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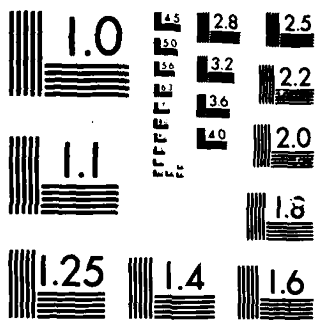

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STATIONARY MOTIONS AND INCOMPRESSIBLE LIMIT FOR COMPRESSIBLE VISCOUS FLUIDS

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STATIONARY MOTIONS AND INCOMPRESSIBLE LIMIT FOR COMPRESSIBLE VISCOUS FLUIDS

H. Beirão da Veiga*

Technical Summary Report #2883
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ABSTRACT

We study the system of equations (1.1), describing the stationary motion of a compressible viscous fluid in a bounded domain Ω of R^3. The total mass of fluid $m|\Omega|$, inside Ω, is fixed (condition (1.2)). We prove that for small f and g, there exists a unique solution (u,ρ) of the above system of equations, in a neighborhood of $(0, m)$. Moreover, by introducing a suitable parameter λ, we prove that the solution of the Navier-Stokes equations (1.14) are the incompressible limit of the solutions of the compressible Navier-Stokes equations ~~(1.13)~~. The proofs given here, apply, without supplementary difficulties, in the context of Sobolev spaces $H^{k,p}$, and other functional spaces. The results can be extended to the heat depending case, too.

AMS(MOS) Subject Classifications: 35G30, 35M05, 35Q10, 76D05, 76N10

Key words: Non-linear partial differential equations, Viscous compressible fluids, Incompressible limit, Stationary solutions

Work Unit Number 1 - Applied Analysis

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SIGNIFICANCE AND EXPLANATION

In this paper ~~we~~ considers the non-linear system of partial differential equations (1.1), describing the barotropic stationary motion of a compressible fluid, in a bounded region Ω, see [4] for details.

We assume that the total mass of fluid inside Ω is fixed, and equal to $m|\Omega|$, where the mean density m is given, (condition (1.2)).

We prove that for small f and g, there exists a unique solution $u(x)$, $\rho(x)$ ~~of (1.1)~~ in a neighborhood of $(0, m)$. Here, $u(x)$ is the field of velocities, $\rho(x)$ the density of the fluid, $p(\rho(x))$ the pressure field, and $f(x)$ the external force field (in the physical interesting case one has $g = 0$).

Moreover, ~~we prove that~~ the solutions of system ~~(1.13)~~ converge to the solution of the Navier-Stokes equation ~~(1.14)~~, as $\lambda \to +\infty$, i.e. when the Mach number becomes small.

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STATIONARY MOTIONS AND INCOMPRESSIBLE LIMIT FOR COMPRESSIBLE VISCOUS FLUIDS

H. Beirão da Veiga*

1. INTRODUCTION AND MAIN RESULTS

In this paper we study the system

$$(1.1)\qquad \begin{cases} -\mu\Delta u - \nu\nabla \mathrm{div}\, u + \nabla p(\rho) = \rho[f - (u\cdot\nabla)u], & \text{in } \Omega, \\ \mathrm{div}(\rho u) = g, & \text{in } \Omega, \\ u_{|\Gamma} = 0, \end{cases}$$

in a bounded, open domain in R^3, locally situated on one side of its boundary Γ, a C^3 manifold (the case $n \neq 3$ can be studied by the same method). As usual,

$$(v \cdot \nabla)u = \sum_{i=1}^{3} v_i \frac{\partial u}{\partial x_i} .$$

System (1.1) describes the stationary motion of a barotropic, compressible fluid; see Serrin [4]. In equation (1.1), $\rho(x)$ is the density of the fluid, $u(x)$ the velocity field, $f(x)$ the assigned external force field, $p = p(\rho)$ the pressure. In the physical equation one has $g = 0$; however, on studying (1.1) from a mathematical point of view, it is not without interest to study the general case.

We assume that the total mass of fluid inside Ω is fixed, i.e., we impose to the solution of (1.1) the contraint

$$(1.2)\qquad \frac{1}{|\Omega|} \int_\Omega \rho(x)dx = m,$$

where the mean density m is a given positive constant. The function ρ will be written in the form $\rho = m + \sigma$, and the new unknown $\sigma(x)$ has to verify the constraint

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$$(1.3) \qquad \tilde{\sigma} \equiv \frac{1}{|\Omega|} \int_\Omega \sigma(x)dx = 0.$$

We assume that the real function $\rho \to p(\rho)$ is defined and has a Lipschitz continuous first derivative $p'(\rho)$ in a neighborhood $I \equiv [m - \ell, m + \ell]$ of m, for some positive $\ell < m/2$. We assume also the (unessential) physical condition $k = p'(m) > 0$. Clearly,

$$(1.4) \qquad p'(\rho) = p'(m + \sigma) = k - \omega(\sigma), \qquad \forall \sigma \in I,$$

where $\omega(\sigma)$ is a Lipschitz continuous function, such that $\omega(0) = 0$. We set

$$S \equiv \sup_{\sigma,\tau\in I} \frac{|\omega(\sigma) - \omega(\tau)|}{|\sigma - \tau|} .$$

Concerning the constants μ and ν, we only assume that

$$(1.5) \qquad \mu > 0 \quad , \quad \nu > -\mu.$$

In the sequel, we write the system (1.1) in the equivalent form

$$(1.6) \qquad \begin{cases} -\mu\Delta u - \nu\nabla \mathrm{div}\, u + k\nabla\sigma = \omega(\sigma)\, \nabla\sigma + \\ \qquad\qquad + (m + \sigma)\, [f - (u\cdot\nabla)u], \ \text{ in } \ \Omega, \\ m\, \mathrm{div}\, u + u \cdot \nabla\sigma + \sigma\, \mathrm{div}\, u = g, \ \text{ in } \ \Omega, \\ u_{|\Gamma} = 0 . \end{cases}$$

Let us introduce some notation. We set

$$|\nabla v|^2 = \sum_{i,k=1}^{3} \left(\frac{\partial v_i}{\partial x_k}\right)^2 , \quad \nabla v : \nabla^2\tau = \sum_{i,k=1}^{3} \frac{\partial v_i}{\partial x_k} \frac{\partial^2 \tau}{\partial x_i \partial x_k} ,$$

where v is a vector and τ a scalar.

We denote by H^k, k integer, the Sobolev space $W^{k,2}(\Omega)$, endowed with the usual norm $\| \ \|_k$, and by $| \ |_p$, $1 < p < +\infty$, the usual norm in $L^p = L^p(\Omega)$. Hence, $\| \ \|_0 = | \ |_2$. For convenience, we utilize the same symbol H^k to denote also the space of vector fields v in Ω such that $v_i \in W^{k,2}(\Omega)$, $i = 1, 2, 3$. This convention applies to all the functional spaces and norms utilized here.

For $k \geq 1$, we define

$$H_0^k = \{v \in H^k : v = 0 \text{ on } \Gamma\} .$$

Moreover,

$$\bar{H}^2 = \{\tau \in H^2 : \bar{\tau} = 0\} , \quad \bar{H}_0^2 = H_0^2 \cap \bar{H}^2,$$

where $\bar{\tau}$ is the mean value in Ω of the scalar field $\tau(x)$. Finally, for vector fields, we define

$$H_{0,d}^3 = \{v \in H_0^3 : \operatorname{div} v = 0 \text{ on } \Gamma\}.$$

In the sequel, c, c_0, c_1, $c_2,\ldots$, denote positive constants depending at most on Ω. Moreover, c', c_0', c_1', ..., denote positive constants depending at most on Ω, μ, ν, k, m, ℓ, and S. The same symbol c (or c') will be utilized to denote different constants, even in the same equation.

In section 3 we prove the following result:

Theorem A. There exists positive constants c_0' and c_1' such that if $f \in H^1$, $g \in \bar{H}_0^2$, and

$$\|f\|_1 + \|g\|_2 \leq c_0', \tag{1.7}$$

then there exists a unique solution $(u,\sigma) \in H_0^3 \times \bar{H}^2$ of problem (1.6), in the ball

$$\|u\|_3 + \|\sigma\|_2 \leq c_1'. \tag{1.8}$$

A crucial tool in order to prove this result will be the study of the linear system

$$\begin{cases} -\mu\Delta u - \nu\nabla \operatorname{div} u + k\nabla\sigma = F, & \text{in } \Omega, \\ m \operatorname{div} u + v \cdot \nabla\sigma + \sigma \operatorname{div} v = g, & \text{in } \Omega, \\ u_{|\Gamma} = 0 \quad , \end{cases} \tag{1.9}$$

for which we will prove the following result:

Theorem B. *Let* $F \in H^1$, $g \in \bar{H}^2_0$, *and* $v \in H^3_{0,d}$ *be given, and assume that* (2.14) *holds. Then, there exists a unique solution* $(u,\sigma) \in H^3_{0,d} \times \bar{H}^2$ *of the linear system* (1.9). *Moreover,*

$$\mu\|u\|_3 + k\|\sigma\|_2 \leq c\left(1 + \frac{\mu + |\nu|}{\mu + \nu} + \frac{\mu + |\nu|}{m}\right) \|F\|_1 + \tag{1.10}$$
$$+ c\,\frac{\mu + |\nu|}{m}\,\|g\|_2 .$$

In section 4, we assume that the function $p(\rho,\lambda)$ depends, in a suitable way, on a parameter λ. By letting $\lambda \to +\infty$, we prove that the solution of the Navier-Stokes equation (1.14) is the incompressible limit of the solutions of system (1.13). For the justification of the physical aspects of the description (i.e., the behavior of $p(\rho,\lambda)$, as $\lambda \to +\infty$) we refer, for instance, to reference [1].

We assume that for each value of the parameter $\lambda \in [\lambda_0, +\infty[$ ($\lambda_0 \in \mathbb{R}$, has no special meaning) the function $p(\rho,\lambda)$ is defined in a neighborhood $I_\lambda \equiv [m-\ell_\lambda, m+\ell_\lambda]$ of m, where $0 < \ell_\lambda < m/2$. Moreover, for each fixed λ, the derivative $dp(\rho,\lambda)/d\rho \equiv p'(\rho,\lambda)$, is Lipschitz continuous on I_λ, with Lipschitz constant S_λ.

We define $k_\lambda \equiv p'(m,\lambda)$, and assume tht $k_\lambda \geq k_0 > 0$ (the constant k_0 has no special meaning, since we will let $k_\lambda \to +\infty$, as $\lambda \to +\infty$). We suppose that there exist positive constants ϕ and ℓ such that

$$S_\lambda \leq \phi\, k_\lambda^2 \quad , \quad \forall\, \lambda \geq \lambda_0 \quad , \tag{1.11}$$

and

$$\ell_\lambda k_\lambda \geq \ell, \quad \forall\, \lambda \geq \lambda_0 \quad . \tag{1.12}$$

By (eventually) defining a smaller ℓ_λ, we assume, without losing generality, that $\ell_\lambda k_\lambda = \ell$. Finally, let $\omega_\lambda(\sigma)$ be defined by $p'(m+\sigma,\lambda) = k_\lambda - \omega_\lambda(\sigma)$.

Consider the stationary compressible Navier-Stokes equation, with state function $p(\rho,\lambda)$,

$$
(1.13) \quad \begin{cases} -\mu\Delta u_\lambda - \nu\nabla \operatorname{div} u_\lambda + \nabla p(\rho_\lambda,\lambda) = \rho_\lambda[f - (u_\lambda\cdot\nabla)u_\lambda] \ , \\ \operatorname{div}(\rho_\lambda u_\lambda) = 0 \quad , \quad \text{in} \quad \Omega \ , \\ (u_\lambda)_{|\Gamma} = 0 \quad , \end{cases}
$$

and the incompressible Navier-Stokes equation

$$
(1.14) \quad \begin{cases} -\mu\Delta u_\infty + \nabla\pi(x) = m[f - (u_\infty\cdot\nabla)u_\infty] \ , \\ \operatorname{div} u_\infty = 0 \quad , \quad \text{in} \quad \Omega \ , \\ (u_\infty)_{|\Gamma} = 0 \quad . \end{cases}
$$

As above, we set $\rho_\lambda(x) = m + \sigma_\lambda(x)$, and we look for solutions of (1.13) verifying assumption (1.2), i.e. such that (1.3) holds.

We denote by $\tilde{c}, \tilde{c}_0, \tilde{c}_1, \tilde{c}_2, \ldots,$ positive constants depending at most on Ω, μ, ν, m, ℓ, ϕ and k_0, and we say that a positive constant is of type $\tilde{c}$ if it depends at most on the above parameters.

In section 4, we prove the following result:

Theorem C. *There exists positive constants* $\tilde{c}_0$ *and* $\tilde{c}_1$ *such that the following statement holds:*

(i) *Let* $f \in H^1$, *belong to the ball*

$$
(1.15) \qquad \|f\|_1 \le \tilde{c}_0 \ .
$$

Then, for each $\lambda > \lambda_0$, *problem* (1.13) *has a unique solution* $(u_\lambda, \sigma_\lambda) \in H_0^3 \times \overline{H}^2$ *in the ball*

$$
(1.16) \qquad \|u_\lambda\|_3 + k_\lambda \|\sigma_\lambda\|_2 \le \tilde{c}_1 \ .
$$

(ii) *If* $\lim\limits_{\lambda\to+\infty} k_\lambda = +\infty$, *then*

$$
(1.17)\quad \begin{cases} u_\lambda \to u_\infty\,, \ \underline{\text{weakly in}}\ H_0^3,\ \underline{\text{strongly in}}\ H_0^s,\ \forall\, s < 3\,, \\ \operatorname{div} u_\lambda \to 0\,, \ \underline{\text{weakly in}}\ H_0^2\,, \ \underline{\text{strongly in}}\ H_0^s,\ \forall\, s < 2\,, \\ \sigma_\lambda \to 0\,, \ \underline{\text{strongly in}}\ \overline{H}^2\,, \\ \nabla p(\rho_\lambda,\lambda) \to \nabla\pi\,, \ \underline{\text{weakly in}}\ H^1,\ \underline{\text{strongly in}}\ H^s,\ \forall\, s < 1\,, \end{cases}
$$

<u>where</u> $(u_\infty,\nabla\pi)$ <u>is the unique solution of problem</u> (1.14).

The existence of the solution $(u_\infty,\nabla\pi)$ of (1.14) is well known. However, it follows from our proof, too.

Note that both problems (1.13), (1.14) are invariant under addition of arbitrary constants to $p(\lambda,\rho)$ and π, respectively.

An existence result for system (1.1) was given first by Padula, in reference [3]. Unfortunately, the (quite simple) proof given there depends in a crucial way on a smallness condition on μ respect to ν (μ and ν positive constants). This condition was dropped in Valli's paper [5], where a result similar to Theorem A is proved, by approximating the stationary solutions with the periodic solutions of the corresponding evolution problem. This technique was applied in [6] to the heat-depending case, and to more general boundary conditions.

The proofs given in our paper are quite simple, and apply as well (without any supplementary difficulty) in the context of other spaces of functions, as for instance Sobolev spaces $H^{k,p}$, $1 < p < +\infty$.

In particular, for small data $(f,g) \in H^{k+1} \times \overline{H}_0^{k+2}$, there exists a unique solution $(u,\sigma) \in H_0^{k+3} \times \overline{H}^{k+2}$, in a neighborhood of the origin, for every $k \geq 0$ (we assume the derivative $p^{(k+1)}$ Lipschitz continuous, and Ω of class C^{k+3}).

Furthermore, all the results hold again in any dimension of space (on dealing with the non-linear problem in space $H^{k,p}$, k must be sufficiently large).

Statements and proofs, in the above general setting up, will be given in a forthcoming paper, where (for completeness) we will consider the heat-conductive-case. In this paper, we state only the counterparts of theorems A and B, in appendix 2. The proofs can be easily done, by following those of theorems A and B. Here, we have preferred considering the main case (1.1) by itself, in order to avoid secondary technicalities. In fact, in the heat-depending case a third equation should be added to system (1.1) (see (6.1)). However, that equation is weakly coupled with its companion equations, in system (6.1). As a matter of fact, the more interesting mathematical problems and the main difficulties, already appear on studying system (1.1).

Finally, it goes by itself that quite obvious assumptions and devices, allow the coefficients μ, ν χ, c_v, γ and γ' on depending on u, ρ and θ.

2. Proof of Theorem B

We start by proving the uniqueness of the solution of the linear system (1.9), under the assumption (2.1) below. Let (u,σ) be a solution, with data $F = 0$, $g = 0$. By multiplying both sides of equation $(1.9)_1$ by mu and of equation $(1.9)_2$ by $k\sigma$, by integrating over Ω_1 and by adding side by side the two equations, one easily shows that

$$m\mu_0 \; \|\nabla u\|_0^2 < \frac{k}{2} \, |\mathrm{div}\; v|_\infty \; \|\sigma\|_0^2 \; ,$$

where

$$\mu_0 = \min\{\mu, \; \mu + \nu\}.$$

Hence,

$$\|u\|_1^2 < c \, \frac{k}{m\mu_0} \, |\mathrm{div}\; v|_\infty \; \|\sigma\|_0^2 \; .$$

Moreover, from $(1.9)_1$ it follows that

$$k\|\sigma\|_0 < ck\|\nabla\sigma\|_{-1} < c(\mu + |\nu|) \; \|u\|_1,$$

since $\tilde{\sigma} = 0$. Consequently,

$$\|u\|_1^2 < c \, \frac{(\mu + |\nu|)^2}{m\mu_0 k} \, |\mathrm{div}\; v|_\infty \; \|u\|_1^2.$$

This proves that the uniqueness holds whenever

$$\|v\|_3 < \frac{m\mu_0 k}{c_0(\mu + |\nu|)^2} \; , \tag{2.1}$$

for a suitable positive constant c_0; recall that $H^2 \subset L^\infty$.

In the remaining of this section we prove the existence of the solution of system (1.9). We assume that $v \in H^3_{0,d}$ verifies the condition

$$(2.2) \qquad |\operatorname{div} v|_\infty + 2|\nabla v|_\infty \le mk/(\mu + \nu) ,$$

and that $F \in H^1$, $g \in \bar{H}^2_0$. Let $\tau \in \bar{H}^2$, and consider the linear problem

$$(2.3) \qquad \frac{mk}{\mu + \nu} \lambda + v \cdot \nabla\lambda = G ,$$

where

$$(2.4) \qquad \begin{aligned} G = \Delta g + \frac{m}{\mu + \nu} \operatorname{div} F - \\ - \left[2\nabla v : \nabla^2\tau + \Delta v \cdot \nabla\tau + \Delta (\tau \operatorname{div} v)\right] . \end{aligned}$$

The significance of equation (2.3) is strongly related to the identity (2.20). It is well known (Lax-Phillips [2]) that there exists a linear map $G \to \lambda$, from all of L^2 into L^2, such that for each $G \in L^2$ the corresponding λ is a weak solution of (2.3), and verifies the estimate

$$(2.5) \qquad \frac{1}{2} \frac{mk}{\mu + \nu} \|\lambda\|_0 \le \|G\|_0 .$$

By a weak solution of (2.3), we mean here a function $\lambda \in L^2$ such that

$$(2.6) \qquad \frac{mk}{\mu + \nu} \int_\Omega \lambda\varphi dx - \int_\Omega \lambda \operatorname{div}(\varphi v) dx = \int_\Omega G\varphi dx, \ \forall\varphi \in H^1 .$$

For the reader's convenience, [illegible]complete proof of this result in the Appendix I.

By using the embeddings $H^1 \subset L^4$ and $H^2 \subset L^\infty$, one verifies that $\|G\|_0$ is bounded by the right hand side of equation (2.7) below. Hence our solution λ of (2.3) verifies

$$\frac{mk}{\mu + \nu} \|\lambda\|_0 \leq c(\|F\|_1 + \|g\|_2 + \|v\|_3 \ \|\tau\|_2) \ . \tag{2.7}$$

Let now $\theta \in H_0^2$ be the solution of the Dirichlet problem

$$\begin{cases} (\mu + \nu)\,\Delta\theta = k\lambda - \operatorname{div} F, & \text{in } \Omega \\ \theta_{|\Gamma} = 0 \ . \end{cases} \tag{2.8}$$

By using (2.7), one has

$$(\mu + \nu)\ \|\theta\|_2 \leq c\,\frac{\mu + \nu}{m}\,(\|F\|_1 + \|g\|_2 + \|v\|_3 \ \|\tau\|_2) + c\|F\|_1 . \tag{2.9}$$

Define now

$$\theta_0(x) = \theta(x) - \bar{\theta} \ . \tag{2.10}$$

Clearly, $\bar{\theta}_0 = 0$. Let (u, σ) be the unique solution in the class $H_0^3 \times \bar{H}^2$ of the following linear Stokes problem, in Ω :

$$\begin{cases} -\mu\Delta u + k\nabla\sigma = F + \nu\nabla\theta_0, \\ \operatorname{div} u = \theta_0, \\ u_{|\Gamma} \ . \end{cases} \tag{2.11}$$

From the L^2 estimates for this problem one has

$$\mu\|u\|_3 + k\|\sigma\|_2 \leq c(\|F\|_1 + |\nu|\ \|\theta_0\|_2 + \mu\|\theta_0\|_2) \ . \tag{2.12}$$

By taking in account that $\|\theta_0\|_2 \leq \|\theta\|_2$, one gets

$$\begin{aligned} \mu\|u\|_3 + k\|\sigma\|_2 &\leq c\left(1 + \frac{\mu + |\nu|}{\mu + \nu} + \frac{\mu + |\nu|}{m}\right) \|F\|_1 + \\ &+ c_1\,\frac{\mu + |\nu|}{m}\,(\|g\|_2 + \|v\|_3 + \|\tau\|_2) \ . \end{aligned} \tag{2.13}$$

Let now c_2 be a positive constant such that

$$|\text{div } w|_\infty + 2|\nabla w|_\infty \leq c_2 \|w\|_3, \quad \forall w \in H_0^3 .$$

In the remaining of this section we assume that the vector field v verifies the condition

$$\|v\|_3 \leq \gamma k \quad , \tag{2.14}$$

where, by definition,

$$\gamma \equiv \min\left\{\frac{\mu_0 m}{c_0(\mu+|\nu|)^2}, \frac{m}{2c_1(\mu+|\nu|)}, \frac{m}{c_2(\mu+\nu)}\right\} . \tag{2.14'}$$

Assumption (2.14), implies, in particular, (2.1) and (2.2).

From (2.13) and (2.14), one gets

$$\begin{aligned} &\mu\|u\|_3 + k\|\sigma\|_2 \leq \frac{1}{2} k\|\tau\|_2 + \\ &+ c\left(1 + \frac{\mu+|\nu|}{\mu+\nu} + \frac{\mu+|\nu|}{m}\right)\|F\|_1 + c\,\frac{\mu+|\nu|}{m}\|g\|_2 . \end{aligned} \tag{2.15}$$

At this point, we call attention to the sequence of <u>linear</u> maps, introduced above:

$$(F, g, \tau) \to (F, \lambda) \to (F, \theta) \to (F, \theta_0) \to (u, \sigma) ,$$

which were defined by equations (2.3) + (2.4), (2.8), (2.9), (2,11), respectively. The product map $(F, g, \tau) \to (u, \sigma)$ is linear and continuous, by (2.15). Hence, if (u_1, σ_1) is the solution corresponding to data (F, g, τ), it follows that $(u - u_1, \sigma - \sigma_1)$ is the solution corresponding to data $(0, 0, \tau - \tau_1)$. Consequently, (2.15) yields, in particular,

$$\|\sigma - \sigma_1\|_2 \leq \frac{1}{2}\|\tau - \tau_1\|_2 .$$

Hence, for fixed F and g, the map $\tau \to \sigma$ is a contraction in $\bar{H}^2$. Consequently, it has a (unique) fixed point $\sigma = \tau$.

In the sequel we prove that the pair (u, σ), corresponding to the fixed point $\sigma = \tau$, solves equation (1.9). Equations $(1.9)_1$ and $(1.9)_3$ follows from (2.11). In order to prove $(1.9)_2$, we start by substituting the expression of λ, obtained from equation $(2.8)_1$, in the first term on the left hand side of (2.3). This yields, since $\tau = \sigma$,

$$m\Delta\theta + v \cdot \nabla\lambda + 2\nabla v : \nabla^2\sigma + \Delta v \cdot \nabla\sigma + \Delta(\sigma \operatorname{div} v) = \tag{2.16}$$
$$= \Delta g \; .$$

On the other hand, by applying the divergence operator to both sides of equation $(2.11)_1$, and by utilizing $(2.11)_2$ one gets $-(\mu + \nu)\Delta\theta + k\Delta\sigma = \operatorname{div} F$, since $\Delta\theta_0 = \Delta\theta$. By comparison with $(2.8)_1$, one shows that $\lambda = \Delta\sigma$. By replacing λ by $\Delta\sigma$ in equation (2.16), it follows that

$$m\Delta \operatorname{div} u + v \cdot \nabla\Delta\sigma + 2\nabla v : \nabla^2\sigma + \Delta v \cdot \nabla\sigma + \tag{2.17}$$
$$+ \Delta(\sigma \operatorname{div} v) - g = 0 \; ,$$

or equivalently,

$$\Delta[m \operatorname{div} u + v \cdot \nabla\sigma + \sigma \operatorname{div} v - g] = 0, \text{ in } \Omega \; . \tag{2.18}$$

The function between square brackets (which belongs to H^1) is equal to the constant $-m\bar{\theta}$ on the boundary, by $(2.8)_2$, (2.10), $(2.11)_2$, and by the assumptions $v = 0$, $\operatorname{div} v = g = 0$ on Γ. Consequently,

$$m \operatorname{div} u + v \cdot \nabla\sigma + \sigma \operatorname{div} v - g = -m\bar{\theta}, \text{ in } \Omega \; .$$

By integrating both sides of this equation in Ω, one shows that it must be $\bar{\theta} = 0$. Hence, equation $(1.9)_2$ is satisfied. Finally, the estimate (1.10) follows from (2.15).

□

<u>Remark</u>. One has to be careful on deducing (2.18) from (2.17), since both equations hold only in a weak sense. The point is to prove the identity

$$-\int_\Omega \nabla(v \cdot \nabla\sigma) \cdot \nabla\varphi dx = -\int_\Omega \Delta\sigma \operatorname{div}(\varphi v)dx + \tag{2.19}$$
$$+ \int_\Omega [2\nabla v : \nabla^2\sigma + \Delta v \cdot \nabla\sigma] \varphi dx, \; \forall\varphi \in C_0^\infty \; ,$$

which is a weak formulation of

$$\Delta(v \cdot \nabla\sigma) = v \cdot \nabla\Delta\sigma + 2\nabla v : \nabla^2\sigma + \Delta v \cdot \nabla\sigma \; . \tag{2.20}$$

For $\sigma \in H^3$, this last identity holds, and yields (2.19). If $\sigma \in H^2$, we approximate it (in the H^2 norm) by a sequence of functions $\sigma_n \in H^3$, and we pass to the limit in equation (2.19) (written with σ replaced by σ_n) as $n \to +\infty$.

3. Proof of Theorem A

For convenience, in this section we will not take case on the explicit dependence of the positive constants respect to the parameters. However, all the constants depend at most on Ω, μ, ν, k, m, ℓ and S.

Let c_3 be a constant such that $|\tau|_\infty \leq c_3 \|\tau\|_2$, for every $\tau \in \bar{H}^2$. We will utilize here the condition

$$\|\tau\|_2 < \frac{\ell}{c_3}, \tag{3.1}$$

which guarantees that $m + \sigma(x)$ belongs to the domain of p, for every $x \in \Omega$, since $-\ell < \tau(x) < \ell$.

Let $v \in H_0^3$ verify (2.14), and $\tau \in \bar{H}^2$ verify (3.1), define

$$F(v, \tau) = (m + \tau)[f - (v \cdot \nabla)v] = \omega(\tau)\nabla\tau, \tag{3.2}$$

and consider the linearized system (1.9) with $F(x)$ given by $F(v, \tau)$, i.e., the system

$$\begin{cases} -\mu\Delta u - \nu\nabla \operatorname{div} u + k\nabla\sigma = F(v, \tau), \text{ in } \Omega, \\ m \operatorname{div} u + (v \cdot \nabla)\sigma + \sigma \operatorname{div} v = g, \text{ in } \Omega, \\ u|_\Gamma = 0 . \end{cases} \tag{3.3}$$

Since $H^1 \subset L^4$, $H^2 \subset L^\infty$ and $|w(\tau)|_\infty \leq S|\tau|_\infty \leq (\ell/c_3) S \|\tau\|_2$, one easily shows that

$$\|F(v, \tau)\|_1 \leq c(\frac{3}{2} m + \frac{\ell}{c_3}) (\|f\|_1 + \|v\|_2^2) + cS\|\tau\|_2^2 . \tag{3.4}$$

This last estimate, together with (1.10), yields the following result:

<u>Theorem 3.1</u>. <u>Let</u> $v \in H^3_{0,d}$, $\tau \in \bar{H}^2$ <u>and let</u> (2.1), (2.14) <u>and</u> (3.1) <u>be satisfied</u>. <u>Then, the unique solution</u> $u, \sigma \in H_0^3 \times \bar{H}^2$ <u>of system</u> (3.3), <u>verifies the estimate</u>

$$\|u\|_3 + \|\sigma\|_2 \leq a(\|\tau\|_2 + \|v\|_2)^2 + b(\|f\|_1 + \|g\|_2) , \tag{3.5}$$

where the positive constants a and b depend only on Ω, μ, ν, k, m, ℓ and S.

The existence and uniqueness of the solution (u, σ) of system (3.3), enables us to define the corresponding map $(u, \sigma) = T(v,\tau)$. The fixed points of the map T are just the solutions of the non-linear system (1.6). In order to prove the existence of these fixed points we assume that

$$\|f\|_1 + \|g\|_2 \leq \frac{1}{2b} \min\{\frac{\delta}{2a}, \gamma k, \frac{\ell}{c_3}\}, \tag{3.6}$$

and that

$$\|v\|_3 + \|\tau\|_2 \leq \min\{\frac{\delta}{2a}, \gamma k, \frac{\ell}{c_3}\} . \tag{3.7}$$

The parameter $\delta \in]0, 1]$, will be fixed later on. Consider the ball

$$B_\delta \equiv \{(v, \tau) \in H^3_{0,d} \times \bar{H}^2: \quad (3.7) \text{ holds}\} .$$

This is a compact set in $H^1_0 \times L^2$. Moreover, by using (3.5), one shows that $TB_\delta \subset B_\delta$, for every $\delta \leq 1$. We want to prove that, for a sufficiently small δ, depending only on Ω, μ, ν, k, m, ℓ and S, the map T is a contraction in B_δ. Hence T has a (unique) fixed point in B_δ, and Theorem A is proved.

Let $(u, \sigma) = T(v, \tau)$, $(u_1, \sigma_1) = T(v_1, \tau_1)$, $F = F(v, \tau)$, $F_1 = F(v_1, \tau_1)$. One has, in Ω,

$$\left\{\begin{aligned} &-\mu\, \Delta(u - u_1) - \nu\nabla \operatorname{div}(u - u_1) + k\nabla(\sigma - \sigma_1) = F - F_1, \\ &m \operatorname{div}(u - u_1) + v \cdot \nabla(\sigma - \sigma_1) + (v - v_1) \cdot \nabla\sigma_1 + \\ &\qquad + \sigma_1 \operatorname{div}(v - v_1) + (\sigma - \sigma_1) \operatorname{div} v = 0 . \end{aligned}\right. \tag{3.8}$$

By multiplying both sides of equation $(3.8)_1$ by $m(u - u_1)$ and both sides of equation $(3.8)_2$ by $k(\sigma - \sigma_1)$, by integrating in Ω, and by adding side by side the two equations obtained in that way, one shows that

$$m(\mu - |\nu|)\ \|\nabla(u - u_1)\|_0^2 \le \frac{1}{2} k|\text{div } v|_\infty \|\sigma - \sigma_1\|_0^2 + \tag{3.9}$$
$$+ ck\|\sigma_1\|_2 \|v - v_1\|_1 \|\sigma - \sigma_1\|_0 + m\|F - F_1\|_{-1} \|u - u_1\|_1 .$$

In proving (3.9), we utilized the Sobolev's immersion theorems $H^2 \hookrightarrow L^\infty$ and $H^1 \hookrightarrow L^4$, and also the inequality $\|\text{div}(u - u_1)\|_0^2 \le \|\nabla(u - u_1)\|_0^2$.

From (3.9) one has

$$\|u - u_1\|_1^2 \le c'|\text{div } v|_\infty \|\sigma - \sigma_1\|_0^2 + \tag{3.10}$$
$$+ c'\|\sigma_1\|_2 \|v - v_1\|_1 \|\sigma - \sigma_1\|_0 + c'\|F - F_1\|_{-1}^2 .$$

On the other hand $\|\sigma - \sigma_1\|_0 \le c\|\nabla(\sigma - \sigma_1)\|_{-1}$, since $\sigma - \sigma_1$ has mean value zero. Hence, by using the expression of $\nabla(\sigma - \sigma_1)$ obtained from equation $(3.8)_1$, (or L^2 estimates for the linear Stokes problem) we show that

$$\|\sigma - \sigma_1\|_0^2 \le c_2'\|u - u_1\|_1^2 + c'\|F - F_1\|_{-1}^2 . \tag{3.11}$$

By multiplying both sides of equation (3.11) by $1/(2c_2')$, by adding (side by side) this equation to equation (3.10), and by using standard devices, we prove that

$$\frac{1}{2} \|u - u_1\|_1^2 + c_3'(1 - c_4' |\text{div } v|_\infty) \|\sigma - \sigma_1\|_0^2 \le \tag{3.12}$$
$$\le c'\|\sigma_1\|_2^2 \|v - v_1\|_1^2 + c'\|F - F_1\|_{-1}^2 ,$$

for some suitable positive constants c_3', c_4', and c'.

On the other hand

$$\|F - F_1\|_{-1} \le \|f\|_1 \|\tau - \tau_1\|_0 + c(1 + \|\tau\|_2)(\|v\|_1 + \|v_1\|_1) \|v - v_1\|_1 + \tag{3.13}$$
$$+ c[\|v_1\|_2^2 + S(\|\tau\|_2 + \|\tau_1\|_2)] \|\tau - \tau_1\|_0 .$$

In fact, by using the immersion $H^1 \hookrightarrow L^4$, one easily shows that

$$\|(\tau - \tau_1)f\|_{-1} \le \|f\|_1 \|\tau - \tau_1\|_0 .$$

Similarly,

$$\|(v\cdot\nabla)v-(v_1\cdot\nabla)v_1\|_{-1} < c(\|v\|_1+\|v_1\|_1)\,\|v-v_1\|_1 ,$$

and

$$\|\tau(v\cdot\nabla)v-\tau_1(v_1\cdot\nabla)v_1\|_{-1} < c\|\tau\|_2\,\|(v\cdot\nabla)v-(v_1\cdot\nabla)v_1\|_{-1} +$$
$$+\,c\|v_1\|_2^2\,\|\tau-\tau_1\|_0 .$$

Furthermore,

$$\|\omega(\tau)\,\nabla\tau-\omega(\tau_1)\,\nabla\tau_1\|_{-1} = \|\nabla_x\int_{\tau_1(x)}^{\tau(x)}\omega(\xi)d\xi\|_{-1} <$$
$$< \|\int_{\tau_1(x)}^{\tau(x)}\omega(\xi)d\xi\|_0 < S(|\tau|_\infty+|\tau_1|_\infty)\,\|\tau-\tau_1\|_0 .$$

The above inequalities yield (3.13).

For $\lambda < a/c_1c_4'$, one has $c_4'|\mathrm{div}\ v|_\infty < c_4'c_1\,\|v\|_3 < \frac{1}{2}$, by (3.7). Hence, from (3.12), (3.13) one gets

$$\|u-u_1\|_1^2+\|\sigma-\sigma_1\|_0^2 < c'\|f\|_1^2\,\|\tau-\tau_1\|_0^2 +$$

$$+\,c'\left[(1+\|\tau\|_2)^2\,(\|v\|_1+\|v_1\|_1)^2+\|\sigma_1\|_2^2\right]\,\|v-v_1\|_1^2 + \tag{3.14}$$

$$+\,c\left[\|v_1\|_2^2+S(\|\tau\|_2+\|\tau_1\|_2)\right]^2\,\|\tau-\tau_1\|_0^2 .$$

By choosing δ sufficiently small, depending only on Ω, μ, ν, k, m, ℓ and S, one has

$$\|u-u_1\|_1^2+\|\sigma-\sigma_1\|_0^2 < \frac{1}{2}\left(\|v-v_1\|_1^2+\|\tau-\tau_1\|_0^2\right) .$$

Hence T is a contraction in B_δ, which proves Theorem A.

□

Remark. B_1 is a compact and convex subset of $H_0^1 \times L^2$, $T : B_1 \to B_1$ is continuous respect to that topology, and $TB_1 \subset B_1$. Hence, we can prove the existence of (at least) a fixed point in B_1 by using Shauder's theorem. The uniqueness follows by using (3.14) (actually, it is quite trivial to obtain more stringent uniqueness results.).

4. Proof of Theorem C. During the proof of part (i) of theorem C, $\mu_\lambda, \sigma_\lambda, k_\lambda, \omega_\lambda$, will be denoted by u, σ, k, ω respectively. Theorem B states that if $F \in H^1$, $g = 0$, and if $v \in H^3_{0,d}$ verifies the condition

(4.1) $$\|v\|_3 < \gamma k \quad ,$$

then there exists a unique solution $(u,\sigma) \in H^3_{0,d} \times \bar{H}^2$ of the linear system (1.9). Moreover,

(4.2) $$\mu\|u\|_3 + k\|\sigma\|_2 < c\left(1 + \frac{\mu+|\nu|}{\mu+\nu} + \frac{\mu+|\nu|}{m}\right) \|F\|_1 \quad .$$

Let us now fix $\tau \in \bar{H}^2$ in the ball

(4.3) $$\|\tau\|_2 < \frac{\ell_\lambda}{c_3} \text{ , or equivalently, } \|k\tau\|_2 < \frac{\ell}{c_3} \quad ,$$

where c_3 was defined in section 3, and ℓ is the positive constant defined in (1.12). Condition (4.3) guarantees that $|\tau(x)| < \ell_\lambda$, $\forall\, x \in \Omega$. In particular $m/2 < m + \tau(x) < (3m)/2$.

By defining $F(v,\tau)$ as in (3.2) (recall that, now, $\omega = \omega_\lambda$) one has, as in section 3,

$$\|F(v,\tau)\|_1 < c\left(\frac{3}{2} m + \frac{\ell_\lambda}{c_3}\right)(\|f\|_1 + \|v\|_2^2) + cS_\lambda \|\tau\|_2^2 \quad .$$

Hence,

(4.4) $$\|F(v,\tau)\|_1 < c\left(\frac{3}{2} m + \frac{\ell}{c_3 k_0}\right)(\|f\|_1 + \|v\|_2^2) + c\phi k^2 \|\tau\|_2^2 \quad ,$$

(recall $k = k_\lambda$). If v and τ verify assumptions (4.1) and (4.3), it follows from (4.2) and (4.4) that the unique solution (u,σ) of system (3.3) verifies the estimate

(4.5) $$\|u\|_3 + \|k\sigma\|_2 < a(\|k\tau\|_2 + \|v\|_2)^2 + b\|f\|_1 \quad ,$$

where now a and b are constants of type $\tilde{c}$ (the above result corresponds to theorem

3.1, in section 3).

The proof goes on as in section 3, by utilizing now $k\sigma$ and $k\tau$ instead of σ and τ, respectively (in this way, inequalities (3.5) and (3.1) becomes (4.5) and $(4.3)_2$, respectively; condition (4.1) remains unchanged).

Following section 3, we denote by T the map $(u,\sigma) = T(v,\tau)$, where the data $(v,\tau) \in H^3_{0,d} \times \bar{H}^2$ verify (4.1), (4.3), and (u,σ) is the (corresponding) solution of system (3.3).

We fix $f \in H^1$ verifying (3.6) (here, $g = 0$), and we consider the restriction of T to the ball B_δ, $0 < \delta \leq 1$, defined by the condition

$$\|v\|_3 + \|k\tau\|_2 \leq \min\left\{\frac{\delta}{2a}, \gamma k, \frac{\ell}{c_3}\right\} \quad . \tag{4.6}$$

The substitution of τ by $k\tau$ transforms (3.7) on (4.6). Arguing as in section 3, and recalling that $k > k_0$, we prove inequalities (3.10), (3.11) and (3.12), if in these inequalities we replace σ, σ_1, τ, τ_1 by $k\sigma$, $k\sigma_1$, $k\tau$, $k\tau_1$ respectively. The constants c', c_2', c_3', c_4' are now of type $\tilde{c}$, hence independent of k.

Inequality (3.13) holds, as written in section 3. Recalling that $S_\lambda \leq \phi k^2$, and that $k > k_0$, we show that (3.13) holds again, if τ, τ_1, and S are replaced by $k\tau$, $k\tau_1$, and ϕ, respectively, and if the right hand side of the inequality is multiplied by $1 + (1/k_0)$.

By choosing δ as in section 3, i.e. $\delta < a/(c_1c_4')$, we get an inequality similar to (3.14), where now τ, σ, τ_1 and σ_1 are multiplied by k, and the constants are of type $\tilde{c}$. By choosing δ sufficiently small (depending only on Ω, μ, ν, m, ℓ, ϕ, k_0) one gets

$$\|u-u_1\|_1^2 + \|k\sigma-k\sigma_1\|_0^2 \leq \frac{1}{2}\left(\|v-v_1\|_1^2 + \|k\tau-k\tau_1\|_0^2\right) \quad . \tag{4.7}$$

Hence T is a contraction in B_δ, which proves the first part of theorem C.

We now prove part (ii) of that theorem. Condition (1.15) guarantees the uniqueness of the solution of problem (1.17), for a sufficiently small $\bar{c}_0$.

Let us write system (1.13) in the form (1.6), i.e.

$$
(4.8)\quad \begin{cases} -\mu\Delta u_\lambda - \nu\nabla \operatorname{div} u_\lambda + k_\lambda \nabla\sigma_\lambda = \omega_\lambda(\sigma_\lambda)\nabla\sigma_\lambda + (m+\sigma_\lambda)\,[f - (u_\lambda\cdot\nabla)u_\lambda]\ , \\ m \operatorname{div} u_\lambda + u_\lambda \cdot \nabla\sigma_\lambda + \sigma_\lambda \operatorname{div} u_\lambda = 0, \quad \text{in } \Omega\ , \\ (u_\lambda)_{|\Gamma} = 0\ . \end{cases}
$$

From (1.16), it follows that there exists $u_\infty \in H_0^3$, such that $(1.17)_1$ holds (here, we consider subsequences of u_λ; the convergence of all the u_λ to u_∞, as $\lambda \to +\infty$, will follow from the uniqueness of the limit u_∞, since we will show that u_∞ is the solution of (1.14)).

The bound (1.16), and the hypothesis $k_\lambda \to +\infty$ as $\lambda \to +\infty$, imply $(1.17)_3$. Furthermore, equation $(4.8)_2$, together with (1.16) and $(1.17)_3$, shows that $\operatorname{div} u_\lambda \to 0$, strongly in H_0^1, as $\lambda \to +\infty$. Since $\|\operatorname{div} u_\lambda\|_2$ is bounded, $(1.17)_2$ follows. In particular, $\operatorname{div} u_\infty = 0$.

Now, we pass to the limit in equation $(1.13)_1$, as $\lambda \to +\infty$. One has $\mu\Delta u_\lambda \to \mu\Delta u_\infty$ and $-\nu\nabla \operatorname{div} u_\lambda \to 0$, weakly in H^1 and strongly in H^s, $0 < s < 1$; and $\rho_\lambda \to m$, strongly in H^2. Moreover, $\rho_\lambda(u_\lambda\cdot\nabla)u_\lambda \to m(u_\infty\cdot\nabla)u_\infty$, weakly in H^2 and strongly in H^s, $0 < s < 2$. By using equation $(1.13)_1$, it follows that $\nabla p(\rho_\lambda,\lambda) \to \mu\Delta u_\infty + m[f - (u_\infty\cdot\nabla)u_\infty]$, weakly in H^1, strongly in H^s, $0 < s < 1$. Obviously, the limit function must be of the form $\nabla\pi(x)$. Theorem C is completely proved.

□

APPENDIX I

For the readers convenience we prove here the result stated at the beginning of section 2, concerning equation (2.3). We assume that the function $v \in H^3$, verifies $v \cdot n = 0$ on Γ, and assumption (2.2). This last condition could be replaced by $\frac{1}{2}|\operatorname{div} v|_\infty + |\nabla v|_\infty < M$. Let $G \in H_0^1$, and λ_ε be the solution of

$$(5.1) \qquad \begin{cases} -\varepsilon\, \Delta\lambda_\varepsilon + M\lambda_\varepsilon + v \cdot \nabla\lambda_\varepsilon = G, & \text{in } \Omega, \\ \lambda_\varepsilon|_\Gamma = 0, \end{cases}$$

where ε is a positive constant. By multiplying both sides of (5.1) by $\Delta\lambda_\varepsilon$, and by integrating over Ω, one easily shows that

$$\varepsilon\|\Delta\lambda_\varepsilon\|_0^2 + \left[M - \left(\tfrac{1}{2}|\operatorname{div} v|_\infty + |\nabla v|_\infty\right)\right] \|\nabla\lambda_\varepsilon\|_0^2 \le$$
$$\le \|\nabla G\|_0 \, \|\nabla\lambda_\varepsilon\|_0 .$$

This estimate, together with (2.2), gives

$$(5.2) \qquad \|\nabla\lambda_\varepsilon\|_0 \le (2/M)\, \|\nabla G\|_0 ,$$

and also

$$(5.3) \qquad \varepsilon\|\Delta\lambda_\varepsilon\|_0 \le \sqrt{2\varepsilon/M}\, \|\nabla G\|_0 .$$

Hence, there exists a subsequence λ_ε such that $\lambda_\varepsilon \to \lambda \in H_0^1$, weakly in H_0^1 and strongly in L^2, moreover, $\varepsilon\, \Delta\lambda_\varepsilon \to 0$ in L^2. By passing to the limit in (5.1), as $\varepsilon \to 0$, one proves that λ is a strong solution of (2.3). In particular, λ verifies (2.6). By multiplying both sides of (2.3) by λ, and by integrating over Ω, one shows that

$$(5.4) \qquad \left(M - \tfrac{1}{2}|\operatorname{div} v|_\infty\right) \|\lambda\|_0 \le \|G\|_0 .$$

This gives, in particular, the uniqueness of the solution λ, in H^1.

Since the linear map $T : H_0^1 \to H_0^1$, defined by $TG = \lambda$, is continuous respect to the L^2 norm, there exists a unique linear continuous map $\tilde{T}$ extending T to all of L^2. Clearly, (5.4) holds again. Furthermore, $\lambda = \tilde{T}G$ is a solution of (2.6).

Remark. The result holds again without assuming that $v \cdot n = 0$ on Γ, and with condition (2.2) replaced by the weaker condition $|\text{div } v|_\infty < M$ (or, more generally, by $|\text{div } v|_\infty < 2M$). In that case, equation (2.6) holds for every text function $\varphi \in H_0^1$. (We start by proving the existence of a solution λ for data belonging to the linear space H generated by an arbitrarily fixed basis $\{G^\ell\}$, $\ell = 1, 2, \ldots,$ on L^2; then, we extend the map $G \to \lambda$ to all of L^2, by continuity.)

APPENDIX II

6. The heat-depending-case.

In the heat-depending-case the equations are

$$(6.1)\quad \begin{cases} -\mu\Delta u - \nu\nabla \operatorname{div} u + \nabla p(\rho,\theta) = \rho[f - (u\cdot\nabla)u] , \\ \operatorname{div}(\rho u) = g , \\ -\chi\Delta\theta + c_v\rho u\cdot\nabla\theta + \theta p_\theta(\rho,\theta)\operatorname{div} u = \rho h + \psi(u,u), \quad \text{in } \Omega , \\ u_{|\Gamma} = 0,\ \theta_{|\Gamma} = n , \end{cases}$$

where

$$(6.2)\qquad \psi(u,u) = \lambda \sum_{i,j} \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)^2 + \lambda'(\operatorname{div} u)^2 .$$

Only for convenience, we will assume that $\mu > 0$, $\mu + \nu > 0$, $\chi > 0$, c_v, λ, λ', and $n > 0$, are constants. As for system (1.1), we impose here the additional condition (1.2).

The function $p(\rho,\theta)$ is defined, and has Lipschitz continuous first derivatives, in an ℓ-neighborhood $[m-\ell,m+\ell] \times [n-\ell,n+\ell]$ of (m,n). By setting $k = p_\rho(m,n)$, $\gamma = p_\theta(m,n)$, one has $p_\rho(m+\sigma,n+\alpha) = k - \omega_1(\sigma,\alpha)$, $p_\theta(m+\sigma,n+\alpha) = \gamma - \omega_2(\sigma,\alpha)$, where ω_i, $i = 1,2$, are Lipschitz continuous (with norms $\leq S$) in the ℓ-neighborhood of $(0,0)$; moreover $\omega_i(0,0) = 0$. We assume the physical condition $k > 0$.

By setting

$$\rho = m+\sigma,\ \theta = n+\alpha ,$$

the system (6.1) becomes

$$(6.3)\quad \begin{cases} -\mu\Delta u - \nu\nabla \operatorname{div} u + k\nabla\sigma + \gamma\nabla\alpha = \rho[f - (u\cdot\nabla)u] + \omega_1(\sigma,\alpha)\nabla\sigma + \omega_2(\sigma,\alpha)\nabla\alpha , \\ m \operatorname{div} u + u\cdot\nabla\sigma + \sigma \operatorname{div} u = g , \\ -\chi\Delta\alpha + \gamma n \operatorname{div} u = \rho h - c_v(m+\sigma)u\cdot\nabla\alpha + \psi(u,u) - \gamma\alpha \operatorname{div} u + \\ \qquad\qquad + \omega_2(\sigma,\alpha)(\alpha+n)\operatorname{div} u , \quad \text{in } \Omega , \\ u_{|\Gamma} = 0,\ \alpha_{|\Gamma} = 0 ; \end{cases}$$

the additional constraint is given by (1.3).

The linearized system is now

$$(6.4)\qquad \begin{cases} -\mu\Delta y - \nu\nabla \operatorname{div} u + k\nabla\sigma + \gamma\nabla\alpha = F \ , \\ m \operatorname{div} u + v\cdot\nabla\sigma + \sigma \operatorname{div} v = g \ , \\ -\chi\Delta\alpha + \gamma n \operatorname{div} u = H, \ \text{in} \ \Omega \ , \\ u_{|\Gamma} = 0, \ \alpha_{|\Gamma} = 0 \ . \end{cases}$$

In the sequel, c', c_0', c_1', β, denote positive constants, depending at most on Ω, on μ, ν, γ, m, χ, n, λ, λ', ℓ and S. The dependence of the constants c' on the above parameters can be easily checked.

One has the following result:

Theorem A'. *There exist positive constants* c_0' *and* c_1' *such that if* $f \in H^1$, $g \in \overline{H}_0^2$, $h \in L^2$, *and*

$$(6.5)\qquad \|f\|_1 + \|g\|_2 + \|h\|_0 \le c_0' \ ,$$

then there exists a unique solution $(u,\sigma,\alpha) \in H_0^3 \times \overline{H}^2 \times H_0^2$ *of problem* (6.1), *in the ball*

$$(6.6)\qquad \|u\|_3 + \|\sigma\|_2 + \|\alpha\|_2 \le c_1' \ .$$

The proof relies on the following result, for the linearized system (6.4):

Theorem B'. *Let* $F \in H^1$, $g \in \overline{H}_0^2$, $H \in L^2$, *and* $v \in H_{0,d}^3$. *There exists a positive constant* β, *such that if*

$$(6.7)\qquad \|v\|_3 \le \beta \ ,$$

then there exists a unique solution $(u,\sigma,\alpha) \in H_0^3 \times \overline{H}^2 \times H_0^2$ *of the linear system* (6.4). *Moreover,*

$$(6.8)\qquad \|u\|_3 + \|\sigma\|_2 + \|\alpha\|_2 \le c'(\|F\|_1 + \|g\|_2 + \|H\|_0) \ .$$

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We study the system of equations (1.1), describing the stationary motion of a compressible viscous fluid in a bounded domain Ω of $\mathbb{R}^3$. The total mass of fluid $m|\Omega|$, inside Ω, is fixed (condition (1.2)). We prove that for small f and g, there exists a unique solution (u,ρ) of the above system of equations, in a neighborhood of $(0,m)$. Moreover, by introducing a suitable parameter λ, we prove that the solution of the Navier-Stokes equations (1.14) are the incompressible limit of the solutions of the compressible Navier-Stokes equations (1.13). The proofs given here, apply, without

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ABSTRACT (continued)

supplementary difficulties, in the context of Sobolev spaces $H^{k,p}$, and other functional spaces. The results can be extended to the heat depending case, too.

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