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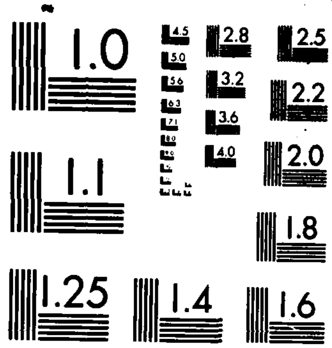

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STOCHASTIC ESTIMATION IN COMBINED ARMS LANCHESTER MODELING OF WARFARE

HERBERT E. COHEN

JULY 1985

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U. S. ARMY MATERIEL SYSTEMS ANALYSIS ACTIVITY
ABERDEEN PROVING GROUND, MARYLAND

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This present paper calls attention to the fact that stochastic estimation theory, as practiced in modern control theory, provides a method for predicting conditions on the battlefield from measurements or observations of the battle area that are themselves uncertain and where the parameters of the mathematical model are not known precisely.

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STOCHASTIC ESTIMATION IN COMBINED ARMS
LANCHESTER MODELING OF WARFARE

Deterministic Lanchester models of warfare with time dependent attrition coefficients has received extensive study [1]. There does not appear to be any analytical effort to incorporate observations of the battlefield into Lanchester models. This is very surprising when one considers the vast amount of effort going on in developing techniques for observing events on the battlefield and integrating intelligence in a "fusion" center. In addition, no analytical work appears to have incorporated uncertainties in the model itself. The objective of this paper is to develop a general methodology, using stochastic estimation theory as practiced in modern control theory, for predicting conditions on the battlefield from measurements or observations of the battle area that are themselves uncertain and where the parameters of the mathematical model are not known precisely. Examples are also provided to show how the general theory can be applied to two specific problems.

Keywords: Kalman Bucy filters.

The fundamental Lanchester equations for homogeneous forces is given by

$$\frac{dx}{dt} = -\alpha y$$
$$\frac{dy}{dt} = -\beta x \tag{1}$$

where x and y are the force strength of each side and α is the attrition of x force due to the weapon effectiveness of the y force and similarly β is the attrition coefficient of the y force due to the weapon effectiveness of the x force. Taking into consideration that the values α and β can change with time and also are not known exactly, we can incorporate the uncertainties in these parameters by adding to our model a gaussian white noise term in each equation so that equation (1) is modified as

$$\frac{dx}{dt} = -\alpha y + w_1(t)$$
$$\frac{dy}{dt} = -\beta x + w_2(t) \tag{2}$$

where w_1 and w_2 are uncorrelated gaussian white noise.

Equation (2) can be written in matrix form

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -\alpha \\ -\beta & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix}$$

or with $\underline{x} = (x, y)^T$ vector

$$\underline{A} = \begin{pmatrix} 0 & -\alpha \\ -\beta & 0 \end{pmatrix} \quad \text{matrix}$$

$$\underline{w} = (w_1, w_2)^T \qquad \text{vector}$$

we have

$$\frac{d\underline{x}}{dt} = \underline{A}\,\underline{x} + \underline{w} \tag{3}$$

where $\underline{w}(t)$ is gaussian white noise such that

$$E[\underline{w}(t)] = 0 \qquad \text{for all } t$$

$$\text{cov}\,[w(t), w(\tau)] = \underline{Q}\delta\ (t-\tau) \tag{3a}$$

i.e., $\underline{w} \sim N\,(\underline{0}, \underline{Q})$

with $\underline{Q} = \underline{Q}^T > 0$

Equation (3) and (3a) represents the vector form of the homogeneous Lanchester model with "plant" noise $\underline{w}$

Observation over the battlefield can be similarly characterized as

$$z_1 = c_1\,x + v_1$$

$$z_2 = c_2\,y + v_2$$

where z_1 is the measurement of x corrupted by measurement noise v_1, and z_2 is the measurement of y corrupted by measurement noise v_2. If we let $\underline{z} = (z_1, z_2)^T$, $\underline{x} = (x, y)^T$ and $\underline{v} = (v_1, v_2)^T$ we have

$$\underline{z} = \underline{C}\,\underline{x} + \underline{v} \tag{4}$$

with $\underline{C} = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$

and $\underline{v}(t) \sim N\,(\underline{0}, \underline{R}\,(t))$ gaussion white noise

such that

$$\left.\begin{aligned} &E\left\{\underline{v}\,(t)\right\} = \underline{0} \quad \text{for all } t \\ &\text{cov}\left\{\underline{v}\,(t), \underline{v}\,(\tau)\right\} = \underline{R}\,\delta\,(t-\tau) \\ &\underline{R}(t) = \underline{R}^T(t) > 0 \end{aligned}\right\} \tag{4a}$$

We also require $\underline{R}$ to be non-singular i.e., $\underline{R}^{-1}$ exists.

Thus the fundamental set of equations which describe the observations over the battlefield for "stochastic" homogeneous Lanchester models of warfare is given by

$$\left.\begin{aligned} &\text{(System model):} \quad \frac{d\underline{x}(t)}{dt} = \underline{A}\ \underline{x}(t) + \underline{w}(t) \\ &\text{(Measurement model):} \quad \underline{z}(t) = \underline{C}\ \underline{x}\ (t) + \underline{v}\ (t) \end{aligned}\right\} \qquad (5)$$

where the initial state $\underline{x}(t_0)$ is modeled as a random vector since the initial conditions are not precisely known. To characterize the initial state $\underline{x}\ (t_0)$ we need

$$E\left\{\underline{x}\ (t_0)\right\} = \underline{\bar{x}}_0$$

$$\text{cov}\ [\ \underline{x}\ (t_0),\ \underline{x}\ (t_0)] = \underline{P}_0 \quad \text{initial state covariance} \qquad (6)$$

$$\underline{P}_0 = \underline{P}_0^T > \underline{0}$$

The "plant" noise $\underline{w}(t)$ is used to model external disturbances and modeling errors in $\underline{A}$. Similarly, the measurement noise $\underline{v}\ (t)$ is used for modeling error in $\underline{C}$.

It is also assumed that $\underline{x}(t_0)$, $\underline{w}(t)$, and $\underline{v}(t)$ are independent for all t_0, t, τ, i.e.,

$$\left.\begin{aligned} &\text{cov}\ [\underline{x}\ (t_0),\ \underline{w}\ (t)] = \underline{0}\ \forall\ t_0,\ t \\ &\text{cov}\ [\underline{x}\ (t_0),\ \underline{v}\ (t)] = \underline{0}\ \forall\ t_0,\ t \\ &\text{cov}\ [\underline{w}\ (t),\ \underline{v}\ (\tau)] = \underline{0}\ \forall\ t_0,\ \tau \end{aligned}\right\} \qquad (7)$$

The flow diagram for equation (5) is shown in Figure 1.

Thus, the stochastic estimation problem is to find an estimate $\hat{\underline{x}}$ for the stochastic homogeneous Lanchester equation (5) subject to equations (3a), (4a), (6), and (7) such that, for a given set of past measurements $\underline{z}(1)$, $\underline{z}(2)$, ... , $\underline{z}(t)$, the estimate $\hat{\underline{x}}(t)$ minimizes

$$J = E\ \left\{\ ||\underline{x}(t) - \hat{\underline{x}}(t)||^2\ |\ \underline{z}(1),\ \underline{z}(2),\ \dots,\ \underline{z}(t)\right\} \qquad (7a)$$

in the least squares sense.

It is now well known that the Kalman-Bucy filter [2, 3, 4] provides such an estimate and that $\hat{\underline{x}}$ satisfies

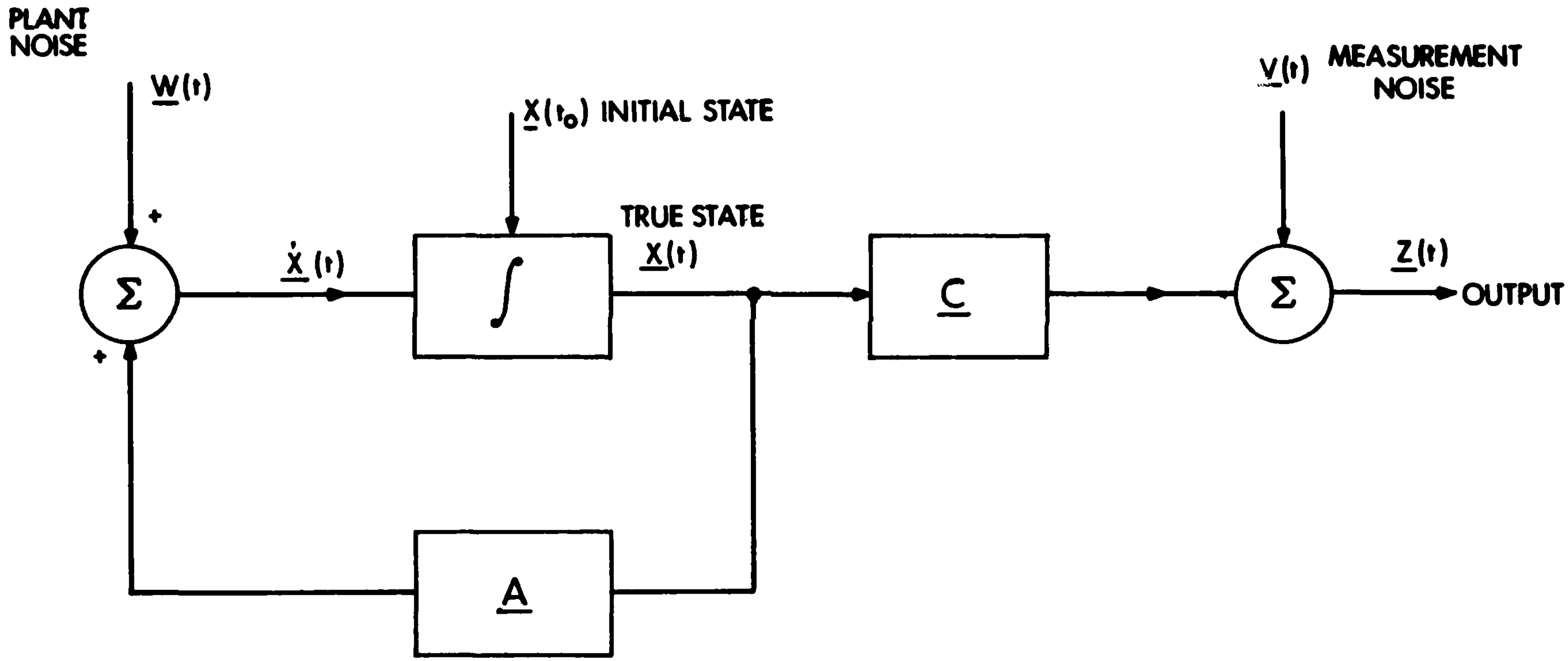
PLANT NOISE
W(t)
+
+
Σ
Ẋ(t)
X(t₀) INITIAL STATE
∫
TRUE STATE
X(t)
A
C
V(t)
MEASUREMENT NOISE
Σ
Z(t)
OUTPUT

Figure 1.

$$\frac{d\underline{\hat{x}}}{dt} = \underline{A}\,\underline{\hat{x}}(t) + \underline{H}\,\left[\underline{z}(t) - \underline{C}\,\underline{\hat{x}}(t)\right] \tag{8}$$

$$\underline{\hat{x}}\,(t_0) = \underline{\bar{x}}_0$$

where the filter gain matrix H is given by

$$H = \underline{P}(t)\,\underline{C}^T\,\underline{R}^{-1}\,(t) \tag{9}$$

with the error covariance matrix $\underline{P}(t)$ satisfies the Riccati equation

$$\frac{d\underline{P}(t)}{dt} = \underline{A}\,\underline{P}(t) + \underline{P}(t)\underline{A}^T + \underline{Q} - \underline{P}(t)\,\underline{C}^T\,\underline{R}^{-1}\,\underline{C}\,\underline{P}(t) \tag{10}$$

with $\underline{P}(t_0) = \underline{P}_0$

The Kalman-Bucy filter can be implemented as shown in Figure 2.

The state estimate $\underline{\hat{x}}$ provides a battlefield commander with a predicted estimate of the future conditions of his force strength and that of his opponents based on available information up to the present time for a homogeneous force.

The heterogeneous or combined arms Lanchester model of warfare can be simply achieved by expanding the definition of x and y to a force structure made up of several distinct weapon systems x_i, y_j for i = 1, 2, ..., n, j = 1, 2, ..., m such that

$$\underline{x} = (x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)^T \qquad n + m \text{ vector}$$

and

$$\underline{w} = (w_1, w_2, \ldots, w_n, w_{n+1} \cdots w_{n+m})^T \qquad n + m \text{ vector}$$

$$\underline{v} = (v_1, v_2, \ldots, v_n, v_{n+1}, \ldots, v_{n+m})^T \qquad n + m \text{ vector}$$

$$\underline{z} = (z_1, z_2, \ldots, z_n, z_{n+1}, \ldots, z_{n+m})^T \qquad n + m \text{ vector}$$

and where the plant matrix F takes on the form

$$\underline{F} = \begin{pmatrix} \underline{0} & \underline{A} \\ \underline{B} & \underline{0} \end{pmatrix} \qquad (n + m) \text{ X } (n + m) \text{ matrix}$$

where $\underline{A}$, $\underline{B}$ are now matrices which represent the attrition coefficients associated with weapon y_j and x_i for j = 1, 2, m and i = 1, 2, ... n.

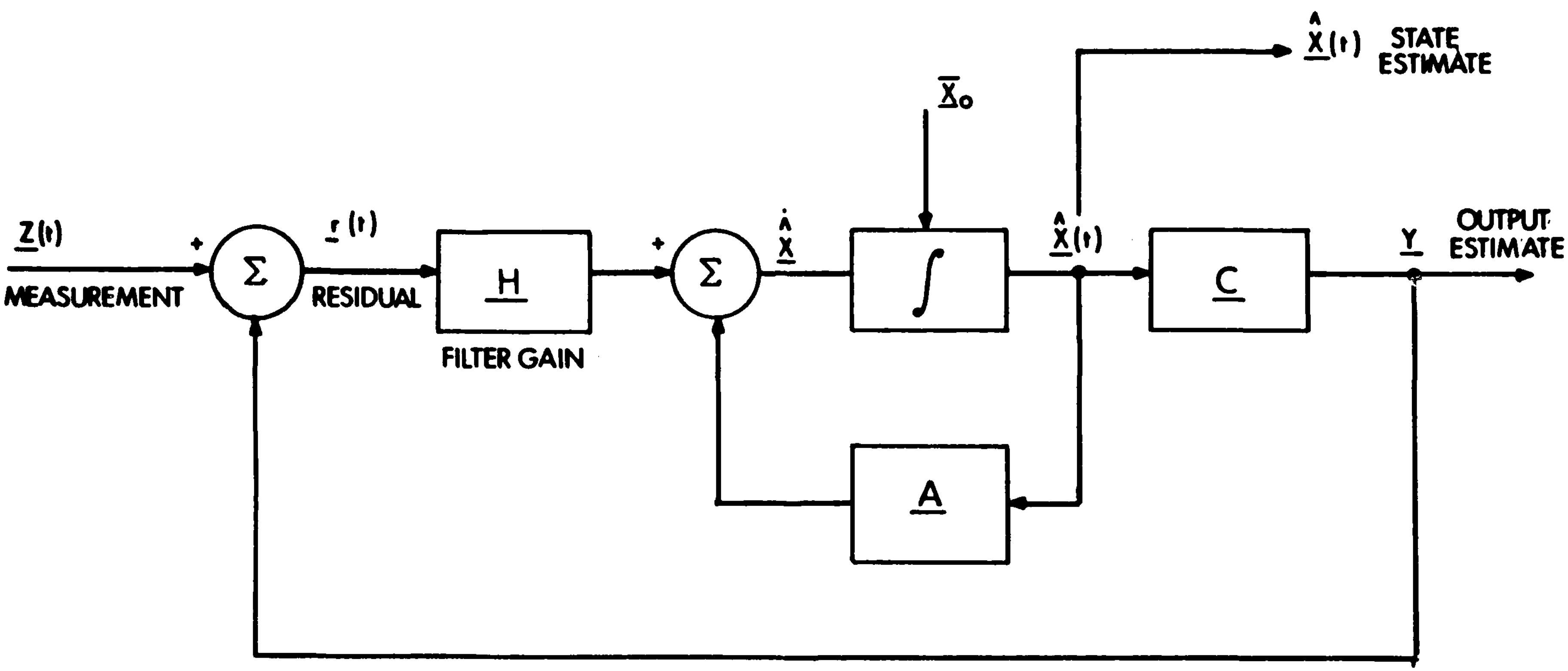

Figure 2. KALMAN - BUCY Filter

Thus for the heterogenous case equation (5) is transformed into

$$\frac{d\underline{x}}{dt} = \underline{F}\ \underline{x} + \underline{w} \qquad (11)$$

$$\underline{z}(t) = \underline{C}\ \underline{x} + \underline{v}$$

Naturally, $\underline{C}$ is similarly defined. The implementation is similar to that shown in Figure 1. All the constraints imposed on $\underline{w}$ and $\underline{v}$ are the same as in the homogeneous case. The optimal estimate, generated by the Kalman-Bucy filter for the heterogeneous systems, is also satisfied for equation (8) through (10). The Kalman-Bucy filter approach can also be applied to the situation in which F(t) and C(t) are time dependent.

Discrete System

We can transform the homogeneous system (equation (5)) to discrete true model if we use

$$\frac{d\underline{x}(t)}{dt} = \lim_{\Delta t \to 0} \frac{\underline{x}(t_K) - \underline{x}(t_{K-1})}{\Delta t}$$

so that

$$\underline{x}(t_K) - \underline{x}(t_{K-1}) = \underline{A}\Delta t\ \underline{x}\ (t_{k-1}) + \Delta t\ \underline{w}\ (t_{k-1})$$

$$\underline{z}(t_k) = \underline{C}\ \underline{x}\ (t_{k-1}) + \underline{v}\ (t_{k-1})$$

which reduces to

$$\left.\begin{aligned} \underline{x}(t_k) &= \underline{\Phi}_{K-1}\ \underline{x}(t_{K-1}) + \Delta t\ \underline{w}\ (t_{K-1}) \\ \underline{z}(t_k) &= \underline{C}\ \underline{x}\ (t_K) + \underline{v}\ (t_k) \end{aligned}\right\} \qquad (12)$$

where

$$\left.\begin{aligned} \Phi_{K-1} &= \underline{I} + \underline{A}\Delta t, \\ \underline{w}_{K-1} &= \Delta t\underline{w}(t_{K-1}) \\ \underline{w}(t_k) &\sim N\ (\underline{0}, \underline{Q}_K) \\ \underline{Q}_K &= \underline{Q}(t)/\Delta t \\ v(t_k) &\sim N\ (\underline{0}, \underline{R}_K) \\ \underline{R}_K &= \underline{R}(t)/\Delta t \end{aligned}\right\} \qquad (13)$$

with initial state $\underline{x}(o)$ gaussian and $\underline{x}(o)$, $\underline{w}(t_{K-1})$, $\underline{v}\ (t_{K-1})$ are independent for all t, τ.

The optimal filtering problem is to determine $\underline{x}(t_{K-1})$ given a sequence of past measurements $\underline{z}(1)$, $\underline{z}(2)$, ..., $z(t_{K-1})$ where the optimal estimate is defined as the conditional mean

(known) $\hat{\underline{x}}(t_{K-1} \mid t_{K-1}) = E\{\underline{x}(t_{K-1}) \mid \underline{z}(1), \underline{z}(2), \ldots, \underline{z}(t_{K-1})\}$ (14)

where $p(\underline{x}(t_{K-1}) \mid \underline{z}(1), z_2(2), \ldots, \underline{z}(t_{K-1}))$, the conditional density function, is gaussian and known. The conditional covariance

(known) $\underline{P}(t_{K-1} \mid t_{K-1}) = \mathrm{cov}\,[\underline{x}(t_{K-1}), \underline{x}(t_{K-1}) \mid \underline{z}(1), \underline{z}(2), \ldots, z(t_{K-1})]$ (15)

We are interested in predicting $\underline{x}(t_K)$ given observations $\underline{z}(1)$, $\underline{z}(2)$, ..., $\underline{z}(t_{K-1})$ before measurement $\underline{z}(t_K)$ is made and then to improve that estimate $\underline{x}(t_K)$ based on expanded information $z(1)$, $z(2)$, ..., $z(t_{K-1})$, $z(t_K)$ after we have made a measurement or observation at t_K i.e., $\underline{z}(t_K)$.

The discrete Kalman filter can be summarized as follows:

State Dynamics Model:

$$\underline{x}(t_K) = \underline{\Phi}_{K-1}\, \underline{x}(t_{K-1}) + \underline{w}(t_{K-1})$$

$$\underline{w}(t_K) \sim N(\underline{0}, \underline{Q}_K)$$

Measurement Model

$$\underline{z}(t_K) = \underline{C}\, \underline{x}(t_K) + \underline{v}(t_K)$$

$$\underline{v}(t_K) \sim N(\underline{0}, \underline{R}_K)$$

Off-Line Calculations

Initialization (t=0):

$$\underline{P}_0 = E\,[(\underline{x}(0) - \hat{\underline{x}}(0))\,(\underline{x}(0) - \hat{\underline{x}}(0))^T]$$

Predict Cycle:

$$\underline{P}(t_K \mid t_{K-1}) = \underline{\Phi}(t_{K-1})\, \underline{P}(t_{K-1})\, \underline{\Phi}^T(t_{K-1}) + \underline{Q}(t_{K-1})$$

Update Cycle:

$$\underline{P}(t_K|t_K) = \underline{P}(t_K|t_{K-1}) - \underline{P}(t_K|t_{K-1})\, \underline{C}^T(t_K) \bullet [\underline{C}(t_K)\, \underline{P}(t_K|t_{K-1})\, \underline{C}^T(t_K) + \underline{R}(t_K)]^{-1}\, \underline{C}(t_K)\, \underline{P}(t_K|t_{K-1})$$

Filter Gain Matrix:

$$\underline{H}(t_K) = \underline{P}(t_K|t_K)\, \underline{C}^T(t_K)\, \underline{R}^{-1}(t_K)$$

On-Line Calculations:

Initialization: (t=0)

$$\hat{\underline{x}}(0|0) = E\{\underline{x}(0)\}$$

Predict Cycle:

$$\hat{x}(t_K|t_{K-1}) = \underline{\Phi}_{K-1}\,\hat{\underline{x}}(t_{K-1} \mid t_{K-1})$$

Update Cycle:

$$\hat{\underline{x}}(t_K|t_K) = \hat{\underline{x}}(t_K|t_{K-1}) + \underline{H}(t_K)\underbrace{\{\underline{z}(t_K) - \underline{C}(t_K)\,\hat{\underline{x}}(t_K|t_{K-1})\}}_{\text{Innovation } \underline{r}(t_K)}$$

System dynamics and measurement flow diagram is given in Figure 3.

The structure of discrete time Kalman filter is shown in Figure 4.

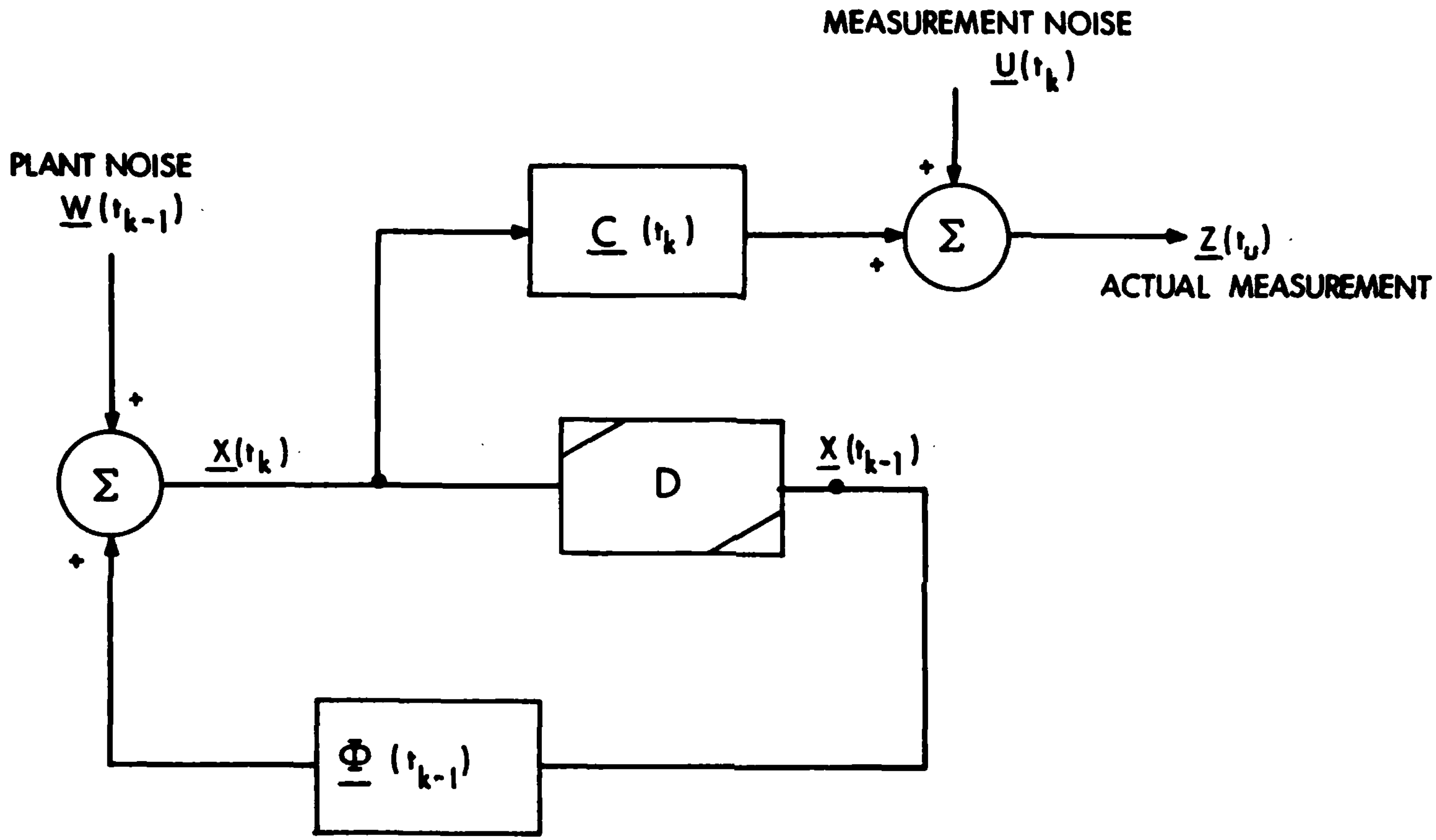

Figure 3. Discrete System Dynamics and Measurements.

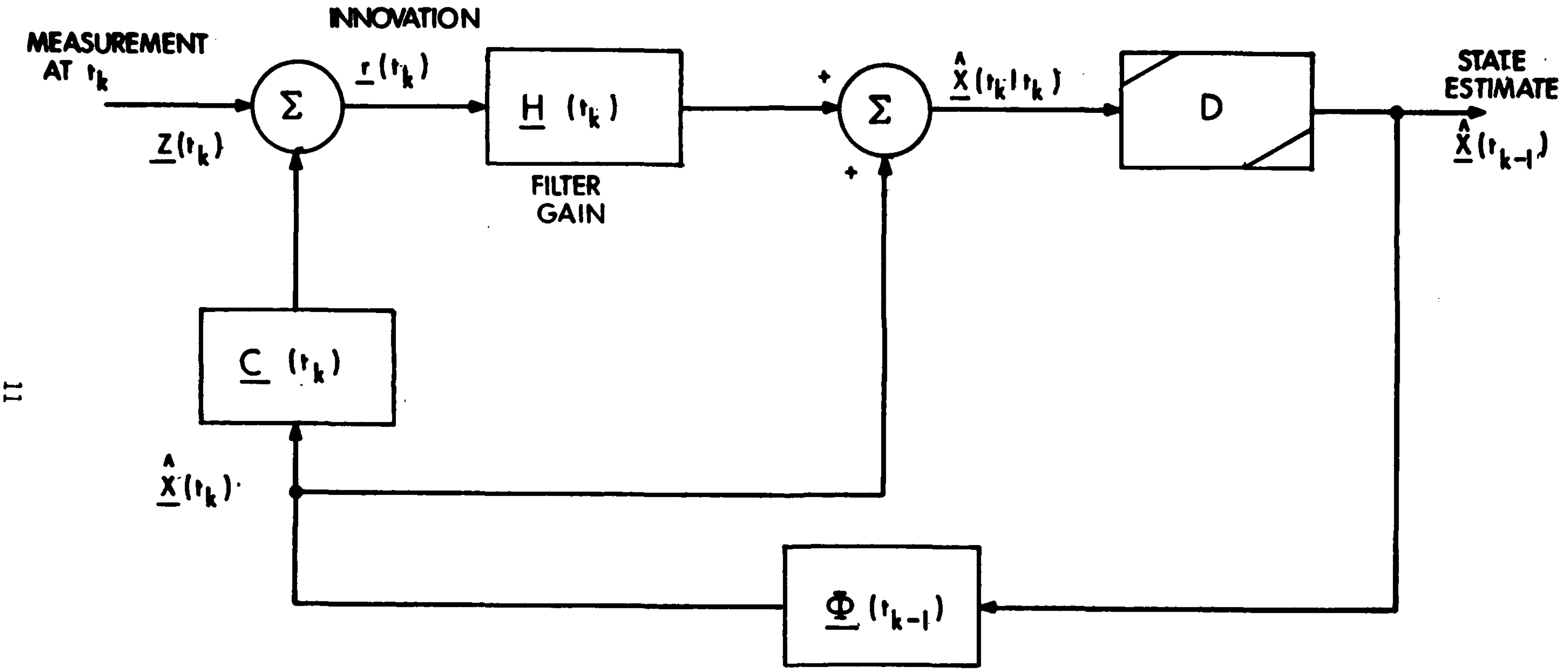

Figure 4. Discrete KALMAN-BUCY Filter.

Several examples will now be considered to demonstrate the method of computation, details of which are presented in the Appendix.

Example 1: Square law with noise in only one of the state variables.

$\dot{x} = 0x - \alpha y$

$\dot{y} = \beta x + 0y + 1\ w$ (noise in y only)

$z = y + v$ (observation of y only)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -\alpha \\ -\beta & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w$$

$$z = (0 \quad 1) \begin{pmatrix} x \\ y \end{pmatrix} + v$$

$$\underline{A} = \begin{pmatrix} 0 & -\alpha \\ -\beta & 0 \end{pmatrix} \qquad \underline{G} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\underline{H} = (0 \quad 1)$$

$$E\ (w(t')) = 0 \qquad E\ (v(t)) = 0$$

$$E\ (w(t)\ w(t')) = q\ \delta\ (t-t') \qquad E\ (v(t)v(t')) = r\ \delta(t-t')$$

$$\underline{Q} = q \qquad \underline{R} = r$$

$$E\ (x\ \ (0)) = 0; \qquad E\ (y\ \ (0)) = 0$$

$$E \begin{pmatrix} x^2 & x\ y \\ y\ x & y^2 \end{pmatrix}_{t=0} = \begin{pmatrix} P_{11}\ (0) & P_{12}\ (0) \\ P_{21}\ (0) & P_{22}\ (0) \end{pmatrix} = \underline{P}(0)$$

$$\dot{\underline{P}} = \underline{A}\ \underline{P} + \underline{P}\ \underline{A}^T + \underline{G}\ \underline{Q}\ \underline{G}^T - \underline{P}\ \underline{H}^T \underline{R}^{-1}\ \underline{H}\ \underline{P} \qquad \text{(Riccati Equation)}$$

$$\underline{P} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \qquad P_{21} = P_{12}$$

$$\dot{P}_{11} = -2\alpha P_{21} - \left(\frac{1}{r}\right) P_{21}^{2} \qquad P_{11}\,(0) \text{ given}$$

$$\dot{P}_{12} = -\alpha P_{22} - \beta P_{11} - \left(\frac{1}{r}\right) P_{21} \;\; P_{22} \qquad P_{12}\,(0) \text{ given}$$

$$\dot{P}_{22} = -2\beta P_{12} + q - \left(\frac{1}{r}\right) P_{22}^{2} \qquad P_{22}\,(0) \text{ given}$$

<u>Steady State</u>:

$$\dot{P}_{11} = \dot{P}_{12} = \dot{P}_{22} = 0$$

$$\dot{P}_{11} = 0 \Rightarrow P_{21} = 0$$

$$\dot{P}_{12} = 0 \Rightarrow 0 = -\left(\alpha + \frac{1}{r} P_{21}\right) P_{22} - \beta P_{11}$$

$$\dot{P}_{22} = 0 \Rightarrow 0 = -2\beta\, P_{12} + q - \frac{1}{r}\; P_{22}^{2}$$

with $P_{12} = P_{21} = 0 \Rightarrow P_{22} = (rq)^{1/2}$

$$P_{11} = -\frac{\alpha}{\beta} P_{22} = -\left(\frac{\alpha}{\beta}\right)(rq)^{1/2}$$

$$\dot{\hat{\underline{x}}} = \underline{A}\,\hat{\underline{x}} + \underline{G}\,\overline{\underline{w}} + \underline{P}\;\underline{H}^{T}\underline{R}^{-1}\;(\underline{z} - \underline{H}\;\hat{\underline{x}}) \qquad \hat{\underline{x}}\;(0) = 0$$

or

$$\dot{\hat{\underline{x}}} = (\underline{A}-\underline{K}\;\underline{H})\;\hat{\underline{x}} + \underline{K}\;\underline{z} \qquad \underline{K} = \underline{P}\;\underline{H}^{T}\underline{R}^{-1}$$

$$\hat{x} = \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}; \quad \underline{A} = \begin{pmatrix} 0 & -\alpha \\ -\beta & 0 \end{pmatrix}; \quad \underline{G} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \underline{H} = (0\ 1)$$

$$\dot{\hat{x}} = -\alpha\hat{y} \quad + \left(\frac{1}{r}\right) P_{21}\;(t)\;(z - \hat{y}) \qquad \text{where } \hat{x}\;(0) = 0$$

$$\dot{\hat{x}} = -\alpha\hat{y} \qquad \text{at steady state}$$

$$\dot{\hat{y}} = -\beta\hat{x} + \left(\frac{1}{r}\right) P_{22}(t)\ (z - \hat{y}) \qquad \hat{y}\,(0) = 0$$

$$= -\beta\hat{x} + \left(\frac{q}{r}\right)^{1/2} (z - \hat{y}) \qquad \text{at steady state}$$

Summarizing the results for the case where only one state variable (y) is corrupted by noise in the systems model and measurement, the general solution satisfies the equations

$$\dot{\hat{x}} = -\alpha\hat{y} + (P_{21}(t)/r)\ (z - \hat{y}) \qquad \hat{x}(0) = 0$$

$$\dot{\hat{y}} = -\ \beta\hat{x} + (P_{22}(t)/r)\ (z - \hat{y}) \qquad \hat{y}(0) = 0$$

while the steady state equations take the form

$$\dot{\hat{x}} = -\alpha\hat{y} \qquad \hat{x}(0) = 0$$

$$\dot{\hat{y}} = -\beta\hat{x} + \frac{q}{r}^{1/2} (z - \hat{y}) \qquad \hat{y}(0) = 0$$

In the next example (example 2) we consider the case where the "plant" noise and measurement noise corrupts <u>both</u> state variables as follows.

Example 2:

$$\dot{x} = 0x - \alpha y + 1\ w_1.$$

$$\dot{y} = -\beta x + 0y + 1\ w_2.$$

$$z_1 = x + v_1$$

$$z_2 = y + v_2 \qquad \underline{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \underline{F} = \begin{pmatrix} 0 & -\alpha \\ -\beta & 0 \end{pmatrix}; \quad \underline{G} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underline{I}$$

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{I}\,\underline{w} \qquad \underline{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$\underline{z} = \underline{I}\,\underline{x} + \underline{v} \qquad \underline{H} = \underline{I};\ \underline{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$E\,(\underline{x}(0)) = \underline{0}$$

$$E\,(\underline{x}\underline{x}^T)_{t=0} = \underline{P}(0)$$

$$E(\underline{w}) = 0;\ E\,(\underline{w}(t)\ \underline{w}(t')^T) = \underline{Q}(t)\ \delta(t-t')$$

$$\underline{Q}(t) = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{21} \end{pmatrix} \text{assumed constant}$$

$$E(\underline{v}) = 0;\ E\,(\underline{v}(t)\ \underline{v}(t')^T) = \underline{R}(t)\ \delta\ (t-t');\quad \underline{R}(t) = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}$$

$$R^{-1} = \frac{1}{|R|}\begin{pmatrix} R_{22} & -R_{12} \\ -R_{21} & R_{11} \end{pmatrix}$$

$\underline{R}$ is assumed constant

$$|R| \neq 0$$

$$\underline{P} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}; \qquad |R| = R_{11}\ R_{22} - R_{21}\ R_{12}$$

$$\underline{K} = \frac{1}{|R|}\begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix} = \underline{P}\ \underline{H}^T\ \underline{R}^{-1}$$

where

$$\xi_{11} = P_{11}\ R_{22} - P_{12}\ R_{21}$$

$$\xi_{12} = P_{12}\ R_{11} - P_{11}\ R_{12}$$

$$\xi_{21} = P_{21} \; R_{22} - P_{22} \; R_{21}$$

$$\xi_{22} = P_{22} \; R_{11} - P_{21} \; R_{12}$$

$$P_{11} = \sqrt{\frac{Q_{11}}{R_{22}} |R|} \qquad \text{Steady state}$$

$$P_{22} = \sqrt{\frac{Q_{22}}{R_{11}} |R|} \qquad \text{Steady state}$$

$$P_{12} = P_{21} \longrightarrow 0$$

The predicted state updated by measurements at an earlier time now is given by

$$\dot{\hat{\underline{x}}} = \underline{A} \; \hat{\underline{x}} + \underline{G} \; \bar{\underline{w}} + \underline{P} \; \underline{H}^T \; \underline{R}^{-1} \; (\underline{z} - \underline{H} \; \hat{\underline{x}}) \qquad \underline{x}(0) = 0$$

or

$$\dot{\hat{\underline{x}}} = (\underline{A}-\underline{K}) \; \hat{\underline{x}} + \underline{K} \; \underline{z}$$

$$\begin{pmatrix} \dot{\hat{x}} \\ \dot{\hat{y}} \end{pmatrix} = \begin{pmatrix} 0 & -\alpha \\ -\beta & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ y \end{pmatrix} + \frac{1}{|R|} \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix} \begin{pmatrix} z_1 - \hat{x} \\ z_2 - \hat{y} \end{pmatrix}$$

$$\dot{\hat{x}} \; (t) = -\alpha \; \hat{y} \; (t) + \frac{1}{|R|} \left\{ \xi_{11} \; (t) \; (z_1 - \hat{x}) + \xi_{12}(t) \; (z_2 - \hat{y}) \right\}$$

$$\dot{\hat{y}} \; (t) = -\beta \; \hat{x} \; (t) + \frac{1}{|R|} \left\{ \xi_{21}(t) \; (z_1 - \hat{x}) + \xi_{22}(t) \; (z_2 - \hat{y}) \right\}$$

where

$$\xi_{11}(t) = P_{11}(t) \; R_{22} - P_{12}(t) \; R_{21}$$

$$\xi_{12}(t) = P_{12}(t) \; R_{11} - P_{11}(t) \; R_{12}$$

$$\xi_{21}(t) = P_{21}(t) \; R_{22} - P_{22}(t) \; R_{21}$$

$$\xi_{22}(t) = P_{22}(t) \; R_{11} - P_{21}(t) \; R_{12}$$

$$\dot{\hat{x}} = \frac{-\xi_{11}(t)}{|R|}\,\hat{x} - \left(\alpha + \frac{\xi_{12}(t)}{|R|}\right)\hat{y} + \frac{\xi_{11}(t)}{|R|}\, z_1 + \frac{\xi_{12}(t)}{|R|}\, z_2$$

$$\dot{\hat{y}} = -\left(\beta + \frac{\xi_{21}(t)}{|R|}\right)\hat{x} \;- -\; \frac{\xi_{22}(t)}{|R|}\,\hat{y} + \frac{\xi_{21}(t)}{|R|}\, z_1 + \frac{\xi_{22}(t)}{|R|}\, z_2$$

$$\underline{F_2}(t) = \begin{pmatrix} \dfrac{\xi_{11}(t)}{|R|} & -\left(\alpha + \dfrac{\xi_{12}(t)}{|R|}\right) \\ -\left(\beta + \dfrac{\xi_{21}(t)}{|R|}\right) & \dfrac{\xi_{22}(t)}{|R|} \end{pmatrix} = \underline{A}(t) - \underline{K}(t)$$

$$\dot{\underline{\hat{x}}} = \underline{F_2}(t)\,\underline{\hat{x}} + \underline{B}\,\underline{z}$$

$$\underline{\hat{x}} = \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \; ; \; \underline{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\underline{B} = \begin{pmatrix} \dfrac{\xi_{11}(t)}{|R|} & \dfrac{\xi_{12}(t)}{|R|} \\ \dfrac{\xi_{21}(t)}{|R|} & \dfrac{\xi_{22}(t)}{|R|} \end{pmatrix} = \underline{K}(t)$$

$$\underline{\hat{x}}(t) = \underline{\Phi}\,(t, t_0)\,\underline{\hat{x}}(t_0) + \int_{t_0}^{t} \underline{\Phi}\,(t, \tau)\,\underline{K}(\tau)\,\underline{z}(\tau)\,d\tau$$

where $\underline{\Phi}\,(t, t_0)$ satisfies the homogeneous equation

$$\dot{\underline{\Phi}}\,(t,t_0) = \underline{F_2}\,(t)\,\underline{\Phi}\,(t,t_0)$$

Accordingly we have obtained either the differential equation for which the predicted estimate $\hat{x}$ needs to satisfy or the integral solution for this estimate in terms of its transition matrix $\underline{\Phi}$ (t, t_0).

In summary, there currently exists engineering techniques from the field of modern control theory that offers the commander in the field the opportunity of predicting or forecasting conditions of the force strength on the battlefield given prior information or observation on status of battle. The approach can handle any number of different weapon systems. The methodology described in this report will be applied to recently acquired data on tank duels and will be reported in a separate report.

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APPENDIX

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APPENDIX

Detailed analysis and derivations will now be presented to demonstrate the method of computation.

Example 1: Square law with noise in only one of the state variables.

$$\dot{x} = 0x - \alpha y$$

$$\dot{y} = \beta x + 0y + 1\ w \quad \text{(noise in y only)}$$

$$z = y + v \quad \text{(observation of y only)}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -\alpha \\ -\beta & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w$$

$$z = (0 \quad 1) \begin{pmatrix} x \\ y \end{pmatrix} + v$$

$$\underline{A} = \begin{pmatrix} 0 & -\alpha \\ -\beta & 0 \end{pmatrix} \qquad \underline{G} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\underline{H} = (0 \quad 1)$$

$$E\,(w(t')) = 0 \qquad E\,(v(t)) = 0$$

$$E\,(w(t)\,w(t')) = q\,\delta\,(t-t') \qquad E\,(v(t)v(t')) = r\,\delta(t-t')$$

$\underline{Q} = q \qquad \underline{R} = r$

$E\,(x\ \ (0)) = 0; \qquad E\,(y\ \ (0)) = 0$

$$E \begin{pmatrix} x^2 & x\,y \\ y\,x & y^2 \end{pmatrix}_{t=0} = \begin{pmatrix} P_{11}\,(0) & P_{12}\,(0) \\ P_{21}\,(0) & P_{22}\,(0) \end{pmatrix} = \underline{P}(0)$$

$$\dot{\underline{P}} = \underline{A}\,\underline{P} + \underline{P}\,\underline{A}^T + \underline{G}\,\underline{Q}\,\underline{G}^T - \underline{P}\,\underline{H}^T\underline{R}^{-1}\,\underline{H}\,\underline{P} \qquad \text{(Riccati Equation)}$$

$$\underline{P} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \qquad P_{21} = P_{12}$$

$$\underline{A}\,\underline{P} = \begin{pmatrix} 0 & -\alpha \\ -\beta & 0 \end{pmatrix} \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = \begin{pmatrix} -\alpha P_{21} & -\alpha P_{22} \\ -\beta P_{11} & -\beta P_{12} \end{pmatrix}$$

$$(\underline{A}\,\underline{P})^T = \underline{P}\,\underline{A}^T = \begin{pmatrix} -\alpha P_{21} & -\beta P_{11} \\ -\alpha P_{22} & -\beta P_{12} \end{pmatrix}$$

$$\underline{G}\,\underline{Q}\,\underline{G}^T = q \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = q \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\underline{H}\,\underline{P} = (0\ 1) \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = (P_{21},\ P_{22})$$

$$(\underline{H}\,\underline{P})^T = \underline{P}\,\underline{H}^T = \begin{pmatrix} P_{21} \\ P_{22} \end{pmatrix}$$

$$\underline{P}\,\underline{H}^T\underline{R}^{-1}\underline{H}\,\underline{P} = \frac{1}{r} \begin{pmatrix} P_{21} \\ P_{22} \end{pmatrix} (P_{21},\ P_{22}) = \frac{1}{r} \begin{pmatrix} P_{21}{}^2 & P_{21}P_{22} \\ P_{21}P_{22} & P_{22}{}^2 \end{pmatrix}$$

$$\dot{P}_{11} = -2\alpha P_{21} - \left(\frac{1}{r}\right) P_{21}{}^2 \qquad P_{11}\,(0) \text{ given}$$

$$\dot{P}_{12} = -\alpha P_{22} - \beta P_{11} - \left(\frac{1}{r}\right) P_{21}\ \ P_{22} \qquad P_{12}\,(0) \text{ given}$$

$$\dot{P}_{22} = -2\beta P_{12} + q - \left(\frac{1}{r}\right) P_{22}{}^2 \qquad P_{22}\,(0) \text{ given}$$

Steady State:

$$\dot{P}_{11} = \dot{P}_{12} = \dot{P}_{22} = 0$$

$$\dot{P}_{11} = 0 \Rightarrow P_{21} = 0$$

$$\dot{P}_{12} = 0 \Rightarrow 0 = -\left(\alpha + \frac{1}{r} P_{21}\right) P_{22} - \beta P_{11}$$

$$\dot{P}_{22} = 0 \Rightarrow 0 = -2\beta\, P_{12} + q - \frac{1}{r}\, P_{22}^2$$

with $P_{12} = P_{21} = 0 \Rightarrow P_{22} = (rq)^{1/2}$

$$P_{11} = -\frac{\alpha}{\beta} P_{22} = -\left(\frac{\alpha}{\beta}\right)(rq)^{1/2}$$

$$\dot{\hat{\underline{x}}} = \underline{A}\,\hat{\underline{x}} + \underline{G}\,\overline{\underline{w}} + \underline{P}\,\underline{H}^T\underline{R}^{-1}\,(\underline{z} - \underline{H}\,\hat{\underline{x}}) \qquad \hat{\underline{x}}\,(0) = 0$$

or

$$\dot{\hat{\underline{x}}} = (\underline{A}-\underline{K}\,\underline{H})\,\hat{\underline{x}} + \underline{K}\,\underline{z} \qquad \underline{K} = \underline{P}\,\underline{H}^T\underline{R}^{-1}$$

$$\hat{x} = \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} ; \quad \underline{A} = \begin{pmatrix} 0 & -\alpha \\ -\beta & 0 \end{pmatrix} ; \quad \underline{G} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \underline{H} = (0\ 1)$$

$$\dot{\hat{x}} = -\alpha\hat{y} + \left(\frac{1}{r}\right) P_{21}\,(t)\,(z - \hat{y}) \qquad \text{where } \hat{x}\,(0) = 0$$

$$\dot{\hat{\underline{x}}} = -\alpha\hat{y} \qquad \text{at steady state}$$

$$\dot{\hat{y}} = -\beta\hat{x} + \left(\frac{1}{r}\right) P_{22}(t)\,(z - \hat{y}) \qquad \hat{y}\,(0) = 0$$

$$= -\beta\hat{x} + \left(\frac{q}{r}\right)^{1/2} (z - \hat{y}) \qquad \text{at steady state}$$

Summarizing the results for the case where only one state variable (y) is corrupted by noise in the systems model and measurement, the general solution satisfies the equations

$$\dot{\hat{x}} = -\alpha\hat{y} + (P_{21}(t)/r)\,(z - \hat{y}) \qquad \hat{x}(0) = 0$$

$$\dot{\hat{y}} = -\beta\hat{x} + (P_{22}(t)/r)\,(z - \hat{y}) \qquad \hat{y}(0) = 0$$

while the steady state equations take the form

$$\dot{\hat{x}} = -\alpha\hat{y} \qquad \hat{x}(0) = 0$$

$$\dot{\hat{y}} = -\beta\hat{x} + \left(\frac{q}{r}\right)^{1/2}(z - \hat{y}) \qquad \hat{y}(0) = 0$$

In the next example (example 2) we consider the case where the "plant" noise and measurement noise corrupts both state variables as follows.

Example 2:

$$\dot{x} = 0x - \alpha y + 1\,w_1.$$

$$\dot{y} = -\beta x + 0y + 1\,w_2.$$

$$z_1 = x + v_1$$

$$z_2 = y + v_2 \qquad \underline{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix} \; ; \quad \underline{F} = \begin{pmatrix} 0 & -\alpha \\ -\beta & 0 \end{pmatrix}; \quad \underline{G} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underline{I}$$

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{I}\,\underline{w} \qquad \underline{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$\underline{z} = \underline{I}\,\underline{x} + \underline{v} \qquad \underline{H} = \underline{I};\ \underline{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$E\,(\underline{x}(0)) = \underline{0}$$

$$E\,(\underline{x}\underline{x}^T)_{t=0} = \underline{P}(0)$$

$$E(\underline{w}) = 0;\; E\,(\underline{w}(t)\;\underline{w}(t')^T) = \underline{Q}(t)\;\delta(t-t')$$

$$\underline{Q}(t) = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{21} \end{pmatrix} \text{assumed constant}$$

$$E(\underline{v}) = 0;\; E\,(\underline{v}(t)\;\underline{v}(t')^T) = \underline{R}(t)\;\delta\;(t-t');\quad \underline{R}(t) = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}$$

$$R^{-1} = \frac{1}{|R|} \begin{pmatrix} R_{22} & -R_{12} \\ -R_{21} & R_{11} \end{pmatrix}$$

$\underline{R}$ is assumed constant

$$|R| \neq 0$$

$$\underline{P} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}; \qquad |R| = R_{11}\,R_{22} - R_{21}\;\;R_{12}$$

$$\underline{G}\;\underline{Q}\;\underline{G}^T = \underline{Q}$$

$$\underline{H}\;\underline{P} = \underline{P}$$

$$\underline{R}^{-1}\;\underline{P} = \frac{1}{|R|} \begin{pmatrix} R_{22} & -R_{12} \\ -R_{21} & R_{11} \end{pmatrix} \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$$

$$= \frac{1}{|R|} \begin{pmatrix} P_{11}\,R_{22} - R_{12}\,P_{21} & R_{22}\,P_{12} - R_{12}\,P_{22} \\ P_{21}\,R_{11} - R_{21}\,P_{11} & R_{11}\,P_{22} - R_{21}\,P_{12} \end{pmatrix}$$

$$\underline{P}\,\underline{R}^{-1}\underline{P} = \frac{1}{|R|}\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}\begin{pmatrix} P_{11} R_{22} - R_{12} P_{21} & R_{22} P_{12} - R_{12} P_{22} \\ P_{21} R_{11} - R_{21} P_{11} & R_{11} P_{22} - R_{21} P_{12} \end{pmatrix}$$

$$= \frac{1}{|R|}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where

$$a_{11} = P_{11} (P_{11} R_{22} - R_{12} P_{21}) + P_{12} (P_{21} R_{11} - R_{21} P_{11})$$

$$a_{12} = P_{11} (R_{22} P_{12} - R_{12} P_{22}) + P_{12} (R_{11} P_{22} - R_{21} P_{12})$$

$$a_{21} = P_{21} (P_{11} R_{22} - R_{12} P_{21}) + P_{22} (P_{21} R_{11} - R_{21} P_{11})$$

$$a_{22} = P_{12} (R_{22} P_{12} - R_{12} P_{22}) + P_{22} (R_{11} P_{22} - R_{21} P_{12})$$

for $R_{12} = R_{21}$ and $P_{21} = P_{12}$ it follows that $a_{12} = a_{21}$.

$$\underline{K} = \underline{P}\,\underline{H}^T\,\underline{R}^{-1}$$

$$= \underline{P}\,\underline{R}^{-1} = \frac{1}{|R|}\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}\begin{pmatrix} R_{22} & -R_{12} \\ -R_{21} & R_{11} \end{pmatrix}$$

$$= \frac{1}{|R|}\begin{pmatrix} P_{11} R_{22} - P_{12} R_{21} & R_{11} P_{12} - R_{12} P_{11} \\ P_{21} R_{22} - P_{22} R_{21} & R_{11} P_{22} - P_{21} R_{12} \end{pmatrix}$$

$$= \frac{1}{|R|}\begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix}$$

where

$$\xi_{11} = P_{11} \; R_{22} - P_{12} \; R_{21}$$

$$\xi_{12} = P_{12} \; R_{11} - P_{11} \; R_{12}$$

$$\xi_{21} = P_{21} \; R_{22} - P_{22} \; R_{21}$$

$$\xi_{22} = P_{22} \; R_{11} - P_{21} \; R_{12}$$

$$\underline{K}^T = \underline{R}^{-1}\,\underline{P}$$

$$\dot{\underline{P}} = \underline{A}\,\underline{P} + \underline{P}\,\underline{A}^T + \underline{G}\,\underline{Q}\,\underline{G}^T - \underline{K}\,\underline{H}\,\underline{P}$$

$$\underline{P}(0) = \underline{P}_0$$

$$\underline{K}\,\underline{H}\,\underline{P} = \underline{P}\,\underline{I}\,\underline{R}^{-1}\,\underline{I}\,\underline{P}$$

$$= \underline{P}\,\underline{R}^{-1}\,\underline{P}$$

The Riccati equation is given by

$$\dot{\underline{P}} = \begin{pmatrix} -2\alpha\, P_{21} + Q_{11} - \frac{1}{|R|}\, a_{11} & -\beta P_{11} - \alpha P_{22} + Q_{12} - \frac{1}{|R|}\, a_{12} \\ -\beta P_{11} - \alpha P_{22} + Q_{21} - \frac{1}{|R|}\, a_{21} & -\beta P_{12} - \alpha P_{22} + Q_{21} - \frac{1}{|R|}\, a_{22} \end{pmatrix}$$

which for steady state $\dot{\underline{P}} = 0$ and $P_{12} \longrightarrow 0$

Evaluating $\underline{P}\,\underline{R}^{-1}$ at steady state

$$\underline{R}^{-1}\,\underline{P} = \frac{1}{|R|}\begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$$

where

$$d_{11} = R_{22}\, P_{11} - R_{12}\, P_{21}$$

$$d_{12} = R_{22}\, P_{12} - R_{12}\, P_{22}$$

$$d_{21} = R_{11}\, P_{21} - R_{21}\, P_{11}$$

$$d_{22} = P_{22}\, R_{11} - R_{21}\, P_{12}$$

$$\underline{P}\,\underline{R}^{-1}\,\underline{P} = \frac{1}{|R|}\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}\begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{12} \end{pmatrix}$$

$$= \frac{1}{|R|}\begin{pmatrix} P_{11}\, d_{11} + d_{12}\, P_{12} & P_{11}\, d_{12} + d_{22}\, P_{12} \\ P_{21}\, d_{11} + d_{21}\, P_{22} & P_{21}\, d_{12} + d_{22}\, P_{22} \end{pmatrix}$$

with $P_{12} \longrightarrow 0$ $\quad d_{11} = R_{22} P_{11}$ $\quad d_{21} = -R_{21} P_{11}$

$d_{12} = -R_{12} P_{22}$ $\quad d_{22} = R_{11} P_{22}$

at Steady state:

$$\underline{P}\ \underline{R}^{-1}\ \underline{P} = \frac{1}{|R|}\begin{pmatrix} R_{22} P_{11}^{2} & -R_{12} P_{22} P_{11} \\ -R_{21} P_{11} P_{22} & R_{11} P_{22}^{2} \end{pmatrix}$$

So that at steady state

$$(1)\ 0 = -2\ \alpha\ P_{21} + Q_{11} - \frac{1}{|R|}\ a_{11}$$

$$(2)\ 0 = -\beta\ P_{11} - \alpha\ P_{22} + Q_{12} - \frac{1}{|R|}\ a_{12}$$

$$(3)\ 0 = -\beta\ P_{11} - \alpha\ P_{22} + Q_{21} - \frac{1}{|R|}\ a_{21}$$

$$(4)\ 0 = -2\beta\ P_{12} + Q_{22} - \frac{1}{|R|}\ a_{22}$$

Steady state: $P_{12} \longleftrightarrow 0$

$a_{11} \longrightarrow P_{11}^{2} R_{22}$

$a_{12} \longrightarrow -R_{12} P_{22} P_{11}$

$a_{21} \longrightarrow -R_{21} P_{11} P_{22}$

$a_{22} \longrightarrow R_{11} P_{22}^{2}$

$$(1a)\quad 0 = Q_{11} - \frac{1}{|R|}\ R_{22} P_{11}^{2}$$

$$(2a)\qquad 0 = -2P_{11} - \alpha P_{22} + Q_{12} + \frac{R_{12}}{|R|}\ P_{11} P_{22}$$

(3a) $$0 = -\beta P_{11} - \alpha P_{22} + Q_{21} + \frac{R_{12}}{|R|} P_{11} P_{22}$$

(4a) $$0 = Q_{22} - \frac{1}{|R|} R_{11} P_{22}^2$$

Thus

$$P_{11}^2 = \frac{Q_{11}}{R_{22}} |R|$$

(1a´) $$P_{11} = \sqrt{\frac{Q_{11}}{R_{22}} |R|}$$ Steady state

(4a´) $$P_{22} = \sqrt{\frac{Q_{22}}{R_{11}} |R|}$$ Steady state

$$P_{12} = P_{21} \longrightarrow 0$$

The predicted state updated by measurements at an earlier time now is given by

$$\dot{\hat{\underline{x}}} = \underline{A}\,\hat{\underline{x}} + \underline{G}\,\overline{\underline{w}} + \underline{P}\,\underline{H}^T\,\underline{R}^{-1}\,(\underline{z} - \underline{H}\,\hat{\underline{x}}) \qquad \underline{x}(0) = 0$$

or

$$\dot{\hat{\underline{x}}} = (\underline{A} - \underline{K})\,\hat{\underline{x}} + \underline{K}\,\underline{z}$$

$$\begin{pmatrix} \dot{\hat{x}} \\ \dot{\hat{y}} \end{pmatrix} = \begin{pmatrix} 0 & -\alpha \\ -\beta & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + \frac{1}{|R|} \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix} \begin{pmatrix} z_1 - \hat{x} \\ z_2 - \hat{y} \end{pmatrix}$$

$$\dot{\hat{x}}\,(t) = -\alpha\,\hat{y}\,(t) + \frac{1}{|R|} \left\{ \xi_{11}\,(t)\,(z_1 - \hat{x}) + \xi_{12}(t)\,(z_2 - \hat{y}) \right\}$$

$$\dot{\hat{y}}(t) = -\beta\,\hat{x}(t) + \frac{1}{|R|}\left\{\xi_{21}(t)\,(z_1 - \hat{x}) + \xi_{22}(t)\,(z_2 - \hat{y})\right\}$$

where

$$\xi_{11}(t) = P_{11}(t)\,R_{22} - P_{12}(t)\,R_{21}$$

$$\xi_{12}(t) = P_{12}(t)\,R_{11} - P_{11}(t)\,R_{12}$$

$$\xi_{21}(t) = P_{21}(t)\,R_{22} - P_{22}(t)\,R_{21}$$

$$\xi_{22}(t) = P_{22}(t)\,R_{11} - P_{21}(t)\,R_{12}$$

$$\dot{\hat{x}} = \frac{-\xi_{11}(t)}{|R|}\,\hat{x} - \left(\alpha + \frac{\xi_{12}(t)}{|R|}\right)\hat{y} + \frac{\xi_{11}(t)}{|R|}\,z_1 + \frac{\xi_{12}(t)}{|R|}\,z_2$$

$$\dot{\hat{y}} = -\left(\beta + \frac{\xi_{21}(t)}{|R|}\right)\hat{x} -- \frac{\xi_{22}(t)}{|R|}\,\hat{y} + \frac{\xi_{21}(t)}{|R|}\,z_1 + \frac{\xi_{22}(t)}{|R|}\,z_2$$

$$\underline{F_2}(t) = \begin{pmatrix} \frac{\xi_{11}(t)}{|R|} & -\left(\alpha + \frac{\xi_{12}(t)}{|R|}\right) \\ -\left(\beta + \frac{\xi_{21}(t)}{|R|}\right) & \frac{\xi_{22}(t)}{|R|} \end{pmatrix} = \underline{A}(t) - \underline{K}(t)$$

$$\dot{\hat{\underline{x}}} = \underline{F_2}(t)\,\hat{\underline{x}} + \underline{B}\,\underline{z}$$

$$\hat{\underline{x}} = \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \; ; \; \underline{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\underline{B} = \begin{pmatrix} \frac{\xi_{11}(t)}{|R|} & \frac{\xi_{12}(t)}{|R|} \\ \frac{\xi_{21}(t)}{|R|} & \frac{\xi_{22}(t)}{|R|} \end{pmatrix} = \underline{K}(t)$$

$$\underline{\hat{x}}(t) = \underline{\Phi}\ (t,\ t_0)\ \underline{\hat{x}}(t_0) + \int_{t_0}^{t} \underline{\Phi}\ (t,\ \tau)\ \underline{K}(\tau)\ \underline{z}(\tau)\ d\tau$$

where $\underline{\Phi}$ $(t,\ t_0)$ satisfies the homogeneous equation

$$\dot{\underline{\Phi}}\ (t,t_0) = \underline{F_2}\ (t)\ \underline{\Phi}\ (t,t_0)$$

Accordingly we have obtained either the differential equation for which the predicted estimate $\hat{x}$ needs to satisfy or the integral solution for this estimate in terms of its transition matrix $\underline{\Phi}$ $(t,\ t_0)$.

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THE PRINCIPAL FINDINGS and recommendations of the work reported herein are as follows:

Combined Arms Lanchester Modeling of Warfare can be enhanced by using techniques of modern control theory to afford battlefield commanders improved forecast of battle.

THE MAIN ASSUMPTIONS on which the work reported herein rests are as follows:
Linear model of warfare corrupted by gaussian white noise with measurement noise constrained to gaussain white noise.

THE PRINCIPAL LIMITATIONS of this work which may affect the findings are as follows:

Need reasonable estimates of initial conditions and properties of noise corrupting both math model and measurements of battle.

THE SCOPE OF THE STUDY

Applicable to wide range of land warfare models including combined arms.

AMSAA Form 43R (18 Jul 85)
Previous editions of this form are obsolete.

THE STUDY OBJECTIVE

Enhance appreciation that modern control can play a key role in modeling land warfare.

THE BASIC APPROACH

Math modeling using Kalman-Bucy filtering of control theory.

THE REASONS FOR PERFORMING THE STUDY

Call attention to the fact that modern control theory provides and improved capability to provide forecasts of status of battle.

STUDY IMPACT STATEMENT (Actual or expected changes in operations realized from implementation of study results -- if possible state in quantitative terms.) May provide improved estimates of status of battle.

THE STUDY SPONSOR

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