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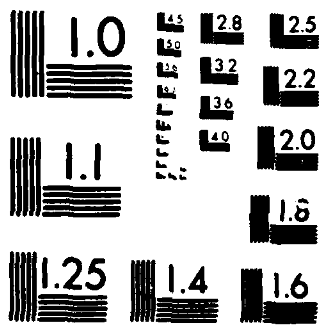

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MRC Technical Summary Report #2874

A PROOF OF GRACE'S THEOREM BY INDUCTION

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October 1985

(Received September 17, 1985)

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ABSTRACT

Two polynomials in $\mathbb{C}[z]$

$$(1) \qquad A(z) = \sum_{k=0}^{n} \binom{n}{k} a_k z^k, \quad B(z) = \sum_{k=0}^{n} \binom{n}{k} b_k z^k$$

are said to be apolar, provided that the equation

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k b_{n-k} = 0$$

Grace's theorem states:

holds. This definition was given at the turn of the century by J.H. Grace who established in [1] the following

Theorem of Grace. Let the polynomials (1) be apolar. If the circular region C contains all the zeros of $A(z)$, then C must contain at least one of the zeros of $B(z)$.

is meant

By a circular region ~~we mean~~ either the closed interior of a circle, or the closed exterior of a circle, or a closed half-plane.

~~Here we~~ give a proof of Grace's theorem by mathematical induction on the degree n.

AMS (MOS) Subject Classifications: 30C10, 30C15

Key Words: Zeros of polynomials; Möbius transformations.

Work Unit Number 3 (Numerical Analysis and Scientific Computing)

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

SIGNIFICANCE AND EXPLANATION

Two polynomials in $\mathbb{C}[z]$

$$(1) \qquad A(z) = \sum_{k=0}^{n} \binom{n}{k} a_k z^k, \quad B(z) = \sum_{k=0}^{n} \binom{n}{k} b_k z^k$$

are said to be apolar, provided that the equation

$$(2) \qquad \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k b_{n-k} = 0$$

holds. This definition was given at the turn of the century by J.H. Grace who established in [1] the following

Theorem of Grace. Let the polynomials (1) be apolar. If the circular region C contains all the zeros of $A(z)$, then C must contain at least one of the zeros of $B(z)$.

By a circular region we mean either the closed interior of a circle, or the closed exterior of a circle, or a closed half-plane.

Here we give a proof of Grace's theorem by mathematical induction on the degree n.

The references [3] and [2] give numerous applications of Grace's theorem. For $n = 2$ the apolarity equation (1.2) is equivalent to the equation

$$\frac{\beta_2-\alpha_2}{\beta_2-\alpha_1} : \frac{\beta_1-\alpha_2}{\beta_1-\alpha_1} = -1,$$

hence the pair of points (β_1,β_2) divides (α_1,α_2) in harmonic ratio.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

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A PROOF OF GRACE'S THEOREM BY INDUCTION

A.W. Goodman and I.J. Schoenberg

1. Introduction. At the turn of the century J.H. Grace [1] introduced the following

Definition 1. Two polynomials

$$A(z) = a_0 + \binom{n}{1}a_1 z + \cdots + \binom{n}{k}a_k z^k + \cdots + a_n z^n \tag{1.1}$$

and

$$B(1) = b_0 + \binom{n}{1}b_1 z + \cdots + \binom{n}{k}b_k z^k + \cdots + b_n z^n \tag{1.2}$$

are said to be apolar provided that their coefficients satisfy the apolarity condition

$$a_0 b_n - \binom{n}{1}a_1 b_{n-1} + \cdots + (-1)^k\binom{n}{k}a_k b_{n-k} + \cdots + (-1)^n a_n b_0 = 0. \tag{1.3}$$

The coefficients of the polynomials may be real or complex. If $a_r \neq 0$ $(r \geq 0)$ and $a_\nu = 0$ for $\nu = r+1, r+2, \ldots, n$, then we regard $z = \infty$ as an $(n-r)$-fold zero of $A(z)$. If all the coefficients of $A(z)$ are zero, then $A(z)$ is not regarded as a polynomial.

Grace discovered the following remarkable

Theorem of Grace. Let the polynomials (1.1) and (1.2) be apolar. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the zeros of $A(z)$ and $\beta_1, \beta_2, \ldots, \beta_n$ be the zeros of $B(z)$. If the circular region C contains all of the α_ν, then C must contain at least one of the β_ν.

By a circular region we mean either the closed interior of a circle, or the closed exterior of a circle, or a closed half-plane.

In [3] G. Szegö gave a proof of Grace's theorem freed of the invariant-theoretic concepts used by Grace in [1], and he also gave a large number of applications. In the present note we establish Grace's theorem by induction on n. Our proof is different from those given earlier.

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2. The invariance of apolarity by Möbius transformations.

By the transform of $A(z)$ under the Möbius transformation

$$z = \frac{aw+b}{cw+d} \quad (ad-bc \neq 0) \tag{2.1}$$

we mean the polynomial

$$A^*(w) \equiv (cw+d)^n A\left(\frac{aw+b}{cw+d}\right) \equiv \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu (aw+b)^\nu (cw+d)^{n-\nu} \equiv \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu^* w^\nu.$$

For example if $A(z) \equiv 1$, then $A^*(w) = (cw+d)^n$ and the n-fold zero of $A(z)$ at $z = \infty$ becomes an n-fold zero of $A^*(z)$ at $w = -d/c$ if $c \neq 0$.

Lemma 1. Let $A(z)$ and $B(z)$ be apolar polynomials. If the Möbius transormation (2.1) changes the polynomials (1.1) and (1.2) into

$$A^*(w) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu^* w^\nu \quad \text{and} \quad B^*(w) = \sum_{\nu=0}^{n} \binom{n}{\nu} b_\nu^* w^\nu, \tag{2.2}$$

then the polynomials (2.2) are also apolar.

Proof. It sufficies to prove Lemma 1 for each of the three special transformations

$$\text{(i)} \quad z = w + h, \quad \text{(ii)} \quad z = kw, \quad \text{(iii)} \quad z = \frac{1}{w} \ . \tag{2.3}$$

(i) $$A^*(w) = A(w + h) = \sum_{\nu=0}^{n} \frac{w^\nu}{\nu!} A^{(\nu)}(h)$$

and therefore

$$A^*(w) = \sum_{\nu=0}^{n} \binom{n}{\nu} \frac{(n-\nu)!}{n!} A^{(\nu)}(h) w^\nu.$$

Similarly

$$B^*(w) = \sum_{\nu=0}^{n} \binom{n}{\nu} \frac{(n-\nu)!}{n!} B^{(\nu)}(h) w^\nu.$$

The apolarity equation for these polynomials is

$$f(h) \equiv \sum_{\nu=0}^{n} (-1)^{\nu}\binom{n}{\nu} \frac{(n-\nu)!}{n!} A^{(\nu)}(h)\frac{\nu!}{n!} B^{(n-\nu)}(h) = 0$$

or

$$\text{(2.4)} \qquad n!\, f(h) = \sum_{\nu=0}^{n} (-1)^{\nu} A^{(\nu)}(h) B^{(n-\nu)}(h) = 0.$$

The apolarity of $A(z)$ and $B(z)$ gives $f(0) = 0$, and we must show that $f(h) = 0$ for all h. This will follow as soon as we show that for all h

$$\text{(2.5)} \qquad f'(h) = 0.$$

From (2.4) we find that

$$n!\, f'(h) = \sum_{\nu=0}^{n} (-1)^{\nu} A^{(\nu+1)}(h) B^{(n-\nu)}(h) + \sum_{\nu=0}^{n} (-1)^{\nu} A^{(\nu)}(h) B^{(n-\nu+1)}(h).$$

Here the νth term $(\nu<n)$ in the first sum cancels with the $(\nu+1)$-st term in the second term, and hence

$$n!\, f'(h) = (-1)^{n} A^{(n+1)}(h) B(h) + A(h) B^{(n+1)}(h)$$

which is evidently zero because $A(z)$ and $B(z)$ are nth degree polynomials. This proves (2.5) and therefore (2.4) for all h.

(ii) For the second transformation in (2.3) we have

$$A^{*}(w) = a_0 + \binom{n}{1} a_1 kw + \cdots + a_n k^n w^n,$$

and

$$B^{*}(w) = b_0 + \binom{n}{1} b_1 kw + \cdots + b_n k^n w^n$$

which are evidently apolar by (1.3).

(iii) Finally, setting $z = 1/w$ gives

$$A^{*}(w) = a_n + \binom{n}{1} a_{n-1} w + \cdots + a_0 w^n$$

and

$$B^*(w) = b_n + \binom{n}{1}b_{n-1}w + \cdots + b_0w^n$$

and these are also apolar by (1.3). ∎

Lemma 2. If α is a zero of the polynomial $A(z)$, then its transform β under (2.1) is a zero of the transformed polynomial $A^*(w)$.

If neither α nor β is ∞, then $\alpha = (a\beta+b)/(c\beta+d)$ and

$$A^*(\beta) = (c\beta+d)^n A\left(\frac{a\beta+b}{c\beta+d}\right) = (c\beta+d)^n A(\alpha) = 0. \tag{2.6}$$

If $\alpha = \infty$ is an r-fold zero of $A(z)$, then $\beta = -d/c$ is clearly an r-fold zero of $A^*(z)$. If $\alpha = a/c$ is an r-fold zero of $A(z)$, then the decomposition used in the proof of Lemma 1 shows that $\beta = \infty$ is an r-fold zero of $A^*(z)$. ∎

It follows from Lemma 2 that if a circular domain C contains all the zeros of $A(z)$ then the transformed domain under (2.1) will contain all the zeros of $A^*(z)$.

3. Proof of Grace's Theorem. We use induction on n. For $n = 1$, the apolarity condition (1.3) gives $a_0b_1 - a_1b_0 = 0$ so $\alpha_1 = \beta_1$ and the theorem is obviously true.

Next we assume the theorem is true for index $n-1$ and wish to prove that it is also true for index n. Here we use the method of contradiction. We shall assume that for some circular domain C and some pair of apolar polynomials $A(z)$ and $B(z)$

$$\alpha_\nu \in C, \quad \nu = 1,2,\ldots,n, \quad \text{and } \beta_\nu \notin C, \ \nu = 1,2,\ldots,n. \tag{3.1}$$

By a transformation we may assume that $\beta_n = \infty$, without loss of generality (use Lemmas 1 and 2). It follows that in (1.2)

$$b_n = 0. \tag{3.2}$$

The second assumption in (3.1) tells us that $\beta_n \notin C$ and hence C is bounded. Therefore all α_ν are finite and hence $a_n \neq 0$. The points $\beta_1, \beta_2, \ldots, \beta_{n-1}$ (finite or not) are the zeros of

$$B(z) = b_0 + \binom{n}{1}b_1z + \cdots + \binom{n}{k}b_kz^k + \cdots + \binom{n}{n-1}b_{n-1}z^{n-1} \tag{3.3}$$

which *we now* regard as a polynomial of degree $n-1$. Now consider the polynomial

$$\frac{1}{n}A'(z) = a_1 + \binom{n-1}{1}a_2z + \cdots + \binom{n-1}{k}a_{k+1}z^k + \cdots + a_nz^{n-1} \tag{3.4}$$

having the zeros $\gamma_1, \gamma_2, \ldots, \gamma_{n-1}$. These zeros are all finite because $a_n \neq 0$.

We claim the two polynomials (3.3) and 3.4) are apolar as polynomials of degree

$n-1$. To confirm this we rewrite (3.3) in the usual form

$$B(z) = b_0' + \binom{n-1}{1} b_1' z + \cdots + \binom{n-1}{k} b_k' z^k + \cdots b_{n-1}' z^{n-1}. \tag{3.5}$$

Then

$$\binom{n}{k} b_k = \binom{n-1}{k} b_k', \quad k = 0,1,2,\ldots,n-1. \tag{3.6}$$

But then our original apolarity condition (1.3)

$$\sum_{k=0}^{n-1} (-1)^k \binom{n}{k} b_k a_{n-k} = 0$$

(since $b_n = 0$ by (3.2)) becomes

$$\sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} b_k' a_{n-k} = 0.$$

This shows that the polynomials (3.4) and (3.5) are apolar.

We now appeal to the Gauss-Lucas Theorem which states that all the zeros $\gamma_1, \gamma_2, \ldots, \gamma_{n-1}$ are in the convex hull of the zeros $\alpha_1, \alpha_2, \ldots \alpha_n$ of $A(z)$. By our first assumption (3.1) we conclude that $\gamma_\nu \in C$, for $\nu = 1,2,\ldots,n-1$. On the other hand $\beta_\nu \notin C$ for $\nu = 1,2,\ldots,n-1$. This contradicts Grace's Theorem for index $n-1$. Hence by the principle of mathematical induction Grace's Theorem is true for every positive integer n. ∎

The reader is referred to Szegös work [3] and the book by Marden [2] for many interesting applications of Grace's Theorem.

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1. J.H. Grace, The zeros of polynomials, Proc. Cambridge Philos. Soc. 11 (1900-1902) 352-357.
2. Morris Marden, Geometry of polynomials, 2nd edition. Math Surveys #3. Amer. Math. Soc. 1966 Providence, Rhode Island.
3. Gabor Szegö, Bemerkungen zu einem Satz von J.H. Grace über die Wurzeln algebraischer Gleichungen, Math. Zeit. 13 (1922) 28-56.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 2874	2. GOVT ACCESSION NO. AD-A163714	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A PROOF OF GRACE'S THEOREM BY INDUCTION		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) A.W. Goodman and I.J. Schoenberg		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53705		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 3 - Numerical Analysis and Scientific Computing
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE October 1985
		13. NUMBER OF PAGES 6
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Zeros of polynomials, Möbius transformations

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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