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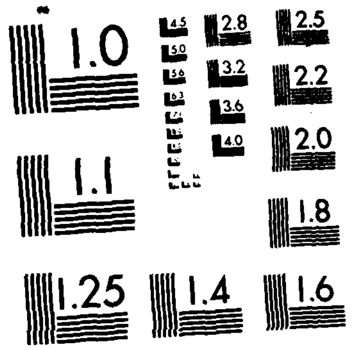

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TECHNICAL REPORT BRL-TR-2702

ON THE EXISTENCE OF LI-YORKE POINTS IN THE THEORY OF CHAOS

Nam P. Bhatia
Walter O. Egerland

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The report establishes an addendum to the Li-Yorke theorem, lists the complete set of Li-Yorke inequalities, and provides in the Equivalence theorem non-periodic conditions equivalent to the known periodic conditions that guarantee chaos.

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I. INTRODUCTION

Li and Yorke[1] introduced in their fundamental paper the term "chaotic" for a class of self-mappings of an interval. The real function f is chaotic if (a) there are points of arbitrarily large periods and (b) there is an uncountable set S such that for every $x_0, y_0 \in S$, $x_0 \neq y_0$, $\lim_{n\to\infty} \sup | f^n(x_0) - f^n(y_0) | > 0$ and $\lim_{n\to\infty} \inf | f^n(x_0) - f^n(y_0) | = 0$. Following the Li-Yorke result that "period three implies chaos," many authors worked on periodic conditions that allow the same conclusion. The best known of these conditions is "period $p \neq 2^n$ implies chaos." Such investigations are summarized in Targonski's monograph.[2]

Li and Yorke also introduced four-point inequalities satisfied by a point and its three successors with respect to the given function f. They showed that these imply the existence of a three-period and hence chaos. Our investigations in this paper show that the Li-Yorke inequalities play a fundamental role in the theory of chaos. We, therefore, formalize the notion of a Li-Yorke point and obtain non-periodic conditions equivalent to the known periodic conditions that guarantee chaos. The importance of the results resides in the fact that the set of Li-Yorke points is always an open set, whereas the number of periodic points of a given period $p \neq 2^n$ is in general countable.

II. DEFINITIONS AND NOTATIONS

Let $f: R \to R$ be continuous. If $x_0 \in R$, the orbit of x_0 under f is defined as the set $\{x : x = f^n(x_0), n = 0, 1, \ldots\}$, where, for every positive integer n, f^n is the n-th iterate of f and $f^0(x_0) = x_0$. We shall write $x_n := f^n(x_0)$ for a given $x_0 \in R$ and call $x_1, x_2 \ldots$ the successors of x_0. A pre-orbit of a given $x_0 \in R$ is any (finite or infinite) sequence $x_0, x_{-1}, x_{-2} \ldots$ such that $f(x_{-n}) = x_{-(n-1)}$ for all n for which x_{-n} is defined. The points $x_{-1}, x_{-2} \ldots$ in any such sequence are called predecessors of x_0. A point x_0 is called critical if $f(x_0) = x_0$, i.e., a critical point of f is a fixed point of f. A periodic point x_0 of period $p > 1$ (p a positive integer) is a point for which the relations $f^p(x_0) = x_0$, $f^k(x_0) \neq x_0$, $1 \leq k < p$, hold.

The following fundamental results are now well-known.

<u>Theorem</u> (Sarkovskii).[3] For $m, n = 0, 1, \ldots$ consider the total ordering of the positive integers:

$$3 < 5 < 7 < \ldots < 2 \cdot 3 < 2 \cdot 5 < 2 \cdot 7 \ldots < 2^n \cdot 3 < 2^n \cdot 5 < 2^n \cdot 7$$
$$< \ldots < 2^m < 2^{m-1} < \ldots < 2^2 < 2 < 1.$$

If a continuous mapping $f : R \to R$ has a periodic point of period p, then it also has a periodic point of period q for every $q > p$ (in the above total order).

<u>Theorem</u> (Li-Yorke): Let $f : R \to R$ be continuous. If there is a point $x_0 \in R$ such that either $x_3 < x_0 < x_1 < x_2$ or $x_3 > x_0 > x_1 > x_2$, then f has a point of period three.

[1] T.-Y. Li and J. A. Yorke, "Period three implies Chaos.," <u>Amer. Math. Monthly</u> 82 (1975), pp. 985-992.

[2] Gyorgy Targonski, "Topics in Iteration Theory," <u>Studia Mathematica, Skript 6</u>, Vandenhoek and Ruprecht, Gottingen and Zurich (1981).

[3] A. N. Sarkovskii, "Coexistence of Cycles of a Continuous Map of a Line into Itself," <u>Ukrain. Mat. Zh. 16</u> (1964), pp. 61-71.

Theorem (Li-Yorke): Let $f : R \to R$ be continuous. If there is a point $x_0 \in R$ such that either $x_3 < x_0 < x_1 < x_2$ or $x_3 > x_0 > x_1 > x_2$, then f has a point of period three. Furthermore, if f has a three periodic point, there exists an uncountable set $S \subset R$ such that for every $x_0, y_0 \in S$, $x_0 \neq y_0$,

$$\lim_{n\to\infty} \sup |x_n - y_n| > 0$$

and

$$\lim_{n\to\infty} \inf |x_n - y_n| = 0.$$

Definition. A point $x_0 \in R$ is called a Li-Yorke point of $f : R \to R$ if x_0 satisfies a Li-Yorke inequality

$$x_3 < x_0 < x_1 < x_2$$

or

$$x_3 > x_0 > x_1 > x_2.$$

III. AN EXAMPLE

The following example shows that the existence of a three periodic point does not guarantee the existence of a Li-Yorke point. The quadratic mapping $g(y) = ay^2 + 2by + c$, $a \neq 0$, b, c, real constants, may be brought in the form $f(x) = x^2 - r$ by setting $y = a^{-1}(x - b)$, $g = a^{-1}(f - b)$, and $r = b^2 - b - ac$. f has a three periodic orbit for $r = 7/4$, but no point $x_0 \in R$ is a Li-Yorke point for $r \leq 7/4$.

IV. THE RELATIONSHIP BETWEEN LI-YORKE POINTS AND POINTS OF PERIOD THREE

We apply frequently the following crossing property of real continuous functions.

Lemma. Let $f(x)$ be continuous on $[a,b]$. If either

$$f(a) \geq a, \quad f(b) \leq b$$

or

$$f(a) \leq a, \quad f(b) \geq b,$$

then there exists $x \in [a, b]$ such that $x = f(x)$. In particular, the existence of such an $x \in [a, b]$ follows from the inclusions $[a,b] \subset [f(a), f(b)]$ or $[a,b] \supset [f(a), f(b)]$.

We shall denote the k-th iterate of x_0 under the function f^m by x_k^m, $k = 0, 1 \ldots$ Thus $x_k^m = (f^m)^k(x_0) = x_{mk}$, and, in particular, $x_0^m = x_k^0 = x_0$ for all nonnegative integers k and m.

Theorem 1. If f has a three periodic point, then f^2 has a Li-Yorke point.

Proof. Let y_0 be a three periodic point of f. Thus $y_0 = y_3$ and $y_i \neq y_k$, $0 \leq i < k \leq 3$. We can assume without loss of generality that $y_0 < y_1 < y_2$. Since $f([y_1, y_2]) \supset [y_0, y_2] \supset [y_1, y_2]$, there is a critical point $c_0 \in (y_1, y_2)$. Hence

$$y_0 < y_1 < c_0 < y_2.$$

Since $f(y_0) = y_1 < c_0$ and $f(y_1) = y_2 > c_0$, we have a $c_{-1} \in (y_0, y_1)$, and since $f(y_2) = y_0 < c_{-1}$ and $f(c_0) = c_0 > c_{-1}$, there is a $c_{-2} \in (c_0, y_2)$, thus yielding the inequality

$$y_0 < c_{-1} < y_1 < c_0 < c_{-2} < y_2 .$$

Similar reasoning establishes points c_{-j}, $j = 3, 4, 5, 6$, and the inequality

$$y_0 < c_{-1} < y_1 < c_{-3} < c_{-5} < c_0 < c_{-6} < c_{-4} < c_{-2} < y_2 .$$

If we identify c_{-6} with x_0, we obtain $c_{-6} = x_0$, $c_{-4} = x_1^2$, $c_{-2} = x_2^2$, $c_0 = x_3^2$ and the inequality $x_3^2 < x_0 < x_1^2 < x_2^2$. This completes the proof.

Theorem 2. If f has a Li-Yorke point x_0, f has at least two distinct three periodic orbits.

Proof. Let $S = \{x_0 : x_3 < x_0 < x_1 < x_2 \text{ or } x_2 < x_1 < x_0 < x_3\}$. Then S is non-empty by hypothesis and open because f is continuous. Each component of S is an open interval. If $I = (a_0, b_0)$ is a component of S, we will show that $-\infty < a_0 < b_0 < \infty$ and that both a_0 and b_0 are three periodic points in different orbits. Fix $x_0 \in I$ and assume that $x_3 < x_0 < x_1 < x_2$. Then the same inequality holds for all $x_0 \in I$. Assuming first that $-\infty < a_0 < b_0 < \infty$, we note that a_0 and b_0 are limit points of points x_0 for which $x_3 < x_0 < x_1 < x_2$. Hence by continuity of f we must have the inequalities $a_3 \leq a_0 \leq a_1 \leq a_2$ and $b_3 \leq b_0 \leq b_1 \leq b_2$ with at least one equality sign holding in each of them. We claim that $a_3 = a_0 < a_1 < a_2$ and $b_3 = b_0 < b_1 < b_2$. Noting that $f(x_1) = x_2 > x_1$ and $f(x_2) = x_3 < x_2$, it follows that there is a critical point $c_0 \in (x_1, x_2)$, so that $x_3 < x_0 < x_1 < c_0 < x_2$. Since $f(x_0) = x_1 < c_0$ and $f(x_1) = x_2 > c_0$ imply the existence of $c_{-1} \in (x_0, x_1)$ with $f(c_{-1}) = c_0$, we have $x_3 < x_0 < c_{-1} < x_1 < c_0 < x_2$. As $a_0 < x_0$ and $f(a_0) = a_1 = a_0 < c_{-1}$, $f(x_0) = x_1 > c_{-1}$, we have $c_{-2} \in (a_0, x_0) \subset I$ with $f(c_{-2}) = c_{-1}$. But $c_{-2} < c_{-1} < c_0 = c_1$ yields by relabelling a $y_0 = c_{-2} \in I$ such that $y_0 < y_1 < y_2 = y_3$, a contradiction. Hence $a_0 < a_1$. If now $a_1 = a_2$, we have $a_0 < a_1 = a_2 = a_3$ which contradicts $a_3 \leq a_0 \leq a_1 \leq a_2$. Therefore, we must have $a_1 < a_2$, and hence $a_3 = a_0 < a_1 < a_2$, i.e., a_0 is a three periodic point. Similarly, b_0 is a three periodic point for which $b_3 = b_0 < b_1 < b_2$ holds. Since $a_0 < b_0$, a_0 is not in the orbit of b_0, and so the orbits of a_0 and b_0 are distinct. It remains to show that $a_0 \neq -\infty$ and $b_0 \neq \infty$. Since $x_0 < b_0$ and $(x_0, b_0) \subset I$ and $c_{-1} \notin S \supset I$, we must have $x_0 < b_0 < c_{-1}$ as $x_0 < c_{-1}$. This shows that $b_0 < \infty$. To see that $a_0 > -\infty$, we note that $x_3 \in f^2[c_{-1}, c_0]$, i.e., x_3 lies in a compact set. If $A = \inf f^2[c_{-1}, c_0]$, then for $x_0 < A$ we conclude $x_0 < x_3$ which contradicts $x_0 \in I$ with $a_0 = -\infty$. This completes the proof.

The theorem extends the Li-Yorke theorem and may be illustrated by the example $f(x) = x^2 - 2$. The point $x_0 = \sqrt{2}$ is a Li-Yorke point of f and $(2\cos\frac{2}{7}\pi, 2\cos\frac{4}{7}\pi, 2\cos\frac{8}{7}\pi)$, $(2\cos\frac{2}{9}\pi, 2\cos\frac{4}{9}\pi, 2\cos\frac{8}{9}\pi)$ are two distinct three periodic orbits.

The following theorem extends the Li-Yorke theorem in another direction. It establishes equivalent companion inequalities to the Li-Yorke inequalities.

Theorem 3. The sets of points $A = \{x_0 : x_3 < x_0 < x_1 < x_2\}$, $B = \{x_0 : x_1 < x_2 < x_0 < x_3\}$, and $C = \{x_0 : x_2 < x_3 < x_0 < x_1\}$ are either all empty or all non-empty. The same statement holds for the sets in which all inequalities are reversed.

Proof. Let $A \neq \phi$. Then for some $x_0 \in R$ we have $x_3 < x_0 < x_1 < x_2$. Since $f(x_1) = x_2 > x_0$ and $f(x_2) = x_3 < x_0$, there is $x_{-1} \in (x_1, x_2)$ such that $x_0 < x_1 < x_{-1} < x_2$. Upon relabelling $y_0 = x_{-1}$, we have $y_1 < y_2 < y_0 < y_3$, so that $B \neq \phi$, i.e., $A \neq \phi$ implies $B \neq \phi$. Let $B \neq \phi$. Then, if $x_1 < x_2 < x_0 < x_3$, it follows from $f(x_0) = x_1 < x_0$ and $f(x_2) = x_3 > x_2$ that there is a critical point $c_0 \in (x_2, x_0)$ such that $x_1 < x_2 < c_0 < x_0 < x_3$. Since $f(c_0) = c_0 < x_0$ and $f(x_2) = x_3 > x_0$, there is $x_{-1} \in (x_2, x_0)$, so that $x_1 < x_2 < x_{-1} < x_0$. Letting $y_0 = x_{-1}$, we have $y_2 < y_3 < y_0 < y_1$. Thus $B \neq \phi$ implies $C \neq \phi$. Finally, assume $C \neq \phi$. If for some $x_0 \in R$ $x_2 < x_3 < x_0 < x_1$, then there exists $x_{-1} \in (x_2, x_0)$ since $f(x_2) = x_3 < x_0$ and $f(x_0) = x_1 > x_0$. From $x_2 < x_{-1} < x_0 < x_1$ follows, setting $x_{-1} = y_0$, $y_3 < y_0 < y_1 < y_2$, and hence that $C \neq \phi$ implies $A \neq \phi$. This completes the proof of the theorem.

Examples show that other four-point inequalities between x_0, x_1, x_2, x_3 do not assure the existence of three-periodic orbits. Therefore, the theorem lists the complete set of Li-Yorke four-point inequalities.

Theorem 4 (Equivalence Theorem). The following statements are equivalent.

(a) f has a point of period $p \neq 2^n$.

(b) There exists $m \geq 1$ such that f^m has a point of period three.

(c) f^{2m} has Li-Yorke points.

(d) f^{2m} has at least two three-periodic orbits.

Proof. If f has a point of period $p \neq 2^n$, then f has a point of period $3 \cdot 2^k$ for some $k \geq 0$ by the theorem of Sarkovskii. Such a point is a three-periodic point of f^m, $m = 2^k$. Hence (a) implies (b). (b) implies (c) by Theorem 1, and (c) implies (d) by Theorem 2. Now let x_0 be a three-periodic point of f^q, $q = 2m$, $x_0 = f^{3q}(x_0)$, $x_0 \neq f^q(x_0)$, $x_0 \neq f^{2q}(x_0)$. x_0 is thus a periodic point of f of minimal period N, $2 \leq N \leq 3q$. This implies that $x_0 = f^{kN}(x_0)$ for every $k \geq 1$ and $x_0 \neq f^s(x_0)$ if $s \neq kN$, $k = 1, 2, \ldots$ Hence $3q = k' N$ for some $k' \geq 1$. This shows that either 3 divides k' or 3 divides N. If 3 divides k', then $q = sN$ and consequently $f^q(x_0) = x_0 = f^{sN}(x_0)$, a contradiction. Hence 3 divides N. But then $N \neq 2^n$ for every $n \geq 0$. Therefore, f has period $N \neq 2^n$ and (d) implies (a). This completes the proof of the Equivalence Theorem.

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1. T.-Y. Li and J. A. Yorke, "Period three implies Chaos.," Amer. Math. Monthly 82 (1975), pp. 985-992.

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