s})^j \exp(-2j\delta n/68) \\ &\leq 5(2n)^s \sum_{j=1}^{\infty} \exp(-j\delta n/68) \\ &= 5(2n)^s e^{-\delta n/68} (1 - e^{-\delta n/68})^{-1} \\ &\leq 5(2n)^s (1 - 2^{-(1+s)})^{-1} e^{-\delta n/68} \\ &\leq 7(2n)^s \exp(-\delta n/68), \end{aligned}$$

where $s \geq 1$ is invoked.

When $\delta n \geq 68(1+s)\log 2$, we have $2^r n \delta_r \geq 2$. By (3)

$$J_{3,n} \leq 2^{r+1}((2^{r+1}n)^{s+1}) \exp(-2^r n \delta_r^2/8). \quad (11)$$

Take $r = r_n$ to be an integer such that $n/2 < 2^r \leq n$. When $\delta n \geq 68(1+s)\log 2$, we have $n^2 \delta_r^2 \geq 2$, $n \delta_r \geq \sqrt{2}$ and $n \delta_r^2 \geq 2\delta$. By (11) we have

$$\begin{aligned} J_{3,n} &\leq 2n((2n^2)^{s+1}) \exp(-n^2 \delta_r^2/16) \\ &\leq 4n(2n^2)^s \exp(-\delta n/8). \end{aligned} \quad (12)$$

Formula (2) follows from (8), (10) and (12). The theorem is proved.

4. APPLICATIONS

Theorem 1 has some applications in strong convergence problems involving the uniform deviation between frequencies and probabilities of a class of events. As an example, we consider the nearest neighbor (NN) density estimates proposed by Loftsgarden and Quesenberry (1965). Suppose that X is a R^d -valued random vectors with distribution μ and unknown density function f . The so called NN estimate of $f(x)$ has the form

$$\hat{f}_n(x) = k / \{n(2a_n(x))^d\}, \quad x = (x^{(1)}, \dots, x^{(d)}) \in R^d, \quad (13)$$

where $k = k_n \leq n$ is a positive integer chosen in advance, $a_n(x)$ is the smallest $a > 0$ such that the cube $[x-a, x+a] = \prod_{i=1}^d [x^{(i)}-a, x^{(i)}+a]$ contains at least k sample points. As an application of Theorem 1, we prove a theorem about the convergence rate of $\sup_{x \in R^d} |\hat{f}_n(x) - f(x)|$

In the sequel, we use $c, \alpha, c_1, c_2, \dots$ for some positive constants independent of n and x . For $x = (x^{(1)}, \dots, x^{(d)}) \in R^d, y = (y^{(1)}, \dots, y^{(d)}) \in R^d$, write $f'(x)(y-x) = \sum_{i=1}^d \frac{\partial f}{\partial x^{(i)}}(x) (y^{(i)} - x^{(i)})$, and take $\|y - x\| = \max_{1 \leq i \leq d} |y^{(i)} - x^{(i)}|$.

We say that the density function f belongs to λ -class for some $\lambda \in (0, 2]$, if $\lambda \in (0, 1]$ and $|f(y) - f(x)| \leq C \|y - x\|^\lambda$ for any $x, y \in R^d$, or $\lambda \in (1, 2]$ and, f are bounded and

$$|f(y) - f(x) - f'(x)(y-x)| \leq C \|y - x\|^\lambda$$

for any $x, y \in R^d$. We have

Theorem 2. Suppose that f belongs to λ -class for some $\lambda \in (0, 2]$. Take $k = o(n)$ and

$$k/n \geq \beta \left(\frac{\log n}{n} \right)^{(d+\lambda)/(d+3\lambda)} \quad (14)$$

where $\beta > 0$ is any given constant. Then

$$\limsup_{n \rightarrow \infty} \{ (n/k)^{\lambda/(d+\lambda)} \sup_x |\hat{f}_n(x) - f(x)| \} \leq C \text{ a.s.} \quad (15)$$

To prove this theorem, we need the following lemma. In the sequel, μ_n denotes the empirical measure of $X_1, \dots, X_n$. Besides, a cube of the form $[x-a, x+a]$ is called a regular cube.

Lemma 3. Let A be a class of regular cubes satisfying the measurability conditions mentioned in paragraph 1 and the condition

$$\sup_{A \in \mathcal{A}} \mu(A) \leq k/n \leq 1/8.$$

Take $k = o(n)$ and

$$k/n \geq \beta \left(\frac{\log n}{n} \right)^{1/(1+2r)}, \quad (16)$$

where $r > 0$ and $\beta > 0$ is any given constant. Then

$$\limsup_{n \rightarrow \infty} \{ \left(\frac{n}{k} \right)^{1+r} \sup_{A \in \mathcal{A}} |\mu_n(A) - \mu(A)| \} \leq C_1 \text{ a.s.}$$

Notice that $\mathcal{A}$ is a V-C class, one can obtain Lemma 3 from Theorem 1 immediately. The proof is omitted.

Proof of Theorem 2. Take $k = o(n)$ and

$$k/n \geq \beta \left(\frac{\log n}{n} \right)^{(d+\lambda)/(d+3\lambda)}$$

Put

$$V_n = \theta_1^{-1} (k/n)^{\lambda/(d+\lambda)}$$

$$q_n = \theta_2 V_n = \theta_1^{-1} \theta_2 (k/n)^{\lambda/(d+\lambda)}$$

$$B_n = \{x: f(x) \geq V_n\}$$

where $\theta_1, \theta_2 \in (0,1)$ will be chosen later.

Let $\mu(x,a)$ and $\mu_n(x,a)$ be the probability measure and empirical measure of $[x-a, x+a]$ respectively. Put $M = \max(\sup_x f(x), 1)$. We have

$$P\{\sup_{x \in B_n} |\hat{f}_n(x) - f(x)| > q_n\} \leq I_n + J_n \quad (17)$$

where

$$I_n = P(U_{x \in B_n} \{\hat{f}_n(x) > f(x) + q_n\}), \quad (18)$$

$$J_n = P(U_{x \in B_n} \{\hat{f}_n(x) < f(x) - q_n\}).$$

Thus

$$I_n \leq P(U_{x \in B_n} \{a_n(x) < b_n(x)\}), \quad (19)$$

where

$$2b_n(x) = \left\{ \frac{k}{nf(x)} (1 + q_n/f(x))^{-1} \right\}^{1/d}.$$

Fix $x \in B_n = \{x: f(x) \geq V_n\}$. Take $\theta_2 < 1/8$, then $q_n/f(x) \leq \theta_2 < 1/8$. Noticing $1/(1+t) < 1 - 7t/8$ for $0 \leq t < 1/8$, we have

$$\begin{aligned} 2b_n(x) &\leq \left\{ \frac{k}{nf(x)} (1 - 7q_n/8f(x)) \right\}^{1/d} \\ &\leq (k/nf(x))^{1/d}. \end{aligned}$$

It follows that

$$\mu(x, b_n(x)) = \int_{x-b_n(x)}^{x+b_n(x)} f(t) dt$$

$$\begin{aligned}
&\leq (2b_n(x))^d f(x) + C_2 (2b_n(x))^{d+\lambda} \\
&= (2b_n(x))^d f(x) [1 + C_2 (2b_n(x))^\lambda / f(x)] \\
&\leq \frac{k}{n} (1 - \frac{7}{8} q_n / f(x)) (1 + C_2 (\frac{k}{nf(x)})^{\lambda/d} / f(x)) \\
&\leq \frac{k}{n} (1 - \frac{7}{8} q_n / f(x) + C_2 (\frac{k}{nf(x)})^{\lambda/d} / f(x)).
\end{aligned}$$

Fix θ_2 , take θ_1 small enough such that $C_2 \theta_1^{(\lambda+d)/d} < \frac{3}{8} \theta_2$, then $C_2 (\frac{k}{nf(x)})^{\lambda/d}$
 $\leq C_2 \theta_1^{\lambda/d} (k/n)^{\lambda/(\lambda+d)} < \frac{3}{8} \theta_1^{-1} \theta_2 (k/n)^{\lambda/(\lambda+d)} = \frac{3}{8} q_n$. It follows that

$$\mu(x, b_n(x)) \leq \frac{k}{n} (1 - \frac{1}{2} q_n / f(x)) < k/n,$$

and

$$\frac{k}{n} - \mu(x, b_n(x)) \geq k q_n / (2nM).$$

Hence, by (19) and Theorem 1, we have

$$\begin{aligned}
I_n &\leq P\{\sup_{x \in B_n} (\mu_n(x, b_n(x)) - \mu(x, b_n(x))) \geq k q_n / (2nM)\} \\
&\leq C_5 n^\alpha \{ \exp(-\frac{n(k q_n / 2nM)^2}{91k/n + 2k q_n / nM}) + \exp(-k/68) \}
\end{aligned}$$

where α is a constant depending only on d . In view of (14), we have for large n

$$\begin{aligned}
I_n &\leq C_5 n^\alpha \{ \exp(-\theta_1^{-1} \theta_2^2 M^{-2} 1 + 2\lambda/(d+\lambda)) \log n / 400) \\
&\quad + \exp(-k/68) \}.
\end{aligned}$$

Take θ_1 small enough, we have

$$\sum I_n < \infty. \quad (20)$$

In the same way, we can take θ_1 and θ_2 such that

$$\sum J_n < \infty \quad (21)$$

By (17), (18), (20) and (21), we have

$$\sum P\{q_n^{-1} \sup_{x \in B_n} |\hat{f}_n(x) - f(x)| > 1\} < \infty.$$

By Borel-Cantelli's lemma,

$$\lim_{n \rightarrow \infty} \sup_{x \in B_n} \{q_n^{-1} \sup_{x \in B_n} |\hat{f}_n(x) - f(x)|\} \leq 1 \text{ a.s.} \quad (22)$$

Fix θ_1, θ_2 , and take $2b_n = C_3(k/n)^{1/(d+\lambda)}$. Fix $x \in B_n^C = \{x: f(x) < V_n\}$. With small C_3 we have

$$\begin{aligned} \mu(x, b_n) &= \int_{x-b_n}^{x+b_n} f(t) dt \\ &\leq (2b_n)^d f(x) + C_2(2b_n)^{d+\lambda} \\ &\leq \frac{k}{n} [\theta_1^{-1} C_3^d + C_2 C_3^{d+\lambda}] < k/2n < k/n. \end{aligned}$$

Taking $r = \lambda/(d+\lambda)$ in Lemma 3, we can assert with probability one that, for n large enough, the inequality

$$\begin{aligned} \mu_n(x, b_n) &\leq \mu(x, b_n) + 2C_1(k/n)^{(d+2\lambda)/(d+\lambda)} \\ &< k/2n + 2C_1(k/n)^{(d+2\lambda)/(d+\lambda)} < k/n \end{aligned}$$

holds uniformly for $x \in B_n^C$. By definition, for $x \in B_n^C$,

$$a_n(x) \geq b_n = \frac{1}{2} C_3 (k/n)^{1/(d+\lambda)},$$

$$\hat{f}_n(x) \geq C_4 (k/n)^{\lambda/(d+\lambda)}$$

It follows that

$$\lim_{n \rightarrow \infty} \sup \{ (n/k)^{\lambda/(d+\lambda)} \sup_{x \in B_n^c} |\hat{f}_n(x) - f(x)| \} \leq C_4 \text{ a.s.} \quad (23)$$

Theorem 2 is proved in view of (22) and (23).

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