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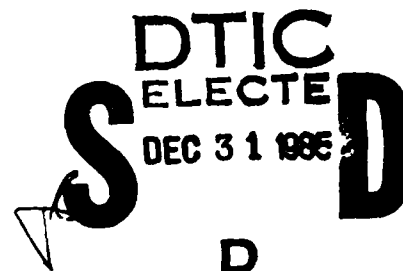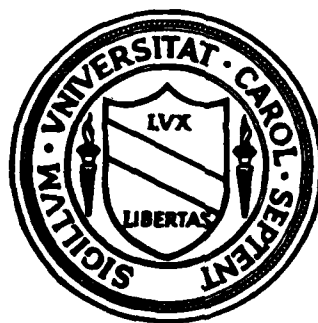
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A MARTINGALE CHARACTERIZATION
OF MIXED POISSON PROCESSES

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Abstract

It is shown that an elementary pure birth process is a mixed Poisson

process *if* the sequence of post-jump intensities forms a martingale

with respect to the σ-fields generated by the jump times of the process.

In this case, the post-jump intensities converge a.s. to the mixing random

variable of the process.

MIXED POISSON PROCESS, MARTINGALES, INTENSITIES

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1. Introduction

Mixed Poisson processes play an important role in many branches of applied probability, for instance in insurance mathematics and physics (see Albrecht (1985) and Pfeifer (1986) for recent surveys). They belong to the class of elementary pure birth processes $\{N(t); t \geq 0\}$ with standard transition probabilities

$$p_{nm}(s,t) = P(N(t) = m \mid N(s) = n), \quad 0 \leq n \leq m, \ 0 \leq s \leq t, \tag{1}$$

possessing right-continuous paths and positive and finite birth rates

$$\lambda_n(t) = \lim_{h \downarrow 0} \frac{1}{h} p_{n,n+1}(t,t+h), \quad n, t \geq 0, \tag{2}$$

and all finite-dimensional marginals of the jump-time sequence $\{T_n; n \geq 0\}$ are absolutely continuous with respect to Lebesgue measure (see Pfeifer (1982)). For such processes, the jump times form a Markov chain with transition probabilities

$$P(T_n > t \mid T_{n-1} = s) = \frac{1 - F_n(t)}{1 - F_n(s)}, \quad 0 \leq s \leq t, \ n \geq 1 \tag{3}$$

and initial distribution function F_0 where

$$1 - F_n(x) = \exp\left(-\int_0^x \lambda_n(u) \, du\right), \quad x \geq 0, \ n \geq 0 \tag{4}$$

(the F_n are in fact all cumulative distribution functions), hence all F_n

are absolutely continuous with densities f_n (say), and the conditional densities for the transition probabilities can be represented as

$$f_n(s \mid t) = \frac{f_n(s)}{1 - F_n(t)} \quad , \quad 0 \le t \le s, \, n \ge 1. \tag{5}$$

Moreover, the birth rates coincide with the hazard rates

$$\lambda_n(t) = \frac{f_n(t)}{1 - F_n(t)} \quad a.e., \quad n,t \ge 0. \tag{6}$$

If especially $\{N(t); \, t \ge 0\}$ is a mixed Poisson process, then also

$$\lambda_n(t) = \int_0^\infty x^{n+1} e^{-xt} \, dG(x) \bigg/ \int_0^\infty x^n e^{-xt} \, dG(x) \; , \quad n,t \ge 0, \tag{7}$$

where G is the cdf of the mixing random variable Λ (say). In fact, Lundberg (1940) has proved that such a representation characterizes the intensities of a mixed Poisson process.

In terms of random variables, a mixed Poisson process behaves like a homogeneous Poisson process with rate λ given $\Lambda = \lambda$, from which it also follows that

$$\lambda_n(t) = E(\Lambda \mid N(t) = n), \quad n,t \ge 0. \tag{8}$$

The following result completes some of Lundberg's (1940) results on the asymptotic behaviour of the intensities for mixed Poisson processes.

Lemma. Let $\{t_n; n \geq 1\}$ be a sequence of positive real numbers converging to $t > 0$ such that $\left|\frac{t_n}{t} - 1\right| = o(n^{-1/3})$, $n \to \infty$. Then, if $1/t$ is a point of increase of G, we have

$$\lim_{n \to \infty} \lambda_n(nt_n) = \frac{1}{t} . \qquad (9)$$

Proof. Let $\varepsilon_n > 0$ be chosen in such a way that $n\varepsilon_n^3 \to 0$, $n\varepsilon_n^2 \to \infty$ and $(\frac{t_n}{t} - 1)/\varepsilon_n \to 0$ for $n \to \infty$. From relation (7) it follows that

$$\lambda_n(nt_n) = \frac{1}{t_n} \int_0^\infty e^n(xt_n)^{n+1}e^{-nxt_n} \, dG(x) \Big/ \int_0^\infty e^n(xt_n)^n e^{-nxt_n} \, dG(x)$$

$$\sim \frac{1}{t_n} \int_{(1-\varepsilon_n)/t_n}^{(1+\varepsilon_n)/t_n} xt_n \exp(-\frac{n}{2}(1-xt_n)^2) \, dG(x) \Big/ \int_{(1-\varepsilon_n)/t_n}^{(1+\varepsilon_n)/t_n} \exp(-\frac{n}{2}(1-xt_n)^2) \, dG(x)$$

$$\sim \frac{1}{t_n} \quad \text{for } n \to \infty . \text{ This proves the Lemma.}$$

The above result is in general not true without further conditions on G as can e.g. be seen by mixing distributions concentrated in a single point $\lambda > 0$; here $\lambda_n(t) = \lambda$ for all n and t, λ being the only point of increase of G. As an example, consider a Pólya-Lundberg process where Λ follows a gamma distribution with mean $\mu > 0$ and variance $\alpha\mu^2$, $\alpha > 0$. Here

$$\lambda_n(t) = \mu \frac{1 + \alpha n}{1 + \mu\alpha t} , \qquad n, t \geq 0, \qquad (10)$$

from which the validity of (9) can be seen explicitly, for all $t > 0$.

2. The martingale characterization

Let $\{\lambda_n(T_{n-1}); n \geq 1\}$ denote the sequence of post-jump intensities.
In the light of (2), the post-jump intensities describe the transition behaviour
of the process immediately after a jump has occured. The following result
gives a characterization of mixed Poisson processes by a martingale property
of this sequence.

Theorem 1 Let $\{N(t); t \geq 0\}$ be an elementary pure birth process with intensities
$\{\lambda_n(t); n,t \geq 0\}$ and jump times $\{T_n; n \geq 0\}$. Let for $n \geq 1$ denote A_n the
σ-field generated by $T_0,\ldots,T_{n-1}$. Then $\{N(t); t \geq 0\}$ is a mixed Poisson process
iff the post-jump intensities $\{\lambda_n(T_{n-1}); n \geq 1\}$ form a martingale with respect
to $\{A_n; n \geq 1\}$.

Proof. Due to the Markov structure of jump times the martingale property of
the post-jump intensities is equivalent to

$$E(\lambda_{n+1}(T_n) \mid T_{n-1} = t) = \lambda_n(t) \quad \text{a.s. for all } n \geq 1 \tag{11}$$

which by (5) and (6) is in turn equivalent to

$$\int_t^\infty \frac{f_{n+1}(s)}{1-F_{n+1}(s)} \frac{f_n(s)}{1-F_n(t)} ds = \frac{f_n(t)}{1-F_n(t)} \quad \text{a.e.} \tag{12}$$

saying that the density f_n is differentiable a.e. with

$$f_n'(t) = - \frac{f_{n+1}(t)}{1-F_{n+1}(t)} f_n(t) \quad \text{a.e.} \tag{13}$$

or equivalently

$$\frac{d}{dt} \log(1-F_{n+1}(t)) = -\frac{f_{n+1}(t)}{1-F_{n+1}(t)} = \frac{f_n'(t)}{f_n(t)} = \frac{d}{dt} \log f_n(t) \quad \text{a.e.} \quad (14)$$

Integration of this last relation shows that there are constants $c_n > 0$ such that

$$1-F_{n+1}(t) = c_n f_n(t), \quad t \geq 0, \quad (15)$$

which in turn implies that f_n is absolutely continuous and the recursive formula

$$f_{n+1}(t) = - c_n f_n'(t) \quad (16)$$

holds everywhere on $[0,\infty)$. By induction, we see that all derivatives of f_n exist on $[0, \infty)$, and that for all $n \geq 1$,

$$f_n(t) = (-1)^n \prod_{k=0}^{n-1} c_k f_0^{(n)}(t), \quad t \geq 0. \quad (17)$$

Since by assumption, the intensities (and hence all f_n) are positive and finite, we have

$$(-1)^n f_0^{(n)}(t) \geq 0, \quad n,t \geq 0. \quad (18)$$

The density f_0 thus is completely monotonic on $[0,\infty)$, hence by Bernstein's (1928) theorem there is a bounded and non-decreasing right-continuous function H such that

$$f_0(t) = \int_0^\infty e^{-xt} \, dH(x), \quad t \geq 0. \quad (19)$$

In fact, since f_0 is a density, we have that $\frac{1}{x} dH(x) = dG(x)$ is a probability measure from which it follows that

$$f_n(t) = \prod_{k=0}^{n-1} c_k \int_0^\infty x^{n+1} e^{-xt} dG(x)$$

$$1-F_n(t) = \prod_{k=0}^{n-1} c_k \int_0^\infty x^n e^{-xt} dG(x), \qquad n,t \geq 0. \tag{20}$$

Hence relation (7) is satisfied, saying that $\{ N(t); t \geq 0 \}$ must be a mixed Poisson process with mixing distribution $dG(x)$.

Conversely, since every mixed Poisson process has intensities of the form (7), it is easily seen that relation (12) holds, hence the post-jump intensities possess the martingale property, which proves the theorem.

It should be pointed out that since $f_0(0) < \infty$ by our assumptions, the mixing random variable must be integrable with

$$E(\Lambda) = f_0(0). \tag{21}$$

A simple application of the Martingale Convergence Theorem (see e.g. Billingsley (1979)) then shows that the post-jump intensities converge a.s. to some integrable random variable since also

$$E(\lambda_1(T_0)) = \int_0^\infty \frac{f_1(t) f_0(t)}{1-F_1(t)} dt = \int_0^\infty - \frac{f_0'(t)}{f_0(t)} f_0(t) dt = f_0(0). \tag{22}$$

The question now is what the possible limits of the post-jump intensities are. The following result gives an answer to this.

Theorem 2. If Λ is the mixing random variable of the process, then the post-jump intensities converge a.s. to Λ .

Proof. For any mixed Poisson process, we have $(n+1)/T_n \to \Lambda$ a.s. by the strong law of large numbers, applied to the Poisson process with rate λ , conditionally on $\Lambda = \lambda$, and by the law of the iterated logarithm,

$$\left| \lambda \frac{T_n}{n+1} - 1 \right| = O\left(\sqrt{\frac{1}{n} \log\log n}\right) = o(n^{-1/3}) \quad \text{a.s. for } n \to \infty .$$ Since also Λ is a.s. concentrated on the points of increase of G, the cdf of Λ ,we have by the above Lemma

$$\lambda_{n+1}(T_n) = \lambda_{n+1}((n+1)(T_n/n+1))) \sim (n+1)/T_n \to \Lambda \qquad \text{a.s.,} \tag{23}$$

which proves the theorem.

For instance, if Λ is concentrated on two points $\nu_1 < \nu_2$ with mass α and $1-\alpha$ each $(\alpha > 0)$, then

$$\lambda_n(nt) \to \begin{cases} \nu_2, & t \leq 1(\nu_1,\nu_2) \\ \alpha\nu_1+(1-\alpha)\nu_2, & t = 1(\nu_1,\nu_2) \\ \nu_1, & t \geq 1(\nu_1,\nu_2) \end{cases} \tag{24}$$

where $1(\nu_1,\nu_2) = (\log \nu_2 - \log \nu_1)/(\nu_2 - \nu_1)$, as can be seen from Lundberg (1940), relation (108). Since always $1/\nu_2 < 1(\nu_1,\nu_2) < 1/\nu_1$, it can explicitly be seen that

$$\lambda_{n+1}(T_n) \to \begin{cases} \nu_1 \text{ with probability } \alpha \\ \\ \nu_2 \text{ with probability } 1-\alpha , \end{cases} \tag{25}$$

i.e. $\lambda_{n+1}(T_n) \rightarrow \Lambda$ a.s.

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