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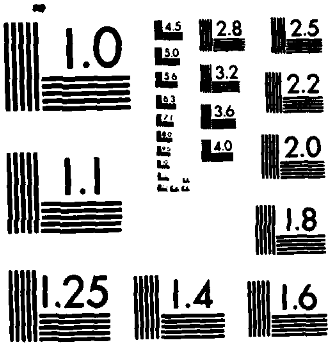

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The Asymptotic Behavior of an Exponential-Type Series

by
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OCTOBER 1985

NAVAL WEAPONS CENTER
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FOREWORD

This report describes mathematical properties of a function arising in a laser backscattering study. The work was performed at the Naval Weapons Center, China Lake, Calif., during 1985 under Program Element 61152N, Task Area ZR000-01-01, Work Unit 138070.

The report has been reviewed for technical accuracy by D. T. Gillespie.

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(U) The limit of a certain function of three variables is shown to be exponential in one variable and independent of the other two. The function is a generalization of an infinite series arising in the calculation of backscattering of sharp laser pulses in an infinite cloud of isotropic scatterers. Convexity of the function in a particular region enables the evaluation of the asymptotic behavior without extensive algebraic manipulations.

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1. INTRODUCTION

The purpose of this report is to show that

$$\lim_{x\to\infty}\left[e^{-x}E(x,a)\right] = 1 \quad \text{for} \quad -1 \le a \le 1 \tag{1}$$

where

$$E(x,a) \equiv \sum_{k=0}^{\infty}\left(\frac{k+1}{x}\right)^{a}\frac{x^k}{k!} \quad \text{for} \quad x > 0 \text{ and all } a$$

More specifically, we will show that

$$e^x + a \le E(x,a) \le e^x + \frac{a}{x}e^x \quad \text{for} \quad 0 \le a \le 1 \tag{2}$$

and

$$e^x + \frac{a}{x}e^x \le E(x,a) \le e^x + a \quad \text{for} \quad -1 \le a \le 0 \tag{3}$$

As a corollary to Equation 1 we have

$$\lim_{x\to\infty}\left[x^{-\frac{1}{2}}e^{-x}S(x)\right] = 1 \tag{4}$$

where

$$S(x) \equiv \sum_{k=0}^{\infty}(k+1)^{\frac{1}{2}}\frac{x^k}{k!} = x^{\frac{1}{2}}E(x,\tfrac{1}{2})$$

Indeed it was an investigation of the asymptotic behavior of $S(x)$ that led to this study.

The series $S(x)$ arose in a calculation of time-dependent backscattering of sharp laser pulses in an infinite cloud of isotropic scatterers (Reference 1). More specifically, letting x denote the time (suitably nondimensionalized) after pulse emission, it turns out that the intensity of the total backscattered signal at times $x >> 1$ is approximately proportional to $x^{-2}e^{-\alpha x}S(x)$ where $\alpha \ge 1$. A sensible interpretation of this result evidently requires a knowledge of the behavior of $S(x)$ for large x. The limit (Equation 4) implies that the backscattered intensity at times $x >> 1$ is proportional to $x^{-3/2}e^{-(\alpha-1)x}$.

In Section 2 we prove Equations 2 and 3 by considering $E(x,\alpha)$ as a function of α for fixed $x > 0$. This turns out to be a convex function which is easily evaluated at $\alpha = -1, 0$ and 1. These values together with the convexity property yield Equations 2 and 3.

In Section 3 we prove the slightly improved inequality

$$E(x,\tfrac{1}{2}) \leq (1+x^{-1})^{\frac{1}{2}}e^{x}$$

This results from manipulating the series for $(E(x,\frac{1}{2}))^2$.

In Section 4 $E(x,\alpha)$ is generalized by the addition of one parameter, and the corresponding generalizations of Equations 1 through 3 are proved. Throughout it will be assumed that $x > 0$.

2. BOUNDS FOR $E(x,\alpha)$

For all α and fixed $x > 0$, the series representation for $E(x,\alpha)$ can be differentiated term by term with respect to α due to uniform convergence.

$$E'(x,\alpha) = \sum_{k=0}^{\infty}\left(\frac{k+1}{x}\right)^{\alpha}\ln\left(\frac{k+1}{x}\right)\frac{x^k}{k!}$$

$$E''(x,\alpha) = \sum_{k=0}^{\infty}\left(\frac{k+1}{x}\right)^{\alpha}\left[\ln\left(\frac{k+1}{x}\right)\right]^2\frac{x^k}{k!}$$

$E(x,\alpha)$ is a convex function of α since $E''(x,\alpha) > 0$ for all α. It is easy to show that $E(x,-1) = e^x - 1$ and $E(x,0) = e^x$. The following steps show the less obvious evaluation of $E(x,1)$:

$$E(x,1) = \sum_{k=0}^{\infty}\left(\frac{k+1}{x}\right)\frac{x^k}{k!}$$

$$= \frac{1}{x} + \sum_{k=1}^{\infty}\left(1+\frac{1}{k}\right)\frac{x^{k-1}}{(k-1)!}$$

$$= \frac{1}{x} + \sum_{k=1}^{\infty}\frac{x^{k-1}}{(k-1)!} + \frac{1}{x}\sum_{k=1}^{\infty}\frac{x^k}{k!}$$

$$= \frac{1}{x} + e^x + \frac{1}{x}(e^x - 1)$$

$$= e^x + \frac{1}{x}e^x$$

For fixed x we have points P(-1), P(0), and P(1) on the curve $E = E(x,\alpha)$ in the αE - plane:

$$P(-1) = (-1, e^x - 1)$$

$$P(0) = (0, e^x)$$

$$P(1) = (1, e^x + e^x/x)$$

Letting L_k denote the line joining P(k $-$ 1) and P(k) for k = 0, 1, we obtain the following equations for L_k:

$$L_0\colon\ E = e^x + \alpha$$

$$L_1\colon\ E = e^x + \alpha e^x/x$$

Since $E(x,\alpha)$ is convex and the three points P(k) lie on the curve $E = E(x,\alpha)$, the curve must lie between L_0 and L_1 in the interval $-1 \le \alpha \le 1$. This proves Equations 2 and 3.

3. AN IMPROVED UPPER BOUND

For the case $\alpha = \frac{1}{2}$, the upper bound in Equation 2 can be slightly improved by computing $(E(x,\frac{1}{2}))^2$.

$$(E(x,\tfrac{1}{2}))^2 = \sum_{k=0}^{\infty}\sum_{r=0}^{k}\left(\frac{r+1}{x}\right)^{\frac{1}{2}}\frac{x^r}{r!}\left(\frac{k-r+1}{x}\right)^{\frac{1}{2}}\frac{x^{k-r}}{(k-r)!}$$

$$= \sum_{k=0}^{\infty}\frac{x^{k-1}}{k!}\sum_{r=0}^{k}\binom{k}{r}[(r+1)(k-r+1)]^{\frac{1}{2}}$$

For $0 \le r \le k$,

$$(r+1)(k-r+1) \le \left(\frac{k+2}{2}\right)^2$$

with equality holding for $r = k/2$. Therefore,

$$\left(E(x,\tfrac{1}{2})\right)^2 < \sum_{k=0}^{\infty} \frac{x^{k-1}}{k!}\left(\frac{k+2}{2}\right)\sum_{r=0}^{k}\binom{k}{r}$$

$$= \sum_{k=0}^{\infty} \frac{x^{k-1}}{k!}\left(\frac{k+2}{2}\right)2^k = \sum_{k=0}^{\infty} \frac{(2x)^{k-1}}{k!}(k+2)$$

$$= \frac{1}{x} + \sum_{k=1}^{\infty} \frac{(2x)^{k-1}}{(k-1)!} + \frac{1}{x}\sum_{k=1}^{\infty} \frac{(2x)^k}{k!}$$

$$= \frac{1}{x} + e^{2x} + \frac{1}{x}(e^{2x}-1) = (1+\frac{1}{x})\, e^{2x}$$

It follows that

$$E(x,\tfrac{1}{2}) \leq \left(1 + \frac{1}{x}\right)^{\frac{1}{2}} e^x < \left(1 + \frac{1}{2x}\right)e^x$$

4. A GENERALIZATION

Let $E_A(x,a)$ be defined by

$$E_A(x,a) = \sum_{k=0}^{\infty}\left(\frac{k+A}{x}\right)^a \frac{x^k}{k!}$$

for $A \geq 1$ and all a, so that $E(x,a) = E_1(x,a)$.

For fixed A and x, $E_A(x,a)$ is a convex function of a, assuming the following values at $a = -1$, 0, and 1:

$$E_A(x,-1) = e^x - F_A(x)$$
$$E_A(x,0) = e^x$$
$$E_A(x,1) = e^x + Ae^x/x$$

where

$$F_A(x) = (A-1)\sum_{k=0}^{\infty}\left(\frac{1}{k+A-1}\right)\frac{x^k}{k!}$$

Defining the three points $P(-1)$, $P(0)$, $P(1)$ and the lines L_0 and L_1 as in Section 2, we have the following equations:

$$L_0\colon\ E = e^x + \alpha F_A(x)$$
$$L_1\colon\ E = e^x + \alpha A e^x/x$$

The curve $E = E_A(x,\alpha)$ is bounded by L_0 and L_1 in the αE-plane. It follows, as in Section 2, that

$$e^x + \alpha F_A(x) \le E_A(x,\alpha) \le e^x + \frac{\alpha A}{x} e^x \quad \text{for } 0 \le \alpha \le 1$$

$$e^x + \frac{\alpha A}{x} e^x \le E_A(x,\alpha) \le e^x + \alpha F_A(x) \quad \text{for } -1 \le \alpha \le 0$$

and

$$\lim_{x\to\infty} \left| e^{-x} E_A(x,\alpha) \right| = 1 \quad \text{for } -1 \le \alpha \le 1 \text{ and } A \ge 1$$

We also obtain the following bound for $F_A(x)$:

$$F_A(x) \le \frac{A}{x} e^x \text{ for } A \ge 1$$

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1. D. T. Gillespie. "A Calculation of n-Scattered Lidar Returns for Large n in an Idealized Cloud," *J. Opt. Soc. Am. A.* (to be published).