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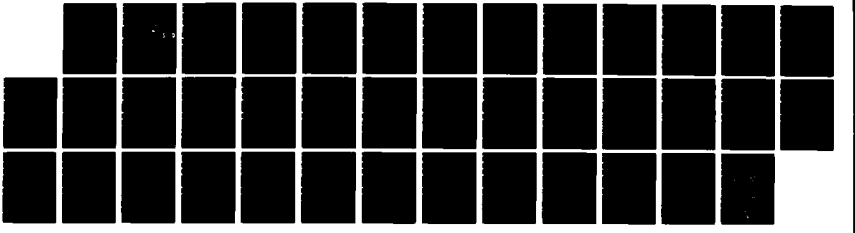

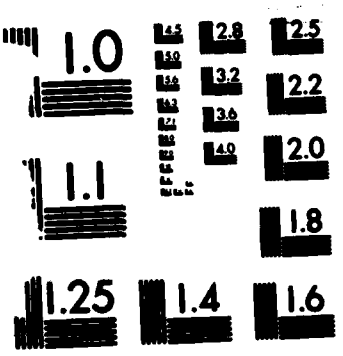

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NAVAL POSTGRADUATE SCHOOL

Monterey, California

GAMMA PROCESSES

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NAVAL POSTGRADUATE SCHOOL
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The Beta-Gamma transformation is described and is used to define a very simple first-order autoregressive Beta-Gamma process, BGAR(1). Maximum likelihood estimation is discussed for this model, as well as moment estimators. The first-order structure is extended to include moving average processes and mixed first-order autoregressive, pth-order moving average processes. It is shown that these Gamma processes are time-reversible and, therefore, too narrow for general physical modelling. A dual process to the BGAR(1) process, DBGAR(1), is introduced, as well as an iterated process which combines the Beta-Gamma process and the GAR(1) process of Gaver and Lewis (1980). Some properties of these extended autoregressive processes are derived. Several highly nonlinear extensions of these processes which produce negative correlation are given.

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GAMMA PROCESSES

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Abstract

The Beta-Gamma transformation is described and is used to define a very simple first-order autoregressive Beta-Gamma process, BGAR(1). Maximum likelihood estimation is discussed for this model, as well as moment estimators. The first-order structure is extended to include moving average processes and mixed first-order autoregressive, pth-order moving average processes. It is shown that these Gamma processes are time-reversible and, therefore, too narrow for general physical modelling. A dual process to the BGAR(1) process, DBGAR(1), is introduced, as well as an iterated process which combines the Beta-Gamma process and the GAR(1) process of Gaver and Lewis (1980). Some properties of these extended autoregressive processes are derived. Several highly nonlinear extensions of these processes which produce negative correlation are given. Use of the processes to model a sequence of times between failures of a computer system is described.

0. INTRODUCTION

The Gamma distribution is used to model a wide variety of positive valued random quantities in fields such as operations analysis, reliability theory, hydrology and meteorology. Thus, service time distributions and interarrival times in queues are often modelled as having Gamma distributions, as are wind velocities (Hugus, 1982; Brown, Katz and Murphy, 1984) measured at successive discrete time points or river flows at successive instants of time (Lawrance and Kottegoda, 1977). In all these cases, the measurements are taken serially in time and are apt to be serially dependent. Thus, development of time series with Gamma distributed marginal distributions and various correlation structures is of great importance.

Gaver and Lewis (1980) showed that the usual linear first-order autoregressive equation, $X_n = \rho X_{n-1} + E_n$, would yield Gamma marginal distributions if the i.i.d. sequence $\{E_n\}$ was chosen with suitable marginal distribution. This Gamma innovation distribution has a positive probability

of being zero, so that the process (GAR(1)) generates sample paths which exhibit 'runs-down' (as seen in river flow data), but which are "defective". The defect lies in the fact that when $E_n = 0$, X_n and X_{n-1} are proportional and ρ can be estimated **exactly** in long enough time series. Moreover, the probability of the defect is higher for k small, which is precisely where the model is needed, since the usual techniques of transforming to normality are then questionable and probably undesirable.

Bernier (1970) introduced the GAR(1) model in a hydrological context and McKenzie (1982) introduced a multiplicative Gamma process called PAR(1) - product autoregression of order one.

In Lewis (1983) and Hugus (1982), a simple linear, random coefficient model called BGAR(1) was introduced. It is based on the Beta-Gamma transformation described in Section 1.

The purpose of this paper is to develop the properties of this BGAR(1) model and to extend the idea to moving average and mixed autoregressive structures. In particular, it is shown that these processes, like the Gaussian ARMA(p,q) models, are time reversible and therefore are very particular.

Several schemes for broadening the structure of Gamma time series are given. In particular, a technique of iteration produces a Gamma autoregressive process with two structional parameters that can model, for given marginal distribution and serial correlation, different kinds of sample path behavior. Some nonlinear schemes that produce negative serial correlation are also introduced.

It is important to note the multiplicity of Gamma processes which can be derived with given first- and second-order structure. Conseqently, in the absence of a 'natural' structure such as exists for Gaussian processes, our aim has been to produce simple structures, i.e., linear, additive, random coefficient processes.

Finally, a series of times between failures of digital computers (Lewis, 1964) is analyzed and fitted with the model. The data is serially correlated with a marginal distribution which is more skewed than an exponential distribution. Although this data is known to be generated by a branching Poisson process (cluster process), the Gamma model is much simpler

and much more tractable than the cluster process, and provides an adequate representation for most purposes.

1. PRELIMINARIES

In what follows we will use B(m,n), or simply B when the parameterization is clear, to stand for a Beta random variable with parameters $m > 0$ and $n > 0$, denoted by Beta(m,n). The probability density function for a Beta(m,n) random variable is

$$f_B(x;m,n) = \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} x^{m-1}(1-x)^{n-1}, \quad 0 \le x \le 1;\ m > 0;\ n > 0, \tag{1.1}$$

where $\Gamma(\cdot)$ is the complete Gamma function.

We will denote by {B(m,n), B'(m',n'),···} an i.i.d. sequence of vector random variables whose components are **independent** Beta random variables.

Let G(k,β) stand for a Gamma random variable with **shape** parameter $k > 0$, and **rate** parameter $\beta > 0$, denoted by Gamma(k,β). The probability density function for a Gamma(k,β) random variable is

$$f_G(x;k,\beta) = \frac{\beta^k x^{k-1} e^{-\beta x}}{\Gamma(k)}, \quad x \ge 0;\ \beta > 0;\ k > 0. \tag{1.2}$$

We will denote by $\{G_n(k,\beta)\}$, an i.i.d. sequence of Gamma variates.

A Gamma(k,β) random variable has moments

$$E(G) = k/\beta = \mu; \quad Var(G) = k/\beta^2; \quad C(G) = s.d(G/\mu) = k^{-1/2}, \tag{1.3}$$

where C(G) is the coefficient of variation, and Laplace-Stieltjes transform

$$L_G(u) = E(e^{-uG}) = \left(\frac{\beta}{\beta + u}\right)^k. \tag{1.4}$$

The Gamma variable is sometimes parameterized in terms of the parameter $\mu = E(G)$. This is useful in statistical work, since the mean is a multiplicative parameter and G can be written as a unit-mean Gamma variate, G*,

times μ, i.e., $G = \mu G^*$. However, in what follows, we will use the fact that two independent Gamma variates **with the same β-parameters**, but possibly different shape parameters, add to give another Gamma variate

$$G''(k+k',\beta) = G(k,\beta) + G'(k',\beta). \tag{1.5}$$

The result is not true if the Gamma variates have the same mean but different shape parameters.

The Gamma family of random variables include the Exponential($k=1$), Erlang(k integer) and Chi-Square($k=r/2$, $r=1,2,\cdots$; $\beta=2$) random variables.

Gamma and Beta variates are intimately related and two of their properties will be used throughout this paper.

(i) A Beta(m,n) variate may be generated as

$$B(m,n) = \frac{G'(m,\beta)}{G'(m,\beta) + G''(n,\beta)}, \tag{1.6}$$

where $G'(m,\beta)$ and $G''(n,\beta)$ are independent. Furthermore, the ratio $B(m,n)$ is independent of the denominator $G'(m,\beta) + G''(n,\beta) = G(m+n,\beta)$ and this property characterizes the Gamma random variable (Johnson and Kotz, 1970a).

(ii) The Beta Gamma transformation. Multiplying a $G(m+n,\beta)$ random variable by an independent $B(m,n)$ random variable gives a $G(m,\beta)$ random variable

$$G(m,\beta) = B(m,n)G(m+n,\beta). \tag{1.7}$$

Thus one can **reduce** the shape parameter of a Gamma random variable (multiply by a Beta random variable), as well as increase it (add an independent Gamma variate, as at (1.5)). A heuristic argument for the result (1.7) is that if we wanted to perform the operation (1.7) on a computer, we could first generate $G'(m,\beta)$ and $G''(m,\beta)$ to form the ratio (1.6) to obtain the $B(m,n)$ variate. There is, however, no need to generate $G(m+n,\beta)$ in (1.7), we can use $G'(m,\beta) + G''(n,\beta)$, which is independent of the Beta variate. Multiplication then gives $G(m,\beta) = G'(m,\beta)$.

(iii) A formal proof of the Beta-Gamma transformation is a special case of the following Lemma, which will be used in the sequel.

Lemma. Let $X(k,\beta)$ be a Gamma random variable and let $B(k\rho,k\bar{\rho})$ be a Beta random variable, which is independent of $X(k,\beta)$, with $\rho = 1-\bar{\rho}$ lying in $(0,1)$. Then

$$E\{e^{-(v+Bu)X}\} = \left(\frac{\beta}{\beta+v}\right)^{k\bar{\rho}} \left(\frac{\beta}{\beta+v+u}\right)^{k\rho} \qquad v \geq 0,\ u \geq 0. \tag{1.8}$$

When $v = 0$, this result proves the Beta-Gamma transformation in the form

$$X(k\rho,\beta) = B(k\rho;k\bar{\rho})X(k,\beta). \tag{1.9}$$

Proof. Using (1.4) and conditioning on B, we get

$$E\{e^{-(v+Bu)X}\} = E_B\left\{\left(\frac{\beta}{\beta+v+Bu}\right)^k\right\} = E_B\left\{\left(\frac{\beta}{\beta+v} \cdot \frac{\beta}{\beta + \frac{\beta u}{\beta+v}B}\right)^k\right\}$$

$$= \left(\frac{\beta}{\beta+v}\right)^k \cdot \sum_{\ell=0}^{\infty} \binom{k+\ell-1}{\ell}\left(\frac{-u}{\beta+v}\right)^{\ell} E(B^{\ell}),$$

where we have used the finiteness of the expectation to take the expectation inside of the binomial expansion.

Now since $E(B^{\ell}) = B(k\rho+\ell,k\bar{\rho})/B(k\rho,k\bar{\rho}) = \Gamma(k\rho+\ell)\Gamma(k\bar{\rho})\Gamma(k)/\{\Gamma(k+\ell)\Gamma(k\bar{\rho})\Gamma(k\rho)\}$ we have that

$$\binom{k+\ell-1}{\ell}E(\beta^{\ell}) = \binom{k+\ell-1}{\ell} \frac{\Gamma(k\rho+\ell)\Gamma(k\bar{\rho})\Gamma(k)}{\Gamma(k+\ell)\Gamma(k\bar{\rho})\Gamma(k\rho)}$$

$$= \frac{\Gamma(k+\ell)\Gamma(k\rho+\ell)\Gamma(k\bar{\rho})\Gamma(k)}{\Gamma(k)\Gamma(\ell+1)\Gamma(k\rho)\Gamma(k\bar{\rho})\Gamma(k+\ell)} = \binom{k\rho+\ell-1}{\ell}.$$

Thus

$$\sum_{\ell=0}^{\infty} \binom{k+\ell-1}{\ell}\left(-\frac{u}{\beta+v}\right)^{\ell} E(B^{\ell}) = \sum_{\ell=0}^{\infty} \binom{k\rho+\ell-1}{\ell}\left(-\frac{u}{\beta+v}\right)^{\ell}$$

$$= \left\{\frac{1}{1+\frac{u}{\beta+v}}\right\}^{k\rho} = \left\{\frac{\beta}{\beta+v+u}\right\}^{k\rho}\left\{\frac{\beta}{\beta+v}\right\}^{-k\rho}$$

and therefore

$$E\{e^{-(v+Bu)X}\} = \left(\frac{\beta}{\beta+v}\right)^{k\bar{\rho}}\left(\frac{\beta}{\beta+v+u}\right)^{k\rho},$$

which was to be proved.

2. THE FIRST-ORDER AUTOREGRESSIVE PROCESS, BGAR(1)

2.1. Construction of the Process.

Using the Beta-Gamma transformation, we can construct a very simple first-order autogressive process $\{X_n(k,\beta,\rho)\}$ with Gamma(k,β) marginal distribution and a single parameter, ρ, that describes the dependency structure of the process. We have

$$X_n(k,\beta,\rho) = B_n(k\rho,k\bar{\rho})X_{n-1}(k,\beta,\rho) + B_n'(k\bar{\rho},k\rho)G_n(k,\beta) \qquad (0 \le \rho < 1) \quad (2.1)$$

$$= B_n(k\rho,k\bar{\rho})X_{n-1}(k,\beta,\rho) + Y_n(k\bar{\rho},\beta) \qquad n=0,\pm1,\ldots, \quad (2.2)$$

where $\{Y_n(k\bar{\rho},\beta)\}$ are i.i.d. Gamma$(k\bar{\rho},\beta)$ variates independent of the $\{B_n(k\rho,k\bar{\rho})\}$ sequence. If $X_{n-1}(k,\beta,\rho)$ has a Gamma(k,β) marginal distribution, then multiplying by $B_n(k\rho,k\bar{\rho})$ reduces it to a Gamma$(k\rho,\beta)$ variate and adding the **innovation** variable $Y_n(k\bar{\rho},\beta)$ creates the Gamma(k,β) variate, $X_n(k,\beta)$. The alternate form (2.1) shows the process as a transformation of an i.i.d. Gamma(k,β) sequence, but clearly generation on a computer would be done with (2.2). In the sequel, we will drop the parametric notation where no confusion is possible.

It is clear that taking X_0 to be a Gamma(k,β) variate will start the process in a stationary mode. Also, the process is Markovian by construction.

2.2. Serial Correlation.

It is easily established, using moments of Beta variables, that

$$\rho(r) = \mathrm{Corr}(X_n, X_{n-r}) = \rho^{|r|}, \qquad r = 0, \pm 1, \pm 2, \cdots . \tag{2.3}$$

Thus, in the three parameter process, the parameters k and β describe the marginal distribution of the process and ρ independently describes the correlation structure. Note that since the process is only defined for $0 \leq \rho < 1$ the correlations are non-negative.

2.3. Joint Laplace-Stieltjes transform.

In the stationary process $\{X_n\}$, let $L_{X_n, X_{n-1}}(u,v)$ denote the joint Laplace-Stieltjes transform of the adjacent variables X_n and X_{n-1}. Then we have

$$\begin{aligned} L_{X_n, X_{n-1}}(u,v) &= E[\exp\{-X_n u - X_{n-1} v\}] \\ &= E[\exp\{-B_n X_{n-1} u - Y_n u - X_{n-1} v\}] \\ &= E(e^{-uY}) E\{e^{-(v+Bu)X}\}, \end{aligned} \tag{2.4}$$

where, in the last step, we have dropped the indices n and n-1 because of stationarity and have used the assumed independence of Y_n and X_{n-1} to write the expectation as the product of two expectations.

Now the second term is evaluated in the Lemma of Section 1 and we have

$$L_{X_n,X_{n-1}}(u,v) = \left(\frac{\beta}{\beta+u}\right)^{k\bar{\rho}}\left(\frac{\beta}{\beta+v}\right)^{k\bar{\rho}}\left(\frac{\beta}{\beta+v+u}\right)^{k\rho}$$

$$= \left(\frac{\beta}{\beta+u}\cdot\frac{\beta}{\beta+v}\right)^{k\bar{\rho}}\left(\frac{\beta}{\beta+v+u}\right)^{k\rho}. \qquad (2.5)$$

Since this transform is symmetric in u and v, the joint distribution of X_n and X_{n-1} is symmetric. Also, since the joint distribution of any set of X_n's can be obtained from (2.5) and the marginal Gamma distribution, the Beta-Gamma process is **time-reversible.**

Note, too, that we have directly from the defining equation (2.2) that the regression of X_n on $X_{n-1} = x$ is linear:

$$E(X_n|X_{n-1}=x) = \rho x + (1-\rho)k/\beta = \rho x + (1-\rho)\mu. \qquad (2.6)$$

The time-reversibility of the process shows that

$$E(X_{n-1}|X_n=y) = \rho y + (1-\rho)\mu. \qquad (2.7)$$

2.4. Convergence to a Gaussian AR(1) Process.

As k gets large, the standardized Gamma(k,1) variate $X' = (X-k)/k^{1/2}$ converges to a standardized Gaussian variate. To prove this, consider the pair X_n and X_{n-1} in the BGAR(1) process. The joint characteristic function for the standardized variables X'_n and X'_{n-1} is, using (2.5),

$$\phi_{X'_n,X'_{n-1}}(s,t) = E\{\exp(-isX'_n - itX'_{n-1})\}$$
$$= e^{ik^{1/2}(s+t)}\left[1+isk^{-1/2}\right]^{-k\bar{\rho}}\left[1+itk^{-1/2}\right]^{-k\bar{\rho}}\left\{1+ik^{-1/2}(s+t)\right\}^{-k\rho}.$$

Taking logarithms and expanding in powers of $k^{-1/2}$ gives

$$\psi_{X'_n,X'_{n-1}}(s,t) = ik^{1/2}(s+t) - k\bar{\rho}\{isk^{-1/2}-(is)^2/(2k)+O(k^{-3/2})\}$$
$$- k\bar{\rho}\{itk^{-1/2}-(it)^2/(2k)+O(k^{-3/2})\}$$
$$- k\rho\{ik^{-1/2}(s+t)-(i)^2(s+t)^2/(2k)+O(k^{-3/2})\}$$

and as $k \to \infty$, this converges to

$$\psi_{X_n', X_{n-1}'}(s,t) = -\frac{1}{2}\{s^2+t^2+2\rho st\}. \qquad (2.8)$$

Thus, since the process is Markovian, the BGAR(1) process is equivalent to a Gaussian AR(1) process when k becomes large.

2.5. Additivity and the GAR(1) Process.

Gaver and Lewis (1980) showed that the usual linear stochastic difference equation

$$X_n = \rho X_{n-1} + E_n, \qquad (2.9)$$

will give a process (GAR(1)) with Gamma(k) marginal distributions if E_n is the Gamma innovation variable $GI_n(k,\rho)$ with Laplace-Stieltjes transform $\{(1+\rho s)/(1+s)\}^k$, where $0 \leq \rho < 1$. This variable can be simulated by methods given by Lawrance (1982) and McKenzie (1986).

Note that the GAR(1) process is a linear additive (constant coefficient) process and adding two independent GAR(1) processes with the same β and ρ values, say $\{X_n^*(k_1)\}$ and $\{X_n^{**}(k_2)\}$ gives a new GAR(1) process with shape parameter k_1+k_2 and dependency parameter ρ. This is not true for the BGAR(1) process which is a **random coefficient**, linear additive process. The process obtained by addition is a process with Gamma marginals and correlation structure $\rho(r) = \rho^{|r|}$, but it is not even Markovian. Additional differences between the processes are that while the BGAR(1) process is time-reversible, the GAR(1) process exhibits 'runs down'. In fact, it is degenerate in the sense that the innovation variable GI_{η_k} is zero with probability ρ^k. Thus, we get $X_n/X_{n-1} = \rho$ with probability ρ^k, and ρ can be estimated exactly in long enough series (Gaver and Lewis, 1980). This degenerate behavior is not exhibited by the BGAR(1) process. A method for combining two processes to obtain a broader process is described in Section 5.

2.6. The Conditional Density for X_n, Given $X_{n-1} = y$.

From the definition (2.2), we have that

$$P\{X_n \le x | X_{n-1} = z\} = P\{B_n(k\rho, k\bar{\rho})z + Y_n(k\bar{\rho}, \beta) \le x\}$$

$$= P\{Y_n(k\bar{\rho}, \beta) \le x - B_n(k\rho, k\bar{\rho})z\}.$$

Now, by definition, $Y_n(k\bar{\rho}, \beta)$ is independent of X_{n-1} and of $B_n(k\rho, k\bar{\rho})$. Thus conditioning on B_n and differentiating with respect to x yields the conditional density for X_n, given $X_{n-1} = z$, as

$$f_{X_n|X_{n-1}}(x,y) = \frac{\Gamma(k)}{\Gamma(k\rho)\{\Gamma(k\bar{\rho})\}^2} \times k^{k\bar{\rho}} e^{-kx} \qquad x \ge 0,\ y \ge 0$$

$$\times \int_0^L w^{k\rho-1}(1-w)^{k\bar{\rho}-1}(x-yw)e^{kyw}dw, \tag{2.10}$$

where

$$L = \begin{cases} 1 & \text{if } x \ge y, \\ x/y & \text{if } x < y, \end{cases} \tag{2.11}$$

and, for simplicity, the scale parameter β has been set equal to one.

This density is a continuous function of x and absolutely continuous except where $x = y$. Its utility is in obtaining maximum likelihood estimates of the parameters β (or μ), k and ρ. This is discussed in the next subsection.

2.7. Moment and Maximum Likelihood Estimates in BGAR(1).

There are "natural" moment estimators for the three parameters in the BGAR(1) model, namely the mean, μ, (or β), the shape parameter, k, and the first-order serial correlation coefficient, ρ. From (2.3) and (1.3), these estimates are, from a sample (observed time series) of length n,

$$\tilde{\mu} = \bar{X} = \sum_{i=1}^{n} X_i/n, \tag{2.12}$$

$$k = (X)^2/S^2, \tag{2.13}$$

where S^2 is the sample variance, and

$$\tilde{\rho} = \sum_{i=1}^{n-1} (X_i-\bar{X})(X_{i+1}-\bar{X})/\{(n-1)S^2\}. \tag{2.14}$$

The variance of $\tilde{\mu}$ is $\text{var}(X)[1+2\sum_{r=1}^{n}\{1-(r/n)\}\rho^r]/n$ and $n\text{Var}(\tilde{\mu})$ is asymptotically equal to $\mu^2(1+\rho)/\{(1-\rho)k\}$. More general properties of the estimates $\tilde{k}$ and $\tilde{\rho}$, however, are hard to derive. But, their distributions are independent of the scale parameter μ.

In Table 2.1, we give the simulated standard deviation and bias of the estimates $\tilde{k}$ and $\tilde{\rho}$ for values of $k = 0.25, 0.75, 1.0, 2.0$ and $\rho = 0.25, 0.60, 0.90$ for various values of n. Note that the values of n differ with ρ, since the "equivalent sample size", $n'=n[(1+\rho)/\{(1-\rho)k\}]^{1/2}$, is different. Here n' is the sample size which would be needed, for given ρ, to achieve the same variance for $\tilde{\mu}$ as in the $\rho=0$ (independence) case. The simulation study was performed with the SUPER-SIMTBED program of Lewis, et al. (1985).

Two other properties of the process may be useful in validating the BGAR(1) model from data.

The first is that difference of successive values in the time series, $D_n = X_{n+1}-X_n$, have an ℓ-Laplace distribution (Dewald, 1985). This result comes from (2.4),

$\tilde{k}$

	ρ = 0.25					ρ = 0.60					ρ = 0.90			
	k					k					k			
n	0.25	0.75	1.00	2.00	n	0.25	0.75	1.00	2.00	n	0.25	0.75	1.00	2.00
17	0.224 (+0.173)	0.492 (+0.286)	0.617 (+0.335)	1.122 (+0.512)	40	0.213 (+0.157)	0.433 (+0.272)	0.524 (+0.312)	0.636 (+0.220)	91	0.392 (+0.290)	0.733 (+0.545)	0.895 (+0.667)	1.531 (+1.116)
34	0.143 (+0.097)	0.313 (+0.157)	0.395 (+0.179)	0.698 (+0.271)	80	0.135 (+0.088)	0.283 (+0.146)	0.346 (+0.170)	0.432 (+0.122)	182	0.218 (+0.159)	0.412 (+0.287)	0.506 (+0.346)	0.923 (+0.577)
52	0.113 (+0.070)	0.247 (+0.105)	0.311 (+0.120)	0.541 (+0.182)	120	0.111 (+0.070)	0.230 (+0.107)	0.280 (+0.113)	0.358 (+0.089)	282	0.161 (+0.116)	0.322 (+0.188)	0.391 (+0.223)	0.685 (+0.358)
67	0.099 (+0.054)	0.217 (+0.084)	0.274 (+0.096)	0.481 (+0.146)	160	0.096 (+0.051)	0.205 (+0.082)	0.249 (+0.097)	0.303 (+0.061)	364	0.131 (+0.084)	0.281 (+0.155)	0.348 (+0.202)	0.619 (+0.288)
83	0.092 (+0.046)	0.197 (+0.072)	0.244 (+0.073)	0.425 (+0.106)	200	0.086 (+0.043)	0.175 (+0.063)	0.221 (+0.075)	0.270 (+0.054)	456	0.115 (+0.082)	0.239 (+0.127)	0.301 (+0.138)	0.539 (+0.267)
100	0.084 (+0.040)	0.175 (+0.052)	0.221 (+0.071)	0.387 (+0.091)	240	0.077 (+0.035)	0.159 (+0.052)	0.205 (+0.065	0.253 (+0.046)	545	0.107 (+0.066)	0.228 (+0.114)	0.283 (+0.112)	0.476 (+0.218)
117	0.075 (+0.033)	0.164 (+0.051)	0.207 (+0.060)	0.361 (+0.079)	280	0.071 (+0.034)	0.152 (+0.047)	0.187 (+0.063)	0.230 (+0.042)	636	0.100 (+0.064)	0.215 (+0.097)	0.267 (+0.125)	0.461 (+0.185)
133	0.073 (+0.029)	0.158 (+0.047)	0.191 (+0.057)	0.333 (+0.072)	320	0.066 (+0.029)	0.152 (+0.039)	0.168 (+0.052)	0.217 (+0.028)	727	0.094 (+0.053)	0.194 (+0.089)	0.253 (+0.101)	0.424 (+0.141)

TABLE 2.1.a

Simulation study of the standard deviation and bias of the moment estimator, $\tilde{k}$, of the shape parameter, k, in a Beta-Gamma process for various values of the parameters k and ρ, and sample sizes n. In each cell, the quantity in parenthesis is the estimated bias of $\tilde{k}$ and the top quantity is the estimated standard deviation of $\tilde{k}$.

$\bar{\rho}$

	$\rho = 0.025$					$\rho = 0.60$					$\rho = 0.90$			
	k					k					k			
n	0.25	0.75	1.00	2.00	n	0.25	0.75	1.00	2.00	n	0.25	0.75	1.00	2.00
17	0.238 (-0.148)	0.234 (-0.136)	0.234 (-0.135)	0.231 (-0.127)	40	0.221 (-0.147)	0.184 (-0.120)	0.171 (-0.113)	0.169 (-0.058)	91	0.137 (-0.096)	0.091 (-0.073)	0.084 (-0.071)	0.070 (-0.063)
34	0.208 (-0.094)	0.191 (-0.074)	0.187 (-0.074)	0.175 (-0.068)	80	0.183 (-0.100)	0.141 (-0.071)	0.132 (-0.066)	0.125 (-0.032)	182	0.100 (-0.059)	0.065 (-0.043)	0.059 (-0.040)	0.049 (-0.034)
52	0.187 (-0.072)	0.165 (-0.052)	0.161 (-0.056)	0.151 (-0.048)	120	0.160 (-0.037)	0.120 (-0.050)	0.111 (-0.046)	0.104 (-0.024)	282	0.085 (-0.041)	0.052 (-0.029)	0.046 (-0.028)	0.038 (-0.021)
67	0.174 (-0.060)	0.150 (-0.046)	0.144 (-0.043)	0.133 (-0.038)	160	0.145 (-0.060)	0.105 (-0.037)	0.010 (-0.035)	0.092 (-0.018)	364	0.075 (-0.040)	0.044 (-0.023)	0.041 (-0.022)	0.035 (-0.018)
83	0.161 (-0.053)	0.140 (-0.037)	0.134 (-0.033)	0.124 (-0.027)	200	0.134 (-0.046)	0.098 (-0.035)	0.089 (-0.028)	0.079 (-0.015)	456	0.057 (-0.029)	0.041 (-0.019)	0.035 (-0.015)	0.029 (-0.015)
100	0.156 (-0.043)	0.130 (-0.032)	0.124 (-0.030)	0.113 (-0.023)	240	0.124 (-0.047)	0.091 (-0.026)	0.083 (-0.025)	0.075 (-0.008)	545	0.056 (-0.024)	0.035 (-0.019)	0.032 (-0.013)	0.028 (-0.012)
117	0.149 (-0.038)	0.121 (-0.031)	0.116 (-0.025)	0.104 (-0.021)	280	0.122 (-0.041)	0.089 (-0.025)	0.077 (-0.022)	0.071 (-0.001)	636	0.055 (-0.025)	0.035 (-0.015)	0.031 (-0.011)	0.025 (-0.012)
133	0.141 (-0.033)	0.115 (-0.026)	0.110 (-0.029)	0.098 (-0.014)	320	0.110 (-0.038)	0.082 (-0.022)	0.074 (-0.018)	0.066 (-0.001)	727	0.050 (-0.020)	0.031 (-0.010)	0.028 (-0.011)	0.023 (-0.010)

TABLE 2.1.b

Simulation study of the standard deviation and bias of the moment estimator, $\bar{\rho}$, of the correlation parameter, ρ, in a Beta-Gamma process for various values of the parameters k and ρ, and sample sizes n. In each cell, the quantity in parentheses is the estimated bias of $\bar{\rho}$ and the top quantity is the estimated standard deviation of $\bar{\rho}$.

after converting to characteristic functions and setting $v = -u$. We have the characteristic function

$$\phi_{D_n}(u) = L_{X_n, X_{n-1}}(iu, -iu)$$

$$= \left\{\frac{\beta}{\beta+iu} \cdot \frac{\beta}{\beta-iu}\right\}^{k\bar{\rho}} \cdot \left\{\frac{\beta}{\beta}\right\}^{k\rho}$$

$$= \left\{\frac{\beta^2}{\beta^2+u^2}\right\}^{k\bar{\rho}}.$$

This is the characteristic function of an ℓ-Laplace random variable with $\ell = k\bar{\rho}$; the distribution is symmetric about zero. It goes to a Normal random variable as $\ell \to \infty$, but for $\ell \leq 1$, the density function is not absolutely continuous at zero. In fact, for $\ell \leq 1/2$, the density is infinite at zero. The fact that the difference has median value 0, irrespective of the value of ρ or β, can be useful in validating the model.

Consider, now, ratios $R_n = X_{n+1}/X_n$; from (2.2) this is

$$R_n = B_{n+1}(k\rho, k\bar{\rho}) + Y_{n+1}(k\bar{\rho}, \beta)/X_n.$$

But $Y_{n+1}(k\bar{\rho}, \beta)$ and X_n are independent Gamma variates, so that, if $k>1$,

$$E(R_n) = E\{B_{n+1}(k\rho, k\bar{\rho}) + E\{Y_{n+1}(k\bar{\rho}, \beta)/X_n\}$$

$$= \rho + (k\bar{\rho}/\beta)/\{(k-1)/\beta\} = 1 + \bar{\rho}/(k-1). \qquad (2.15)$$

Higher moments can also be obtained.

These results could be of use in validating the model. Another possibility for validating models is the higher-order residual analysis of Lawrance and Lewis (1986).

Joint-maximum likelihood estimates for k and ρ (and perhaps μ) can be obtained from the conditional density (2.10) and the formula for the joint density of $X_n, X_{n-1}, \cdots, X_1$ is

$$f(x_n;x_{n-1};\cdots;x_1) = f_{X_n|X_{n-1}}(x_n;x_{n-1}) \times f_{X_{n-1}|X_{n-2}}(x_{n-1};x_{n-2}) \times \cdots \times f_{X_1}(x_1), \tag{2.16}$$

where $f_{X_1}(x_1)$ is a Gamma(k,1) density.

Hugus (1982) used (2.16) to obtain joint-maximum likelihood estimates for k and ρ. Three cases are shown in Figure 2.1. Generally, when k is greater than 1, moment estimates of k and ρ are quite efficient, which agrees with results for independent Gamma(k,1) variates (Bartlett and Kendall, 1946). However, when k is less than one, the maximum likelihood estimates become much more efficient than the moment estimators.

The moment estimators, $\tilde{\mu}$, $\tilde{\kappa}$, $\tilde{\rho}$, given at (2.12), (2.13) and (2.14) serve as good starting points for numerical evaluations to find the maximum likelihood estimates, $\hat{\mu}$, $\hat{k}$, $\hat{\rho}$, of k and ρ. Techniques for the numerical integration of (2.10) are given in Hugus (1982).

3. MOVING AVERAGE AND HIGHER ORDER AUTOREGRESSIVE STRUCTURES

The Beta-Gamma transformation can be used to generate dependency structures other than first-order autoregressive structures for Gamma disbributed time series. Several of these structures are given in this section.

3.1. The First-Order Moving Average Process, BGMA(1).

The first-order moving average process, the BGMA(1), is constructed in essentially the same way as the BGAR(1) above. If $\{G_n\}$ is a sequence of i.i.d. Gamma(k,β) random variables and $\{B_n\}$ is an independent sequence of i.i.d. Beta(kα,k$\bar{\alpha}$) random variables, where $\bar{\alpha} = 1-\alpha$, we define the (backward) BGMA(1) process $\{X_n\}$ by

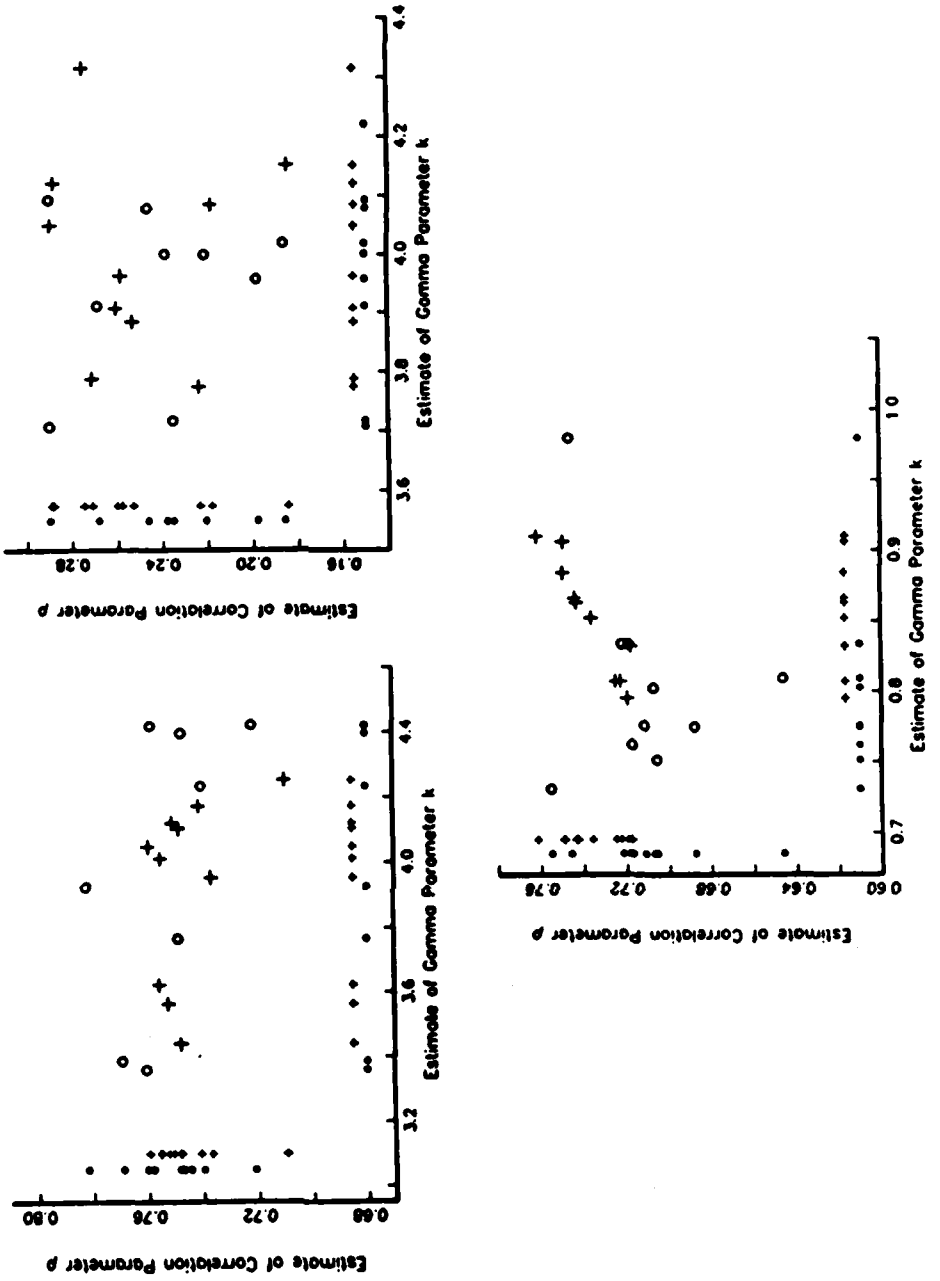

Figure 2.1

Simulations of joint maximum likelihood estimates for k and ρ and joint moment estimates for k and ρ, for three different sets of values of the parameters. Ten replications for each case. Top left figure: ρ=0.75, k=4.0; top right figure: ρ=0.25, k=4.0; bottom figure: ρ=0.75, k=0.75. The symbol (+) refers to estimates of ρ; the symbol (o) refers to estimates of k. Large symbols are joint estimates; small symbols are marginal estimates.

$$X_n = G_n + B_n G_{n-1}, \qquad 0 \le \alpha \le 1. \tag{3.1}$$

Evidently, $\{X_n\}$ is a stationary process and X_n and X_{n-r} are independent if $|r| > 1$. The marginal distribution of X_n may be derived by noting that the right-hand side of (3.1) is the sum of a $G(k,\beta)$ random variable and an independent $G(k\alpha,\beta)$ random variable. Thus, X_n is a $G\{k(1+\alpha),\beta\}$ random variable. This process has the same structure as the usual Gaussian MA(1) process, except that here the coefficient, B_n, is a random variable rather than a constant. An immediate effect of this construction is that the observed and innovation processes, $\{X_n\}$ and $\{G_n\}$, respectively, have different Gamma marginal distributions. This is in contrast to the structure of the EARMA processes (Lawrance and Lewis, 1980), where it was deliberately arranged that they should have the same distribution. However, as we shall see shortly, this disparity in marginal structure has some advantages.

From the viewpoint of modelling, it is more useful to determine the parameters of the innovation process in terms of those of the observed process. For this reason, we reparameterize (3.1) slightly. We consider $\{G_n\}$ to be i.i.d. Gamma$\{k/(1+\alpha),\beta\}$ r.v.s. and $\{B_n\}$ to be i.i.d. Beta$\{k\alpha/(1+\alpha), k\bar{\alpha}/(1+\alpha)\}$ random variables. This yields an observed process $\{X_n\}$ which is Gamma(k,β).

We may note that if we write $\rho = \rho_X(1) = \alpha/(1+\alpha)$, then G_n is Gamma$(k\bar{\rho},\beta)$, the same innovation process as for the BGAR(1) process.

3.2. Autocorrelation Function for the BGMA(1) Process.

The autocorrelation function for the moving average process may be determined directly. Thus, $\mathrm{Cov}(X_n,X_{n-1}) = \mathrm{Cov}(B_n G_{n-1}, G_{n-1}) = E(B)\mathrm{Var}(G)$, and so

$$\rho_X(r) = \begin{cases} \dfrac{\alpha}{1+\alpha}, & |r| = 1, \\ 0, & |r| > 1. \end{cases} \tag{3.2}$$

For $|k| = 1$, the attainable range of correlation is $0 \le \rho_X(1) \le 0.5$, which is the full possible range of positive correlation. This is because, for a

first-order moving average, $|\rho_X(1)| \leq 0.5$; see, e.g. Hugus (1982). The fact that the whole positive range is available is important, because it is in contrast to the EMA models (Lawrance and Lewis, 1977; 1980), where correlation is bounded above by 0.25. The greater flexibility in (3.2) is a result of the innovation and observed processes having different distributions. Since for $k = 1$, the Gamma distribution is an exponential distribution, the BGMA(1) process is then a broader first-order exponential moving average process than the EMA(1) process.

3.3. Joint Distributions.

The bivariate Laplace transform of (X_{n+1}, X_n) can be derived by using (1.8). Thus, again, using the notation $\bar{\alpha}=1-\alpha$, we have

$$\begin{aligned} L(u,v) &= E\{\exp(-uX_{n+1}-vX_n\} = E\{\exp(-uG_{n+1}-uB_{n+1}G_n-vG_n-vB_nG_{n-1}) \\ &= L_G(u)E\{L_G(v+uB)\}L_{BG}(v) \\ &= \left(\frac{\beta}{\beta+u}\right)^{k\bar{\alpha}}\left(\frac{\beta}{\beta+v}\right)^{k\bar{\alpha}}\left(\frac{\beta}{\beta+u+v}\right)^{k\alpha} \end{aligned} \tag{3.3}$$

using the Lemma above. This has exactly the same form as the joint transform, (2.5), of (X_{n+1}, X_n) for the BGAR(1) process with α corresponding to ρ. This, too, corresponds to the behavior in Gaussian processes, where the joint distributions for the autoregressive and the moving average processes have the same form and differ only in their autocorrelation functions. An immediate consequence is that the conditional distribution of X_{n+1} given X_n, and X_n given X_{n+1}, have exactly the same form as for the BGAR(1). We note the somewhat unusual result for a non-Gaussian process that regression is linear in both directions, even though the process is a moving average. In fact, $E(X_{n+1}|X_n=x) = E(X_n|X_{n+1}=x) = \alpha x+k\bar{\alpha}/\beta = \alpha x+\bar{\alpha}E(X)$.

The joint Laplace transform of any finite set of consecutive observations can be obtained by the procedure that yielded (3.3). Thus, the joint transform of $(X_n, X_{n-1}, \cdots, X_{n-r+1})$ is given by

$$L(u_1,u_2,\cdots,u_r) = \left\{\frac{\beta}{(\beta+u_1)}\right\}^{k\bar{\alpha}} \times \prod_{i=2}^{r-1} \left\{\frac{\beta}{(\beta+u_i)}\right\}^{k(1-2\alpha)} \times \left\{\frac{\beta}{(\beta+u_r)}\right\}^{k\bar{\alpha}} \times \prod_{i=2}^{r} \left\{\frac{\beta}{(\beta+u_i+u_{i-1})}\right\}^{k\alpha}. \tag{3.4}$$

Note that this is not the r-dimensional transform for the BGAR(1) process. Equality holds for only r = 2.

One consequence of (3.4) is that, since $L(u_1,u_2,\cdots,u_r) = L(u_r,u_{r-1},\cdots,u_1)$, the process is time-reversible.

The bivariate Gamma distribution whose transform is given at (3.3) is well known (Johnson and Kotz, 1970b, p. 219) and is called by Ghirtis (1967) the **double Gamma** distribution. Since the multivariate form of this bivariate Gamma distribution arises as the individual sums of m independent Gamma variates with a common, independent Gamma variate, it is doubtful that triples, say X_{n+2}, X_{n+1}, X_n, in the BGAR(1) process would have this multivariate distribution. In fact, (3.4) shows that this is not so for the moving average process.

Another result that we can immediately derive from the joint transform is the distribution of the sum of n consecutive observations. This has a particularly simple form for the BGMA(1) process. If $T_n = \sum_{i=1}^{r} X_i$ then $L_T(u) = L(u,u,\cdots,u)$, which, from (3.4), is given by

$$L_T(u) = \left\{\frac{\beta}{(\beta+u)}\right\}^{k\{r-2(r-1)\alpha\}} \left\{\frac{\beta}{(\beta+2u)}\right\}^{k(r-1)\alpha}. \tag{3.5}$$

Further, since $0 \leq \alpha \leq 0.5$, we can rewrite (3.5) in terms of random variables as

$$T_r = G[k\{r-2(r-1)\alpha\},\beta] + 2G\{k(r-1)\alpha,\beta\},$$

where the two Gamma random variables are independent.

3.4. Higher Order Moving Average Processes.

Higher order moving average processes may be constructed by extending the BGMA(1) in an obvious way. Thus, the GBMA(q) is given by

$$X_n = G_n + \sum_{i=1}^{q} B_{n,i} G_{n-i}, \qquad (3.6)$$

where $\{G_n\}$ is a sequence of i.i.d. Gamma$\{k/(1+\sum_1^q \alpha_i), \beta\}$ random variables and $\{B_{n,i}\}$ is an independent sequence of i.i.d. vector random variables with $B_{n,i}$, $i = 1,2,\cdots,q$, being a $\{k\alpha_i/(1+\sum_1^q \alpha_i), k\bar{\alpha}_i/(1+\sum_1^q \alpha_i)\}$ random variable. In this case, $\{X_n\}$ is a stationary Gamma(k,β) process with X_n, X_{n-r} independent when $|r| > q$. The autocorrelation function is given by

$$\rho_X(r) = \begin{cases} \left\{ \alpha_r + \sum_{j=1}^{q-1} \alpha_j \alpha_{j+r} \right\} \Big/ \left\{ 1 + \sum_{j=1}^{q} \alpha_j \right\} & r=1,2,\ldots,q \\ 0 & r > q, \end{cases}$$

which, again, is a close analogue of the usual autocorrelation function for the Gaussian MA(q) process. The major difference is that all the correlations are non-negative.

4. THE MIXED AUTOREGRESSIVE, MOVING AVERAGE PROCESS, BGARMA(1,1)

A more complicated dependence structure in Gamma distributed variables that is the analog of the usual linear ARMA(p,q)-type process is now given.

4.1. Structure of the Mixed Process, BGARMA(1,1).

We can construct an ARMA-type process with a Gamma marginal distribution by combining the two first-order processes we have discussed above. For convenience, we write each in a slightly different form. The moving average component is given by

$$X_n = Y_{n-1} + B_n G_n, \qquad (4.1)$$

where $\{G_n\}$, $\{B_n\}$ are as given in Section 3.1 above, i.e., independent sequences of i.i.d. Gamma$\{k/(1+\alpha), \beta\}$ random variabes and i.i.d.

Beta{$k\alpha/(1+\alpha)$, $k\bar{\alpha}/(1+\alpha)$} random variables, respectively. Notice that B_n is the coefficient of G_n in (4.1), whereas it was associated with G_{n-1} in (3.1). Clearly, this change will make no distributional difference and, as we shall see, renders the parameters of the ARMA model more readily interpretable. The sequence $\{Y_n\}$ is generated from a BGAR(1) process given by

$$Y_n = A_n Y_{n-1} + A_n' G_n. \tag{4.2}$$

The process $\{G_n\}$ is as above and $\{A_n\}$ and $\{A_n'\}$ are independent sequences of i.i.d. Beta{$k\rho/(1+\alpha)$,$k\bar{\rho}/(1+\alpha)$} and i.i.d. Beta{$k\bar{\rho}/(1+\alpha)$,$k\rho/(1+\alpha)$} random variables, respectively. If Y_{n-1} is G{$(k/(1+\alpha),\beta$}, then so also is Y_n and the required stationary process $\{Y_n\}$ results. The product $A_n'G_n$ is, of course, the Gamma{$k\bar{\rho}/(1+\alpha)$,β} random variable used as the innovation process in Section 2.1 above, but is written in this form here to make explicit the dependence on the innovation sequence G_n.

4.2. The Autocorrelation Function of the BGARMA(1,1) Process.

The autocorrelation function of $\{X_n\}$ may be derived in the usual way. Thus, $\mathrm{Cov}(X_n,X_{n-r}) = \mathrm{Cov}(Y_{n-1},Y_{n-r-1}) + \mathrm{Cov}(Y_{n-1},B_{n-r}G_{n-r})$.

Now, $\mathrm{Cov}(Y_{n-1},Y_{n-r-1}) = \rho^r \mathrm{Var}(Y)$, and $\mathrm{Cov}(Y_{n-1},B_{n-r}G_{n-r}) = \alpha\bar{\rho}\rho^{r-1}\mathrm{Var}(G)$, so that we obtain

$$\rho_X(r) = \frac{\rho + \alpha\bar{\rho}}{1+\alpha}\,\rho^{r-1}, \qquad r = 1,2,\cdots. \tag{4.3}$$

This is the form of the autocorrelation function of the ARMA(1,1) process. Note, too, that the choice of the slightly different structure for (4.1) and (4.2) has endowed the two parameters (ρ,α) with physical significance. Choosing $\alpha=0$ effectively sets B_n to zero, so that the process is now simply the BGAR(1) process, and we can see from (4.3) that $\rho_X(r)$ becomes ρ^r, as expected. If we choose $\rho=0$, then Y_n becomes G_n and $\{X_n\}$ is the BGMA(1) process, and (4.3) reduces to $\alpha/(1+\alpha)$, as it should. The reduction of the mixed model to its simpler forms when the parameters vanish is a consequence of the structure we have chosen.

4.3. Joint Distributions in the BGARMA(1,1) Process.

The joint distribution of two consecutive observations of the process $\{X_n\}$ can be derived in the form of the corresponding Laplace transform. Thus,

$$L(u,v) = E\{\exp(-uX_{n+1} - vX_n)$$

$$= E[\exp\{-u(A_nY_{n-1} + A_n'G_n) - uB_{n+1}G_{n+1} - vY_{n-1} - vB_nG_n\}]$$

$$= L_{BG}(u).E\{L_Y(v+uA)\}.E\{L_G(uA' + vB)\}.$$

The first two terms of this product have already been evaluated and we now consider the third. By considering appropriate series expansions, we can evaluate the third term in the form

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-u)^m}{m!} \cdot \frac{(-v)^n}{n!} \cdot \frac{\Gamma(\alpha\Theta+n)\Gamma(\bar{\rho}\Theta+m)\Gamma(\Theta+m+n)\Gamma(\Theta)}{\Gamma(\alpha\Theta)\Gamma(\bar{\rho}\Theta)\Gamma(\Theta+n)\Gamma(\Theta+m)},$$

where $\Theta = k/(1+\alpha)$. Hence, we can show that $L(u,v)$ is given by

$$\left\{\frac{1}{(1+u)(1+v)}\right\}^{\Theta\bar{\rho}} \left\{\frac{1}{1+u+v}\right\}^{\Theta(\alpha+\rho)} {}_2F_1\left(\Theta\rho;\Theta\alpha,\Theta,\frac{-uv}{1+u+v}\right), \qquad (4.5)$$

where ${}_2F_1$ is the Hypergeometric function, defined by

$${}_2F_1(a,b,c,z) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+n)} \cdot \frac{z^n}{n!}.$$

The behavior and properties of this function are detailed in Abramowitz and Stegun (1964, Ch. 15). It is easily verified that when $\rho=0$ or $\alpha=0$, the appropriate forms of $L(u,v)$ result from (4.5). The transform corresponding to higher-dimensional distributions are more difficult to obtain in closed form, although, in principle, series expansions of the form of (4.4) can be derived.

Further, the symmetry of $L(u,v)$ in u and v implies that the conditional distributions of X_{n+1} given X_n and X_n given X_{n+1} are identical. In particular, we can recover the conditional moments from (4.5) and it is found that regression is linear and the conditional variance is quadratic in X_n.

4.4. Higher Order Mixed Processes.

Higher order BGARMA processes can be derived by suitable extensions of the BGARMA(1,1). In particular, it is straightforward to construct a BGARMA(1,q) process by replacing (4.1) by an MA(q) form as in (3.6). Thus,

$$X_n = \sum_{i=0}^{q-1} B_{n-i} G_{n-i} + Y_{n-q},$$

replaces (4.1) and (4.2) is as before. The more general problem of extending to higher order AR forms is more difficult. One way of achieving it, however, is to use mixtures (random indexing). For details, the reader is referred to Lewis (1985).

5. DUAL AND ITERATED GAMMA PROCESSES

The first-order autoregressive Beta-Gamma processes given in equation (2.1) has been shown to be time-reversible. This can be a handicap in modelling phenomena such as water run-offs, which tend to have 'runs down' in their sample paths. This is modelled, as noted, in a defective way by the GAR(1) process of Gaver and Lewis (1980). We, therefore, look for other Gamma processes, possibly with more than one parameter to model dependency structure, which broaden the BGAR(1) process.

The first process to consider is the dual of the BGAR(1). The duality refers to the fact that where in (2.1) we decrease the shape parameter, k, of X_{n-1} by using the Beta-Gamma transform and then bring it up to k by adding an independent Gamma variable, we now increase k in X_{n-1} by adding an independent Gamma variate and then decrease the parameter to k by using the Beta-Gamma transform. Thus, we have

$$X_n(k,\beta) = B(k,q)\{X_{n-1}(k,\beta) + G_n(q,\beta)\}. \qquad (5.1)$$

However, it can be shown that the joint transform of X_n and X_{n-1} is $(1+u)^{-q}(1+v)^{-q}(1+u+v)^{q-k}{}_2F_1\{q,q;k+q;uv/(1+u)(1+v)\}$, so that the process is time-reversible. We thus have nothing new by way of broadening the BGAR(1) process.

Another approach to broadening the BGAR(1) structure is to iterate the process. Thus, in (2.1), the left hand side is a Gamma(k,β) random variable and the procedure in (2.1) can be reapplied. However, a time-reversible process is again obtained. A better way to iterate is to apply the GAR(1) procedure to (2.1) and obtain a combination of the BGAR(1) and GAR(1) processes:

$$X_n(k,\beta) = \gamma\{B_n'(k\rho,k\bar{\rho})X_{n-1} + Y_n(k\bar{\rho},\beta)\} + GI_n(k\gamma,\beta) \qquad (5.2)$$

$$= \gamma B_n'(k\rho,k\bar{\rho})X_{n-1} + \gamma Y_n(k\bar{\rho},\beta) + GI_n(k\gamma,\beta), \qquad (5.3)$$

where $0 \le \gamma \le 1$, $0 \le \rho \le 1$, $\rho = \gamma \ne 1$, $k > 0$ and $\{GI_n(k\gamma,\beta)\}$ is a sequence of i.i.d. Gamma innovation random variables with Laplace-Stieltjes transform $\{(\beta+\gamma s)/(\beta+s)\}^k$, independent of $\{B_n'\}$ and $\{Y_n(k\bar{\rho},\beta)\}$. The condition that ρ and γ do not both equal one is necessary to obtain an ergodic process. Note that (5.2) is different from the combination given in Lawrance and Lewis (1982) and we denote it by GBGAR(1).

Now in (5.2), the case $\gamma=1$ gives the BGAR(1) process, $\gamma=0$ and/or $\rho=0$ gives an i.i.d. sequence $\{X_n\}$ while $\rho=1$ gives the GAR(1) process. Thus, we should find sample path behavior running from time-reversibility to 'runs down' behavior. Also, the process is Markovian and has serial correlation

$$\rho(r) = (\gamma\rho)^{|r|} \qquad r = 0,\pm1,\pm2,\cdots. \qquad (5.4)$$

For the joint Laplace-Stieltjes transform of X_n and X_{n-1}, we have

$$L_{X_n,X_{n-1}}(u,v) = E\{\exp(-uX_n - vX_{n-1})\}$$

$$= E[\exp\{-uYB_n'(k\rho,k\bar{\rho})X_{n-1} - uYY_n(k\bar{\rho},\beta) - uGI_n(kY,\beta) - vX_{n-1}\}]$$

$$= \left\{\frac{\beta+Yu}{\beta+u}\right\}^k \left\{\frac{\beta}{\beta+uY} \cdot \frac{\beta}{\beta+v}\right\}^{k\bar{\rho}} \left\{\frac{\beta}{\beta+v+uY}\right\}^{k\rho}.$$

Thus, it can be seen that the process is not time-reversible unless $Y=1$ (the BGAR(1) case). The regression of X_n on $X_{n-1} = x$ is

$$E(X_n|X_{n-1}=x) = \rho(1)x + \{1-\rho(1)\}E(X),$$

which is linear in x, but it is not the same as $E(X_{n-1}|X_n=x)$.

To separately identify the parameters Y and ρ in this process, one must go beyond second-order properties of the process. This is because the parameters enter into the correlation (5.4) as a product. Higher order residual analyses (Lawrance and Lewis, 1986) and maximum likelihood estimation will be considered elsewhere.

6. NEGATIVE CORRELATION AND NON-LINEAR PROCESSES

All of the processes described above are limited by their serial correlations being non-negative. There are a number of ways of extending the processes to give negative valued serial correlations and we discuss one of them in some detail. All methods involve non-linear functions of, say, X_{n-1} in a first-order autoregressive process. This is necessary because of the non-negativity and lack of symmetry of Gamma disributed variates.

6.1. Antithetic Variates.

Let X be a continuous random variable with c.d.f. $F_X(x)$ and inverse c.d.f. $F_X^{-1}(\alpha)$, $0 < \alpha < 1$. Then the random variable $X^* = F_X^{-1}\{1-F_X(X)\}$ is called the **antithetic** variable to X. For symmetric two-sided random variables centered at zero, $X^* = -X$. For positive valued variables such as Gammas, X^* has the maximum attainable negative correlation for bivariate

Gamma pairs (Moran, 1967). In particular, if k=1 (Exponential), $X^* = -\ln(1-e^{-X})$, but if k≠1 the transformation is difficult to compute.

If X_{n-1} in (2.1) is replaced by X^*_{n-1} in (2.1), then a very non-linear, Markovian first-order autoregressive process is obtained. Serial correlations beyond lag one are difficult to compute.

6.2. Coupling.

Gaver and Lewis (1980) introduced a scheme in the context of the GAR(1) processes for cross-coupling two Gamma processes so that the marginal processes will have negative serial correlations. It is actually easier to implement this scheme for the BGAR(1) process than for the GAR(1) process, because the random, Beta distributed coefficients are continuous. We do not pursue this here.

6.2. Inverse Processes.

A direct scheme for obtaining negative correlation in a Gamma process is now given. It is a generalization of a scheme given by Lewis (1983) to generate negatively correlated bivariate Gamma pairs. Its utility lies in the fact that the sequence can be generated with nothing but i.i.d. Gamma variates, no numerical inversions of inverse distribution functions are required.

Thus, let $B_n(k;q-k)$, for $q > k$, be a sequence of independent Beta$(k;q-k)$ variates, independent of $G'_n(q)$ and $G''_n(k+q)$, which are independent sequences of independent Gamma variates, $n=1,2,\cdots$. Also, let $X_0(k)$ be a Gamma(k) variate, where $k > 0$. The idea is that we want $X_n(k)$ to be small when $X_{n-1}(k)$ is large, while retaining the Gamma(k) marginal structure of the process. We have

$$X_n(k) = B_n(k,q-k)\,\frac{G'_n(q)}{X_{n-1}(k)+G'_n(q)}\,G''(k+q) \qquad q \geq k,\ n = 1,2,\cdots. \qquad (6.1)$$

Note that the ratio is Beta$(q;k)$ and, by the Beta-Gamma transform (1.7), the product of this ratio with the independent Gamma$(k+q)$ variable is

a Gamma(q) variate. The multiplier $B_n(k;q-k)$ reduces the shape parameter from q to k.

To obtain $\rho(1)$, the correlation between $X_n(k)$ and $X_{n-1}(k)$, we need

$$E\{X_n(k)X_{n-1}(k)\} = E\{B_n(k;q-k)\}E\left\{\frac{G_n'(q)X_{n-1}(k)}{X_{n-1}(k) + G_n'(q)}\right\}E\{G''(k+q)\}$$

$$= \frac{k}{q}\,\frac{(k+q)}{\beta}\,E\left\{\frac{G_n'(q)X_{n-1}(k)}{X_{n-1}(k) + G_n'(q)}\right\}. \qquad (6.2)$$

To evaluate the remaining expectation in (6.2), we use the fact that in an expression such as $X_{n-1}(k)/\{X_{n-1}(k) + G_n'(q)\}$, the denominator is independent of the ratio. Then we have, after some manipulation, that

$$E\left\{\frac{G'(q)X_{n-1}(k)}{X_{n-1}(k) + G_n'(q)}\right\} = \frac{kq}{\beta(k+q+1)}. \qquad (6.3)$$

Combining (6.2) and (6.3), we get the suprisingly simple result that

$$\mathrm{Corr}(X_n,X_{n-1}) = \rho(1) = -\frac{k}{k+q+1} \qquad q > k. \qquad (6.4)$$

This correlation is always negative and if k = 1 (the Exponential case), $\rho(1)$ has a minimum attainable value of -1/3 when q = k. This is about halfway to the minimum attainable correlation for bivariate Exponential variables of -0.61.

The scheme can be iterated to achieve greater negative serial correlation. In this and the scheme (6.1), serial correlations of higher order are difficult to obtain. However, since the process is Markovian, the decay of the absolute values of the serial correlations is geometrically bounded.

7. AN ANALYSIS OF TIMES BETWEEN FAILURES OF A COMPUTER SYSTEM

Hugus (1982) used the Beta-Gamma process to analyze a long sequence of wind speeds. This sequence is very non-stationary, containing yearly cycles. The model actually used is $\mu(n)G_n^*$, where $\mu(n)$ is a log-linear function of n and G_n^* is a unit-mean BGAR(1) process.

A simpler, stationary series of times between failures of a computer system is analyzed here. Although this data is known to be generated by a branching Poisson process (Lewis, 1964), the Gamma model is much simpler and much more tractable than the branching Poisson process and provides an adequate representation for most purposes. Modelling of these times between failures is important because, for example, they represent times at which requests for service to the computer are made.

In Figure 7.1, we give a Gamma probability plot for the 256 times between failures. The fit appears adequate, but the goodness of fit statistics in the table in Figure 7.1 must be used with caution, since the data is serially correlated and two parameters have been estimated from the data. The parameter k (ALPHA in the table) is estimated as $k = 0.704$. Note the two outliers in the Gamma probability plot. Since these are serially adjacent, they probably represent a lapse in recording of computer failures.

In Figure 7.2, we show a correlation analysis of the data. The decay in correlation from $\tilde{\rho}(1) = 0.353$ to higher lags is consistent with first-order autoregressive correlation structure. The bands in the figure are approximate confidence intervals for each $\tilde{\rho}(k)$ under the assumption the true correlation is zero for lag greater than k. (See Box and Jenkins, 1976, p. 35 for details). Clearly the times between failures are correlated and thus a renewal model, say, for these times between failures would be inadequate.

The partial autocorrelation plot in Figure 7.2 also confirms the first-order autoregressive nature of the data. The last panel in Figure 7.2 shows the autocorrelation function for the estimated residuals, $R_n = X_n - \tilde{\rho}(1)X_{n-1}$; there is no significant correlation in this series.

Finally, in Figure 7.3, we give empirical density functions (kernel density estimates) for the successive differences, $X_n - X_{n-1}$, $n=2,3,\cdots$ and successive ratios X_n / X_{n-1}, $n=2,3\cdots$, which were discussed in Section 2.7.

The differences show a highly symmetric and long-tailed density function which is consistent with an ℓ-Laplace distribution. Note that the median of the differences is estimated to be -9. Given the range of the differences, this is probably not significantly different from the value of zero which would hold for the time-reversible Beta-Gamma process. The ratios, X_n/X_{n-1}, have an estimated mean of 10.812 with estimated standard deviation of $42.432/(255)^{1/2} = 2.66$. Thus, there is nothing here to suggest any inadequacy in the Beta-Gamma model for characterizing the data.

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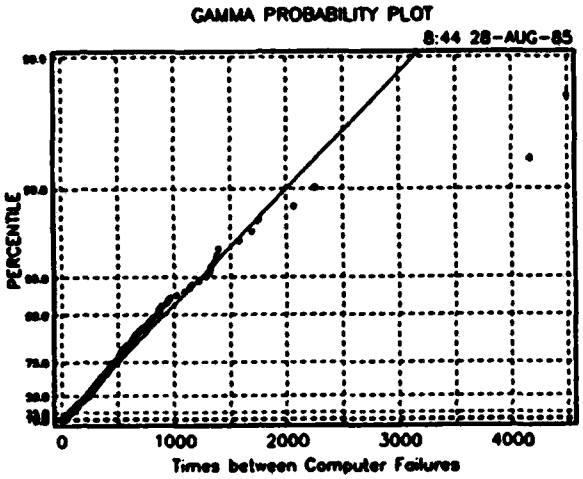

GAMMA DISTRIBUTION

X	: TBFCOMP1
SELECTION	: ALL
LABEL	: TIMES BETWEEN COMPUTER FAILURES
SAMPLE SIZE	: 256
MINIMUM	: 1.000
MAXIMUM	: 4494.000
CENSORING	: NONE
EST. METHOD	: MAXIMUM LIKELIHOOD

	SAMPLE	FITTED
MEAN	: 362.95	362.95
STD DEV	: 515.01	432.58
SKEWNESS	: 4.3088	2.3637
KURTOSIS	: 30.005	11.523

COVARIANCE MATRIX OF PARAMETER ESTIMATES

	ALPHA	BETA
ALPHA	0.0030816	-2.0823
BETA	-2.0823	2085.4

PERCENTILES	SAMPLE	FITTED
5:	7	8.4381
10:	15	17.535
25:	65	67.88
50:	230	211.97
75:	449	498.73
90:	830	909.86
95:	1284	1232.9

GOODNESS OF FIT

CHI-SQUARE :	0.88084
DEG FREED:	1
SIGNIF :	0.4083
KOLM-SMIRN :	0.037550
SIGNIF :	0.88303
CRAMER-V M :	0.055407
SIGNIF :	> .15
ANDER-DARL :	0.45805
SIGNIF :	> .15

KS, AD, AND CV SIGNIF. LEVELS NOT EXACT WITH ESTIMATED PARAMETERS

0.95 CONFIDENCE INTERVALS

PARAMETER	ESTIMATE	LOWER	UPPER
ALPHA	0.70387	0.58084	0.808
BETA	515.37	408.46	622.88

Figure 7.1

Gamma probability plot for times between failures of a computer system

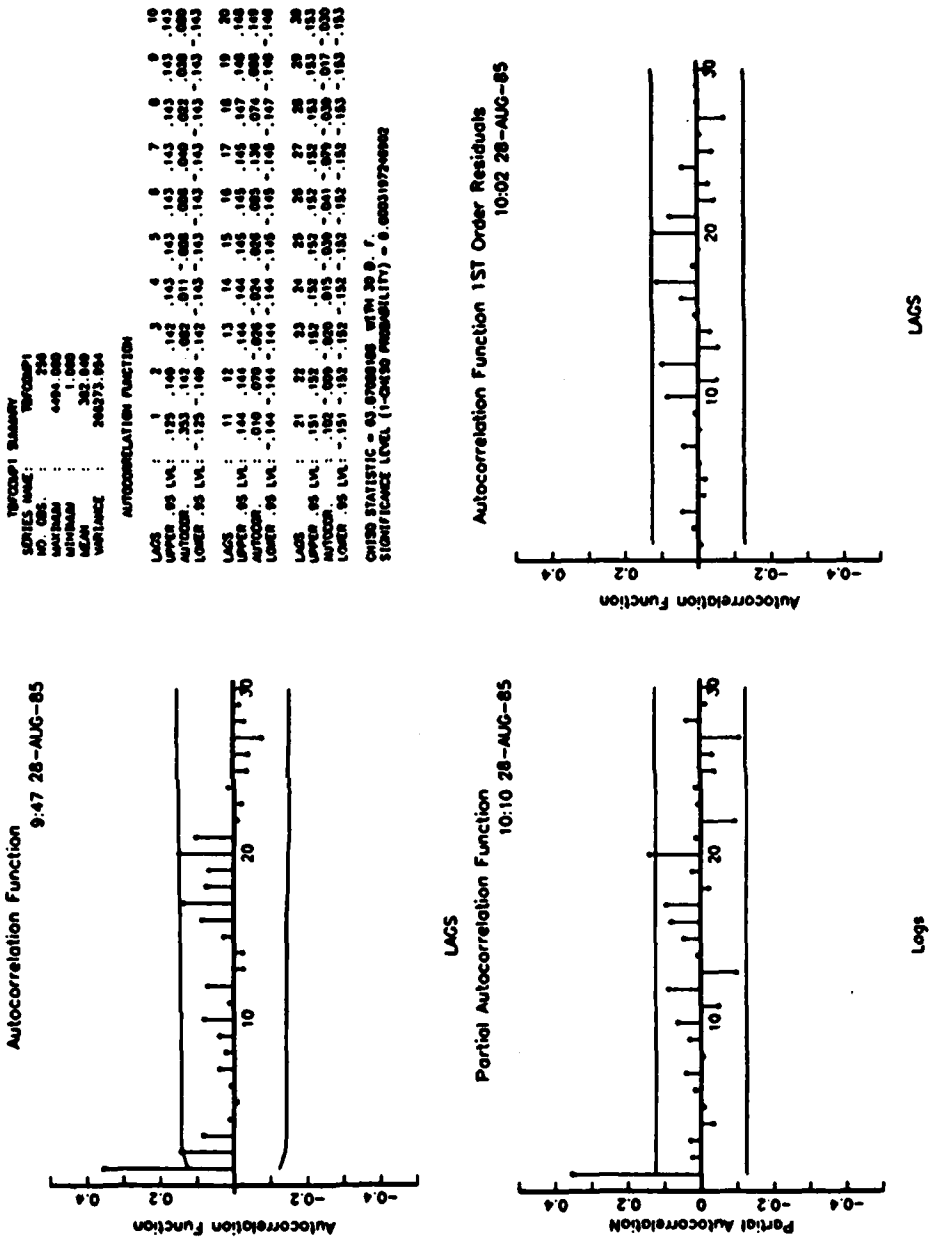

Figure 7.2

Autocorrelation structure of times between failures of a computer system

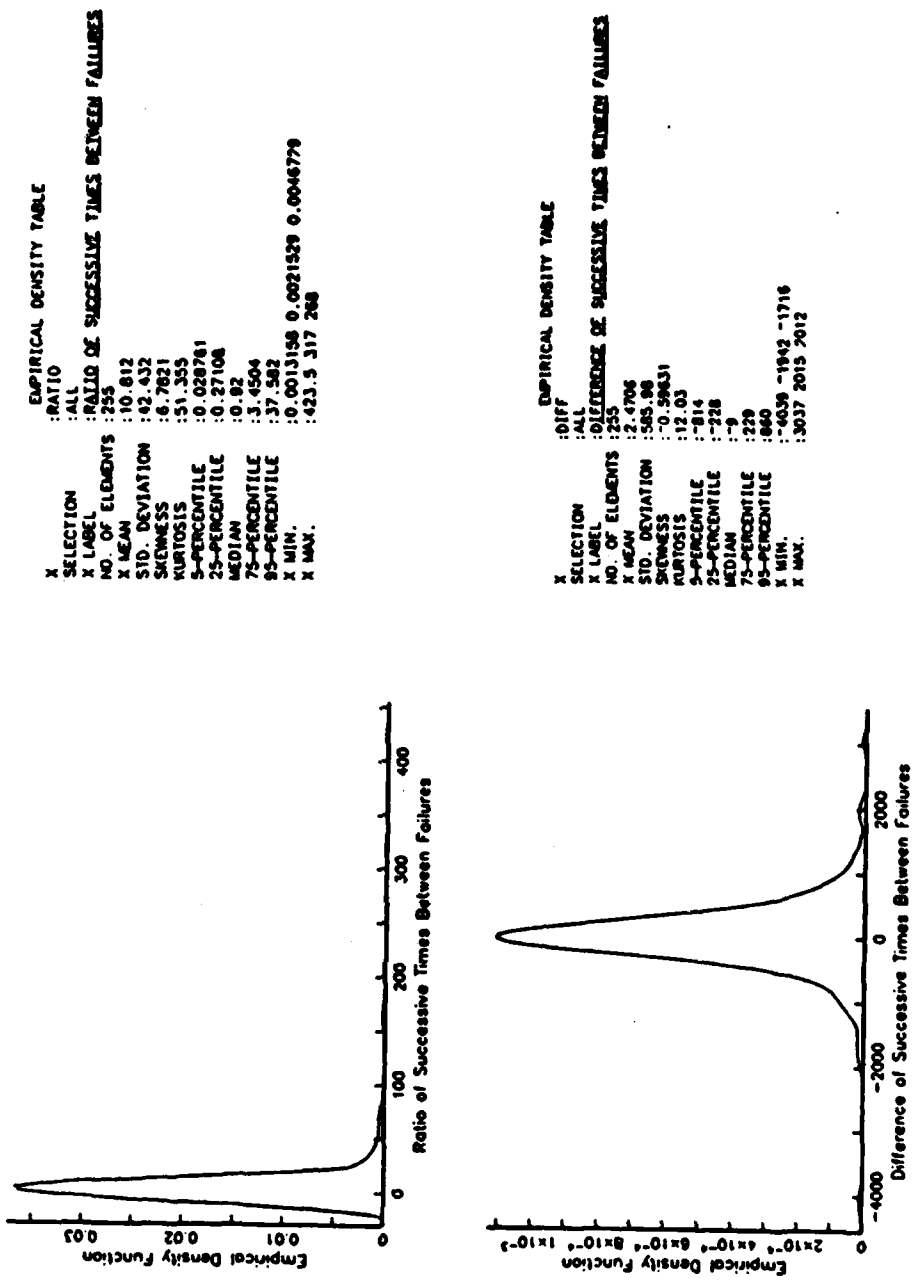

Figure 7.3

Empirical density functions for ratios and differences of successive times between failures

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