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SOLUTIONS OF

NONLINEAR MATRIX EQUATIONS

Thesis

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Captain, USAF

AFIT/GOR/MA/85D-1

DEPARTMENT OF THE AIR FORCE

AIR UNIVERSITY

# AIR FORCE INSTITUTE OF TECHNOLOGY

Wright-Patterson Air Force Base, Ohio

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Thesis

Presented to the Faculty of the School of Engineering

of the Air Force Institute of Technology

Air University

in Partial Fulfillment of the

Requirements for the Degree of

Master of Science

by

Bruce W. Colletti, B.S.

Captain, USAF

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My earliest memory of mathematics recalls writing
numbers in the spaces of a grid.  Little did the first
grader know that by creating this "box of numbers", he was
laying the groundwork for subsequent study in matrices,
themselves very special "boxes of numbers."  After studying
matrix theory as a high school freshman, I remember hoping
for a chance someday to pursue it in great depth.  Though
the hope has now been realized, it will by no means be put
to rest:  for my master's in mathematics, I will pick up
where this thesis left off.

Fortune doesn't always favor us, and as Machiavelli
says, will strike after being wooed and cajoled.  However,
she must have been in a good mood in the Fall of '84 because
our class had Dr. John Jones Jr for Numerical Analysis.  His
presentation of matrix algebraic concepts delighted me, and
upon discovering his abiding interest and renown with matrix
equations, decided to study under him.  Undoubtedly,
Dr Jones was one of two teachers having a profound impact on
my mathematical development.  I couldn't have wished for a
finer advisor and coach than him.  His patience and positive
outlook sustained me during those times when I was setting
back mathematical science instead of advancing it.  For him,
theory and applications are one, though primacy goes to
theory.  Just as it should be.  Thanks for your interest in
me, Dr Jones.  I will return your concern years hence to one

of my students.

     To LTC (and Dr.) Charles Ebeling, my reader in the Operations Research Department, I want to thank for being (I believe) a mathematician at heart, and hence an ally in the department.  If I couldn't have done a mathematical thesis, I would have done a simulation one and sought you out as an advisor.  Thanks for not discouraging me from pursuing an interest begun years ago.

# Contents

## List of Figures

## List of Symbols

| | |
|---|---|
| $O_{m \times n}$ | an m X n matrix whose entries are all zero |
| $A \blacktriangle B$ | tensor product of matrices A and B |
| $A_{m \times n}$ | the m X n matrix A |
| $\mathcal{C}$ | the set of complex numbers |
| $\mathcal{C}(\Gamma)$ | the set of polynomials with coefficients in the complex numbers and variables in the components of the n-tuple $\Gamma$ |
| $\mathcal{C}^{m \times n}(\Gamma)$ | the set of m X n matrices whose entries are polynomials with coefficients in the complex numbers and variables in the components of the n-tuple $\Gamma$ |
| $\mathcal{C}(x)$ | the set of polynomials in the variable x |
| $\mathcal{C}(x,y)$ | the set of polynomials in the variables x,y |
| det(A) | the determinant of the matrix A |
| iff | if and only if |
| $I_{n \times n}$ | the n X n identity matrix |
| rank(A) | the rank of the matrix A |
| $S_1 - S_2$ | the set of elements which are in the set $S_1$ but not in the set $S_2$ |
| $\Gamma$ | an arbitrary n-tuple $(\Gamma_1, \ldots, \Gamma_n)$ |
| $x \in S$ | the entity x is contained in the set S |
| $Z^+$ | the set of positive integers $\geq 0$ |

## Abstract

This paper seeks the solutions to a system of equations (equalities) in n variables by expressing the system in matrix algebraic form. Properties of the solutions to the ensuing matrix equation are investigated using similarity transformations. The three types of matrix equations to be studied are the linear equation -

$$AX = b$$

the Lypanov equation

$$AX - XB = C$$

the second-order Riccati equation —

$$XDX + AX + XB + C = 0$$

and the third-order Riccati equation —

$$XAXBX + XCX + DX + XE + F = 0$$

The entries of all matrices, including the solution X, are restricted to being polynomials in $\Gamma$ having complex coefficients, where $\Gamma$ is the n-tuple of indeterminates. That is, all matrices are elements of the ring $\mathbb{C}^{m \times n}(\Gamma)$ for m and n of appropriate size.

Because adding and multiplying matrices (having multivariate polynomial entries) is tedious in practice, an interactive BASIC program is presented in the appendix. This program, which can be used on a personal computer, permits the user to perform operations on matrices having multivariate polynomial entries. Via menu selections, the user may perform

-weighted addition between two matrices

-multiplication between two matrices
-create matrices, with an option of building a diagonal
 matrix whose diagonal entries are all equal
-view matrices
-transpose a matrix
-extract special submatrices (U,M,V,N of Chapter IV)
 from a given matrix.

# I   Overview and Literature Review

**Overview.**   This paper seeks the solutions of the third-order Riccati matrix equation

$$XAXBX + XCX + DX + XE + F = 0 \qquad (1.1)$$

where the entries of the matrices A,B,C,D,E,F,X are multivariate polynomials.   In solving (1.1) the linear matrix equation

$$AX = b$$

the Lypanov matrix equation

$$AX - XB = C$$

and the second-order Riccati matrix equation

$$XDX + AX + XB + C = 0$$

will be addressed because the form of their solutions will hint at the nature of those to (1.1).   Because cursory knowledge of matrix algebra is sufficient to motivate the paper's thesis—the solution to (1.1)—the rigorous (and lengthy) definition of terms and statement of objective will be made in the next chapter.

Why a matrix equation (such as those given above) is worthy of attention, let alone finding its solution, is a legitimate concern.   Practical problems often arise which require the solution to a system of m equations in n unknowns, e.g., the system of equations

$$ax + by = c \qquad (1.2)$$
$$dx + ey = f$$

where all variables except x and y are known.   One way to

1.1

find the solutions x and y is to express the system in its
equivalent matrix form

$$\begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix}$$

(which is a linear matrix equation) and solve for x and y,
i.e.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix}^{-1} \begin{bmatrix} c \\ f \end{bmatrix}$$

Thus, a system of equations may give rise to a matrix
equation whose solution in turn gives the answer to the
original system.

Other more complex systems may have a matrix
representation. For instance

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \end{bmatrix} \qquad (1.3)$$

$$= \begin{bmatrix} xa + yc & xb + yd \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} x(xa + yc) + y(xb + yd) \end{bmatrix}$$

$$= \begin{bmatrix} x^2a + xy(c + b) + y^2d \end{bmatrix}$$

which means that the system consisting of the one equation

$$x^2a + xy(c + b) + y^2d = e$$

is represented by the matrix equation (1.3). These two
examples show that matrix equations warrant attention if for
no other reason than they can help solve systems of
equations arising in practical problems. Indeed, given the
nonlinear optimization challenge

Maximize $f(x)$ subject to                              (1.4)

equality constraint$_1$
.
.
equality constraint$_n$

the constraint set may be expressable as a matrix equation.
If so, this equation can be solved to identify the feasible
region determined by the constraints and thence the optima.
This approach is an alternative to solving (1.4) using, say,
Lagrangian multipliers and partial differential calculus.

In examples (1.2) and (1.4), the variables are often
taken to represent real or complex numbers. However, there
is nothing stopping the variables from assuming functional
values. For instance (1.2) may assume the form

$$a(u,v)x + b(u,v)y = c(u,v)$$
$$d(u,v)x + e(u,v)y = f(u,v)$$   (1.5)

where a,b,c,d,e,f are functions of the parameters u and v.
Though the solutions x and y of (1.5) wouldn't necessarily
have numeric values, this is all right because all other
variables are functions:  x and y will likely take on
functional forms. To illustrate,

$$\begin{bmatrix} u^2v & u \\ uv & v \end{bmatrix} \begin{bmatrix} u + v \\ u - v \end{bmatrix} = \begin{bmatrix} u^2v(u + v) + u(u - v) \\ uv(u + v) + v(u - v) \end{bmatrix}$$

$$= \begin{bmatrix} u^3v + u^2v^2 + u^2 - uv \\ u^2v + uv^2 + uv - v^2 \end{bmatrix}$$

and so a solution to the matrix equation

$$\begin{bmatrix} u^2v & u \\ uv & v \end{bmatrix} \begin{bmatrix} x(u,v) \\ y(u,v) \end{bmatrix} = \begin{bmatrix} u^3v + u^2v^2 + u^2 - uv \\ u^2v + uv^2 + uv - v^2 \end{bmatrix}$$

is $x(u,v) = u + v$  and $y(u,v) = u - v$. As given in (1.1),
this paper will address matrix equations whose entries

1.3

assume the functional form of a polynomial in several variables.

Another area where matrix equations appear (where the matrices' entries are multivariate functions) is control theory, the science addressing the orbital natures of objects in space.  Matrix equations frequently arise from systems of differential equations which describe a satellite's orbit.  For example, given X(t) a columnar matrix whose entries (each describing an aspect of a satellite's orbit) are functions in the parameter t, a theorem from control theory states that the equilibrium of the system of differential equations

$$\frac{dX}{dt} = FX$$

is asymptotically stable if the Lypanov matrix equation

$$F^T P + PF = -C \qquad (1.6)$$

has a positive-definite solution P for any matrix C > 0. Though the terms in this theorem won't be defined here (see [4:144-152]), the important item is that a satellite's orbital stability depends on the solution of a Lypanov matrix equation (1.6).

Matrix equations can also describe the path taken by an X-ray passing through matter.  This is a concern of the medical community since the quality of CAT scan pictures depends on the manner in which the X-rays penetrate the skull.  A good description of the path may allow finding a

brain tumor, whereas a poorly chosen angle of entry may not.

R. Vasudevan [32] develops higher order Riccati equations

describing how beam particles scatter upon hitting matter,

e.g. the skull.  Simply put, scattering is modelled by

including higher powers of a solution matrix X in a Riccati

equation.  For instance, scattering may be roughly described

by the second order Riccati equation

$$XDX + AX + XB + C = 0$$

and more completely by the third order Riccati equation

$$XAXBX + XCX + DX + XE + F = 0$$

where the coefficient matrices A,B,C,D,E,F reflect known

characteristics of the environment in which the X-rays

behave.  Higher powers of X will more accurately describe

scattering.  Bellman and Vasudevan [2] describe techniques

reducing a given Riccati equation to one of lower form.

This reduction (quasi-linearization) leads to a series which

converges to the actual solution of the original equation.

   <u>Literature Review</u>.  The literature supporting this

paper addresses three topics:

   I   generalized inverses of matrices
   II  algebraic structures of matrices over fields and
       rings
   III the Lypanov and Riccati equations

   <u>Category I</u>.  An understanding of matrix

generalized inverses launched the research behind this

paper.  Matrix generalized inverses encompass the

traditional notion of an inverse by assigning these to

non-square matrices.  The theory is well developed for

<div align="center">1.5</div>

matrices whose entries are complex numbers, and is described below.

The physicist R. Penrose [28] proved in 1955 that for any matrix A whose entries are complex numbers, there exists matrices U,Y,Z,W having complex entries such that:

1.  AUA = A                                                    (1.7)
2.  YAY = Y
3.  AZ equals its conjugate transpose (i.e., AZ is Hermitian)
4.  WA is Hermitian

The matrices U,Y,Z,W are known as the "generalized inverses" of A. Penrose also showed the existence of a matrix X which simultaneously satisfied conditions (1) thru (3), and a Y simultaneously satisfying (1), (2) and (4). Penrose also showed that if a matrix X satisfied all conditions (given a matrix A), then X was unique. From the traditional matrix algebra viewpoint, this unique X corresponds to the familiar inverse of a square matrix A whose determinant is not equal to zero. Though Penrose's work was original, he was unaware that the mathematician E.H. Moore [24] had proved (using abstract algebra) the existence of generalized inverses for arbitrary rings more than three decades earlier.

Rao and Mitra's textbook [29] is devoted to the study of generalized inverses, and is a frequently cited authority in this area. The first two chapters of [6] present techniques (with the justifying theorems) generating various types of generalized inverses for a given matrix A, while Captain Craig Murray, AFIT Class GCS 85D, has recently

1.6

completed the programming of these techniques.  Two papers,
[12] and [21], use generalized inverses to prove theorems
about special types of matrix equations whose entries are
complex numbers.

The power of generalized inverses weakens as one moves
from matrices with complex entries to those with
multivariate entries, the subject of this paper.  This is
because the theory in the latter is still very young.
Sontag [31] discusses the existence of some generalized
inverses for special matrices, while Jones [11] uses these
generalized inverses to solve certain matrix equations.

Category II.  Most of this paper's research dealt
with the decisive influence on the existence and form of
solutions to matrix equations:  the algebraic structure of
matrices.  An illustration of the importance of algebraic
structure is readily given:  the solution of the equation

$$x + 7 = 5 \tag{1.8}$$

cannot be found among the set of positive integers $Z^+$ (even
though the coefficients 7 and 5 are positive integers)
because its structure doesn't include negative numbers.
Since $-2$ is the solution, (1.8) must be placed in a larger
algebraic structure in order for a solution to exist.  Thus
one moves from $Z^+$ to the set of all integers Z.  Though Z
has a larger algebraic structure than $Z^+$ it still isn't
large enough to contain the solution of the equation

$$2x - 8 = 7 \tag{1.9}$$

1.7

even though the coefficients 2,-8,7 are found in Z.  It now becomes necessary to move to a larger structure (the set of rational numbers) to find the solution to (1.9), 15/2. Likewise, to solve the equations $x^2 = 2$ and $x^2 + 1 = 0$  one must move to the larger algebraic structures found in the irrational and complex numbers respectively.

Matrices too lend themselves to different structures. However, not only must the form of a matrix be addressed . (e.g., whether it's invertible, whether it's similar to another matrix), but the structure in which individual entries are found must also be considered.  For instance, decomposing a matrix into a product of other matrices may depend upon whether or not an entry is irreducible in its own setting.  A trivial example is decomposing the 1 X 1 matrix

$$[x^2 + x - 1] \tag{1.10}$$

into a product of two other 1 X 1 matrices each having polynomial entries whose coefficients are integer, as is the entry in (1.10).  Such a decomposition is impossible because $x^2 + x - 1$  cannot be expressed as a product of two polynomials with integer coefficients.  Indeed, the roots of this equation are  $(-1 \pm \sqrt{5}) / 2$ , which are irrational.

In general, this paper allows matrix entries to be multivariate polynomials (having complex coefficients, e.g., $x^2 + xy$, $xy^2z - (2 + 3i)x^6yz$), who in turn have complex algebraic structures.  In fact, a solid understanding of

1.8

abstract algebra would have helped make several articles intelligible.  Though Fraleigh's text [7] was found to be an excellent primer on abstract algebra, time did not allow sufficient study of a subject key to this paper.  Wang [33] gives an algorithm for irreducible factoring of multivariate polynomials having coefficients in an arbitrary algebraic number field.  McClellan [23] presents methods for solving systems of equations involving univariate polynomials with rational coefficients (this paper highlights his doctoral dissertation).  Two papers from the early twentieth century, [3] and [27], discuss the form of factors between polynomials.

The algebraic structures of matrices was addressed by Frost and Storey [8] and Lee and Zak [20].  These special structures, called Smith Forms, are discussed in the next chapter.  Unfortunately, only the Smith Forms of matrices having bivariate polynomial entries can be addressed:  the Smith Form involving multivariate entries remains an unresolved issue in mathematics.

Category III.  Working from a base rooted in generalized inverses and recognizing the critical role of algebraic structures on matrices (and again among their entries), a study of matrix equations can proceed.  Though much has been published on finding solutions to systems of equations, many authors succeed in finding only particular solutions.  After all, finding the general forms of all

solutions may have either been a far too ponderous task or
the mathematical approach proved elusive.  In any case,
viewing a system of multivariate equations as a single
matrix equation (having a solution in its own right) is not
commonplace.  Depending on the nature of the system, the
entire solution set may be found by solving a matrix
equation representing the system.

Work has been done on matrix equations having complex
entries.  Roth [30] (a frequently cited paper) wrote on the
Lypanov matrix equation in 1952, and had his work extended
in 1972 by Jones [12] who also addressed the second order
Riccati equation, using generalized inverses to identify
solutions.  Morris and Odell [25] attempted to find the
common solutions to a set of linear matrix equations
$\{A_k X = B_k\}$ by using generalized inverses.  Lancaster [18]
provides several approaches (none using generalized
inverses) to solving the matrix equation

$$\sum A_k X B_k = C$$

As matrix equations assume multivariate polynomial
entries, the constraints cited in the previous section
seriously handicap the search for solutions.  As a result,
pioneering work in these equations is still underway.
The papers [11], [16] and [17] give recent results connected
with the Lypanov and second/third order Riccati equations.
Collectively, these papers address the roles of generalized
inverses and matrix algebraic structures in the search for

1.10

solutions to these matrix equations.

## II  Foundations

This chapter lays the groundwork for later developments.  Because this paper seeks solutions to matrix equations, the latter's nature will be addressed first.

Matrix Equations.  A matrix equation is an algebraic expression whose arguments involve matrices.   Matrix equations are frequently used to represent a kindred system of equations.  For instance, the system of equations

$$2x + 3y = 7$$
$$4x - 2y = 1$$

is represented by the matrix equation

$$\begin{bmatrix} 2 & 3 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

which has the familiar form

$$AX = b \qquad\qquad (2.1)$$

where

$$A = \begin{bmatrix} 2 & 3 \\ 4 & -2 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \end{bmatrix} \qquad b = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

The reason for expressing a system of equations in matrix algrebraic form is, of course, so that the machinery of matrix algebra can be used to find a solution to the given initial system.

Other systems of equations have matrix representations. For instance, the system

$$x + 2u = 5$$
$$3x + 4u = 7$$
$$9x + 10u = 23$$
$$y + 2v = 6$$
$$3y + 4v = 8$$
$$9y + 10v = 24$$

2.1

has the matrix form

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 9 & 10 \end{bmatrix} \begin{bmatrix} x & y \\ u & v \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \\ 23 & 24 \end{bmatrix}$$

which in turn has the form (2.1) where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 9 & 10 \end{bmatrix} \quad X = \begin{bmatrix} x & y \\ u & v \end{bmatrix} \quad b = \begin{bmatrix} 5 & 6 \\ 7 & 8 \\ 23 & 24 \end{bmatrix}$$

Thus in viewing (2.1), it isn't necessary that b and X be one-columnar matrices and A square, as is traditionally presented.

Lypanov Equation. The linear matrix equation (2.1) leads to the slightly more complex equation

$$AX + XB = C \qquad\qquad (2.2)$$

which is known as the Lypanov equation. In order for (2.2) to be defined, the matrices A,B must be square (though not necessarily of the same size) and the matrices X,C must share the same size (though they may be rectangular). To see why this is true, let the following hold:

```
MATRIX      A       X       B
SIZE      a X b   c X d   e X f
```

For AX to be defined, b must equal c: b = c. For XB to be defined, d = e. AX is a X d and XB is c X f. For AX + XB to be defined, a = c and d = f. Since b = c and a = c, a = b and A is square c X c. Similarly for B: d = e and d = f and so e = f which makes B square d X d. It follows that AX + XB is c X d which is the size of X.

To illustrate (2.2), the system

2.2

$$( x + 2u) + (5x + 8y + 11z) = 14 \qquad (2.3)$$
$$( y + 2v) + (6x + 9y + 12z) = 15$$
$$( z + 2w) + (7x + 10y + 13z) = 16$$
$$(3x + 4u) + (5u + 8v + 11w) = 17$$
$$(3y + 4v) + (6u + 9v + 12w) = 18$$
$$(3z + 4w) + (7u + 10v + 13w) = 19$$

has the Lypanov form

$$
\overset{(A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}
\overset{X}{\begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix}}
\; + \;
\overset{X}{\begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix}}
\overset{B)}{\begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \\ 11 & 12 & 13 \end{bmatrix}}
$$

$$
= \overset{(C)}{\begin{bmatrix} 14 & 15 & 16 \\ 17 & 18 & 19 \end{bmatrix}}
$$

Of course, (2.3) can be expressed as (2.1) by combining like terms, thereby creating six equations in six unknowns. However, it will be seen (Chapter IV) that by solving the Lypanov equation instead of its equivalent linear form (2.1), solutions of a higher order matrix equation can be derived, one in which the Lypanov is embedded.   This higher equation is known as the second-order Riccati equation.

Riccati Equation and Higher.   In high school algebra, a student moves from solving the simple linear equation  bx = c  to the higher order quadratic equation $ax^2 + bx = c$   (in which is embedded the linear equation bx = c).   In the present setting, the former equation is likened to the Lypanov matrix equation and the second-order Riccati matrix equation to the quadratic.   The Riccati equation has the form

$$XDX + AX + XB + C = 0 \qquad (2.4)$$

For (2.4) to be defined, A and B must be square (possibly

2.3

not the same size), D must have the size of $X^T$, and X,C

share the same size.   The argument for this follows that of

the Lypanov equation.

To illustrate (2.4), the Riccati form of the system

$$
\begin{aligned}
x^2 + uy + 2x &= -1 \qquad\qquad (2.5)\\
xy + yv + 2y &= -2\\
xz + yw + 2z &= -3\\
ux + vu + 2u &= -4\\
uy + v^2 + 2v &= -5\\
uz + vw + 2w &= -6
\end{aligned}
$$

is

$$
\overset{(X}{\begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix}}
\overset{D}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}
\overset{X}{\begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix}} +
\overset{A}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}
\overset{X)}{\begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix}} +
$$

$$
\overset{(X}{\begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix}}
\overset{B}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} +
\overset{D}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} =
\overset{0)}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}
$$

As with the Lypanov equation, there may exist many solutions

to the second-order Riccati equation.   In fact it will be

seen in Chapters III and IV that the solutions of (2.4) will

satisfy pairs of equations determined by the matrices

A,B,C,D in (2.4).

As it is an easy matter to create a polynomial of a

given degree (e.g., $3x^5 + 3x^2 - 7$), so is it also to create

a higher order matrix equation.   For instance, the third

order Riccati equation

$$XAXBX + XDX + AX + XB + C = 0 \qquad\qquad (2.6)$$

or the related matrix equation

$$EXAXBXF + GXDXH + AXJ + KXB + C = 0$$

where the matrices not equal to X are known and so can be

2.4

thought of as coefficients of a polynomial in x. However,
the non-commutativity of matrix multiplication gives the
placement of each coefficient matrix a decisive influence on
the solution.

A natural question that arises is whether or not a
given system has an accompanying matrix form. At present,
the only known way to answer this question is to experiment,
an understandably unattractive task. However, if a matrix
representation is derived (or stumbled upon), the concepts
presented in this paper will help to identify the solutions
to the given matrix equation, and thus to its underlying
system of equations.

Matrices Over a Ring. Two items determine the nature
of a matrix: its size and entries. Concerning the latter,
a matrix is said to be "over a set S" if and only if its
entries come from the set S. For example if S = {1,2,3}
then the 1 X 4 matrix

$$A \quad = \quad [3 \ 1 \ 1 \ 2]$$

is "over S" because each of its entries belong to S. However,
the 2 X 3 matrix

$$B \quad = \quad \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & \wedge \end{bmatrix} \qquad\qquad (2.7)$$

is not "over S" because B(2,3) = 0, which is not a member of
S. If however, S = {the real numbers} , then B is "over S".
Likewise, if

2.5

$$A = \begin{bmatrix} 2 + 3i & 7 \\ 9 & 8 - 7i \end{bmatrix} \qquad (2.8)$$

then A is not a matrix over the real numbers, but is a matrix over the complex numbers.  In this paper, the set of complex numbers will be given by the symbol ¢ (see the "List of Symbols").

These examples lead to a notation used throughout this paper:  given a set S, the set of all m X n matrices whose entries are in S is symbolically given by

$$S^{m \times n} \qquad (2.9)$$

If A is an m X n matrix whose entries are in S, then the statement "A is an element of $S^{m \times n}$" is represented by

$$A \in S^{m \times n}$$

Example:  From (2.8),  $A \in ¢^{2 \times 2}$  because A is a 2 X 2 matrix whose entries are complex.  However from (2.7), B is not in $¢^{2 \times 2}$ because B is a 2 X 3, despite the fact that the entries of B are elements of the complex numbers (the set of real numbers is a subset of the complex numbers).  However, $B \in ¢^{2 \times 3}$ ∎

The concept of "overness" will now be extended.  Let

$$S(x) = \{\text{all polynomials in the variable } x \text{ with coefficients in the set } S\} \qquad (2.10)$$

Thus if S = {1,2,3}, then

$$f(x) = x^2 + 2x + 3 \in S(x)$$

because f(x) is a polynomial in x with coefficients in S. However

$$f(y) = y^2 + 2y + 3$$

2.6

is not in $S(x)$ because $f(y)$ is a polynomial in y.  Likewise,

$$f(x) = 7x^2 + 2x + 3$$

is not in $S(x)$ because the coefficient of $x^2$ is seven, which

's not found in $S = \{1,2,3\}$.  It follows that $¢(x)$ is the

set of polynomials (in the variable x) with complex

coefficients.

Similar to (2.10), let

$$S(x,y) = \{\text{polynomials in the variables } x,y \\ \text{whose coefficients are in the set } S\}$$

Thus if $S = \{4,5,2\}$, then

$$f(x,y) = 4x^{10}y^5 + 2x^2y + 5x + 5y \ \notin \ S(x,y)$$

because $f(x,y)$ is a polynomial in x,y with coefficients in

S.  It follows that $¢(x,y)$ is the set of polynomials in the

variables x,y with complex coefficients.

In general, let $\Gamma$ represent the n-tuple $(x_1,x_2,\ldots,x_n)$.

Then given a set S, $S(\Gamma)$ is defined to be the set of all

polynomials (in the variables found in $\Gamma$) with coefficients

in the set S.

Example:  If $\Gamma = (x,y,z,v,w)$, then $¢(\Gamma)$ is the set of

all polynomials in the variables x,y,z,v,w with complex

coefficients.  Thus

$$f(x,y,z,v,w) = (4+5i)x^2 + xyzvw - (2+12i)w^7 \ \in \ ¢(\Gamma).$$

However if S is the set of real numbers, then $f(x,y,z,v,w)$

is not in $S(\Gamma)$.  ∎

Now to extend notation to matrices.  Define

$$S^{m \times n}(\Gamma) = \{\text{m X n matrices whose} \qquad (2.11) \\ \text{entries are in } S(\Gamma)\}$$

2.7

Thus if $S = \{1,2,4,8,9\}$ and $\Gamma = (x,y,z)$ and

$$A = \begin{bmatrix} x^2yz + 9z & x \\ 2x & 4yz + 8z^2 \end{bmatrix}$$

then $A \in S^{2\times 2}(\Gamma)$ because A is 2 X 2, and each entry of A is

a polynomial in $\Gamma = (x,y,z)$ with coefficients in S.  However

if $S = \{1,2,4,8\}$, then A would no longer be in $S^{2\times 2}(\Gamma)$

because a coefficient of A(1,1) isn't in S (namely, 9).

From (2.11), $\mathcal{C}^{m\times n}(\Gamma)$ is the set of all  m X n  matrices

whose entries are polynomials in $\Gamma$ having complex

coefficients.

    With this final definition, the object of this paper

can be stated:  find the solution $X \in \mathcal{C}^{m\times n}(\Gamma)$ of the

third-order Riccati equation

$$XAXBX + XCX + DX + XE + F = 0 \qquad (2.12)$$

given $\Gamma$ the tuple of indeterminates in the polynomial

entries of $A,B,C,D,E,F \in \mathcal{C}^{k\times j}(\Gamma)$ for k,j,m,n of appropriate

size.

    One may feel that in (2.12), mandating  $X \in \mathcal{C}(\Gamma)$  is

unduly restrictive.  After all, forcing the solutions of

$x^2 - 2 = 0$  (a polynomial in x over the integers, i.e., a

polynomial with integer coefficients) to again be a

polynomial over the integers isn't possible, since $\sqrt{2}$ is

irrational.  Also, given the matrix equation  AX = b  with

the entries of A,b integer and A square and invertible, it

may not be possible to have the solution  $X = A^{-1}b$  to again

have integer entries.  Why then require X in (2.12) to have

entries in $\mathcal{C}(\Gamma)$ when (like these two analogies) it may be

necessary to leave $\mathfrak{C}(\Gamma)$? The argument is well posed, since

a given matrix equation may indeed not have a solution in a

given space. In fact, practical solutions could be missed

by restricting solutions for theoretical reasons.

The reasons for mandating $X \in \mathfrak{C}^{m \times n}(\Gamma)$ are in part

pragmatic. First of all, polynomial entries don't lend

themselves to singularities: $X$ would be defined at all

values of $\Gamma$. This is important because given an entry that

is a rational function (e.g., $1/x$), there may exist a

singularity or discontinuity at a value which in fact does

have physical significance. Because of the rational entry,

the form of the solution may prove untenable. Secondly,

many functions can be approximated by a series of

polynomials, e.g., Chebysev polynomials [5:239]. In fact,

computational considerations may force the analyst to use

polynomial approximations of a function. Restricting the

entries of $X$ to $\mathfrak{C}(\Gamma)$ would help generate these polynomials

to be used in approximating the solutions to a phenomena.

The driving force, however, for requiring $X \in \mathfrak{C}^{m \times n}(\Gamma)$

is theoretical: is it possible to find some solutions in

the same space as the coefficient matrices in a given matrix

equation? If so, how would the solutions be obtained? It

may very well be that solutions thus found may shed light on

the nature of solutions which lie outside the required

space.

A word needs to be said on the setting for addition and

2.9

multiplication on matrices over $\mathcal{C}(\Gamma)$. Though these operations seem to hardly merit concern, there is in fact an extremely important algebraic structure behind them. Some of the current literature on matrix equations refers to this "ring" structure of $\mathcal{C}^{m \times n}(\Gamma)$.

In the broad setting, a set of objects S (e.g., matrices) is given, and two operations, + and *, are defined (e.g., matrix addition and multiplication) between a pair of elements of S. These operations have the property that if $a, b \in S$ then: $a + b$, $a * b \in S$ and $a + b$, $a * b$ assume unique values (e.g., $a + b$ has one value and one value only). Operations with these closure and uniqueness properties are called binary operations. Depending on the nature of the operations, an algebraic structure (denoted by $\langle S, +, * \rangle$) is defined on the set S taken together with the operations. For S the set of matrices over $\mathcal{C}(\Gamma)$ and the customary operations of matrix addition and subtraction, $\langle S, +, * \rangle$ is known as a ring.

A ring $\langle R, +, * \rangle$ is a set R together with two binary operations + and * defined on R such that the following axioms are satisfied [7:195]:

        R1.   $\langle R, + \rangle$ is an abelian group.
        R2.   * is associative.
        R3.   For all $a, b, c \in R$,
                    $a*(b+c) = (a*b) + (a*c)$
                                    and
                    $(a+b)*c = (a*c) + (b*c)$

It is an easy matter to confirm that the operations of matrix addition and multiplication satisfy the ring axioms.

2.10

<u>Rank and Determinant of a Matrix</u>.  This chapter
closes by addressing two familiar properties of a matrix:
its determinant and rank.  Both will be cited throughout
this paper.

The rank of an  m X n  matrix A (denoted by rank(A)) is
the number of linearly independent columns in A.  Chapter III
will show that rank(A) affects the solution of the matrix
equation  AX = b, where A,X,b are matrices over $\phi(\Gamma)$.  In
finding rank(A), the determinant of a matrix will come into
play.

Despite the many techniques which find the determinant
of a square matrix, later proofs will refer to the first
historical definition of a determinant, computationally
inefficient though it is.  This definition, later referred
to as the "permutational" form of a determinant, derives its
name from its indexing (of matrix elements) on a permutation
of a set [26:89].

Given the set of integers S = {1,2,...,N}  a
permutation on S is a one-to-one and onto mapping $\sigma$ from S
into itself, and is represented by a  2 X N  matrix whose
top row is S and whose bottom row is a specific permutation
of S.  For instance, the two permutations of {1,2} are given
by

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

For S = {1,2,3,4} one permutation is given by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

(where $\sigma(1) = 2$, $\sigma(2) = 4$, $\sigma(3) = 3$ and $\sigma(4) = 1$) and

another by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

where $\sigma(1) = 4$, $\sigma(2) = 3$, $\sigma(3) = 2$ and $\sigma(4) = 1$.

If $A = [a_{kj}]$ is an $N \times N$ matrix, then the

determinant of A is given by

$$\det(A) = \sum_{p \in S} [\text{sgn}(p) \ \Pi_{j=1,n}(a_{j,p(j)})] \qquad (2.13)$$

where

S is the set of all permutations on $\{1,2,...N\}$

and

sgn(p) = ±1, depending on the number of transpositions
which comprise the particular permutation p  [7:47-48].

# III   Solving the Linear Equation AX = b

The key to solving the Lypanov and Riccati equations involves finding the solutions of the linear equation

$$AX = b \qquad\qquad (3.1)$$

As before, matrices may be non-square but entries must be in $\mathbb{C}(\Gamma)$.

Though the matrices of (3.1) may be rectangular, one is first led to investigating properties of a square matrix $A \in \mathbb{C}^{n \times n}(\Gamma)$—in particular, whether or not A is invertible. As in the real case, if det(A) equals zero, then A has no inverse.  In general however,  the determinant of A is a polynomial over $\Gamma$ and thus has an inverse except at the roots of det(A).  For example, if

$$A = \begin{bmatrix} x + y & x \\ y & xy \end{bmatrix} \in \mathbb{C}^{2 \times 2}(x,y) \qquad (3.2)$$

then

$$\det(A) = (x + y)xy - xy = xy(x + y - 1)$$

and A has an inverse except in the root set of det(A)

$$\{(x,y) : x = 0 \text{ or } y = 0 \text{ or } x + y = 1\}$$

If A must be invertible for all values of $\Gamma$ (as will soon be required), then det(A) can only be a non-zero complex number. In this case $A^{-1}$ is easily found using the well known formula [26:96]

$$A^{-1} = |A|^{-1} \text{ [matrix-of-cofactors of A]}^{\top}$$

That  $A^{-1} \in \mathbb{C}^{n \times n}(\Gamma)$  can be seen by recalling that the cofactor of an entry $a_{ij}$ of A is the determinant of a

3.1

submatrix of A (obtained by deleting the i'th row and j'th column of A). Because the entries of A are in $\mathbb{C}(\Gamma)$, the cofactors are also in $\mathbb{C}(\Gamma)$, and thus the transpose of the matrix of cofactors is in $\mathbb{C}^{n \times n}(\Gamma)$. Since det(A) is assumed to be a non-zero complex number, so too will $|A|^{-1}$ which implies $A^{-1} \in \mathbb{C}^{n \times n}(\Gamma)$. With this in mind, a specific form of (3.1) can be solved for X: if $A \in \mathbb{C}^{n \times n}(\Gamma)$ and det(A) $\in \mathbb{C} - \{0\}$ then there exists an $A^{-1} \in \mathbb{C}^{n \times n}(\Gamma)$ and $X = A^{-1}b$ (the general setting for the familiar result involving real matrices). Finally, since the entries of $A^{-1}$ and b are in $\mathbb{C}(\Gamma)$, so too are the entries in their product $X = A^{-1}b$ as mandated in (3.1).

The picture becomes more complicated when A doesn't have an inverse. This may be because det(A) = 0, det(A) $\in \mathbb{C}(\Gamma) - \mathbb{C}$ , or det(A) isn't defined, i.e. A rectangular. Nevertheless, (3.1) may still have a solution and in fact, may have infinitely many. However, before the general solution of (3.1) can be derived, a more universal setting for 'inverses' of a matrix will be addressed.

The basic concept here involves elementary row and column operations on a matrix $A \in \mathbb{C}^{m \times n}(\Gamma)$. Similar to their counterparts for real matrices, elementary row (column) operations on A are limited to:

1. interchanging two rows (columns)
2. adding a polynomial multiple of a row (column) to another row (column)
3. multiplying a row (column) by a complex *scalar* not equal to zero.

3.2

As with real matrices, an elementary row (column) operation on $A \in \mathbb{C}^{m \times n}(\Gamma)$ is equivalent to left (right) multiplication of A by the corresponding elementary row (column) matrix. For example, interchanging the rows of (3.2) can be accomplished by multiplying A on the left by

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

obtained by interchanging the rows of $I_{2 \times 2}$. Another example: multiplying the first column of (3.2) by $x^2$ and adding it to the second column can be accomplished by multiplying A on the right by

$$\begin{bmatrix} 1 & x^2 \\ 0 & 1 \end{bmatrix}$$

obtained by multiplying the first column of $I_{2 \times 2}$ by $x^2$ and adding it to the second column. Again, as with real matrices, the determinant of an elementary matrix over $\mathbb{C}(\Gamma)$ is a nonzero complex number, and thus the elementary matrix always has an inverse. It is for this reason that the third elementary operation is limited to multiplication by a *scalar* for if polynomial multiples were allowed, the determinant of the resulting elementary matrix may be a polynomial with degree $\geq 1$. The matrix would then fail to have an inverse at the roots of its determinant. A square matrix whose determinant equals $\pm 1$ is called unimodular, and so is invertible.

Both elementary row and column operations are used on A to find the general solution to (3.1). Though two different

3.3

cases must be considered--A does and doesn't have constant

rank--both follow the same approach:  find unimodular

matrices P and Q that reduce A to a form yielding the

general solution of (3.1).  The two cases will be analyzed

separately.

$\underline{AX = b, rank(A) constant}$.  The aim here is to transform

$A \in \math4{c}^{m \times n}$, using elementary operations, into the m X n matrix

$$A_1 \; = \; \begin{bmatrix} I_{p \times p} & 0_{p \times n-p} \\ 0_{m-p,p} & 0_{m-p,n-p} \end{bmatrix} \qquad (3.3)$$

where p is the rank of A.  The main question thus

becomes,"What are the unimodular matrices P and Q such that

$PAQ = A_1$?"  Once P and Q are found, the general solution of

(3.1) will follow, as this chapter will show.  That such P

and Q do exist was recently proven by E. Sontag [31]:

$\underline{THEOREM\ 3.0\ (Sontag)}$:  The following statements are

equivalent for a matrix $A = A(\Gamma)$ over $R = \math4{c}(\Gamma)$:

    a.  There exists a matrix B over R such that
        ABA = A and BAB = B.  B is called the weak
        generalized inverse (or the {1,2}-generalized
        inverse) of A.

    b.  There exists square unimodular matrices P and Q
        over R such that $A_1 = PAQ$.

    c.  As a function of the the complex variables
        $\Gamma = (\Gamma_1,...,\Gamma_n)$, rank(A) is constant.

 *Proof*:  See [31].  ∎

A method for determining P and Q has been developed by

Dr. John Jones Jr., AFIT, and works by keeping a

'cumulative' record of all the elementary row and column

operations performed in reducing A to $A_1$.  The "Jones ST

Method" begins by forming the matrix

$$A_2 = \left[\begin{array}{c|c} A_{m \times n} & I_{m \times m} \\ \hline I_{n \times n} & O_{n \times m} \end{array}\right] \qquad (3.4)$$

Subsequent elementary operations are done on $A_2$ until $A$ becomes $A_1$. Once this point is reached, the matrix occupying the $I_{m \times m}$ block is $P$, and the matrix occupying the $I_{n \times n}$ block is $Q$. That these two blocks do indeed contain $P$ and $Q$ is easily seen: $P$ and $Q$ will be the respective products of the elementary row and column transformations done in reducing $A$ to $A_1$; since $I_{m \times m}$ is row adjacent to $A$, $P$ will be the left multiplier of $A$, and since $I_{n \times n}$ is column adjacent to $A$, $Q$ will be $A$'s right multiplier. Also, both $P$ and $Q$ are invertible, since their determinants are nonzero complex. To see this, consider $P = \Pi R_1$ where the $R_1$'s are the elementary row transformations done in reducing $A$ to $A_1$. It follows that

$$\det(P) = \det(\Pi R_1) = \Pi \det(R_1)$$

Since each $\det(R_1)$ is nonzero complex (the nature of an elementary matrix), so will their product be, and thus $\det(P)$. A similar approach holds for $Q$.

Example: Reduce to the form of (3.3) the matrix

$$A = \left[\begin{array}{ccc} 1 & x & y \\ 0 & 0 & 1 \end{array}\right] \ \xi \ \mathcal{C}^{2 \times 3}(x,y)$$

Begin by augmenting $A$ by the identity matrices $I_{2 \times 2}$ and $I_{3 \times 3}$ to form per (3.4)

3.5

$$
\left[
\begin{array}{ccc|cc}
1 & x & y & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
\hline
1 & 0 & 0 & & \\
0 & 1 & 0 & \multicolumn{2}{c}{0_{3\times 2}} \\
0 & 0 & 1 & &
\end{array}
\right]
$$

The next step is to get an identity matrix in the upper left hand corner of A, per (3.3). First, interchange the second and third columns to form

$$
\left[
\begin{array}{ccc|cc}
1 & y & x & 1 & 0 \\
0 & 1 & 0 & 0 & 1 \\
\hline
1 & 0 & 0 & & \\
0 & 0 & 1 & \multicolumn{2}{c}{0_{3\times 2}} \\
0 & 1 & 0 & &
\end{array}
\right]
$$

Next, add -y times the second row and add it to the first to form

$$
\left[
\begin{array}{ccc|cc}
1 & 0 & x & 1 & -y \\
0 & 1 & 0 & 0 & 1 \\
\hline
1 & 0 & 0 & & \\
0 & 0 & 1 & \multicolumn{2}{c}{0_{3\times 2}} \\
0 & 1 & 0 & &
\end{array}
\right]
$$

Next, add -x times the first column to the third column obtaining

$$
\left[
\begin{array}{ccc|cc}
1 & 0 & 0 & 1 & -y \\
0 & 1 & 0 & 0 & 1 \\
\hline
1 & 0 & -x & & \\
0 & 0 & 1 & \multicolumn{2}{c}{0_{3\times 2}} \\
0 & 1 & 0 & &
\end{array}
\right]
$$

The process is now complete since A has been reduced to the form of (3.3). The P and Q elementary row and column matrices are read off to be

$$
P = \begin{bmatrix} 1 & -y \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad Q = \begin{bmatrix} 1 & 0 & -x \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}
$$

3.6

Thus

$$PAQ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Since det(P) = 1 and det(Q) = -1, P and Q are unimodular and hence invertible. ∎

The above example illustrates a theorem helping to identify the general solution of (3.1) for A of constant rank:

<u>THEOREM 3.1</u>: Let $A \in \mathbb{C}^{m \times n}(\Gamma)$ and rank(A) = p a constant. If P and Q are unimodular and

$$PAQ = \begin{bmatrix} I_{p \times p} & O_{p \times n-p} \\ O_{m-p \times p} & O_{m-p \times n-p} \end{bmatrix}$$

where

$$P = \begin{bmatrix} T \\ M \end{bmatrix} \in \mathbb{C}^{m \times m}(\Gamma) \quad , \quad Q = [S \quad N] \in \mathbb{C}^{n \times n}(\Gamma),$$

$$T \in \mathbb{C}^{p \times m}(\Gamma) \quad , \quad M \in \mathbb{C}^{m-p \cdot m}(\Gamma) \quad , \quad S \in \mathbb{C}^{n \times p}(\Gamma),$$

$$N \in \mathbb{C}^{n \times n-p}(\Gamma)$$

then

$$\begin{bmatrix} A & I_{m \times m} \\ I_{n \times n} & O_{n \times m} \end{bmatrix} \quad \text{is similar to} \quad \begin{bmatrix} I_{p \times p} & O_{p \times n-p} & T_{p \times m} \\ O_{m-p \times p} & O_{m-p \times n-p} & M_{m-p \times m} \\ S_{n \times p} & N_{n \times n-p} & O_{n \times m} \end{bmatrix}$$

$$PROOF: \quad \begin{bmatrix} P_{m \times m} & O_{m \times n} \\ O_{n \times m} & I_{n \times n} \end{bmatrix} \begin{bmatrix} A_{m \times n} & I_{m \times m} \\ I_{n \times n} & O_{n \times m} \end{bmatrix} \begin{bmatrix} Q_{n \times n} & O_{n \times m} \\ O_{m \times n} & I_{m \times m} \end{bmatrix}$$

$$= \begin{bmatrix} (PA)_{m \times n} & P_{m \times m} \\ I_{n \times n} & O_{n \times m} \end{bmatrix} \begin{bmatrix} Q_{n \times n} & O_{n \times m} \\ O_{m \times n} & I_{m \times m} \end{bmatrix}$$

$$= \left[ \begin{array}{c|c} (PAQ)_{m \times n} & P_{m \times m} \\ \hline Q_{n \times n} & O_{n \times m} \end{array} \right] \quad \begin{array}{c} \text{(from given and)} \\ = \\ \text{(substitute for P \& Q)} \end{array}$$

$$= \begin{bmatrix} I_{p \times p} & O_{p \times n-p} & T_{p \times n} \\ O_{m-p \times p} & O_{m-p \times n-p} & M_{m-p \times m} \\ \hline S_{n \times p} & N_{n \times n-p} & O_{n \times m} \end{bmatrix}$$

Since P and Q are unimodular,

$$\begin{bmatrix} P_{m \times m} & O_{m \times n} \\ O_{n \times m} & I_{n \times n} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} Q_{n \times n} & O_{n \times m} \\ O_{m \times n} & I_{m \times m} \end{bmatrix}$$

are also unimodular and thus invertible.  This is seen by
noting that if A and B are square matrices, then

$$\det \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \det(A) * \det(B)$$

a consequence of the permutational definition of a
determinant (2.13).  ∎

Example of Theorem 3.1:  Let

$$A = \begin{bmatrix} x & y & x^2 + y + 1 \\ x^2 y & xy^2 & x^3 y + xy^2 + xy \end{bmatrix}$$

Matrices P and Q are sought that reduce A to the form (3.3).
First, augment A by the identity matrices $I_{2 \times 2}$ and $I_{3 \times 3}$ to
get

| x | y | $x^2 + y + 1$ | 1 | 0 |
|---|---|---|---|---|
| $x^2 y$ | $xy^2$ | $x^3 y + xy^2 + xy$ | 0 | 1 |
| 1 | 0 | 0 | | |
| 0 | 1 | 0 | $O_{3 \times 2}$ | |
| 0 | 0 | 1 | | |

Next, add -xy times row 1 and add to row 2 to obtain:

| x | y | $x^2 + y + 1$ | 1 | 0 |
|---|---|---|---|---|
| 0 | 0 | 0 | -xy | 1 |
| 1 | 0 | 0 | | |
| 0 | 1 | 0 | $O_{3 \times 2}$ | |
| 0 | 0 | 1 | | |

Next, add -1 times column 2 to column 3 to get:

3.8

$$
\begin{array}{ccc|cc}
x & y & x^2 + 1 & 1 & 0 \\
0 & 0 & 0 & -xy & 1 \\
\hline
1 & 0 & 0 & & \\
0 & 1 & -1 & & 0_{3\times 2} \\
0 & 0 & 1 & &
\end{array}
$$

Now add $-x$ times column 1 to column 3 to obtain:

$$
\begin{array}{ccc|cc}
x & y & 1 & 1 & 0 \\
0 & 0 & 0 & -xy & 1 \\
\hline
1 & 0 & -x & & \\
0 & 1 & -1 & & 0_{3\times 2} \\
0 & 0 & 1 & &
\end{array}
$$

Switching columns 1 and 3:

$$
\begin{array}{ccc|cc}
1 & y & x & 1 & 0 \\
0 & 0 & 0 & -xy & 1 \\
\hline
-x & 0 & 1 & & \\
-1 & 1 & 0 & & 0_{3\times 2} \\
1 & 0 & 0 & &
\end{array}
$$

Adding $-y$ times column 1 to column 2, followed by $-x$ times

column 1 to column 3 yields:

$$
\begin{array}{ccc|cc}
1 & 0 & 0 & [\ 1 & 0\ ] \gg T \\
0 & 0 & 0 & [-xy & 1\ ] \gg M \\
\hline
-x & xy & 1 + x^2 & & \\
-1 & 1 + y & x & & 0_{3\times 2} \\
1 & -y & -x & & \\
\wedge & \wedge & \wedge & & \\
\wedge & & \setminus\ / & & \\
S & & N & &
\end{array}
$$

Since A has been reduced to the form (3.3), P and Q have

been found, and thus S,T,M,N of Theorem 3.1.  It's easy to

verify the unimodularity of P and Q.  ■


The groundwork is now in place to identify the general

form of the solution to (3.1).  The following theorem is due

to Jones [11]:

3.9

<u>THEOREM 3.2</u>: Let $A \in \mathbb{C}^{m \times n}(\Gamma)$ of constant rank p and $b \in \mathbb{C}^{m \times \nu}(\Gamma)$. Then $AX = b$ has a solution $X \in \mathbb{C}^{n \times \nu}(\Gamma)$ if and only if $Mb = 0_{m-p \times \nu}$, M given in Theorem 3.1. The general form of X is given by

$$X = STb + NZ$$

S,T,N as given in Theorem 3.1, and $Z \in \mathbb{C}^{n-p \cdot \nu}(\Gamma)$ arbitrary.

*PROOF*: Let $P \in \mathbb{C}^{m \times m}(\Gamma)$ and $Q \in \mathbb{C}^{n \times n}(\Gamma)$ be unimodular matrices such that

$$PAQ = \begin{bmatrix} I_{p \times p} & 0_{p \times n-p} \\ 0_{m-p,p} & 0_{m-p,n-p} \end{bmatrix}$$

as given in Theorem 3.0. Then $AX = b$ has a solution X
                                   iff
$PAX = Pb$ has a solution X (by virtue of the unimodularity of P and hence the existence of $P^{-1}$)
                                   iff
$PA(QQ^{-1})X = Pb = (PAQ)(Q^{-1}X)$ has a solution X (since Q is unimodular, there exists $Q^{-1}$)
                                   iff
$(PAQ)Y = Pb$ has a solution $Y = Q^{-1}X$                          (3.5)

Let

$$Y = \begin{bmatrix} W_{p \times \nu} \\ Z_{n-p \times \nu} \end{bmatrix} \quad , \quad P = \begin{bmatrix} T_{p \times m} \\ M_{m-p \times m} \end{bmatrix} \quad , \quad Q = [S_{n \times p} \quad N_{n \times n-p}]$$

Then substituting for PAQ, P and Y in (3.5):

$$\begin{bmatrix} I_{p \times p} & 0_{p \times n-p} \\ 0_{m-p,p} & 0_{m-p,n-p} \end{bmatrix} \begin{bmatrix} W \\ Z \end{bmatrix} = \begin{bmatrix} T \\ M \end{bmatrix} b$$

$$\begin{bmatrix} W \\ 0_{m-p \times \nu} \end{bmatrix} = \begin{bmatrix} Tb \\ Mb \end{bmatrix} \tag{3.6}$$

Since $Y = Q^{-1}X$ from (3.5),

$$X = QY = [S \quad N] \begin{bmatrix} W \\ Z \end{bmatrix} = SW + NZ = STb + NZ$$

Note that (3.6) allows Z to be arbitrary since Z is zeroed out by the second row of PAQ. (3.6) also says that if a

3.10

solution exists to $AX = b$ , then $Mb$ must equal $O_{m-p \times v}$.
This mandate is referred to as the 'consistency condition'.

■

### EXAMPLE (THEOREM 3.2):

Given

$$A = \begin{bmatrix} x & y & x^2 + y + 1 \\ x^2y & xy^2 & x^3y + xy^2 + xy \end{bmatrix}$$

$$b = \begin{bmatrix} x + xy + y + x^2y + y^2 \\ x^2y + xy^2 + x^2y^2 + x^3y^2 + xy^3 \end{bmatrix}$$

solve $AX = b$.

The example for Theorem 3.1 (which reduced A) found
that

$T = [1 \quad 0]$ , $M = [-xy \quad 1]$ , $S = [-x \quad -1 \quad 1]^t$

and

$$N = \begin{bmatrix} xy & 1 + x^2 \\ 1 + y & x \\ -y & -x \end{bmatrix}$$

It is a simple matter to show $Mb = 0$ thereby confirming
the existence of a solution for $AX = b$. From Theorem 3.2
it follows that the general solution is

$$X = STb + NZ =$$

$$\begin{bmatrix} -x^2 - x^2y - xy - x^3y - xy^2 \\ -x - xy - y - x^2y - y^2 \\ x + xy + y + x^2y + y^2 \end{bmatrix} + a \begin{bmatrix} xy \\ 1 + y \\ -y \end{bmatrix} + \beta \begin{bmatrix} 1 + x^2 \\ x \\ -x \end{bmatrix}$$

where $a, \beta \in ¢(x,y)$ ■

A benefit of Theorem 3.2 is that it easily identifies
the basis for the kernal of the linear transformation
represented by the m X n matrix A, a task generally quite
tedious. Recall that the kernal of a linear transformation

represented by the matrix A is the set

$$K_A = \{ X : AX = 0_{m \times 1} \}$$

To find the basis of $K_A$, one must solve the equation

$$AX = 0_{m \times 1}$$

which is a special case of $AX = b$ where $b = 0_{m \times 1}$ From Theorem 3.2, it follows that

$$X = STb + NZ = ST0_{m \times 1} + NZ = NZ$$

Thus the span of the columns of N is $K_A$. However, since $Q = [S \ N]$ and Q is unimodular, the columns of Q are linearly independent, and thus the columns of N. Since the columns of N is a linearly independent spanning set of $K_A$, it follows that the columns of N are the basis for the kernal of the linear transformation represented by the matrix A.

AX = b, rank(A) not constant. Consider the following matrix $A \in \mathbb{C}^{3 \times 3}(x,y,z)$ which doesn't have constant rank:

$$A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix} \qquad (3.7)$$

Rank(A) $\in \{0,1,2,3\}$ , depending on the values assumed by $(x,y,z)$: if $x = y = z = 0$, then rank(A) = 0; if $x,y,z \neq 0$, then rank(A) = 3; if two are 0 while the third isn't, rank(A) = 1; if two aren't zero while the third is, then rank(A) = 2. This section addresses the solution of the equation $AX = b$ given rank(A) not constant.

A result similar to Theorem 3.2 holds for such a matrix A. The approach is practically identical to the

3.12

constant rank case:  find unimodular matrices P and Q which

reduce A to the form

$$PAQ = \begin{bmatrix} I_{p \times p} & O_{p \times n-p} \\ O_{m-p \times p} & \Omega_{m-p \times n-p} \end{bmatrix} , \quad p \geq 1 \qquad (3.8)$$

where p is as large as possible and $\Omega \in \mathfrak{C}^{m-p \times n-p}(\Gamma)$.  As

before, P and Q are found using the Jones ST Method.  If

such P and Q are found (they may not exist), the following

theorem applies:

THEOREM 3.3:  Let $A \in \mathfrak{C}^{m \times n}(\Gamma)$  If P and Q are

unimodular and

$$PAQ = \begin{bmatrix} I_{p \times p} & O_{p \times n-p} \\ O_{m-p \times p} & \Omega_{m-p \times n-p} \end{bmatrix}$$

where

$$P = \begin{bmatrix} T \\ M \end{bmatrix} \in \mathfrak{C}_{m \times m}(\Gamma) \quad , \quad Q = [S \; N] \in \mathfrak{C}^{n \times n}(\Gamma),$$

$T \in \mathfrak{C}^{p \times m}(\Gamma)$   , $M \in \mathfrak{C}^{m-p \cdot m}(\Gamma)$   , $S \in \mathfrak{C}^{n \times p}(\Gamma)$   ,

$N \in \mathfrak{C}^{n \times n-p}(\Gamma)$   , $\Omega \in \mathfrak{C}^{m-p \times n-p}(\Gamma)$

then

$$\begin{bmatrix} A & I_{m \times m} \\ I_{n \times n} & O_{n \times m} \end{bmatrix} \quad \text{is similar to} \quad \begin{bmatrix} I_{p \times p} & O_{p \times n-p} & T_{p \times m} \\ O_{m-p \times p} & \Omega_{m-p \times n-p} & M_{n-p \times m} \\ S_{n \times p} & N_{n \times n-p} & O_{n \times m} \end{bmatrix}$$

*PROOF:*  identical to that of Theorem 3.1.  ∎

Once P and Q have been found, the following theorem is

applied:

THEOREM 3.4:  Let  $A \in \mathfrak{C}^{m \times n}(\Gamma)$  and  $b \in \mathfrak{C}^{m \times \vee}(\Gamma)$.  Then

$AX = b$  has a solution $X \in \mathfrak{C}^{n \times \vee}(\Gamma)$  if and only if there

exists a  $Z \in \mathfrak{C}^{n-p \cdot \vee}(\Gamma)$  such that  $Mb = \Omega Z$  M,$\Omega$ given in

Theorem 3.3.  In this case, the general form of X is given

3.13

by

$$X = STb + NZ$$

for all Z such that $Mb = \Omega Z$ and S,T,N given in Theorem 3.3.

*PROOF*: Let $P \in \mathbb{C}^{m \times m}(\Gamma)$ and $Q \in \mathbb{C}^{n \times n}(\Gamma)$ be

unimodular matrices such that

$$PAQ = \begin{bmatrix} I_{p \times p} & O_{p \times n-p} \\ O_{m-p,p} & \Omega_{m-p,n-p} \end{bmatrix}$$

AX = b has a solution X

iff

PAX = Pb has a solution X (by virture of the unimodularity
of P and hence the existence of $A^{-1}$)

iff

$PA(QQ^{-1})X = Pb = (PAQ)(Q^{-1}X)$ has a solution X (since Q is
unimodular, there exists $Q^{-1}$)

iff

$(PAQ)Y = Pb$ has a solution $Y = Q^{-1}X$ $\qquad$ (3.9)

Let

$$Y = \begin{bmatrix} W_{p \times v} \\ Z_{n-p \times v} \end{bmatrix}, \qquad P = \begin{bmatrix} T_{p \times m} \\ M_{m-p \times m} \end{bmatrix}$$

and

$$Q = [S_{n \times p} \quad N_{n \times n-p}]$$

Then substituting for PAQ, P and Y in (3.9):

$$\begin{bmatrix} I_{p \times p} & O_{p \times n-p} \\ O_{m-p,p} & \Omega_{m-p,n-p} \end{bmatrix} \begin{bmatrix} W \\ Z \end{bmatrix} = \begin{bmatrix} T \\ M \end{bmatrix} b$$

$$\begin{bmatrix} W \\ \Omega Z \end{bmatrix} = \begin{bmatrix} Tb \\ Mb \end{bmatrix} \qquad (3.10)$$

Since $Y = Q^{-1}X$,

$$X = QY = [S \quad N] \begin{bmatrix} W \\ Z \end{bmatrix} = SW + NZ = STb + NZ$$

subject to

$$\Omega Z = Mb \qquad (3.11)$$

per (3.10). This latter mandate is the consistency

condition. That $Z \in \mathbb{C}^{n-p \times v}(\Gamma)$ proceeds from (3.9): since

3.14

Q has polynomial entries and is unimodular, so too is $Q^{-1}$.
Since X is also required to be in $\mathbb{C}^{n \times \nu}(\Gamma)$ the product of
$Q^{-1}$ and X must be in $\mathbb{C}^{n \times \nu}(\Gamma)$. Since Z is a submatrix of
$Y = Q^{-1}X$ , $Z \in \mathbb{C}^{n-p \times \nu}(\Gamma)$.  ∎

In general, there is no assurance that the consistency
condition (3.11) can be met for a given equation  AX = b:
though A may be reduced to the form (3.8), there may not
exist a Z OVER $\mathbb{C}(\Gamma)$ satisfying the consistency condition
$\Omega Z = Mb$.  It may be necessary to leave $\mathbb{C}(\Gamma)$ in order to find
a Z satisfying the consistency condition, an act contrary to
this paper's aim.

What then can be said about the solvability over $\mathbb{C}(\Gamma)$
of  AX = b  when rank(A) isn't constant?  It's possible that
A isn't reducible to the form (3.8).  For instance the
matrix in (3.7):  reducing A to the form (3.8) implies that
A has a rank no less than p (the first p columns of (3.8)
are linearly independent despite the form of $\Omega$).  However,
the rank of A may be zero if x = y = z = 0 , and so A isn't
reducible to the form (3.8).

This example, however, is symptomatic of a larger
problem.  Assume that a given matrix A has been reduced as
much as possible (no known method exists that insures
maximum reducibility of a matrix over $\mathbb{C}(\Gamma)$, $\Gamma$ arbitrary),
i.e. the maximum value of p has been found.  One is then
left with finding a Z satisfying the consistency condition.
$\Omega$ in turn cannot be reduced, since this would violate the

3.15

assumption that A has been reduced as far as is possible.
If $\Lambda$ is square, $\det(\Lambda)$ cannot be a nonzero complex number
(if it were, $\text{rank}(\Lambda) = m - p$  and $\Lambda$ could be reduced to
$I_{m-p \times m-p}$, thus giving A constant rank, a contradiction).  If
$\det(\Lambda) \neq 0$ repeated use of Cramer's rule may identify some
solutions Z, but these would be local since the roots of
$\det(\Lambda)$ must be omitted.  In general, however, $\Lambda$ will be
rectangular.

Thus finding the general solution of (3.1) given
$\text{rank}(A)$ not constant ultimately requires solving (3.11)
where $\Lambda$ cannot be transformed into the form (3.8).  The
search for Z may be aided by matrices related to $\Lambda$, the
topic of the next section.

SMITH FORM OF A MATRIX.  A matrix  $A \in \mathbb{C}^{m \times n}(\Gamma)$  is said
to be in Smith Form (SF) if and only if

$$A = \left[ \begin{array}{ccc|c} f_1 & & 0 & \\ & \ddots & & 0_{j \times n-j} \\ 0 & & f_j & \\ \hline 0_{m-j \times j} & & & 0_{m-j \times n-j} \end{array} \right] \qquad (3.12)$$

where all off diagonal entries equal zero and $f_k$ is a factor
of $f_{k+1}$ .  For example

$$A = \left[ \begin{array}{ccc} x & 0 & 0 \\ 0 & x^2 y & 0 \\ 0 & 0 & x^2 y^3 \end{array} \right] \in \mathbb{C}^{3 \times 3}(x,y)$$

is in SF (3.12) because all off diagonal entries are zero, x
is a factor of $x^2 y$, and $x^2 y$ is a factor of $x^2 y^3$.  Another
example is

3.16

$$A = \begin{bmatrix} x & 0 \\ 0 & x(x+1) \\ 0 & 0 \end{bmatrix} \quad \in \quad \mathbb{C}^{3 \times 2}(x,y)$$

which is in SF (3.12) because all off diagonal entries are zero, and x is a factor of x(x+1). The solving of matrix equation (3.1) may be aided if A can be transformed (using unimodular matrices P' and Q') into a matrix having SF. In fact, Smith Forms have already appeared: (3.3) is a special case where $f_k = 1$, $k \leq p$    The main aim here is to use SFs to solve the equation (3.11) $\Omega Z = Mb$  of the last section.

To illustrate, assume that in solving $\Omega Z = Mb$  (derived from some hypothetical original equation  AX = b), unimodular matrices P' and Q' have been found that give $P' \Omega Q'$ a SF. Thus

$$\Omega Z = Mb$$
$$P' \Omega Z = P'Mb$$
$$P' \Omega (Q'Q'^{-1})Z = P'Mb$$
$$(P' \Omega Q')(Q'^{-1}Z) = P'Mb$$

Letting

$$Y = Q'^{-1}Z$$
$$H = P'Mb$$
$$F = P' \Omega Q' \qquad\qquad (3.13)$$

the last equality becomes

$$FY = H$$

Since F is in SF, there exists at most one nonzero entry in each row and column of F. Because F and H are known, some entries in Y can be found by dividing the respective F and H factors (if however a given entry of Y isn't compatible with two or more equations, or if the entry doesn't turn out to

3.17

be a polynomial, then $\Omega Z = Mb$ has no solution Z.)
Depending on the structure of F, there may be entries of Y
which can assume arbitrary values. Y is then substituted
into (3.13) to yield $Z = Q'Y$, thus solving (3.11) and in
turn (3.1).

Given $A \in \mathbb{C}^{m \times n}(\Gamma)$, if there exists unimodular
matrices $P' \in \mathbb{C}^{m \times m}(\Gamma)$ and $Q' \in \mathbb{C}^{n \times n}(\Gamma)$ such that
$A' = P'AQ'$ is in SF, then A is said to be equivalent to its
Smith Form. It has been proven [1:188] that for every
$A \in \mathbb{C}^{m \times n}(x)$, A is equivalent to a unique Smith Form (an
algorithm exists [1:192] which finds this). However, such
is not always the case for multi-dimensional $\Gamma$. In fact,
Lee and Zak [20] prove that a matrix $A \in \mathbb{C}^{m \times n}(x,y)$ is
equivalent to its SF iff a certain system of linear
polynomial equations has a solution. The conditions under
which a matrix in three or more variables is equivalent to
its Smith Form remains an open question.

Although the existence of a SF for a given matrix A
must be known before a search for it can begin, a method is
needed which identifies the unimodular P' and Q' which
yield a particular SF of A. As before, the Jones ST Method
can be used.

EXAMPLE: Find the Smith Form of the matrix (Frost and
Storey [8])

$$A = \begin{bmatrix} s + z & 0 & 1 \\ 0 & s + z & 0 \\ 0 & 0 & s \end{bmatrix} \in \mathbb{C}^{3 \times 3}(s,z)$$

3.18

Begin by augmenting A with $I_{3\times3}$:

$$
\left[
\begin{array}{ccc|ccc}
s + z & 0 & 1 & 1 & 0 & 0 \\
0 & s + z & 0 & 0 & 1 & 0 \\
0 & 0 & s & 0 & 0 & 1 \\
\hline
1 & 0 & 0 & & & \\
0 & 1 & 0 & & & \\
0 & 0 & 1 & & &
\end{array}
\right]
$$

Next, add $-(z + s)$ times the third column to the first

column to get:

$$
\left[
\begin{array}{ccc|ccc}
0 & 0 & 1 & 1 & 0 & 0 \\
0 & s + z & 0 & 0 & 1 & 0 \\
-s(s+z) & 0 & s & 0 & 0 & 1 \\
\hline
1 & 0 & 0 & & & \\
0 & 1 & 0 & & & \\
-(s+z) & 0 & 1 & & &
\end{array}
\right]
$$

Add $-s$ times row 1 to row 3 obtaining:

$$
\left[
\begin{array}{ccc|ccc}
0 & 0 & 1 & 1 & 0 & 0 \\
0 & s + z & 0 & 0 & 1 & 0 \\
-s(s+z) & 0 & 0 & -s & 0 & 1 \\
\hline
1 & 0 & 0 & & & \\
0 & 1 & 0 & & & \\
-(s+z) & 0 & 1 & & &
\end{array}
\right]
$$

Finally, interchange the first and third columns:

$$
\left[
\begin{array}{ccc|ccc}
1 & 0 & 0 & 1 & 0 & 0 \\
0 & s + z & 0 & 0 & 1 & 0 \\
0 & 0 & -s(s+z) & -s & 0 & 1 \\
\hline
0 & 0 & 1 & & & \\
0 & 1 & 0 & & & \\
1 & 0 & -(s+z) & & &
\end{array}
\right]
$$

Since the upper left hand matrix is in Smith Form (i.e.,

1 is a factor of $s + z$, in turn a factor of $-s(s + z)$ and

off diagonal entries are zero)

$$
P' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -s & 0 & 1 \end{bmatrix}
\qquad
Q' = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -(s+z) \end{bmatrix}
$$

3.19

are the two unimodular (easily verifiable) matrices which

reduce A to its Smith Form. ∎

## IV   Solutions to Lypanov and Riccati Equations

This chapter will consider the solutions to the Lypanov matrix equation

$$AX - XB = C \qquad\qquad (4.1)$$

and the second order Riccati matrix equation

$$XDX + AX + XB + C = 0 \qquad\qquad (4.2)$$

where the entries of A,B,C,D are in $\mathbb{C}(\cap)$.  It will be shown that solving these two equations for X hinges upon solving· equations of the form  AX = b  the subject of Chapter III.

Lypanov Equation.  (4.1) may be solved using tensor products or similarity transformations.  The former approach will be addressed first because it uses the results of Chapter III at the onset.  The approach using similarity transformations will then be presented as a lead-in to solving the Riccati equation (4.2), whose solution relies exclusively on these tranformations.  Whichever method is taken though, the linear equation (3.1) will demand resolution.

Lypanov Equation:  Tensor Solution.  Multiplication between two matrices is easily extended to that of their tensor product.  Given  $A = [a_{i j}]$  the tensor product of the matrices A and B, represented by A$_4$B, is given by $[a_{i j}B]$. Unlike matrix multiplication, the number of columns in A doesn't have to equal the number of rows in B.

4.1

Example: If

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

then

$$A_\blacktriangle B = [a_{ij}B] = \begin{bmatrix} 1B & 2B & 3B \\ 4B & 5B & 6B \end{bmatrix}$$

$$= \left[ \begin{array}{cc|cc|cc} 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 2 & 2 & 4 & 3 & 6 \\ \hline 4 & 4 & 5 & 5 & 6 & 6 \\ 4 & 8 & 5 & 10 & 6 & 12 \end{array} \right] \qquad \text{a } 4 \times 6$$

It follows that if A is m X n and B is r X s, then $A_\blacktriangle B$ is mr X ns. ■

The next step in solving (4.1) is to identify the solution of the equation

$$AXB = C \qquad\qquad (4.3)$$

where $A_{i,r}$ denotes the i'th row of A and $A_{i,c}$ its i'th column (the i-jth entry of A is given by $a_{ij}$ with no comma between subscripts).

THEOREM 4.1: Solving the matrix equation

$$A_{m \times n} X_{n \times p} B_{p \times q} = C_{m \times q}$$

is equivalent to solving the equation

$$Gu = c$$

where

$$G = A_\blacktriangle B^\top \ , \ u = [X_{1,r} \ldots X_{n,r}]^\top \ , \ c = [C_{1,r} \ldots C_{m,r}]^\top$$

*PROOF:* Let $A = [a_{ij}]$ , $B = [\beta_{ij}]$ ,

$$C = [c_{ij}] \ , \ X = [x_{ij}]$$

Since $AXB = C$ ,

4.2

$$C_{IJ} = A_{I,r}XB_{J,c}$$
$$= A_{I,r}[X_{1,r}B_{J,c}...X_{n,r}B_{J,c}]^T$$
$$= \Sigma_k[a_{Ik}\Sigma_L(x_{kL}\beta_{LJ})]$$
$$= \Sigma_{k,L}(a_{Ik}\beta_{LJ}x_{kL})$$
$$= \Sigma_L(a_{I1}\beta_{LJ}x_{1L}) + \cdots + \Sigma_L(a_{In}\beta_{LJ}x_{nL})$$
$$= a_{I1}\Sigma_L(\beta_{LJ}x_{1L}) + \cdots + a_{In}\Sigma_L(\beta_{LJ}x_{nL})$$
$$= [\ a_{I1}[\beta_{1J}...\beta_{pJ}]\ ...\ a_{In}[\beta_{1J}...\beta_{pJ}]\ ]$$
$$* [\ X_{1,r}...X_{n,r}\ ]^T$$
$$= [a_{I1}B^T_{J,c}\ ...\ a_{In}B^T_{J,c}]$$
$$* [X_{1,r}...X_{n,r}]^T \quad (4.4)$$

Since $c = [C_{1,r}...C_{m,r}]^T$ is given and from (4.4), the

theorem follows. ∎

The following theorem is due to P. Lancaster [18]:

Theorem 4.2: Given the matrix equation

$$A_1XB_1 + A_2XB_2 = C \qquad (4.5)$$

where $A_I$ are m X m , $B_I$ are n X n , X,C are m X n (with $X_I$

and $C_I$ the rows of X and C respectively), (4.5) is equivalent

to solving the equation $Gx = c$ where

$$G = A_1 \triangle B_1^T + A_2 \triangle B_2^T$$
$$x = [X_1...X_m]^T$$
$$c = [C_1...C_m]^T$$

*Proof:* Similar to that of Theorem 4.1. ∎

The restriction on square $A_I$ and $B_I$ in Theorem 4.2 is

easily modified to include rectangular matrices. With this

modification, Theorem 4.2 allows for the solution of the

Lypanov equation (4.1) which is a special case of (4.5) with

$B_1 = I$ and $A_2 = -I$. Once a given Lypanov equation has

been reduced to an equation of the form $AX = b$ using

tensor products, the results of Chapter III can be applied to

this latter equation.

Example: Solve the Lypanov equation

$$AX + XB = C = AXI + IXB$$

where

$$A = \begin{bmatrix} s+z & s \\ z & s-z \end{bmatrix}, \quad B = \begin{bmatrix} -z & -s \\ -z-s & s-z \end{bmatrix}$$

$$C = \begin{bmatrix} sz & s^2 \\ 2s^2-2z^2-sz & 4sz-2z^2-3s^2 \end{bmatrix}$$

$$X = \begin{bmatrix} y & z \\ v & w \end{bmatrix} \qquad\qquad (4.6)$$

The first step is to reduce the given Lypanov equation to the form $Gx = c$ so that Theorem 4.2 may be used. Thus

$G = A \blacktriangle I^T + I \blacktriangle B^T =$

$$\begin{bmatrix} s+z & s \\ z & s-z \end{bmatrix} \blacktriangle \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \blacktriangle \begin{bmatrix} -z & -z-s \\ -s & s-z \end{bmatrix} =$$

$$\begin{bmatrix} s+z & 0 & s & 0 \\ 0 & s+z & 0 & s \\ z & 0 & s-z & 0 \\ 0 & z & 0 & s-z \end{bmatrix} + \begin{bmatrix} -z & -z-s & 0 & 0 \\ -s & s-z & 0 & 0 \\ 0 & 0 & -z & -z-s \\ 0 & 0 & -s & s-z \end{bmatrix} =$$

$$\begin{bmatrix} s & -z-s & s & 0 \\ -s & 2s & 0 & s \\ z & 0 & s-2z & -z-s \\ 0 & z & -s & -2s-2z \end{bmatrix}$$

Let $x = [y \; z \; v \; w]^T$ and

$c = [sz \quad s^2 \quad 2s^2-2z^2-sz \quad 4sz-2z^2-3s^2]^T$. Thus the solution

$$X = \begin{bmatrix} y & z \\ v & w \end{bmatrix}$$

to the given Lypanov equation is the solution to the linear equation

4.4

$$\begin{bmatrix} s & -z-s & s & 0 \\ -s & 2s & 0 & s \\ z & 0 & s-2z & -z-s \\ 0 & z & -s & -2s-2z \end{bmatrix} \begin{bmatrix} y \\ z \\ v \\ w \end{bmatrix} = \begin{bmatrix} sz \\ s^2 \\ 2s^2-2z^2-sz \\ 4sz-2z^2-3s^2 \end{bmatrix}$$

Depending on the nature of the G matrix obtained using tensor products, the given Lypanov equation may or may not be easily solvable. For instance, if G cannot be reduced to the form (3.3), one then tries for form (3.8). Here however, finding the Z's satisfying the consistency condition (3.11) $\Omega Z = Mb$ may be difficult. In any event, the use of tensor products creates a large matrix G, thereby compounding calculations.

The second approach to solving the Lypanov equation (4.1) uses similarity transformations. Although this method may not explicitly identify the solution to (4.1), it will identify a set of matrices within which the solutions of (4.1) will be found.

<u>Lypanov Equation: Solution via Similarity Transformations</u>. The second approach to solving (4.1) proceeds from a few simple observations unrelated to solving a matrix equation. Given an n X n matrix X and $I = I_{n \times n}$ the matrix

$$E = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \qquad (4.7)$$

always has det(E) = 1 despite the nature of X (this follows from the permutational definition of a determinant (2.13). Since E is unimodular, it has the unimodular inverse

4.5

$$E' = \begin{bmatrix} I & -X \\ O & I \end{bmatrix} \qquad (4.8)$$

which is verified below:

$$EE' = \begin{bmatrix} I & X \\ O & I \end{bmatrix} \begin{bmatrix} I & -X \\ O & I \end{bmatrix} = \begin{bmatrix} I + XO & -IX + XI \\ OI + IO & -OX + I^2 \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix}$$

$(E'E = I_{2n \times 2n}$ likewise follows).

The next observation uses the matrix

$$R = \begin{bmatrix} A & C \\ O & B \end{bmatrix} \qquad (4.9)$$

where A,B,C are all nXn matrices:

$$ERE^{-1} = \begin{bmatrix} I & X \\ O & I \end{bmatrix} \begin{bmatrix} A & C \\ O & B \end{bmatrix} \begin{bmatrix} I & -X \\ O & I \end{bmatrix} = \begin{bmatrix} A & C+XB \\ O & B \end{bmatrix} \begin{bmatrix} I & -X \\ O & I \end{bmatrix}$$

$$= \begin{bmatrix} A & -AX + C + XB \\ O & B \end{bmatrix} \qquad (4.10)$$

The above two observations can now be brought together to solve (4.1): the 1-2'th entry of $ERE^{-1}$ above is itself a Lypanov equation. If it equalled zero, then X would be a solution to the equation

$$-AX + XB + C = O \quad \text{or} \quad AX - XB = C$$

which is of the form (4.1). The following theorem due to Jones [11] has now been proven:

Theorem 4.3: If X is a solution to the Lypanov equation

$$AX - XB = C$$

where $A,B,C,X \in \mathbb{C}^{n \times n}(r)$ then

$$R = \begin{bmatrix} A & C \\ O & B \end{bmatrix} \quad \text{is similar to} \quad R' = \begin{bmatrix} A & 0 \\ O & B \end{bmatrix} \quad (4.11)$$

*Proof:* previous discussion. ∎

The following more powerful theorem was proven by Roth [30] for matrices whose entries are complex numbers:

<u>Theorem 4.4</u>: If $A,B,C \in \mathfrak{C}^{n \times n}$, then $AX - XB = C$ has a solution $X \in \mathfrak{C}^{n \times n}$ if and only if $R$ and $R'$ in (4.11) are similar.

*Proof:* See [30]. ∎

Although Theorem 4.3 is the core result leading to a solution of (4.1), the final assault upon (4.1) will use polynomials of a matrix. For this reason, the following theorems are presented.

<u>Theorem 4.5</u>: Let $A,B$ be $n \times n$ matrices such that $B^{-1}$ exists. Then for $n \in Z^{+}$

$$(BAB^{-1})^{n} = BA^{n}B^{-1} \quad (4.12)$$

*Proof:* Inductive reasoning will be used. (4.12) is trivially true for $n = 1$. For $n = 2$,

$$(BAB^{-1})^{2} = (BAB^{-1})(BAB^{-1}) = BA(B^{-1}B)AB^{-1}$$
$$= BA(I)AB^{-1} = BA^{2}B^{-1}$$

and (4.12) is again true. Assume (4.12) is true for $n = k$
Thus

$$(BAB^{-1})^{k} = BA^{k}B^{-1}$$

and so

$$(BAB^{-1})^{k+1} = (BAB^{-1})^{k}(BAB^{-1}) = BA^{k}B^{-1}(BAB^{-1})$$
$$= BA^{k}(B^{-1}B)AB^{-1} = BA^{k+1}B^{-1}$$

and so (4.12) holds true for $k + 1$ and therefore for all

4.7

positive integers. ∎

Theorem 4.6: Let $f(x)$ be a polynomial in the variable x. Let A,B be n X n matrices such that $B^{-1}$ exists. Then

$$f(BAB^{-1}) = B f(A) B^{-1}$$

Proof: Let $f(x) = a_0 + \Sigma_{k=1,n} a_k x^k$ Then

$$
\begin{aligned}
f(BAB^{-1}) &= a_0 I + \Sigma a_k (BAB^{-1})^k = \text{(from Theorem 4.5)} \\
&= a_0 BB^{-1} + \Sigma a_k BA^k B^{-1} \\
&= B a_0 B^{-1} + B (\Sigma a_k A^k) B^{-1} \\
&= B (a_0 I + \Sigma a_k A^k) B^{-1} \\
&= B f(A) B^{-1} \quad ∎
\end{aligned}
$$

Theorem 4.7: Let

$$R = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$$

A,B,C nXn matrices. Then for $K \in Z^+$

$$R^k = \begin{bmatrix} A^k & 0 \\ * & C^k \end{bmatrix} \qquad (4.13)$$

* an expression in A,B,C.

Proof: Inductive reasoning will be used. (4.13) is trivially true for i = 1, where $* = B = 0A + 1B + 0C$

For i = 2 ,

$$R^2 = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} = \begin{bmatrix} A^2 & 0 \\ BA+CB & C^2 \end{bmatrix}$$

and (4.13) holds true again. Let (4.13) be true for n = k Then

$$R^{k+1} = R^k R = \begin{bmatrix} A^k & 0 \\ * & C^k \end{bmatrix} \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} = \begin{bmatrix} A^{k+1} & 0 \\ * & C^{k+1} \end{bmatrix}$$

and thus (4.13) is true for all positive integers. ∎

<u>Theorem 4.8</u>: Let

$$R = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$$

A,B,C nXn matrices. If f(x) is a polynomial in x, then

$$f(R) = \begin{bmatrix} f(A) & 0 \\ * & f(C) \end{bmatrix} \tag{4.14}$$

* an expression in A,B,C

*Proof*: Inductive reasoning will be used. Let

$f(x) = a_0 + \Sigma_{k=1,n}a_k x^k$   For n = 1, f(R) = $a_0 + a_1 R$ =

$$a_0 \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + a_1 \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} = \begin{bmatrix} a_0 I + a_1 A & 0 \\ a_1 B & a_0 I + a_1 C \end{bmatrix}$$

$$= \begin{bmatrix} f(A) & 0 \\ * & f(C) \end{bmatrix}$$

where

$$* = a_1 B = a_1 B + 0A + 0C$$

and so (4.14) holds.

For n = 2, f(R) = $a_0 I + a_1 R + a_2 R^2$

$$= a_0 \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + a_1 \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} + a_2 \begin{bmatrix} A^2 & 0 \\ * & C^2 \end{bmatrix} \text{ (from Theorem 4.7)}$$

$$= \begin{bmatrix} a_0 I + a_1 A + a_2 A^2 & 0 \\ * & a_0 I + a_1 C + a_2 C^2 \end{bmatrix}$$

$$= \begin{bmatrix} f(A) & 0 \\ * & f(C) \end{bmatrix}$$

and (4.14) remains true. Let (4.14) be true for n = v

If  $f_1(x) = [ a_0 + \Sigma_{k=1,v}a_k x^k ]$  and

$f(x) = f_1(x) + a_{v+1}x^{v+1}$      then

$$f(R) = \begin{bmatrix} f_1(A) & 0 \\ * & f_1(C) \end{bmatrix} + a_{v+1}R^{v+1}$$

$$= \begin{bmatrix} f_1(A) & 0 \\ * & f_1(C) \end{bmatrix} + a_{v+1} \begin{bmatrix} A^{v+1} & 0 \\ * & C^{v+1} \end{bmatrix}$$

4.9

$$= \begin{bmatrix} f_1(A) + a_{\nu+1}A^{\nu+1} & 0 \\ * & f_1(C) + a_{\nu+1}C^{\nu+1} \end{bmatrix}$$

$$= \begin{bmatrix} f(A) & 0 \\ * & f(C) \end{bmatrix}$$

and so (4.14) is true for all positive integers. ∎

One more theorem remains to be cited before the assault on (4.1) can begin. The Hamilton-Cayley Theorem will permit the identification of a pair of linear equations whose common solutions will contain those of (4.1).

Theorem 4.9 (Hamilton-Cayley): Given A an nXn matrix and $\mathfrak{F}(\mu) = \det(A - \mu I_{n \times n})$ the characteristic equation of A, then

$$\mathfrak{F}(A) = 0_{n \times n}$$

*Proof:* see Nering [26:100] ∎

The tools are now in place with which to solve (4.1). The following theorem is due to Jones [11]:

Theorem 4.10: Given the Lypanov equation

$$AX - XB = C \quad A,B,C \in \mathbb{C}^{n \times n}(\Gamma) \tag{4.15}$$

let R and R' be defined as in (4.11) and $f_A$ the characteristic equation of A. If

$$f_A(R) = \begin{bmatrix} U & M \\ V & N \end{bmatrix} \tag{4.16}$$

then

$$U + XV = 0 \tag{4.17}$$
$$M + XN = 0$$

Thus a solution to (4.15) will be found among the common solutions of the pair of equations (4.17).

*Proof*: From the discussion preceeding Theorem 4.3, it was seen that

$$ERE^{-1} = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = R'$$

for X satisfying (4.15). By substitution,

$$f_A(ERE^{-1}) = f_A(R') \qquad (4.18)$$

From Theorem 4.8,

$$f_A(R') = \begin{bmatrix} f_A(A) & 0 \\ * & f_A(B) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ * & f_A(B) \end{bmatrix} \quad \text{(Hamilton-Cayley Theorem 4.9)}$$

From Theorem 4.6

$$f_A(ERE^{-1}) = E f_A(R) E^{-1} = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} U & M \\ V & N \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} U+XV & M+XN \\ V & N \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} U+XV & -(U+XV)X + (M+XN) \\ V & -VX+N \end{bmatrix} \qquad (4.19)$$

$$= \begin{bmatrix} 0 & 0 \\ * & f_A(B) \end{bmatrix} \quad \text{(from (4.18))} \qquad (4.20)$$

Since two matrices are equal iff respective entries are equal, the 1-1th entries of (4.19) and (4.20) imply

$$U + XV = 0 \qquad (4.21)$$

and the 1-2th entries imply

$$0 = -(U + XV)X + (M + XN)$$
$$= 0X + (M + XN) \quad \text{(from (4.21))}$$
$$= M + XN$$

Thus a solution of (4.15) will satisfy the pair of equations (4.17). ∎

The example used to illustrate the tensor solution to (4.1) will serve to illustrate Theorem 4.10:

4.11

<u>Example Theorem 4.10</u>:  Solve the Lypanov equation

$$AX + XB = C \qquad\qquad (4.22)$$
$$= AX - X(-B)$$

where

$$A = \begin{bmatrix} s+z & s \\ z & s-z \end{bmatrix}, \quad B = \begin{bmatrix} -z & -s \\ -z-s & s-z \end{bmatrix}$$

$$C = \begin{bmatrix} sz & s^2 \\ 2s^2-2z^2-sz & 4sz-2z^2-3s^2 \end{bmatrix}$$

From (4.11),

$$R = \begin{bmatrix} A & C \\ 0 & -B \end{bmatrix} = \begin{bmatrix} s+z & s & sz & s^2 \\ z & s-z & 2s^2-2z^2-sz & 4sz-2z^2-3s^2 \\ 0 & 0 & z & s \\ 0 & 0 & z+s & z-s \end{bmatrix}$$

Next, find $f_A(\mu)$, the characteristic equation of A.

$$f_A(\mu) = \det \begin{bmatrix} s + z - \mu & s \\ z & s - z - \mu \end{bmatrix}$$

$$= (s+z-\mu)(s-z-\mu) - sz$$
$$= s^2 - sz - s\mu + zs - z^2 - z\mu - \mu s + \mu z + \mu^2 - sz$$
$$= s^2 - sz - z^2 + \mu^2 - 2s\mu$$

Now evaluate $f_A(R)$ (this was done using the program in

Appendix A, a BASIC program performing operations on

matrices whose entries are multivariate polynomials):

$$f_A(R) = \begin{bmatrix} 0 & 0 & -s^2z + 3s^3 & -5s^3 + 7s^2z - 2sz^2 \\ 0 & 0 & 5sz^2 - 5s^3 - 2z^3 + 2s^2z & 2sz^2 + 8s^3 - 8s^2z \\ 0 & 0 & 2s^2 - 2sz & -3s^2 + 2sz \\ 0 & 0 & -sz - 3s^2 + 2z^2 & 5s^2 - 4sz \end{bmatrix}$$

$$= \begin{bmatrix} U & M \\ V & N \end{bmatrix}$$

According to Theorem 4.10, the solution to (4.22) will be

found among the common solutions to the pair of equations

4.12

$$U + XV = 0 \qquad\qquad (4.23)$$
$$M + XN = 0 \qquad\qquad (4.24)$$

Since $U = V = 0$, (4.23) doesn't contribute to the solution.

So the search restricts itself to solutions of the equation

$$M = -XN \quad \text{or}$$

$$\begin{bmatrix} -s^2z + 3s^3 & -5s^3 + 7s^2z - 2sz^2 \\ 5sz^2 - 5s^3 & -2z^3 + 2s^2z \quad 2sz^2 + 8s^3 - 8s^2z \end{bmatrix}$$

$$= \begin{bmatrix} w & x \\ y & z \end{bmatrix} \begin{bmatrix} -2s^2 + 2sz & 3s^2 - 2sz \\ sz + 3s^2 - 2z^2 & -5s^2 + 4sz \end{bmatrix}$$

However, in order to apply the techniques of

Chapter III, one must have an equation of the form

$$AX = b$$

It is an easy matter to convert (4.24) into the form of

(3.1) by observing that $M = -XN$ implies $M^T = -N^TX^T$ This

latter equation is in the form (3.1), and the techniques of

Chapter III can be applied to this transposed equation. ∎

It is possible to further restrict the set of equations

within which the solution of (4.15) belongs:

<u>Theorem 4.11</u>: Given the Lypanov equation

$$AX - XB = C \quad A,B,C \in \mathbb{C}^{n \times n}(\Gamma)$$

let R and R' be defined as in (4.11) and $f_B$ the

characteristic equation of B. If

$$f_B(R) = \begin{bmatrix} U & M \\ V & N \end{bmatrix}$$

then

$$N - VX = 0 \qquad\qquad (4.25)$$
$$M - UX = 0$$

Thus a solution to (4.15) will be found among the common

4.13

solutions of the pair of equations (4.25).

     *Proof*:    The proof is similar to that of Theorem 4.10.

From the discussion preceeding Theorem 4.3, it was seen that

$$ERE^{-1} = \begin{bmatrix} I & X \\ O & I \end{bmatrix} \begin{bmatrix} A & C \\ O & B \end{bmatrix} \begin{bmatrix} I & -X \\ O & I \end{bmatrix} = \begin{bmatrix} A & O \\ O & B \end{bmatrix} = R'$$

for X satisfying (4.15).  By substitution,

$$f_B(ERE^{-1}) = f_B(R') \qquad\qquad (4.26)$$

From Theorem 4.8,

$$f_B(R') = \begin{bmatrix} f_B(A) & O \\ * & f_B(B) \end{bmatrix}$$

$$= \begin{bmatrix} f_B(A) & O \\ * & O \end{bmatrix} \quad \text{(Hamilton-Cayley Theorem 4.9)} \quad (4.27)$$

From Theorem 4.6

$$f_B(ERE^{-1}) = E\, f_B(R)\, E^{-1} = \begin{bmatrix} I & X \\ O & I \end{bmatrix} \begin{bmatrix} U & M \\ V & N \end{bmatrix} \begin{bmatrix} I & -X \\ O & I \end{bmatrix}$$

$$= \begin{bmatrix} U+XV & M+XN \\ V & N \end{bmatrix} \begin{bmatrix} I & -X \\ O & I \end{bmatrix}$$

$$= \begin{bmatrix} U+XV & -(U+XV)X + (M+XN) \\ V & -VX+N \end{bmatrix}$$

$$= \begin{bmatrix} U+XV & X(N-VX) + M - UX \\ V & N - VX \end{bmatrix} \begin{array}{l} \text{(rearrange 1-2th} \\ \text{entry)} \qquad (4.28) \end{array}$$

$$= \begin{bmatrix} f_B(A) & O \\ * & O \end{bmatrix} \qquad \text{(from (4.27))} \qquad (4.29)$$

the 2-2th entries of (4.28) and (4.29) imply

$$N - VX = O \qquad\qquad (4.30)$$

and for the 1-2th entries,

$$\begin{aligned} O &= X(N-VX) + M - UX \\ &= \quad OX + (M - UX) \quad \text{(from (4.30))} \\ &= M - UX \end{aligned}$$

Thus a solution of (4.15) will satisfy the pair of equations (4.25). ∎

The example for Theorem 4.10 will serve to illustrate Theorem 4.11:

Example of Theorem 4.11. From (4.22) begin by finding the characteristic equation for

$$-B = \begin{bmatrix} z & s \\ z+s & z-s \end{bmatrix}$$

$$\begin{aligned} f_{-B}(\mu) = \det(-B - \mu I) &= \det \begin{bmatrix} z-\mu & s \\ z+s & z-s-\mu \end{bmatrix} \\ &= (z-u)(z-s-\mu) - s(z+s) \\ &= z^2 - zs - z\mu - \mu z + \mu s + \mu^2 - sz - s^2 \\ &= (z^2 - s^2 - 2sz) + (s - 2z)\mu + \mu^2 \end{aligned}$$

Using the program in Appendix A,  $f_{-B}(R) =$

$$\left[ \begin{array}{cc|cc} s^2 & 3s^2-2sz & -2sz^2+2s^2z+3s^3 & 5s^2z-2s^3-2sz^2 \\ -2z^2+3sz & 4z^2-6sz+s^2 & -5s^2z+2z^3+s^3+sz^2 & -12sz^2+4z^3+10s^2z-s^3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$= \begin{bmatrix} U & M \\ V & N \end{bmatrix}$$

The pair of equations which need to be satisfied are

$$N - VX = 0 \qquad\qquad (4.31)$$
$$M - UX = 0 \qquad\qquad (4.32)$$

(4.31) doesn't contribute to the solution, and so the

solution to (4.15) will be found in the solution space of

(4.32)   M = UX or

$$\begin{bmatrix} -2sz^2+2s^2z+3s^3 & 5s^2z-2s^3-2sz^2 \\ -5s^2z+2z^3+s^3+sz^2 & -12sz^2+4z^3+10s^2z-s^3 \end{bmatrix}$$

$$= \begin{bmatrix} s^2 & 3s^2-2sz \\ -2z^2+3sz & 4z^2-6sz+s^2 \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} \quad \blacksquare$$

To summarize, given the Lypanov equation

$$AX - XB = C \quad A,B,C,X \in \mathbb{C}^{n \times n}(\Gamma)$$

and

$$R = \begin{bmatrix} A & C \\ O & B \end{bmatrix}$$

and $f_A(\mu)$, $f_B(\mu)$ the characteristic equations of A and B respectively, then X will be a common solution to the pair of equations

$$M = -XN$$
$$M' = U'X$$

where

$$f_A(R) = \begin{bmatrix} U & M \\ V & N \end{bmatrix} \qquad \text{and} \qquad f_B(R) = \begin{bmatrix} U' & M' \\ V' & N' \end{bmatrix}$$

Riccati Equation. Attention will now be turned towards (4.2), where $A,B,C,D,X \in \mathbb{C}^{n \times n}(\Gamma)$  As with the Lypanov equation, a few insights (not necessarily connected with solving the Riccati equation) will serve to motivate the techniques used to solve (4.2).

If X is an n X n matrix and $I = I_{n \times n}$, then the matrix

$$E = \begin{bmatrix} X & I \\ I & O \end{bmatrix} \qquad (4.33)$$

always has a determinant equal to -1 (this follows from the permutational definition of the determinant), and is thus uni..odular.  E's unimodular inverse is

$$E' = \begin{bmatrix} O & I \\ I & -X \end{bmatrix}$$

since $EE' = \begin{bmatrix} X & I \\ I & O \end{bmatrix} \begin{bmatrix} O & I \\ I & -X \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix}$

and $E'E = I$ likewise follows.

Given the matrix

$$R = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{4.34}$$

where A,B,C,D are nXn matrices,

$$ERE^{-1} = \begin{bmatrix} X & I \\ I & 0 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 0 & I \\ I & -X \end{bmatrix}$$

$$= \begin{bmatrix} XA+C & XB+D \\ A & B \end{bmatrix} \begin{bmatrix} 0 & I \\ I & -X \end{bmatrix}$$

$$= \begin{bmatrix} XB+D & XA+C-XBX-DX \\ B & A-BX \end{bmatrix}$$

$$= \begin{bmatrix} XB+D & -XBX-DX+XA+C \\ B & A-BX \end{bmatrix} \tag{4.35}$$

It can be seen that the 1-2th entry of $ERE^{-1}$ is in the form of the Riccati equation (4.2). However, to get the 1-2th entry to agree exactly with (4.2) which is

$$XDX + AX + XB + C = 0$$
$$= -XDX - AX - XB - C \tag{4.36}$$

it will be necessary to modify the R matrix (4.34). The modification is implemented by comparing the form of the 1-2th entry of (4.35)

$$-XBX - DX + XA + C \qquad \text{(from R)}$$

with the the second equation in (4.36)

$$-XDX - AX - XB - C \qquad \text{(original)}$$

By matching the coefficient matrices A,B,C,D for like expressions involving X, the suggested change is from

$$R = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \text{to} \quad R = \begin{bmatrix} -B & D \\ -C & A \end{bmatrix} \tag{4.37}$$

So with this new form of R,

$$ERE^{-1} = \begin{bmatrix} X & I \\ I & 0 \end{bmatrix} \begin{bmatrix} -B & D \\ -C & A \end{bmatrix} \begin{bmatrix} 0 & I \\ I & -X \end{bmatrix}$$

4.17

$$= \begin{bmatrix} -XB-C & XD+A \\ -B & D \end{bmatrix} \begin{bmatrix} 0 & I \\ I & -X \end{bmatrix}$$

$$= \begin{bmatrix} XD+A & -XB-C-XDX-AX \\ D & -B - DX \end{bmatrix} \qquad (4.38)$$

If X is a solution of the 1-2th entry, then

$$-XB - C - XDX - AX = 0 = XDX + AX + XB + C$$

and X is a solution of (4.2).

The next insight stems from the procedure followed in solving the Lypanov equation

$$AX - XB = C$$

In Theorems 4.10 and 4.11, a solution to the Lypanov equation was found among the solutions to the pair of equations (4.17)

$$U + XV = 0$$
$$M + XN = 0$$

(derived from using the characteristic equation of A $f_A$) and (4.25)

$$N' - V'X = 0$$
$$M' - U'X = 0$$

(derived from using the characteristic equation of B $f_B$). Although only one equation from each pair turned out to be significant in solving (4.1), the original pairs point to the solution of the Riccati equation. To obtain (4.17), notice that if

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}\begin{bmatrix} U & M \\ V & N \end{bmatrix} = \begin{bmatrix} U + XV & M + XN \\ V & N \end{bmatrix}$$

is set equal to

$$\begin{bmatrix} 0 & 0 \\ V & N \end{bmatrix}$$

4.18

then matching the top rows will yield (4.17). The matrix

$$\begin{bmatrix} I & X \\ O & I \end{bmatrix}$$

was picked because it helped generate the Lypanov equation
in (4.10). As before,

$$f_A(R) = \begin{bmatrix} U & M \\ V & N \end{bmatrix} \quad , \quad R = \begin{bmatrix} A & C \\ O & B \end{bmatrix}$$

The development of the pair (4.25) is similar:  set

$$\begin{bmatrix} U & M \\ V & N \end{bmatrix} \begin{bmatrix} I & -X \\ O & I \end{bmatrix} = \begin{bmatrix} U & -UX + M \\ V & -VX + N \end{bmatrix}$$

equal to

$$\begin{bmatrix} U & O \\ V & O \end{bmatrix} \quad \text{where} \quad f_B(R) = \begin{bmatrix} U & M \\ V & N \end{bmatrix}$$

to obtain the pair (4.25).

The same approach will now be used to generate
solutions to the Riccati equation.  Let $f(x)$ be any
polynomial in the indeterminate $x$ with degree $\ge 1$.
From (4.37)

$$R = \begin{bmatrix} -B & D \\ -C & A \end{bmatrix}$$

Let $f(R) = \begin{bmatrix} U & M \\ V & N \end{bmatrix}$ (4.39)

Now the crucial step:  as if developing the pair (4.17), let
X be such that

$$Ef(R) = \begin{bmatrix} X & I \\ I & O \end{bmatrix} \begin{bmatrix} U & M \\ V & N \end{bmatrix}$$

$$= \begin{bmatrix} XU+V & XM+N \\ U & M \end{bmatrix} = \begin{bmatrix} O & O \\ U & M \end{bmatrix} = S \quad (4.40)$$

That such an X exists is motivated by the previous
discussion concerning the Lypanov equation.  From (4.40)

4.19

$$f(R) = E^{-1}S \text{ or}$$

$$f(R) = \begin{bmatrix} U & M \\ V & N \end{bmatrix} = \begin{bmatrix} O & I \\ I & -X \end{bmatrix} \begin{bmatrix} O & O \\ U & M \end{bmatrix} = \begin{bmatrix} U & M \\ -XU & -XM \end{bmatrix} \quad (4.41)$$

Because $f(x)$ is a polynomial in $x$, $xf(x) = f(x)x$ and so from (4.41)

$$Rf(R) = \begin{bmatrix} -B & D \\ -C & A \end{bmatrix} \begin{bmatrix} U & M \\ -XU & -XM \end{bmatrix} = \begin{bmatrix} -BU-DXU & -BM-DXM \\ -CU-AXU & -CM-AXM \end{bmatrix} \quad (4.42)$$

and

$$f(R)R = \begin{bmatrix} U & M \\ -XU & -XM \end{bmatrix} \begin{bmatrix} -B & D \\ -C & A \end{bmatrix} = \begin{bmatrix} -UB-MC & UD+MA \\ XUB+XMC & -XUD-XMA \end{bmatrix} \quad (4.43)$$

Since (4.42) and (4.43) are the same matrix

$$\begin{aligned} BU + DXU &= UB + MC & (4.44) \\ BM + DXM &= -UD - MA & (4.45) \\ CU + AXU &= -XUB - XMC & (4.46) \\ CM + AXM &= XUD + XMA & (4.47) \end{aligned}$$

Within the above four equations lie two solutions to the Riccati equation (4.2). From (4.44) and (4.46),

$$CU + AXU = -X(UB + MC) = -X(BU + DXU) = -XBU - XDXU$$

and so

$$\begin{aligned} CU + AXU + XBU + XDXU &= O \\ &= (C + AX + XB + XDX)U \quad (4.48) \end{aligned}$$

If $\det(U)$ is a nonzero complex number, then $U^{-1}$ exists and from (4.48)

$$XDX + AX + XB + C = O$$

and $X$ satisfies the Riccati equation (4.2). The second solution uses (4.45) and (4.47):

$$CM + AXM = X(UD + MA) = X(-BM - DXM) = -XBM - XDXM$$

and so

$$CM + AXM + XBM + XDXM = O = (C + AX + XB + XDX)M$$

4.20

If det(M) is a nonzero complex number, $M^{-1}$ exists and so

$$XDX + AX + XB + C = 0$$

another solution to (4.2).

The following theorem due to Jones has now been proven:

__Theorem 4.12__: Given $A,B,C,D \in \mathbb{C}^{n \times n}(\Gamma)$ and $f(x)$ a polynomial with degree $\geq 1$, let

$$R = \begin{bmatrix} -B & D \\ -C & A \end{bmatrix} \quad \text{and} \quad f(R) = \begin{bmatrix} U & M \\ V & N \end{bmatrix}$$

If $X \in \mathbb{C}^{n \times n}(\Gamma)$ satisfies

$$\begin{aligned} XU + V &= 0 \\ XM + N &= 0 \end{aligned} \qquad (4.49)$$

and $\det(M) \in \mathbb{C} - \{0\}$, then $XDX + AX + XB + C = 0$. Also, if $X \in \mathbb{C}^{n \times n}(\Gamma)$ satisfies the pair (4.49) and $\det(U) \in \mathbb{C} - \{0\}$ then $XDX + AX + XB + C = 0$.

*Proof:* previous discussion. ∎

A third solution exists for (4.2) but requires a different setup akin to developing the pair (4.25). Let R be as in (4.37) and $f(R)$ as in (4.39). Let X satisfy

$$f(R) E = \begin{bmatrix} U & M \\ V & N \end{bmatrix} \begin{bmatrix} 0 & I \\ I & -X \end{bmatrix} = \begin{bmatrix} M & U-MX \\ N & V-NX \end{bmatrix}$$

$$= \begin{bmatrix} M & 0 \\ N & 0 \end{bmatrix} \qquad (4.50)$$

That such an X exists is motivated by the development of the pair (4.25) for the Lypanov equation. (4.50) implies

$$f(R) = \begin{bmatrix} U & M \\ V & N \end{bmatrix} = \begin{bmatrix} M & 0 \\ N & 0 \end{bmatrix} \begin{bmatrix} X & I \\ I & 0 \end{bmatrix} = \begin{bmatrix} MX & M \\ NX & N \end{bmatrix} \qquad (4.51)$$

Since $Rf(R) = f(R)R$

4.21

$$Rf(R) = \begin{bmatrix} -B & D \\ -C & A \end{bmatrix} \begin{bmatrix} MX & M \\ NX & N \end{bmatrix}$$

$$= \begin{bmatrix} -BMX + DNX & -BM + DN \\ -CMX + ANX & -CM + AN \end{bmatrix} \quad (4.52)$$

and

$$f(R)\,R = \begin{bmatrix} MX & M \\ NX & N \end{bmatrix} \begin{bmatrix} -B & D \\ -C & A \end{bmatrix}$$

$$= \begin{bmatrix} -MXB - MC & MXD + MA \\ -NXB - NC & NXD + NA \end{bmatrix} \quad (4.53)$$

Equating last rows of (4.52) and (4.53)

$$NXB + NC = CMX - ANX \quad (4.54)$$
$$NXD + NA = -CM + AN \quad (4.55)$$

It follows that

$$NXB + NC = (CM - AN)X = (-NXD - NA)X$$
$$= -NXDX - NAX$$

and so

$$NXDX + NAX + NXB + NC = 0 = N(XDX + AX + XB + C)$$

If det(N) is a nonzero complex number, $N^{-1}$ exists and so

$$XDX + AX + XB + C = 0$$

and a third solution to (4.2) has been found and thus

Theorem 4.13: Given $A,B,C,D \in \mathbb{C}^{n \times n}(\Gamma)$ and $f(x)$ a

polynomial with degree $\geq 1$, let

$$R = \begin{bmatrix} -B & D \\ -C & A \end{bmatrix} \quad \text{and} \quad f(R) = \begin{bmatrix} U & M \\ V & N \end{bmatrix}$$

If $X \in \mathbb{C}^{n \times n}(\Gamma)$ satisfies

$$U - MX = 0 \quad (4.56)$$
$$V - NX = 0$$

and det(N) $\in \mathbb{C} - \{0\}$, then $XDX + AX + XB + C = 0$.

Proof: previous discussion. ∎

The next theorem presents a companion Riccati equation

4.22

to that of (4.2) given a particular polynomial $f(x)$:

Theorem 4.14: Let $f(x)$ be a polynomial of degree $\geq 1$ and

$$XDX + AX + XB + C = 0 \qquad A,B,C,D,X \in \mathcal{C}^{n \times n}(\Gamma)$$

with R and $f(R)$ given by (4.37) and (4.39). Then

$$XMX + NX - XU - V = 0$$

Proof: It has already been shown that

$$E = \begin{bmatrix} X & I \\ I & 0 \end{bmatrix} \quad \text{and} \quad E' = \begin{bmatrix} 0 & I \\ I & -X \end{bmatrix}$$

are both unimodular and inverses of one another. From (4.38) and the given,

$$ERE^{-1} = \begin{bmatrix} XD + A & 0 \\ D & -B - DX \end{bmatrix}$$

From Theorem 4.8,

$$f(ERE^{-1}) = \begin{bmatrix} f(XD+A) & 0 \\ * & f(-B-DX) \end{bmatrix} \qquad (4.57)$$

$$= Ef(R)E^{-1} \quad \text{(from Theorem 4.6)}$$

$$= \begin{bmatrix} X & I \\ I & 0 \end{bmatrix} \begin{bmatrix} U & M \\ V & N \end{bmatrix} \begin{bmatrix} 0 & I \\ I & -X \end{bmatrix}$$

$$= \begin{bmatrix} XU + V & XM + N \\ U & M \end{bmatrix} \begin{bmatrix} 0 & I \\ I & -X \end{bmatrix}$$

$$= \begin{bmatrix} XM + N & XU + V - XMX - NX \\ M & U - MX \end{bmatrix} \qquad (4.58)$$

Equating the 1-2th entries of (4.57) and (4.58),

$$XU + V - XMX - NX = 0 = XMX + NX - XU - V \quad \blacksquare$$

## V   Solutions of the Third Order Riccati Equation
## XAXBX + XCX + XD + EX + F = 0

This chapter will discuss some of the approaches which may yield solutions to the third order Riccati equation

$$XAXBX + XCX + XD + EX + F = 0 \qquad (5.1)$$

where $A,B,C,D,E,F,X \in \mathbb{C}^{n \times n}(\Gamma)$, $\Gamma$ the tuple of variates. The approach for (5.1) will be less general than that for the second order Riccati and Lypanov equations because of the form of (5.1). Difficulties stemming from this form are discussed in the next chapter.

One approach to solving (5.1) is akin to that taken for the second order Riccati. Let $A,B,C,D,E,F$ be known matrices in $\mathbb{C}^{n \times n}(\Gamma)$ and

$$R = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

and

$$f(R) = \begin{bmatrix} U & M \\ V & N \end{bmatrix}$$

where $f(x)$ is a given polynomial with degree $\geq 1$. As seen previously, if $X \in \mathbb{C}^{n \times n}(\Gamma)$,

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} U & M \\ V & N \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} = \begin{bmatrix} U + XV & -(U + XV)X + (XN + M) \\ V & N - VX \end{bmatrix}$$

Suppose $U + XV = 0$ (motivated by its appearance in solving for the Lypanov and second order Riccati equations). Thus

$$U = -XV \qquad (5.2)$$

Because $f(R)$ is a polynomial in R, R and $f(R)$ commute. From (5.2),

$$f(R) = \begin{bmatrix} -XV & M \\ V & N \end{bmatrix}$$

It follows that

$$Rf(R) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} -XV & M \\ V & N \end{bmatrix} = \begin{bmatrix} BV - AXV & AM + BN \\ DV - CXV & CM + DN \end{bmatrix}$$

and

$$f(R)R = \begin{bmatrix} -XV & M \\ V & N \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} MC - XVA & MD - XVB \\ VA + NC & VB + ND \end{bmatrix}$$

Equating the first columns,

$$BV - AXV = MC - XVA \qquad (5.3)$$
$$DV - CXV = VA + NC \qquad (5.4)$$

VA appears in the right hand sides of (5.3) and (5.4). If

MC in (5.3) involved XNC, then a substitution from (5.4)

could be made in (5.3). Accordingly, let

$$MC = -XNC - \Sigma \qquad (5.5)$$

Then (5.3) becomes

$$\begin{aligned} BV - AXV &= (-XNC - \Sigma) - XVA \\ &= -X(NC + VA) - \Sigma \\ &= -X(DV - CXV) - \Sigma \quad \text{(from (5.4))} \\ &= -XDV + XCXV - \Sigma \end{aligned}$$

or

$$XCXV - XDV + AXV - BV - \Sigma = 0 \qquad (5.6)$$

Because all terms except $\Sigma$ end in V, let

$$\Sigma = e_1 V$$

(5.6) then becomes

$$(XCX - XD + AX - B - e_1)V = 0$$

If det(V) is a complex, non-zero number (easily verified

since f(R) is known), then the inverse of V exists and so

$$XCX - XD + AX - B - e_1 = 0 \qquad (5.7)$$

Since solutions of the third order Riccati equation are

5.2

sought, let $e_1 = -XEXFX$. (5.7) then becomes

$$XEXFX + XCX + AX - XD - B = 0 \qquad (5.8)$$

and X is a solution to this third order Riccati equation.

However, since $e_1 = -XEXFX$ and $\Sigma = e_1 V$, it follows that

$\Sigma = -XEXFXV$ and from (5.2), $\Sigma = XEXFU$. From (5.5) it

follows that

$$MC = -XNC - \Sigma = -XNC - XEXFU$$

To summarize, if

$$R = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad , \qquad f(R) = \begin{bmatrix} U & M \\ V & N \end{bmatrix}$$

det(V) a non-zero complex number and X satisfies the pair of

equations

$$\begin{aligned} U + XV &= 0 \\ -XEXFU - XNC &= MC \end{aligned} \qquad (5.9)$$

then X is a solution to the third order Riccati equation

$$XEXFX + XCX + AX - XD - B = 0 \qquad (5.10)$$

If det(FU) is a non-zero complex number, then (5.9) can be

reduced to a second order Riccati equation of the form

solved in the previous chapter.

Another solution to (5.10) can be gleened using the

above approach. Let $XN + M = 0$. Then

$$Rf(R) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} U & -XN \\ V & N \end{bmatrix} = \begin{bmatrix} AU + BV & BN - AXN \\ CU + DV & DN - CXN \end{bmatrix} \qquad (5.11)$$

and

$$f(R)R = \begin{bmatrix} U & -XN \\ V & N \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} UA - XNC & UB - XND \\ VA + NC & VB + ND \end{bmatrix} \qquad (5.12)$$

Equating the second columns of (5.11) and (5.12),

$$-AXN + BN = UB - XND \qquad (5.13)$$
$$-CXN + DN = VB + ND \qquad (5.14)$$

Noticing that ND appears in the right hand sides, let

$$UB = -XVB + \Sigma \qquad (5.15)$$

Then (5.13) becomes

$$
\begin{aligned}
-AXN + BN &= (-XVB + \Sigma) - XND \\
&= -X(VB + ND) + \Sigma \\
&= -X(DN - CXN) + \Sigma \\
&= -XDN + XCXN + \Sigma \qquad (5.16)
\end{aligned}
$$

and so

$$XCXN - XDN + AXN - BN + \Sigma = 0 \qquad (5.17)$$

Let $\Sigma = e_1 N$. Then (5.17) becomes

$$(XCX - XD + AX - B + e_1)N = 0$$

If det(N) is a non-zero complex number, then N has an inverse, and so

$$XCX - XD + AX - B + e_1 = 0 \qquad (5.18)$$

If $e_1 = XEXFX$ then

$$XCX - XD + AX - B + XEXFX = 0$$

and another solution to the third order Riccati has been found. Since $\Sigma = e_1 N$, $\Sigma = XEXFXN = -XEXFM$. (5.15) then becomes

$$UB = -XVB - XEXFM$$

Thus, if X is a common solution to the pair of equations

$$
\begin{aligned}
0 &= XN + M \\
UB &= -XVB - XEXFM
\end{aligned}
$$

and det(N) is non-zero complex, then

$$XEXFX + XCX + AX - XD - B = 0$$

Another approach to solving (5.1) extends the solution via similarity transformation used in the matrix equations presented herein. Though this approach fails to generate

5.4

sets of equations (whose common solutions satisfy (5.1)), it does present relations among the coefficient matrices for a restricted case of (5.1).

Let $A,B,C,D,E,F,G,H \in \mathbb{C}^{n \times n}(\Gamma)$. Then given $X \in \mathbb{C}^{n \times n}(\Gamma)$,

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} E+XG & F+XH \\ G & H \end{bmatrix} \begin{bmatrix} A+XC & -AX-XCX+B+XD \\ C & D-CX \end{bmatrix}$$

$$= \mathfrak{F}$$

By multiplying these matrices, the following entries of $\mathfrak{F}$ are obtained:

1-1: $EA + EXC + XGA + XGXC + FC + XHC$       (5.19)

1-2: $-XGXCX - X(GA+HC)X - (EA+FC)X + X(GB+HD) +$
                              $(EB+FD) + (EXD-EXCX+XGXD)$    (5.20)

1-3: $GA + GXC + HC$                                   (5.21)

1-4: $GB + GXD - GAX - GXCX + HD - HCX$

Of the entries, the only one in which the "cubic" of the third order Riccati appears is (5.20), that is, XGXCX. If it weren't for the last term in paranthesis in (5.20), the form of (5.20) would be the same as that of (5.1). In order to eliminate this last term and preserve the structural integrity of (5.20), let $D = E = 0$. It follows that

$$\mathfrak{F} = \begin{bmatrix} XGA+XGXC+FC+XHC & -XGXCX-X(GA+HC)X-FCX+XGB \\ GA+GXC+HC & GB-GAX-GXCX-HCX \end{bmatrix}$$    (5.22)

If X were a solution to $\mathfrak{F}(1,2)$, then X would be a solution to a third order Riccati equation of the form

$$XAXBX + XCX + XD + EX = 0$$         (5.23)

which is a special case of (5.1). From the remarks preceeding (5.19), it follows that (since $D = E = 0$)

$$\xi = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & F \\ G & H \end{bmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} \qquad (5.24)$$

$$= \begin{bmatrix} XGA+XGXC+FC+XHC & 0 \\ GA+GXC+HC & GB-GAX-GXCX-HCX \end{bmatrix} \qquad (5.25)$$

Multiplying the three middle matrices in (5.24),

$$\begin{bmatrix} 0 & F \\ G & H \end{bmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} FC & 0 \\ GA + HC + GXC & GB \end{bmatrix} \qquad (5.26)$$

$$= \xi_1$$

If $f(x)$ is any polynomial of degree $\geq 1$, then from (5.24) and Theorem 4.6,

$$f(\xi) = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} f(\xi_1) \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}$$

and from Theorem 4.8

$$f(\xi) = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} f(FC) & 0 \\ * & f(GB) \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} \qquad (5.27)$$

$$= \begin{bmatrix} f(XGA+XGXC+FC+XHC) & 0 \\ * & f(GB-GAX-GXCX-HCX) \end{bmatrix}$$

Since $f(\xi)$ and $f(\xi_1)$ are similar matrices, they have the same determinants. From the permutational definition of the determinant (2.13),

$$det(f(\xi)) = det[f(XGA+XGXC+FC+XHC)] *$$
$$det[f(GB-GAX-GXCX-HCX)] \qquad (5.28)$$

and

$$det(f(\xi_1)) = det[f(FC)] * det[f(GB)] \qquad (5.29)$$

and thus

$$det[f(FC)] * det[f(GB)] = det[f(XGA+XGXC+FC+XHC)] *$$
$$det[f(GB-GAX-GXCX-HCX)]$$

The following theorem has now been proven:

Theorem 5.1: Given $A,B,C,F,G,H \in \mathbb{C}^{n \times n}(\Gamma)$ and $X \in \mathbb{C}^{n \times n}(\Gamma)$ a solution to

$$-XGXCX-X(GA+HC)X-FCX+XGB = 0$$

then

$$det[f(FC)] * det[f(GB)] = det[f(XGA+XGXC+FC+XHC)] * det[f(GB-GAX-GXCX-HCX)]$$

*Proof:* previous discussion. ∎

Theorem 5.1 may be extended to a Riccati equation of
the form

$$XA_1XA_2X + XA_3X + A_4X + XA_5 = 0 \qquad (5.30)$$

To get this equation in the form (of the given in

Theorem 5.1)

$$-XGXCX-X(GA+HC)X-FCX+XGB = 0$$

equating coefficient matrices is done.   Thus

$$A_1 = -G \qquad (5.31)$$
$$A_2 = C \qquad (5.32)$$
$$A_3 = -(GA + HC) = A_1A - HA_2 \qquad (5.33)$$
$$A_4 = -FC = -FA_2 \qquad (5.34)$$
$$A_5 = GB = -A_1B \qquad (5.35)$$

Since G and C are readily known, it remains to determine

A,H,F and B.   From the above equations, one may have

considerable latitude in choosing A,H,F,B.   From (5.35), B

satisfies the equation $A_1B = -A_5$.   Likewise, F satisfies

the equation $FA_2 = -A_4$.   Lastly, A and H satisfy the Lypanov

equation (5.33)

$$A_3 = A_1A - HA_2$$

By proper choice of A,B,F,H, the challenge of finding

solutions to (5.30) via Theorem 5.1 may be made easier.

## VI Conclusions and Recommendations

This paper sought the solution X to four types of

matrix equations:  the linear equation

$$AX = b$$

the Lypanov equation

$$AX - XB = C$$

the second-order Riccati equation

$$XDX + AX + XB + C = O$$

and the third-order Riccati equation

$$XAXBX + XCX + XD + EX + F = O$$

where the entries of all matrices are restricted to being

multivariate polynomials over the complex numbers.  The

method of solution involved two phases:

1.  identify a similarity transformation on a matrix $\Sigma$
    in which is embedded the coefficient matrices of
    the above equations.  The transformation gives a
    matrix whose entries include the equation being
    solved.

2.  identify a polynomial which, when its argument is
    the matrix $\Sigma$ in (1), gives a matrix whose entries
    yield pairs of linear equations.  The common
    solutions of these pairs of equations will contain
    those of the given matrix equation.  The choice of
    polynomial may yield different solutions.

Since the search for solutions ends in solving a linear

equation, a method was presented which may identify the

general solution of the given linear equation.  The method's

success depends on if the matrix A in $AX = b$ is reducible to

a Smith Form.  The similarity transformation of (1) is given

by $E \Sigma E^{-1}$ where E equals

$$\begin{bmatrix} I & X \\ O & I \end{bmatrix}$$

(which is unimodular) and $\Sigma$ for the Lypanov equation equals

$$\begin{bmatrix} A & C \\ O & B \end{bmatrix}$$

and $\Sigma$ for the second order Riccati equals

$$\begin{bmatrix} -B & D \\ -C & A \end{bmatrix}$$

The third order Riccati presented difficulties because its form wasn't adaptable to the method of similarity transformations. Nevertheless, a transformation could be done which might yield some solutions. When successful, the solution to the third order Riccati was a common solution to a linear and a second order Riccati equation. The polynomial in (2) can assume any form and when its argument is $\Sigma$,

$$f(\Sigma) \quad = \quad \begin{bmatrix} U & M \\ V & N \end{bmatrix} \qquad (6.1)$$

The submatrices U,M,V,N yield pairs of equations which depend on the matrix equation being solved. These pairs in turn yield solutions of the original matrix equation.

Recommendations. The biggest challenge is also the central concern: develop the mathematical theory which will identify whether or not a given matrix, whose entries are multivariate polynomials, is reducible to Smith Form. Though the answer for the two-variate polynomial case is known, the three or more variate case remains an open question. It's felt that this challenge will involve

6.2

creative work involving number theory, abstract algebra, and matrix theory. Once the theory identifying the conditions under which reducibility to Smith Form is known, the next hurdle will be to develop a computer algorithm performing the reductions (the Smith Form may not be unique). The papers [8] and [20] give approaches to the two-variate case, and the papers [23] and [33] give insights dealing with the "computer programming algebra" of matrix forms and polynomials. Perhaps there is an underlying linear programming or network formulation to the Smith Form reduction.

In solving the Lypanov and Riccati equations, it was necessary to assume the existence of the inverse to one or more of the submatrices in (6.1). Of course, (6.1) depends on the polynomial $f(x)$ chosen, which in turn dictates the types of solutions found to the original matrix equation. Research needs to be done into the types of polynomials yielding one or more invertible submatrices of (6.1). Perhaps there are equivalence classes of polynomials. Or families of polynomials. Maybe there is no structure. Or perhaps there is a "minimum" polynomial which (in some way) generates all polynomials yielding invertible submatrices. Again, a knowledge of abstract algebra may prove valuable: much literature dealing with polynomial structures is couched in abstract algebraic terms.

Much effort was spent in finding the "proper embedding"

6.3

for the third order Riccati equation. Disappointingly
limited success was realized. To see what is meant by
"proper embedding", notice that in

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} = \begin{bmatrix} A + XC & -AX - XCX + B + XD \\ C & -CX + D \end{bmatrix}$$

a second order Riccati equation is found in the 1-2th
position of the matrix on the right. If C = 0, then this
equation becomes the Lypanov. Thus the Lypanov and second
order Riccati equations are "embedded" in the matrix on the
right, which in turn comes from the similarity
transformation on the left. Since this approach has proven
so successful in generating linear equations whose solutions
are those to the Lypanov and second order Riccati equations,
it is natural to extend the similarity transformation to
handle the third order Riccati equation.

Unfortunately, this is more easily said than done.
Since the third order Riccati equation

$$XAXBX + XCX + XD + EX + F = 0 \tag{6.2}$$

has six coefficient matrices, the above approach would have
to be modified to handle these six. As it stands, it is
geared to the four coefficients of the second order Riccati
equation. Is it possible to work with matrices which are
two sub-matrices deep and wide, or is it necessary to go to
higher order matrices, perhaps three submatrices in
dimension. If so, what would the unimodular matrix be that
effects the similarity transformation? Is the restriction
of unimodularity unduly restrictive, mandatory though it

6.4

seems?  Also, the first term XAXBX of (6.2) presents a

challenge:  how to introduce the middle X from the

transformation $E \; \Sigma \; E^{-1}$ while hopefully embedding a third

order Riccati equation in the resulting matrix of

transition.  Perhaps a different type of transformation

needs to be done.  Instead of a single transform, a string

of transforms may be the answer, for example

$$E_1 \Sigma_1 E_2 \Sigma_2 E_3$$

where the $\Sigma_j$ are matrices whose entries include the

coefficient matrices, and the $E_j$ contain the solution X.  If

this approach is indeed the way to go, how are the $E_j$'s

related.  Again, the third order Riccati should be embedded

somewhere in the resulting matrix.

Or should it?  If not embedded, could other matrix

equations which are embedded as a matter of course yield, in

some combination, some (or all!) solutions of the third

order Riccati?  Another concern arises:  in the approaches

presented herein, a specially selected polynomial $f(x)$

played the crucial role of generating pairs of equations

whose common solutions included those of the matrix equation

in question.  For the third order Riccati, how is the

polynomial to be selected?  How should the pairs be

generated, if indeed this pattern holds?  Perhaps triplets

of equations arise instead of pairs.

The unsettling thought occurs:  maybe similarity

transforms are not powerful enough for the third order

6.5

Riccati, and whole new approaches are in order.  After all, current research into these matrix equations is still in the pioneering stages.  Hopefully, the successful resolution to the third order Riccati equation will point the way to solving higher order Riccati equations.  However, a spectre appears:  in solving for the roots of a polynomial in one variable, the insolvability of the quintic was demonstrated by Abel in the early nineteenth century.  Could there be a similar obstacle ahead for matrix equations?  In any case, finding solutions to matrix equations will prove challenging and endlessly fascinating--as well as having immediate practical applications.

The last recommendation is a minor one:  create a new segment (in the computer program of Appendix A) which will compute the determinant of a given square matrix having multivariate polynomial entries.  The method would have to keep calculations down to a minimum, since finding determinants is computationally intensive.  One approach, which seems to be the quickest way to find a matrix's determinant, was published by G. Macloskie [22] in 1904.  This method could be adapted to that of finding determinants for matrices with polynomial entries.

## Appendix A:   BASIC program performing matrix operations

The following BASIC program performs operations on
rectangular matrices having multivariate polynomial entries
over the real numbers.   Via menu options, the interactive
program allows the user to

        -create a matrix
        -view a matrix
        -transpose a matrix
        -add two matrices
        -multiply two matrices
        -extract the U,M,V,N matrices from a given matrix

The size of matrices which the program can handle is limited
only by the amount of memory available on the computer and
the parameters in the DIMENSION statement.   The program was
designed on an IBM PC (DOS 3.0) and was intended to assist
the thesis effort.

The program's logic hinges upon the way it recognizes a
polynomial in n-variates:   as a set of (n+1)-tuples each of
whose entries come from the coefficients and exponents of
the polynomial.   To obtain the tuples, the entire polynomial
is re-expressed in "standard form" as a sum of terms, each
having all variate and exponent positions appear.   For
instance, the polynomial in the variates x and y

$$2x^2y - 17xy^5 - 3y + 7 \qquad\qquad (A.1)$$

is re-expressed in standard form as

$$2x^2y^1 + (-17x^1y^5) + (-3x^0y^1) + (7x^0y^0) \qquad (A.2)$$

and is the sum of the four terms $2x^2y^1$, $-17x^1y^5$, $-3x^0y^1$,
$7x^0y^0$.   Though the order of the variates (i.e., x before y

versus y before x) is arbitrary, the order must be
consistent.

Once in standard form, each term is put into its
tuple representation

$$(coefficient, exponent_1,...,exponent_n) \qquad (A.3)$$

with the polynomial represented by the collection of the
tuples.  For the above example:

| TERM | 3-tuple |
|------|---------|
| $2x^2y^1$ | ( 2,2,1) |
| $-17x^1y^5$ | (-17,1,5) |
| $-3x^0y^1$ | (-3,0,1) |
| $7x^0y^0$ | ( 7,0,0) |

and so the polynomial

$$2x^2y - 17xy^5 - 3y + 7$$

is represented by the set

$$\{ (2,2,1), (-17,1,5), (-3,0,1), (7,0,0) \}$$

Similarly, the polynomial

$$42xyuvz - 2z^2yx$$

is re-expressed in the standard form (variate order is
x,y,z,u,v)

$$42x^1y^1z^1u^1v^1 + (-2x^1y^1z^2u^0v^0)$$

and thus the set of 6-tuples

$$\{ (42,1,1,1,1,1), (-2,1,1,2,0,0) \}$$

Addition and multiplication between two polynomials are
easily expressed in terms of their tuple representations
provided

A.2

-all tuples have the same number of entries
-coordinate positions all correspond to the same
 variable

Addition. Given two terms in n-variates, let $(a,\alpha)$ and $(b,\beta)$ be their respective $(n+1)$-tuple representations where $a,b$ are the coefficients of each term and $\alpha,\beta$ the exponent vectors. For example, if the terms are $3x^2y$ and $7x^9y^5$, then

$$3x^2y \approx (3,2,1) \;, \; a = 3, \; \alpha = (2,1)$$

$$7x^9y^5 \approx (7,9,5) \;, \; b = 7, \; \beta = (9,5)$$

Tuple addition is defined by

$$(a,\alpha) \not\# (b,\beta) = \begin{cases} \{ (a,\alpha), (b,\beta) \} & \alpha \neq \beta \\ (a+b,\alpha) & \alpha = \beta \end{cases} \qquad (A.4)$$

The operation $\not\#$ is clearly commutative. To add two polynomials, the tuples representing the terms are added according to (A.4). For example:

$$(2x^2y \quad - \quad 9xy^2) \quad + \quad (4xy^2 \quad + \quad x^2y)$$
$$\approx [(2,2,1) \not\# (-9,1,2)] \not\# [(4,1,2) \not\# (1,2,1)]$$

then, rearranging terms

$$= [(2,2,1) \not\# (1,2,1)] \not\# [(-9,1,2) \not\# (4,1,2)]$$
$$= (3,2,1) \not\# (-5,1,2)$$
$$= \{ (3,2,1), (-5,1,2) \}$$
$$\approx 3x^2y - 5xy^2 \qquad \blacksquare$$

Multiplication. Multiplication between two tuples is defined by

$$(a,\alpha) \cdot (b,\beta) = (ab,\alpha+\beta) \qquad (A.5)$$

where $\alpha + \beta$ is the familiar operation of addition between two vectors in Euclidean n-space. For example,

A.3

$$(2x^3y^5)(6x^7y^8) \approx (2,3,5)\cdot(6,7,8)$$
$$= (2*6,3+7,5+8)$$
$$= (12,10,13)$$
$$\approx 12x^{10}y^{13} \qquad \blacksquare$$

Using (A.5) the tuple representation of the product between two polynomials is quickly found. Consider the product of the n-term polynomial

$$TERM_{11} + \ldots + TERM_{1n} \qquad (A.6)$$

and the m-term polynomial

$$TERM_{21} + \ldots + TERM_{2m} \qquad (A.7)$$

which is

$$TERM_{11}(TERM_{21} + \ldots + TERM_{2m}) + \ldots \qquad (A.8)$$
$$+ TERM_{1n}(TERM_{21} + \ldots + TERM_{2m})$$

$$= (TERM_{11}TERM_{21} + \ldots + TERM_{11}TERM_{2m}) + \ldots$$
$$+ (TERM_{1n}TERM_{21} + \ldots + TERM_{1n}TERM_{2m})$$

If the value of $TERM_{jk}$ is now changed to the tuple form of the polynomial element $TERM_{jk}$, then the product of (A.6) and (A.7) expressed in tuple form is

$$= (TERM_{11}\cdot TERM_{21} \# \ldots \# TERM_{11}\cdot TERM_{2m}) \# \ldots$$
$$\# (TERM_{1n}\cdot TERM_{21} \# \ldots \# TERM_{1n}TERM_{2m})$$

EXAMPLE:

$$(x^2y^3 + 12x^4y^5)(x^6y^7 + 11x^8y^9)$$
$$= (x^2y^3)(x^6y^7) + (x^2y^3)(11x^8y^9)$$
$$+ (12x^4y^5)(x^6y^7) + (12x^4y^5)(11x^8y^9)$$
$$\approx [(1,2,3)\cdot(1,6,7)] \# [(1,2,3)\cdot(11,8,9)]$$
$$\# [(12,4,5)\cdot(1,6,7)] \# [(12,4,5)\cdot(11,8,9)]$$
$$= (1,8,10) \# (11,10,12)$$
$$\# (12,10,12) \# (132,12,14)$$
$$= (1,8,10) \# (23,10,12) \# (132,12,14)$$
$$= \{ (1,8,10), (23,10,12), (132,12,14) \}$$
$$\approx x^8y^{10} + 23x^{10}y^{12} + 132x^{12}y^{14} \qquad \blacksquare$$

Network. Put in a nutshell, the program keeps track of tuple operations using a system of pointers (program lines 1150-1490). That is, a network tree is constructed which

A.4

represents the results of tuple operations.  This section

assumes knowledge of networks as given in [10:91-124].

The fundamental insight is that a polynomial can be

represented by a tree.  Specifically, the results of tuple

operations is maintained in a network tree whose nodes

represent a particular variate in a polynomial term.  Nodal

"potentials" are the exponent of the corresponding variate.

Terminal nodes of the polynomial's tree have a second

potential whose value is that of the coefficient of a

particular term.  The i'th depth of the tree corresponds to

the i'th variate.  For example, the polynomial

$$3x^2yz^{13} - 7x^{12}y^7z + 8x^{12}y^7z^{22}$$

is represented by the tree in Figure 1.  All the nodes at

depth one are x variates; nodes at depth two are y variates;

nodes at depth three are z variates.  The numbers in [] are

node potentials, and numbers in []* are the coefficient of

the given polynomial term.  From this example, it can be

seen that a polynomial term corresponds to a unique path

from node 0 to a given terminal node at depth three, e.g.,

the term $8x^{12}y^7z^{22}$ has the nodal path 0-4-5-7.

In performing a tuple operation, the order in which the

branches are built are term by term.  The following example

illustrates the sequence in which the program builds

branches.

EXAMPLE:  Construct the tree representing

$$x^2(1 + y) + y(x^2 + y)$$

A.5

$$x^2(1 + y) + y(x^2 + y) = x^2 + x^2y + yx^2 + y^2$$
$$= x^2y^0 + x^2y^1 + y^1x^2 + x^0y^2 \qquad (A.9)$$

Since there are 2 variates x,y the tree will have 2 depths.

The first term is $x^2y^0$ which contributes the first branch

0-1-2 in Figure 2. The second term $x^2y^1$ contributes the

branch 0-1-3 in Figure 3. Note that the second potential of

node 3 ([]*) is 1: the coefficient of the second term $x^2y^1$.

Because the third term of (A.9) is also $x^2y^1$, it doesn't

contribute a branch to the tree. However, the second

potential at node 3 is increased by 1: the coefficient in

the third term $x^2y^1$ (Figure 4). The fourth term $x^0y^2$

contributes the branch 0-4-5 in Figure 5. Since there are

no more terms in (A.9), the tree in Figure 5 represents the

polynomial $x^2(1 + y) + y(x^2 + y)$
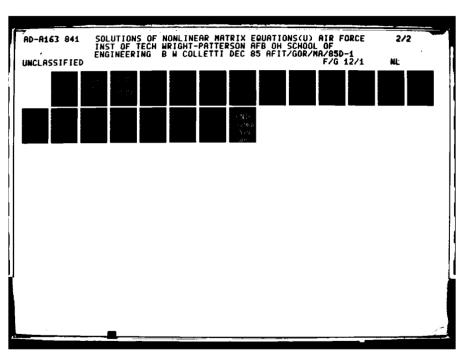
Manual input of data. When creating a matrix via the

menu prompt to "build a matrix", polynomials are entered

term by term in their tuple form with no commas between

tuple entries. Each tuple MUST remain on the same line, and

each tuple except the last ends with a carriage return; the

last tuple ends with a semicolon followed by a carriage

return. The program will then prompt for the next

polynomial entry and the above protocol is repeated. A

sample session inputing the matrix
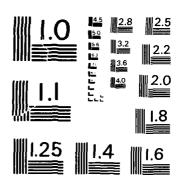
$$Q = \begin{bmatrix} x^2 + 3xyz + z^2 & 4 + 2x - 3y \\ z^6 + 17y^{78} & x^2y^3z^4 \end{bmatrix} \qquad (A.10)$$

is given in Figure 6.

Output. The program outputs a polynomial in its tuple

form.  Figure 7 is a sample session which printed the matrix
fA(R) preceeding (4.23).

A.7

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z $[/3]$ (3) $(3)^*$　$[/]$ (6) $[-7]^*$　$[22]$ (7) $[8]^*$

y $[/]$ (2)　$[9]$ (5)

x $[2]$ (1)　$[/2]$ (4)

(0)
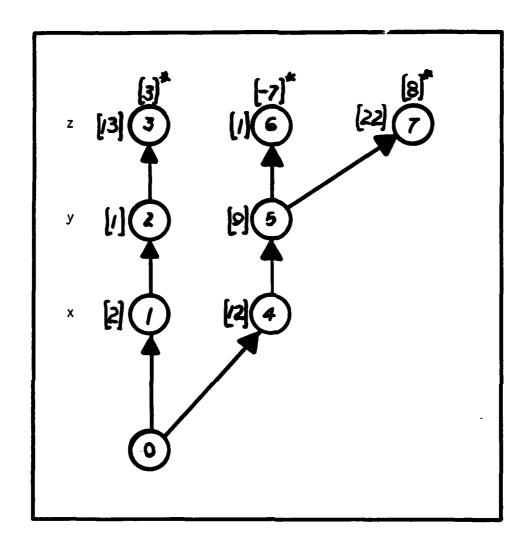
Figure 1.  Network tree representing the polynomial
$$3x^2yz^{13} - 7x^{12}y^9z + 8x^{12}y^9z^{22}$$
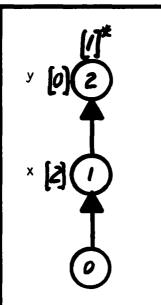
Fig. 2. Branch for $x^2$
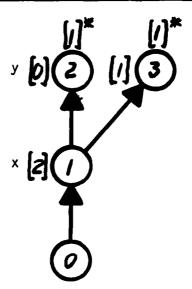
Fig. 3. Branch for $x^2 + x^2y$
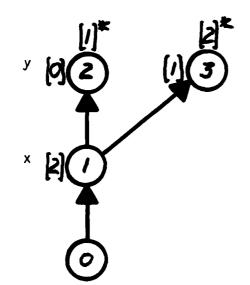
Fig. 4. Branch for $x^2 + 2x^2y$
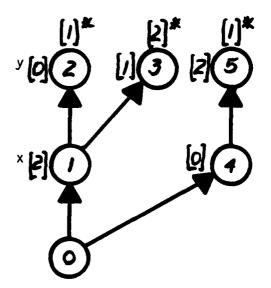
Fig. 5. Branch for
$x^2 + 2x^2y + y^2$

```
****************************************************
MENU:
        B   Build a new matrix"
        V   View a matrix"
        '   Transpose a matrix"
        +   Add 2 matrices"
        *   Multiply 2 matrices"
        U   Extract U,M,V,N matrices
****************************************************
? B
Matrix name? B:Q
ROW,COL dimension? 2,2
Number of polynomial variates? 3
Build a square diagonal matrix whose diagonal entries
    are equal Y/N? N
*** COEF EXP1 EXP2 ... EXPn ***
 a semicolon ; represents end of polynomial
(1,1):
? 1 2 0 0
? 3 1 1 1
? 1 0 0 2;

(1,2):
? 4 0 0 0
? 2 1 0 0
? -3 0 1 0;

(2,1):
? 1 0 0 6
? 17 0 98 0;

(2,2):
? 1 2 3 4;
```

Figure 6.   Session building matrix (A.10)

```
*******************************************************
MENU:
        B   Build a new matrix"
        V   View a matrix"
        '   Transpose a matrix"
        +   Add 2 matrices"
        *   Multiply 2 matrices"
        U   Extract U,M,V,N matrices
*******************************************************
? V
Screen output Y/N?   Y
Matrix to view?  B:fA(R)

MATRIX B:fA(R) IS  4 X 4 and has  2 polynomial variates
B:fA(R)( 1, 1):                      B:fA(R)( 3, 1):
0  0  0                              0  0  0


B:fA(R)( 1, 2):                      B:fA(R)( 3, 2):
0  0  0                              0  0  0


B:fA(R)( 1, 3):                      B:fA(R)( 3, 3):
-1  2  1                             2  2  0
 3  3  0                            -2  1  1


B:fA(R)( 1, 4):                      B:fA(R)( 3, 4):
-5  3  0                            -3  2  0
 7  2  1                             2  1  1
-2  1  2


B:fA(R)( 2, 1):                      B:fA(R)( 4, 1):
0  0  0                              0  0  0


B:fA(R)( 2, 2):                      B:fA(R)( 4, 2):
0  0  0                              0  0  0


B·fA(R)( 2, 3):                      B:fA(R)( 4, 3):
 5  1  2                            -1  1  1
-5  3  0                            -3  2  0
-2  0  3                             2  0  2
 2  2  1


B:fA(R)( 2, 4):                      B:fA(R)( 4, 4):
 2  1  2                             5  2  0
 8  3  0                            -4  1  1
-8  2  1
```

Figure 7.   Session viewing the matrix fA(R) preceeding
            (4.23)

A.11

```
10          'CPT Bruce W. Colletti, AFIT/EN; 1 August 1985
20          'The input file for this program (which multiplies 2
30          'matrices having multivariate polynomial entries)
            'has the following
40          'structure:
50          '    line 1:    row, column dimension of matrix A;
            '               number of variates
60          '    line 2:    number of terms in the A(1,1)
            '               polynomial
70          '    line 3:    tuple corresponding to the 1st term of
            '               A(1,1)
80          '    line 4:    tuple corresponding to the 2nd term of
            '               A(1,1)
90          '    line n:    tuple corresponding to the last term
            '               of A(1,1)
100         ' line n+1:   number of terms in the A(1,2)
            '               polynomial
110         ' line n+2:   tuple corresponding to the 1st term of
            '               A(1,2)
120         ' line n+3:   tuple corresponding to the 2nd term of
            '               A(1,2)
130         '    line m:    tuple corresponding to the last term
            '               of A(1,2)
140         ' line m+1:   number of terms in the A(1,3)
            '               polynomial
150         '     etc:    matrix entries are entered row at a
            '               time
160         CLS : CLEAR
170         ROWS = 5 : COLS = 5 : TERM = 25 : VAR = 5
180         DIM A(ROWS,COLS,TERM,VAR), B(ROWS,COLS,TERM,VAR),
            B$(20)
190         DIM F(90),POT(90),R(90),BACK(90),M(90),E(VAR)
200         DIM COEFL(ROWS,COLS),COEFR(ROWS,COLS)
210         'A contains the elements of the left matrix and B
            'the elements of
220         'the right (e.g., A + B, A*B):
230         'A(I,J,K,L) the exponent of the L'th variate of the
            'K'th term of the
240         '               I-J'th polynomial entry of A
250         'B(I,J,K,L) the exponent of the L'th variate of the
            'K'th term of the
260         '               I-J'th polynomial entry of B
270         '    BACK(i) the node-back pointer for node i
280         'COEFL(i,j) the number of terms in the polynomial
            'located at
290         '               A(i,j)
300         'COEFR(i,j) the number of terms in the polynomial
            'located at
```

```
310      '              B(i,j)
320      '   E(i)       contains the exponent of the i'th
         '              variable of the
330      '              product between 2 polynomial terms
340      '   F(i)       forward pointer for node i
350      '   M(i)       the "summed coefficient".  This is a
         '              node in the final
360      '              tree with no forward pointer.  It is
         '              the coefficient of
370      '              that term of the polynomial having the
         '              path from given
380      '              terminal node to source node
390      '   POT(i)     potential of node i; that is, the
         '              exponent of the
400      '              variable represented by the node
410      '   R(i)       the right pointer for node i
420      PRINT "**************************************"
430      PRINT "MENU:"
440      PRINT "        B  Build a new matrix"
450      PRINT "        V  View a matrix"
460      PRINT "        '  Transpose a matrix"
470      PRINT "        +  Add 2 matrices"
480      PRINT "        *  Multiply 2 matrices"
490      PRINT "        U  Extract U,M,V,N matrices
500      PRINT "**************************************"
510      INPUT OPT$
520      IF OPT$ = "B" THEN 2410
530      IF OPT$ = "V" THEN 2790
540      IF OPT$ = "U" THEN 3470
550      IF OPT$ = "'" THEN 3130
560      IF ((OPT$ <> "*") AND (OPT$ <> "+")) THEN 420
570      INPUT "Left matrix"; FILEL$
580      INPUT "Right matrix"; FILER$
590      INPUT "Output matrix"; FILEO$
600      IF OPT$ = "+" THEN 620
610      INPUT "α ( R   in αLEFT*RIGHT"; ALPHA : GOTO 630
620      INPUT "α,β ( R  in αLEFT + βRIGHT"; ALPHA,BETA
630      FILE$ = FILEL$ : GOSUB 950
640      ROW1 = ROW : COL1 = COL : V1 = V
650      FILE$ = FILER$ : GOSUB 950
660      ROW2 = ROW : COL2 = COL : V2 = V
670      IF V1 <> V2 THEN 1730
680      IF ((OPT$ = "+") AND
         ((ROW1 <> ROW) OR (COL1 <> COL))) THEN 1560
690      IF ((OPT$ = "*") AND (COL1 <> ROW)) THEN 1560
700      OPEN FILEO$ FOR OUTPUT AS #2
710      IF OPT$ = "+" THEN GOSUB 2140
720      PRINT #2, ROW1; COL2; V
730      FOR I1 = 1 TO ROW1
740        FOR I2 = 1 TO COL2
750        PRINT "("; I1; ","; I2; ")"
760          GOSUB 1800
770          FOR I3 = 1 TO COL1
```

```
780              T1 = COEFL(I1,I3) : T2 = COEFR(I3,I2)
790             FOR I4 = 1 TO T1
800               FOR I5 = 1 TO T2
810                 PROD = ALPHA * A(I1,I3,I4,0) *
                                    B(I3,I2,I5,0)
820                   FOR I6 = 1 TO V
830                     E(I6) = A(I1,I3,I4,I6) + B(I3,I2,I5,I6)
840                     NEXT I6
850                     GOSUB 1150
860                   NEXT I5
870                 NEXT I4
880               NEXT I3
890             GOSUB 1870
900             NEXT I2
910           NEXT I1
920      ' CLOSE #2
930       STOP
940       'Subroutine reads a matrix for +/* routines
950       OPEN FILE$ FOR INPUT AS #1
960       INPUT #1, ROW,COL,V
970       IF ((ROW > ROWS) OR (COL > COLS)) THEN 1620
980       FOR I = 1 TO ROW
990         FOR J = 1 TO COL
1000            INPUT #1, TERMS
1010            IF TERMS > TERM THEN 1680
1020            IF FILE$ = FILEL$ THEN COEFL(I,J) = TERMS
1030            IF FILE$ = FILER$ THEN COEFR(I,J) = TERMS
1040            FOR K = 1 TO TERMS
1050              FOR L = 0 TO V
1060                IF FILE$ = FILER$ THEN 1080
1070                INPUT #1, A(I,J,K,L) : GOTO 1090
1080                INPUT #1, B(I,J,K,L)
1090                NEXT L
1100              NEXT K
1110            NEXT J
1120        NEXT I
1130      CLOSE #1
1140      RETURN
1150      'Pointer ordering subroutine
1160      I = 1 : CNODE = F(0) : BK = 0
1170      IF CNODE = 0 THEN 1370
1180      BASE = BK
1190      IF POT(CNODE) = E(I) THEN 1430
1200      BK = CNODE
1210      CNODE = R(CNODE)
1220      IF CNODE <> 0 THEN 1190
1230      NODES = NODES + 1
1240      POT(NODES) = E(I)
1250      R(BK) = NODES
1260      BACK(NODES) = BASE
1270      BK = NODES
1280      FOR J = I+1 TO V
1290          NODES = NODES + 1
```

```
1300          POT(NODES) = E(J)
1310          BACK(NODES) = BK
1320          F(BK) = NODES
1330          BK = NODES
1340          NEXT J
1350     M(BK) = M(BK) + PROD
1360     GOTO 1490
1370     NODES = NODES + 1
1380     F(NODES) = 0
1390     F(BK) = NODES
1400     BACK(NODES) = BK
1410     CNODE = NODES
1420     POT(CNODE) = E(I)
1430     BASE = CNODE
1440     I = I + 1    'I is the same as the Depth of a node
1450     IF (I > V) THEN M(CNODE) = M(CNODE) + PROD :
         GOTO 1490
1460     BK = CNODE
1470     CNODE = F(CNODE)
1480     GOTO 1170
1490     RETURN
1500       'Error routine for too many polynomial variates
1510       PRINT "******************************************"
1520       PRINT CHR$(7); "DIM statement allows only "; VAR;
           "variables.";
1530       PRINT "  Modify DIM statement";
1540       STOP
1550       'Error routine for ill-dimensioned matrices
1560       PRINT "******************************************"
1570       PRINT CHR$(7); "Dimensions incompatable"
1580       PRINT "    "; FILEL$ ; " is "; STR$(ROW1); " X ";
           STR$(COL1)
1590       PRINT "    "; FILER$ ; " is "; STR$(ROW); " X ";
           STR$(COL)
1600       STOP
1610       'Error routine for matrices whose dimensions are
           'too large
1620       PRINT "******************************************"
1630       PRINT CHR$(7); "Modify the dimension statement in
           the program"
1640       PRINT ; "The dimensions of "; FILE$; "is ";
           STR$(ROW);
1650       PRINT " X "; STR$(COL)
1660       STOP
1670       'Error routine for a polynomial having too many
           terms
1680       PRINT "******************************************"
1690       PRINT CHR$(7); "There are "; TERMS; "terms in
           the ";
1700       PRINT "polynomial, whereas the DIM statement
           allocates "
1710       PRINT TERM; " terms in the polynomial.  Modify
           DIM stmt."
```

```
1720      STOP
1730      PRINT "***********************************************"
1740      PRINT CHR$(7); "Polynomial variates in matrices
          aren't the same"
1750      STOP
1760      'Error for finding U,M,V,N matrices
1770      PRINT CHR$(7); "Source matrix "; FILE1$; " doesn't
          have even";
1780      PRINT " row and column dimensions"
1790      STOP
1800      'Subroutine to clear tree building variables
1810      FOR I = 0 TO NODES
1820          F(I) = 0 : POT(I) = 0 : R(I) = 0
1830          BACK(I) = 0 : M(I) = 0
1840          NEXT I
1850      NODES = 0
1860      RETURN
1870      'Subroutine prints polynomial term
1880      S = 0
1890      FOR I = 1 TO NODES   'find # of terms in polynomial
1900          IF M(I) <> 0 THEN S = S + 1
1910          NEXT I
1920      IF S = 0 THEN 1940
1930      PRINT #2,S : GOTO 2000
1940      PRINT #2,"1";
1950      FOR I = 0 TO V   'handles a zero entry in result
1960          PRINT #2,0;
1970          NEXT I
1980      PRINT #2,""
1990      GOTO 2130
2000      FOR I = 1 TO NODES
2010          IF M(I) = 0 THEN 2120
2020          PRINT #2, M(I);                    'coefficient of term
2030          PTR = I
2040          FOR J = V TO 1 STEP -1 'because we are going
                                       from the
2050              ZZ(J) = POT(PTR)    'top of the tree down
                                       but are
2060              PTR = BACK(PTR)     'writing the exponents
                                       in the
2070              NEXT J              'reverse order
2080          FOR J = 1 TO V
2090              PRINT #2, ZZ(J);
2100              NEXT J
2110          PRINT #2,""
2120          NEXT I
2130      RETURN
2140      'Addition subroutine
2150      PRINT #2,ROW1; COL1; V1
2160      FOR I1 = 1 TO ROW1
2170        FOR I2 = 1 TO COL2
2180          PRINT "("; I1; ","; I2; ")"
2190          GOSUB 1800                     'clear tree
```

```
2200        'Send A(I1,I2) up through tree
2210             FOR I4 = 1 TO COEFL(I1,I2)
2220                PROD = ALPHA * A(I1,I2,I4,0)
2230                FOR I6 = 1 TO V
2240                    E(I6) = A(I1,I2,I4,I6)
2250                    NEXT I6
2260                    GOSUB 1150
2270             NEXT I4
2280        'Send B(I1,I2) up through tree
2290             FOR I4 = 1 TO COEFR(I1,I2)
2300                PROD = BETA * B(I1,I2,I4,0)
2310                FOR I6 = 1 TO V
2320                    E(I6) = B(I1,I2,I4,I6)
2330                    NEXT I6
2340                    GOSUB 1150
2350             NEXT I4
2360          GOSUB 1870
2370          NEXT I2
2380        NEXT I1
2390      CLOSE #2
2400      STOP
2410     'Matrix input subroutine
2420     INPUT "Matrix name"; FILE$
2430     INPUT "ROW,COL dimension"; ROW,COL
2440     INPUT "Number of polynomial variates"; V
2450     OPEN FILE$ FOR OUTPUT AS #1
2460     PRINT "Build a square diagonal matrix whose
         diagonal entries ";
2470     PRINT "are equal Y/N"; : INPUT OPT1$
2480     IF ((OPT1$ <> "Y") AND (OPT1$ <> "N")) THEN 2460
2490     PRINT CHR$(7); "*** COEF EXP1 EXP2 ... EXPn ***"
2500     PRINT " a semicolon ; represents end of polynomial"
2510     PRINT #1, ROW; COL; V
2520     FOR I = 1 TO ROW
2530         FOR J = 1 TO COL
2540             S = 0 : A$ = ""
2550             IF ((OPT1$ = "N") OR ((I=1) AND
                 (J=1))) THEN 2610
2560             IF I = J THEN S = SS : PRINT #1,S :
                 GOTO 2680
2570             PRINT #1,1;
2580             FOR K = 0 TO V : PRINT #1,0; : NEXT K
2590             PRINT #1,""
2600             GOTO 2750
2610             PRINT "("; STR$(I); ","; STR$(J); "):      "
2620             WHILE (RIGHT$(A$,1) <> ";")
2630                 S = S + 1
2640                 INPUT A$
2650                 B$(S) = A$ : BB$(S) = A$
2660                 WEND
2670             PRINT #1,S  :  SS = S
2680             FOR K = 1 TO S-1
2690                 IF OPT1$="Y" THEN B$(K)=BB$(K)
```

```
2700                    PRINT #1,B$(K)
2710                    NEXT K
2720                IF OPT1$ = "Y" THEN B$(S) = BB$(S)
2730                PRINT #1,LEFT$(B$(S),LEN(B$(S))-1)
2740                PRINT
2750                NEXT J
2760            NEXT I
2770        CLOSE #1
2780        GOTO 420
2790        'subroutine views a matrix file
2800        INPUT "Screen output Y/N"; OPT1$
2810        IF ((OPT1$ <> "Y") AND (OPT1$ <> "N")) THEN 2800
2820        INPUT "Matrix to view"; FILE$
2830        OPEN FILE$ FOR INPUT AS #1
2840        INPUT #1, ROW,COL,V
2850        IF OPT1$ = "Y" THEN 2890
2860        LPRINT "MATRIX "; FILE$; " IS "; STR$(ROW);
            " X"; STR$(COL);
2870        LPRINT " and has "; STR$(V); " polynomial variates"
2880        GOTO 2920
2890        PRINT "MATRIX "; FILE$; " IS "; STR$(ROW); " X";
            STR$(COL);
2900        PRINT " and has "; STR$(V); " polynomial variates"
2910        PRINT
2920        FOR I = 1 TO ROW
2930            FOR J = 1 TO COL
2940            IF OPT1$ = "Y" THEN 2970
2950            LPRINT FILE$; "("; STR$(I); ","; STR$(J); "): "
2960            GOTO 2980
2970            PRINT FILE$; "("; STR$(I); ","; STR$(J); "):   "
2980            INPUT #1,S
2990                FOR K = 1 TO S
3000                    LINE INPUT #1, A$
3010                    IF OPT1$ = "Y" THEN 3030
3020                    LPRINT A$ : GOTO 3040
3030                    PRINT A$
3040                    NEXT K
3050                IF OPT1$ = "Y" THEN 3080
3060            LPRINT
3070            GOTO 3090
3080            PRINT
3090            NEXT J
3100        NEXT I
3110        CLOSE #1
3120        GOTO 420
3130        'Subroutine transposes a matrix
3140        INPUT "Matrix to transpose"; FILE$
3150        INPUT "Matrix to store transpose in"; FILEO$
3160        OPEN FILE$ FOR INPUT AS #1
3170        INPUT #1, ROW,COL,V
3180        IF ((ROW > ROWS) OR (COL > COLS)) THEN 1620
3190        FOR I = 1 TO ROW
3200            FOR J = 1 TO COL
```

```
3210              INPUT #1, TERMS
3220              IF TERMS > TERM THEN 1680
3230              COEF(I,J) = TERMS
3240              FOR K = 1 TO TERMS
3250                  FOR L = 0 TO V
3260                      INPUT #1, A(I,J,K,L)
3270                      NEXT L
3280                  NEXT K
3290              NEXT J
3300          NEXT I
3310      CLOSE #1
3320      OPEN FILEO$ FOR OUTPUT AS #1
3330     PRINT #1,COL;ROW;V
3340     FOR I = 1 TO COL
3350         FOR J = 1 TO ROW
3360             PRINT #1, COEF(J,I)
3370             FOR K = 1 TO COEF(J,I)
3380                 FOR L = 0 TO V
3390                     PRINT #1, A(J,I,K,L);
3400                     NEXT L
3410                     PRINT #1,""
3420                 NEXT K
3430             NEXT J
3440         NEXT I
3450      CLOSE #1
3460     GOTO 420
3470     'Extracts U,M,V,N matrices
3480     INPUT "Source matrix of U,M,V,N"; FILE1$
3490     INPUT "Disk to output to A/B"; D$
3500     IF ((D$ <> "A") AND (D$ <> "B")) THEN 3490
3510     D$ = D$ + ":"
3520     PRINT CHR$(7); "***  WARNING:   files will be
         output to "; D$;
3530     PRINT "U,M,V,N ***    ";
3540     INPUT "OK Y/N"; OPT$
3550     IF OPT$ <> "Y" THEN STOP
3560     FILE2$ = D$ + "U"
3570     FILE3$ = D$ + "M"
3580     FILE4$ = D$ + "V"
3590     FILE5$ = D$ + "N"
3600     OPEN FILE1$ FOR INPUT AS #1
3610     OPEN FILE2$ FOR OUTPUT AS #2
3620     OPEN FILE3$ FOR OUTPUT AS #3
3630     OPEN FILE4$ FOR OUTPUT AS #4
3640     OPEN FILE5$ FOR OUTPUT AS #5
3650     INPUT #1,ROW,COL,V
3660     ROW1 = ROW/2 : COL1 = COL/2
3670     IF ((ROW1 <> INT(ROW1)) OR
         (COL1 <> INT(COL1))) THEN 1760
3680     FOR I = 2 TO 5 : PRINT #I,ROW1,COL1,V : NEXT I
3690     FOR I = 1 TO ROW
3700         FOR J = 1 TO COL
3710             INPUT #1,S
```

```
3720            FIL = 5
3730            IF (I <= ROW1) THEN FIL = 2
3740            IF ((FIL = 2) AND (J > COL1)) THEN FIL = 3
3750            IF ((FIL = 5) AND (J <= COL1)) THEN FIL = 4
3760            PRINT #FIL,S
3770            FOR K = 1 TO S
3780                LINE INPUT #1,A$
3790                PRINT #FIL,A$
3800                NEXT K
3810            NEXT J
3820        NEXT I
```

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This paper seeks the solutions to a system of equations (equalities) in n variables by expressing the system in matrix algebraic form. Properties of the solutions to the ensuing matrix equation are investigated using similarity transformations. The three types of matrix equations to be studied are the linear equation $AX = b$, the Lypanov equation $AX - XB = C$, the second-order Riccati equation $XDX + AX + XB + C = 0$, and the third-order Riccati equation $XAXBX + XCX + DX + XE + F = 0$. The entries of all matrices, including the solution X, are restricted to being polynomials in $\Gamma$ having complex coefficients, where $\Gamma$ is the n-tuple of indeterminates. That is, all matrices are elements of the ring

$$C^{m \times n}(\Gamma)$$

for m and n of appropriate size.

Because adding and multiplying matrices (having multivariate polynomial entries) is tedious in practice, an interactive BASIC program is presented in the appendix. This program, which can be used on a personal computer, permits the user to perform operations on matrices having multivariate polynomial entries. Via menu selections, the user may perform

-weighted addition between two matrices
-multiplication between two matrices
-create matrices, with an option of building a diagonal matrix whose diagonal entries are all equal
-view matrices
-transpose a matrix
-extract special submatrices (U,M,V,N of Chapter IV) from a given matrix

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