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Best Approximation of Signal Amplitude and Delay in a Narrowband Radar

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY
LINCOLN LABORATORY

**BEST APPROXIMATION OF SIGNAL AMPLITUDE AND
DELAY IN A NARROWBAND RADAR**

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ABSTRACT

The range estimation problem is generally solved by assuming a low order target mass center motion description (such as constant velocity or constant acceleration) or by postulating a well defined maneuver. Assumptions are often made that require the receiver signal associated with a well tracked target to have a narrow bandwidth. These assumptions are unreasonable for certain range estimation problems. An approach general enough for use with virtually any pulsed narrowband transmitter waveform and a variety of finite parameter descriptions of time varying target range and cross section is developed. The associated best approximation problem is nonlinear but has a special structure which permits a computable solution in applications of interest involving thousands of unknowns. An Appendix provides an example of estimating a polynomial propagation delay from observations of the radar receiver signal.

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1. INTRODUCTION

One way to extend the capabilities of a radar to targets at greater range or targets of lower cross section (keeping other factors constant) is to base parameter estimates on a larger number of receiver signal samples collected over a longer observation time. Popular observation models might not adequately describe the observed process over the longer observation times. For example, when the signal-to-noise ratio is small, from several seconds to a few minutes of data might be required in order to perform meaningful parameter estimation. In this problem many targets do not appear to have the simple constant velocity or constant acceleration mass center motion proposed in the literature (Refs. [2], [4], [10], [11]). In addition, even small target aspect angle change results in complex dynamics of the observed cross section over the observation time (Ref. [2]). Accurate models for the observed receiver signal over increased time intervals are correspondingly complex.

This complexity severely limits the usefulness of an ordinary "matched filter" receiver. The observation model should be known exactly except for an unknown amplitude scalar and unknown constant time delay to design the optimal matched filter. The model proposed here is considerably more complex.

One possible approach to estimating the parameters of these complex time dependent models is to exploit the model's smoothness. When a sufficient number of observations can be accumulated in a short enough time, linearizations of the analytic models for the processes involved adequately describe the observations. Then a certain notion of optimal estimation is realized by the "Kalman filter" (Refs. [1] p. 36, [6] p. 107).

Unfortunately, as the observation time is increased such linearizations of smooth models must eventually become inadequate (Refs. [5], [14]). If the nonlinearities are structured, then these popular estimators are generalized by various schemes and are said to be "extended" or "adaptive" (e.g., Refs. [1] p. 195, [4], [15]). But in these extended and adaptive forms much of the original optimality of the estimator is obscured (Ref. [19], p. 25).

Any of the optimal estimators can be arranged as nodes in a "tree" or "bank" of estimators by discretely parameterizing the hypotheses under which a particular node of the tree is shown to be optimal (Refs. [10], [15], [17]). These schemes can be implemented in a way that preserves the original notion of optimality. Typically though, a problem arises in choosing the parameter values which define the nodes of the tree. If too many nodes are defined the computation time becomes excessive.

If the wrong nodes are defined then good approximations of the optimal estimates are not obtained.

The notion of minimum norm (or best) approximation is of great importance in establishing the optimality of estimators. Instead of contributing to the proliferation of extended and adaptive techniques, this report examines the problem of best approximation of the time-varying receiver signal from the viewpoint of modern computational methods. The resulting estimator, when interpreted in the least squares sense, is also the maximum likelihood estimator for a Gaussian estimation problem.

In this report best approximation techniques are adapted to solve a general radar surveillance problem. Let r be a vector in a normed vector space H containing the subset S . A best approximation of r on S is defined as a vector \hat{r} in S such that $\|r - \hat{r}\| \leq \|r - y\|$ for all y in S . The set S consists of functions which are called ideal receiver signals in this report. The vector r in the definition is the function whose values are observed over time in the radar receiver, called the observed receiver signal in this report.

The ideal receiver signals are constructed from prior knowledge of the noiseless radar observation process and are parameterized by functions which characterize the attributes of the target. Because of modeling errors, hardware factors, thermal and other noise, an observed receiver signal is

generally not contained in S . Thus the best approximation problem in this context becomes the problem of choosing the functions (characterizing the attributes of the target) which minimize the norm of the error between the observed receiver signal and an ideal receiver signal.

Many schemes involving significantly more assumptions than those made in this report fit within the general framework which is to be described. The purpose of this report is to describe an estimator which can be implemented to provide a very general narrowband radar surveillance capability without precluding the use of other constraints to handle special cases.

First, nonlinear functions are proposed for the ideal receiver signals. Their form is general enough to approximate observations of signals having time-varying range and cross section over extended time intervals. The formulation is kept general so that any of various popular narrowband transmitter waveforms could be utilized. The resulting ideal receiver signals involve time-varying delay and amplitude functions.

The principal problem is to constructively characterize the best approximation of an observed receiver signal on the set S of ideal receiver signals. A typical parameterization of S can involve thousands of variables. Although the best approximation problem is nonlinear, its special structure exposes a linear subproblem which can be solved very efficiently on modern

floating point processors. The solution of the linear subproblem is used to define two other nonlinear problems which have the same solution as the principal best approximation problem. After parameterization, the two nonlinear problems are of significantly lower dimension than the original problem and their solution can be approximated numerically in a number of ways.

Finally, an Appendix provides some details concerning the solution of the ordinary least squares approximation problem. In the example it appears that commercial hardware is adequate for real time implementation of an estimator based on the techniques developed in this report as part of a target tracking system for a very general class of targets.

2. CHARACTERIZATION OF THE SET OF IDEAL RECEIVER SIGNALS

In this section the ideal receiver signals are characterized as complex valued functions of time. These functions define the set from which a best approximation of the observed receiver signal is chosen. By definition, all prior knowledge concerning the radar observation process should be utilized. Then it is natural, using other assumptions, to propose a signal characterization which is different from the one presented here. The purpose of this section is therefore restricted to present just one characterization which motivates the definition of the approximating functions used in the discussion of the best approximation problem in Section 3.

So that the discussion does not become academic, the proposed characterization is designed to be practical for most narrowband radar surveillance missions. Because the radar receiver signal is naturally observed in the time domain, the signals are characterized as time domain functions. The ideal signals are also characterized as deterministic, although a stochastic approach is possible.

Throughout this section, the signals encountered in the radar and the effects of channel/target modulation are generally represented by complex valued functions of continuous time. The usual definitions of pointwise addition and multiplication of such functions apply. The development of this section closely follows Fig. 2-1, which associates a signal amplitude at time t

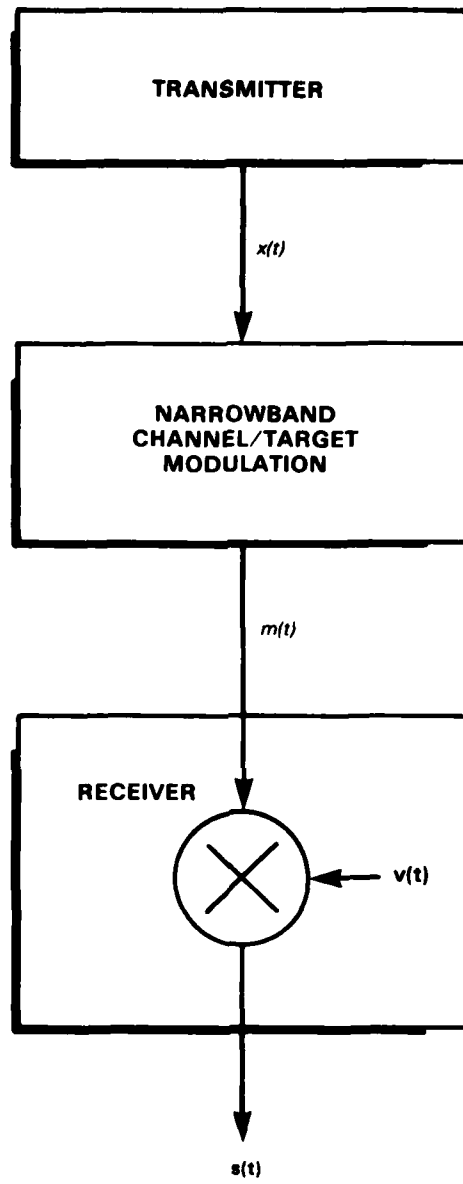


Figure 2-1. The ideal receiver signal model is derived with a narrowband assumption.

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with each of several major steps in the radar observation process.

As usual, each signal is associated with some real disturbance, such as the electric field at a point in space or a voltage in the transmitter/receiver. The signal characterization is derived by supposing that the transmitted signal $x:t \rightarrow x(t)$ is transformed by the channel/target into the modulated signal $m:t \rightarrow m(t)$. In the receiver at each time t , $m(t)$ is multiplied by a locally generated signal value $v(t)$. The resulting signal $s:t \rightarrow s(t)$ has a bandwidth small enough to permit sampling of the continuous-time functions at low sampling rates for subsequent digital processing.

2.1 Narrowband Channel/Target Modulation

First the modulated signal $m:t \rightarrow m(t)$ is constructed. In order to simplify the derivation of the modulated signal a simple linear channel assumption is used. It is assumed without loss of generality that, except for a time delay proportional to the propagation path length, the medium effects are negligible. Attenuation, polarization, refraction or other medium effects can be modeled if desired. The additional parameters might be estimated jointly with other parameters or their values might be available as prior knowledge.

The value of the modulated signal at a given time t is obtained from a linear transformation involving a complex valued

range image function $a: (\theta(t), r) \rightarrow a(\theta(t), r)$. The function $a: (\theta(t), r) \rightarrow a(\theta(t), r)$ depends on the relative real range r and on the value at a time t of the real vector-valued function $\theta: t \rightarrow \theta(t)$ which determines the aspect of the target relative to the radar line-of-sight. Figure 2-2 illustrates the geometry of the problem. Define

$$m(t) \triangleq \int_{R_t - \tau(t)/2 - r/c}^{\tau(t)/2 - r/c} a(\theta(t - \tau(t)/2 - r/c), r) x(t - \tau(t) - 2r/c) dr \quad (2.1-1)$$

where

$$R_t \triangleq \{r | a(\theta(t), r) \neq 0\}. \quad (2.1-2)$$

The function $\tau: t \rightarrow \tau(t)$ is a real valued function of time. The value $\tau(t)$ is usually interpreted as the *round trip propagation time* of an imaginary particle traveling at real velocity c and reaching (at time t) the end of its propagation path. The propagation path extends from a point in the transmitter (where $x(t)$ is defined) to a point on the target which corresponds to one half the path length and back to a point in the receiver (where $m(t)$ is defined). This is an idealized nonrelativistic argument. The set R_t will be called the range extent of the target at time t . After (2.1-1) is accepted as an adequate representation, the remaining development follows readily.

Suppose that $x: t \rightarrow x(t)$ is Fourier transformable, specifically that $X: \omega \rightarrow X(\omega)$ exists with

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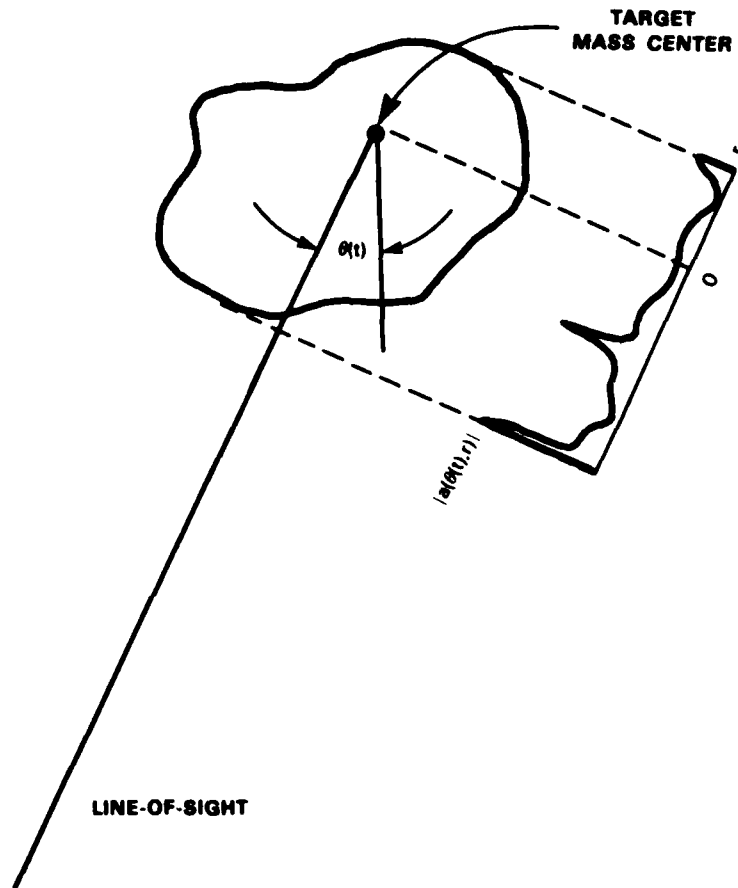


Figure 2-2. The effect of target modulation is obtained from a simple linear transformation.

$$x(t) = \int_B X(\omega) e^{j\omega t} d\omega \quad (2.1-3)$$

where

$$B = \{\omega | X(\omega) \neq 0\}. \quad (2.1-4)$$

The set B will be called the transmitter bandwidth. In practice there will be a bounded set B which is sufficient to adequately approximate $x(t)$ by (2.1-3).

Using (2.1-3), equation (2.1-1) becomes

$$\begin{aligned} m(t) = & \int_{R_{t-\tau(t)/2-r/c}} a(\theta(t - \tau(t)/2 - r/c), r) \\ & \cdot \int_B X(\omega) e^{j\omega(t-\tau(t)-2r/c)} d\omega dr. \end{aligned} \quad (2.1-5)$$

Formally interchanging the order of integration yields

$$\begin{aligned} m(t) = & \int_B X(\omega) e^{j\omega(t-\tau(t))} \\ & \cdot \int_{R_{t-\tau(t)/2-r/c}} a(\theta(t - \tau(t)/2 - r/c), r) e^{-j2\omega r/c} dr d\omega \\ = & \int_B X(\omega) e^{j\omega(t-\tau(t))} A(\theta(t), \omega) d\omega. \end{aligned} \quad (2.1-6)$$

Here the definition

$$A(\theta(t), \omega) \triangleq \int_{R_{t-\tau(t)/2-r/c}} a(\theta(t - \tau(t)/2 - r/c), r) e^{-j2\omega r/c} dr \quad (2.1-7)$$

is made to write (2.1-6) in compact form. For each time t , equation (2.1-7) defines an unnormalized Fourier transformation of a range domain function.

Suppose that $\alpha(t)$ approximates $A(\theta(t), \omega)$ on the set B . That is there exists $\alpha: t \rightarrow \alpha(t)$ such that

$$\alpha(t) = A(\theta(t), \omega) \quad (2.1-8)$$

for ω in B and each time t . For each time t , $A(\theta(t), \omega)$ is approximately constant on B . The assumption (2.1-8) is usually called a narrowband assumption, probably because it is a useful approximation whenever the transmitter bandwidth B and the target range extent R_t are appropriately small.

Using (2.1-8), equation (2.1-6) becomes

$$\begin{aligned} m(t) &= \alpha(t) \int_B X(\omega) e^{j\omega(t-\tau(t))} d\omega \\ &= \alpha(t) x(t - \tau(t)). \end{aligned} \quad (2.1-9)$$

This defines the final form of the modulated signal $m: t \rightarrow m(t)$. The result indicates that the modulated signal is approximately a delayed and amplitude scaled version of the transmitted signal. The simple form for $m(t)$ is a direct result of the narrowband assumption and the linear channel assumption, although other arguments may be used to obtain the result (2.1-9).

Two special cases are of interest where the expression for $m(t)$ can be given exactly in closed form. If R_t collapses to $\{0\}$ (a point target) at each time t then $A(\theta(t), \omega)$ does not depend on ω . Then $\alpha: t \rightarrow \alpha(t)$ exists such that (2.1-8), and hence (2.1-9), become exact regardless of the transmitter bandwidth. Also when B collapses to $\{\omega_0\}$ (continuous wave transmission) then

$$m(t) \propto A(\theta(t), \omega_0) e^{j\omega_0(t-\tau(t))} \quad (2.1-10)$$

regardless of the target range extent.

2.2 The Ideal Receiver Signal

Having constructed the modulated signal $m: t \rightarrow m(t)$, the ideal receiver signal $s: t \rightarrow s(t)$ is defined by

$$s(t) \stackrel{\Delta}{=} m(t)v(t) = \alpha(t)x(t - \tau(t))v(t). \quad (2.2-1)$$

The ideal receiver might have an estimate $\bar{\tau}$ of τ and set

$$v(t) = x^*(t - \bar{\tau}(t)), \quad (2.2-2)$$

the conjugate of $x(t - \bar{\tau}(t))$, to keep the bandwidth of (2.2-1) small. Then

$$s(t) = \alpha(t)x(t - \tau(t))x^*(t - \bar{\tau}(t)). \quad (2.2-3)$$

It is customary in radar technology to suppose that the transmitted signal consists of $2N + 1$ "pulses" each of time duration T . Then the transmitted signal $x: t \rightarrow x(t)$ can be represented by

$$x(t) = \sum_{n=-N}^N p_n(t)z(t - n\Delta). \quad (2.2-4)$$

Here Δ is a real constant, the function $z:t \rightarrow z(t)$ characterizes the signal for a single pulse and

$$p_n(t) = \begin{cases} 1, & -T/2 + n\Delta \leq t \leq T/2 + n\Delta \\ 0, & \text{otherwise.} \end{cases} \quad (2.2-5)$$

It is assumed that $T \ll \Delta$ so that, for example, the sets $\xi_n = \{t | p_n(t) \neq 0\}$ for $-N \leq n \leq N$ are disjoint. That is, the pulses do not overlap.

Equation (2.2-3) becomes

$$s(t) = \alpha(t) \left(\sum_{n=-N}^N p_n(t - \tau(t))z(t - n\Delta - \tau(t)) \right) \\ \cdot \left(\sum_{m=-N}^N p_m(t - \bar{\tau}(t))z^*(t - m\Delta - \bar{\tau}(t)) \right). \quad (2.2-6)$$

If $\bar{\tau}(t) = \tau(t)$, then the assumption $T \ll \Delta$ also implies that $p_n(t - \tau(t))p_m(t - \bar{\tau}(t)) = 0$ when $n \neq m$. With this result, equation (2.2-6) will simplify to

$$s(t) = \alpha(t) \sum_{n=-N}^N q_n(t)z(t - n\Delta - \tau(t)) \\ \cdot z^*(t - n\Delta - \bar{\tau}(t)) \quad (2.2-7)$$

where

$$q_n(t) \triangleq p_n(t - \tau(t))p_n(t - \bar{\tau}(t)). \quad (2.2-8)$$

As a practical matter, the simple form (2.2-7) will always be valid.

The expression (2.2-7) is representative of the ideal narrowband receiver signal. It provides a concrete example for further comments in the Appendix related to implementation of estimators for solving the problems which are discussed in Section 3. It also serves to motivate the structure imposed on the set S of approximating functions for the discussion of the best approximation problem. Using expression (2.2-7), define the functions $s:(\alpha, \tau) \rightarrow s(\alpha, \tau)$ and $s(\alpha, \tau):t \rightarrow s(\alpha, \tau)(t)$ by

$$s(\alpha, \tau)(t) \triangleq \alpha(t) \sum_{n=-N}^N q_n(t)z(t - n\Delta - \tau(t)) \cdot z^*(t - n\Delta - \bar{\tau}(t)) \quad (2.2-9)$$

Then for each scalar 'a' and function $\alpha:t \rightarrow \alpha(t)$ and $\beta:t \rightarrow \beta(t)$,

$$\begin{aligned}
s(a\alpha + \beta, \tau)(t) &= (a\alpha(t) + \beta(t)) \sum_{n=-N}^N q_n(t) \\
&\quad \cdot z(t - n\Delta - \tau(t))z^*(t - n\Delta - \bar{\tau}(t)) \\
&= a\alpha(t) \sum_{n=-N}^N q_n(t)z(t - n\Delta - \tau(t)) \\
&\quad \cdot z^*(t - n\Delta - \bar{\tau}(t)) + \beta(t) \sum_{n=-N}^N q_n(t) \\
&\quad \cdot z(t - n\Delta - \tau(t))z^*(t - n\Delta - \bar{\tau}(t)) \\
&= as(\alpha, \tau)(t) + s(\beta, \tau)(t). \quad (2.2-10)
\end{aligned}$$

That is, the function $s:(\alpha, \tau) \rightarrow s(\alpha, \tau)$ is linear in the first variable. This linearity property is typical of models for the ideal receiver signal and is a key property abstracted to the approximating functions of Section 3.

Consistent with the stated objectives, the expression (2.2-7) is a reasonable model for the ideal receiver signal associated with a large class of targets and radars. The most restrictive assumption utilized in its derivation is probably the narrowband assumption (2.1-8). In practice this restriction constrains the bandwidth occupied by the transmitted signal $x:t \rightarrow x(t)$. Otherwise any form for $x:t \rightarrow x(t)$ can be used, including all of the popular phase coded waveforms utilizing Barker codes (Ref. [20], p. 316) or linear frequency

modulation (Ref. [20], pp. 290, 292). Equation (2.2-7) is general enough to model virtually all targets, including simple point targets (Ref. [20], chapter 10) and, at the other extreme, rotating objects whose dimensions are significant compared to the wavelength of the transmitted waveform. In the Appendix an example of particular parametric forms for $\alpha:t \rightarrow \alpha(t)$, $\tau:t \rightarrow \tau(t)$ and $z:t \rightarrow z(t)$ is given. For now sufficient structure has been imposed to discuss the best approximation problem.

3. BEST APPROXIMATION OF AN OBSERVED RECEIVER SIGNAL

The previous section characterizes the ideal receiver signals defining the set from which a best approximation of the observed receiver signal is chosen. In this section the best approximation problem is defined and discussed. First a few definitions are made to establish the underlying structure for the problem. The definitions are motivated by the properties of the characterization of the ideal receiver signals developed in the previous section but remain general enough to ensure compatibility with many other characterizations which might be proposed.

Let H be a Hilbert space of complex valued functions of the real (time) line with the usual pointwise operations of addition and scalar multiplication. In the space H the norm is denoted by $\|\cdot\|$ and the inner product is denoted by $\langle \cdot, \cdot \rangle$. Let A be a subspace of H and let T be some subset of H . Suppose that there exists a function $s: A \times T \rightarrow H$ which abstracts the relevant properties of the model developed in Section 2. Specifically $s: (\alpha, \tau) \rightarrow s(\alpha, \tau)$ satisfies the following linearity

Property: $s(a\alpha + \beta, \tau) = as(a, \tau) + s(\beta, \tau)$ for all complex scalars a, α and β in A and τ in T . (3.0-1)

This property imposes the required structure for the discussion of the best approximation of an element of H on the set

$$S \triangleq \{s(\alpha, \tau) | \alpha \text{ in } A, \tau \text{ in } T\}. \quad (3.0-2)$$

A subset S_τ of S also plays an important role. The set S_τ is defined for each τ in T by

$$S_\tau \triangleq \{s(\alpha, \tau) | \alpha \text{ in } A\}. \quad (3.0-3)$$

The property (3.0-1) implies that for each τ in T , S_τ is a subspace of H . Note also that

$$S = \bigcup_{\tau \in T} S_\tau. \quad (3.0-4)$$

The element of H which is to be approximated is called the observed receiver signal, while the approximating set S consists of elements called ideal receiver signals.

The best approximation of the observed receiver signal r in H on the set S is the ideal receiver signal \hat{r} in S such that $\|r - \hat{r}\| \leq \|r - y\|$ for all y in S . It is customary to consider the questions (i) when does the best approximation exist, (ii) when is it unique, and (iii) how is it characterized. The first two issues are briefly addressed before considering the third, which is the subject of primary concern to this report. In particular a constructive characterization is desired, which is also readily computable in special cases of interest.

3.1 Existence and Uniqueness of the Best Approximation

The theory of best approximation is well developed when the approximating set is a subspace of H . Although many special cases of nonlinear approximating sets have been considered, it appears that a unified theory is lacking (Ref. [18], p. 359). The most popular nonlinear set in approximation theory is probably the convex set. It is well known that when the approximating set is a closed convex subset of H , then the existence and uniqueness of the best approximation is guaranteed.

Furthermore, for many parametric best approximation problems the existence of a unique best approximation is effectively equivalent to convexity of the approximating set. This happens when the approximating set is naturally compact. To see this consider the result attributed to Efimov and Steckin in Ref. [18], p. 368. This result asserts (in a more general form) that if for every element r in H there exists a unique best approximation of r on a compact set S , then S is convex. It is generally not too restrictive in parametric estimation theory to suppose that S is the continuous image of a compact parameter set. Then the compactness of S is guaranteed and the result of Efimov and Steckin applies. In this case, given the compactness of S , the existence of a unique best approximation to an arbitrary element r in H is equivalent to the convexity of S .

The particular choices of functions $s:(\alpha, \tau) \rightarrow s(\alpha, \tau)$ which arise in the Appendix are constrained by existing radar hardware and physical models. As a result, the approximating set S of functions of the form (2.2-7) is not convex, although it is easily made compact. Thus, the uniqueness of the best approximation cannot be guaranteed for arbitrary r in H . The observation r in H must be structured somehow if the best approximation problem is to have a unique solution.

A common method is to assume that each r is ultimately of the form

$$r = s + n \quad (3.1-1)$$

for some s in S and some n in S^\perp . The set S^\perp is defined by

$$S^\perp = \{x \text{ in } H \mid \langle x, s \rangle = 0 \text{ for all } s \text{ in } S\}. \quad (3.1-2)$$

Thus the observed receiver signal consists of an ideal receiver signal plus an orthogonal error (or noise) function. Now let y be any element of S . It follows from the Pythagorean property that

$$\|r - y\|^2 = \|r - s + s - y\|^2 = \|n\|^2 + \|s - y\|^2. \quad (3.1-3)$$

Because (3.1-2) implies that $\|r - y\| > \|n\|$ whenever $y \neq s$ it may be concluded that condition (3.1-1) implies that s is the unique best approximation of r on S .

There are various devices for imposing the condition (3.1-1). These often involve spaces of stochastic processes or random variables. In addition, other geometries can be used to resolve the existence and uniqueness issues. The specific devices used in a particular problem are, to some extent, a matter of personal preference. The detailed structure of an estimator based on the characterizations of the best approximation which follow will ultimately determine how the existence and uniqueness issues must be resolved.

3.2 Construction of the Best Approximation

Four problems are defined in this section. Each one is either a best approximation problem or is closely related to a best approximation problem. It is assumed that each problem has at least one solution.

The best approximation problem of interest is defined by the

Principal Problem: Given r in H , find a best approximation of r on S , that is, find $\hat{\alpha}$ in A and $\hat{\tau}$ in T such that

$$\|r - s(\hat{\alpha}, \hat{\tau})\| \leq \|r - s(\alpha, \tau)\|$$

for all α in A and τ in T . (3.2-1)

The function $s(\hat{\alpha}, \hat{\tau})$ will be called a solution of the principal problem (3.2-1).

By a well known projection theorem (Ref. [13], page 51) the error $r - s(\hat{\alpha}, \hat{\tau})$ is orthogonal to the subspace $S_{\hat{\tau}}$. In particular

$$\langle r - s(\hat{\alpha}, \hat{\tau}), s(\hat{\alpha}, \hat{\tau}) \rangle = 0. \quad (3.2-2)$$

The function $s:(\alpha, \tau) \rightarrow s(\alpha, \tau)$ of two variables is nonlinear (in the second variable). Otherwise many efficient methods would be available to solve the principal problem. The first step in solving such a nonlinear problem might be to reduce the problem to a finite dimensional one. A reduction in the dimensionality can be accomplished by introducing a finite dimensional parameterization of the sets A and T. A finite dimensional parameterization of the set A is a function $f:P_A \rightarrow A$ from the finite dimensional set P_A onto the set A. Just as $f:P_A \rightarrow A$ denotes a finite dimensional parameterization of the set A, let $g:P_T \rightarrow T$ finitely parameterize the set T.

Then the most obvious numerical solution of the general nonlinear best approximation problem is to attempt numerical minimization of $\|r - s(\alpha, \tau)\|$ on $P_A \times P_T$. Alternately, necessary conditions associated with the optimal solution of the general problem can be formulated, yielding a new problem. If the original problem has a solution, it is contained in the set of solutions to the new problem. In special cases, this new problem obtained from necessary conditions might have a closed form solution, or it might otherwise be easier to solve than the

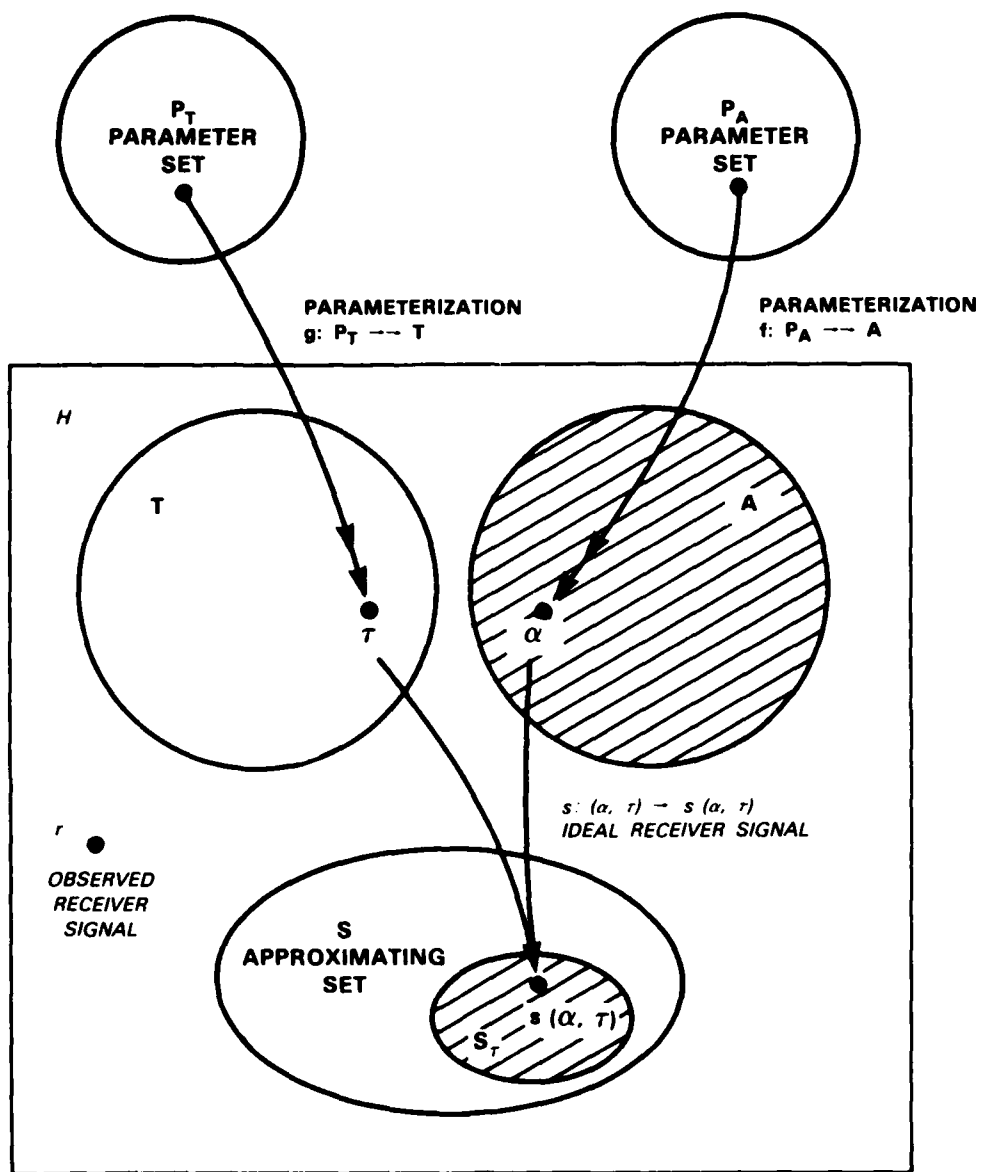


Figure 3-1. The point, set and functional relationships associated with the best approximation problem are illustrated. Cross-hatched sets are subspaces of H .

original problem. But when reformulation of the original best approximation problem leads to another nonlinear problem, numerical optimization is usually more effectively applied to the original problem (Ref. [13], p. 271). In any case, numerical success is elusive when large numbers of variables are involved.

Instead of solving the principal problem as if it were a general nonlinear problem, it is advantageous to exploit the linearity property (3.0-1) to effectively reduce the dimension of the numerical solution of such problems (Refs. [7], [9], [16]). In this case the principal problem contains a

Linear Subproblem: Given r in H and τ in T , find a best approximation of r on S_τ , that is find $\hat{\alpha}_\tau$ in A such that

$$\|r - s(\hat{\alpha}_\tau, \tau)\| \leq \|r - s(\alpha, \tau)\|$$

for all α in A .

(3.2-3)

The function $s(\hat{\alpha}_\tau, \tau)$ will be called the solution of the linear subproblem (3.2-3). In this case, a projection theorem guarantees that

$$\langle r - s(\hat{\alpha}_\tau, \tau), s(\hat{\alpha}_\tau, \tau) \rangle = 0. \quad (3.2-4)$$

The approximating set S_τ is a subspace of H . Best approximation on a subspace is a standard linear problem, and the linear subproblem can be solved efficiently by a number of techniques which are not suitable for the general nonlinear problem. The number of unknowns in the parameterization of A (the dimension of P_A) which can be practically accommodated is also larger than for a general nonlinear problem. The Appendix contains additional remarks concerning a closed form solution of the linear subproblem for a particular example.

In order to use the solution $s(\hat{\alpha}_\tau, \tau)$ of the linear subproblem to construct a solution of the principal problem, define the

Alternate Problem: Given r in H , find $\hat{\tau}$ in T such that

$$\|r - s(\hat{\alpha}_{\hat{\tau}}, \hat{\tau})\| \leq \|r - s(\hat{\alpha}_\tau, \tau)\|$$

for all τ in T . (3.2-5)

Thus $s(\hat{\alpha}_{\hat{\tau}}, \hat{\tau})$ denotes the solution of the alternate problem (3.2-5).

A simple inequality chain

$$\|r - s(\hat{\alpha}_T, \hat{\tau})\| \leq \|r - s(\hat{\alpha}_T, \tau)\| < \|r - s(\alpha, \tau)\| \quad (3.2-6)$$

which holds for each α in A and τ in T implies that the solution of the alternate problem also solves the principal problem. Thus it is possible to construct a solution to the principal problem by numerical optimization over T , utilizing the solution of the linear subproblem. This is extremely important in practical applications, as illustrated by the example in the Appendix, when long observation times are required. In that example the increase in observation time leads directly to an increase in the dimensionality of both the nonlinear principal problem and the linear subproblem, but does not directly increase the numerical dimensionality of the nonlinear alternate problem.

The alternate problem is not the only nonlinear optimization problem over T which also solves the principal problem. Suppose $\bar{\tau}$ in T satisfies

$$\|s(\hat{\alpha}_T, \bar{\tau})\|^2 \geq \|s(\hat{\alpha}_T, \tau)\|^2 \quad (3.2-7)$$

for all τ in T . The orthogonality condition (3.2-4) permits the application of the Pythagorean property to both sides of

$$\|r\|^2 - \|s(\hat{\alpha}_{\tau}^-, \tilde{\tau})\|^2 \leq \|r\|^2 - \|s(\hat{\alpha}_{\tau}, \tau)\|^2 \quad (3.2-8)$$

which follows from (3.2-7). Then

$$\|r - s(\hat{\alpha}_{\tau}^-, \tilde{\tau})\|^2 \leq \|r - s(\hat{\alpha}_{\tau}, \tau)\|^2 \quad (3.2-9)$$

for all τ in T .

Therefore a solution of the alternate problem (and hence the principal problem) can also be found by solving the

Final Problem: Given r in H , find $\hat{\tau}$ in T such that

$$\|s(\hat{\alpha}_{\hat{\tau}}, \hat{\tau})\| \geq \|s(\hat{\alpha}_{\tau}, \tau)\| \quad (3.2-10)$$

for all τ in T .

Thus it is possible to solve the principal problem by each of the following methods.

1. Solve the alternate problem (3.2-5) by optimization over T utilizing the solution of the subproblem (3.2-3).
2. Solve the final problem (3.2-10) by optimization over T utilizing the solution of the subproblem (3.2-3).
3. Solve the principal problem (3.2-1) by optimization over $T \times A$.

Because of the linearity, there are many interesting cases where a closed form solution of the linear subproblem is easily computed. One case is discussed in the Appendix. Then by using methods 1 or 2 numerical optimization over the set P_A of parameters characterizing A may be avoided. Thus computation based on methods 1 or 2 may be considerably faster than computation based on method 3.

It is possible to restrict the assumptions used in this report in such a way that $\|s(\hat{\alpha}_T, \tau)\|$ becomes the ordinary "ambiguity function" of radar technology (Ref. [20]). Thus, $\|s(\hat{\alpha}_T, \tau)\|$ may be regarded as a generalized ambiguity function, and the final problem is analogous to numerical maximization of the ambiguity function. This suggests that when properly implemented, estimators based on the final problem will perform at least as well as the canonical radar receivers associated with the usual ambiguity functions. The estimator will produce the same estimate as many of these receivers in appropriate special cases, but will accommodate significantly more general target assumptions when necessary.

4. SUMMARY

An approach to solving the range estimation problem for a very general class of targets has been presented. The technique can be applied to virtually any pulsed narrowband system. The penalty for such generality is that the approach requires the solution of a nonlinear best approximation problem.

Furthermore, the Appendix presents a least squares application where typically hundreds or thousands of parameters are to be estimated, making a direct numerical solution impractical.

A partitioning of the variables exposes two additional nonlinear problems with solutions which also solve the original best approximation problem. Each new problem contains a linear best approximation subproblem which can be solved in closed form and efficiently implemented on modern floating point processors. This effectively reduces the dimension of the nonlinear problem to the point where, as in an example in the Appendix, a numerical solution is possible.

The Appendix presents some additional details concerning the least squares solution of the problem. Particular parameterizations of the transmitter, propagation delay and amplitude functions yield an example which is particularly suited to the problem of observing targets at low signal-to-noise ratios. In this case the long observation time

required to obtain enough signal energy for meaningful parameter estimation is associated with relatively complex dynamics of the observed target cross section.

APPENDIX

In this appendix the signal characterization of Section 2 is specialized to particular forms typical of a narrowband radar. The best approximation concepts of Section 3 are applied to the least squares approximation of the observed multi-pulse receiver signal associated with the target.

First particular forms for the sets A and T and the functions $x:t \rightarrow x(t)$ and $v:t \rightarrow v(t)$ are chosen. Then a solution of the linear subproblem (3.2-3) is proposed. As in Section 2, an informal approach is used in the Appendix, but for concreteness H can be taken to be $L^2([-1,1])$. It is easy to formally adapt the results to a finite dimensional space E^K by sampling continuous time functions. In E^K a numerical solution is possible. A simple estimate of the floating point computation rate required for the proposed solution suggests that commercial hardware is adequate to implement an estimation scheme based on the techniques described in this report as part of a real-time tracking system.

The set A is defined by assuming that the value of $\alpha:t \rightarrow \alpha(t)$ in (2.1-9) is approximately constant over a time interval corresponding to a transmitter pulse width. Thus A can be defined by

$$A \triangleq \{\alpha: t \rightarrow \alpha(t) \mid \alpha(t) = \sum_{n=-N}^N A_n q_n(t), A_n \text{ in } C\} \quad (A-1)$$

where C is the field of complex numbers and $q_n: t \rightarrow q_n(t)$ is defined by (2.2-8). Throughout the remainder it is assumed that n takes on the integer values between $-N$ and N inclusive.

Definition (A-1) implies the existence of a parameterization from C^{2N+1} onto A .

The propagation time delay due to the mass center motion of the target relative to the radar is often a smooth function of time and so, for the sake of argument, define T by

$$T \triangleq \{\tau: t \rightarrow \tau(t) \mid \tau(t) = \sum_{m=0}^M \tau_m t^m, \tau_m \text{ in } R\} \quad (A-2)$$

where R is the field of real numbers. As with definition (A-1), definition (A-2) implies the existence of a parameterization from R^{M+1} onto T . Constraints arising from external force effects could be included if desired. The additional parameters might be estimated jointly with other parameters or their values might be available as prior knowledge.

Define $z: t \rightarrow z(t)$ by

$$z(t) \triangleq e^{2\pi j(f_n t - b_n t^2/2T)} \quad (A-3)$$

so that the transmitted signal becomes

$$x(t) = \sum_{n=-N}^N p_n(t) e^{2\pi j(f_n(t-n\Delta) - b_n(t-n\Delta)^2/2T)} \quad (A-4)$$

from (2.2-4). The change in instantaneous frequency within each pulse, called the instantaneous bandwidth, is $2\pi b_n$ and the center frequency of the pulse is $2\pi f_n$.

Consistent with (2.2-2) and (A-4), define the receiver's local oscillator signal $v:t \rightarrow v(t)$ by

$$v(t) \triangleq \sum_{n=-N}^N p_n(t - c_n) e^{2\pi j(\bar{f}_n(t-n\Delta - c_n) - \bar{b}_n(t-n\Delta - c_n)^2/2T)}. \quad (A-5)$$

In order that the bandwidth of $s:t \rightarrow s(t)$ as given by (2.2-7) be small, it is sufficient that

$$\bar{f}_n = -f_n,$$

$$\bar{b}_n = -b_n,$$

and

$$c_n = \tau(n\Delta). \quad (A-6)$$

It is perhaps best to compute \bar{f}_n , \bar{b}_n and c_n as a function of $\bar{\tau}:t \rightarrow \bar{\tau}(t)$, the radar system's estimate of $\tau:t \rightarrow \tau(t)$.

The expressions (A-4) and (A-5) describe the most general signals (except for specifying the phase at a given time) that can be synthesized by a typical narrowband digital waveform generator.

Now the receiver signal $s:t \rightarrow s(t)$ analogous to (2.2-7) becomes

$$s(t) = \sum_{n=-N}^N q_n(t) A_n e^{2\pi j \phi(t)} \quad (\text{A-7})$$

where

$$\begin{aligned} \phi(t) = & (-\bar{f}_n c_n - \bar{b}_n c_n^2 / 2T - f_n \tau(t) - b_n \tau^2(t) / 2T) \\ & + (\bar{f}_n + f_n + \bar{b}_n c_n / T + b_n \tau(t) / T)(t - n\Delta) \\ & + (-\bar{b}_n / 2T - b_n / 2T)(t - n\Delta)^2. \end{aligned} \quad (\text{A-8})$$

For the specific forms (A-7) and (A-8) of $s:t \rightarrow s(t)$ the linear subproblem (3.2-3) is easy to solve. There is a set $\{u_n(\tau):t \rightarrow u_n(\tau)(t)\}_n$ of orthogonal functions which span the set S_τ of ideal receiver signals. Each $u_n(\tau):t \rightarrow u_n(\tau)(t)$ is defined by

$$u_n(\tau)(t) \triangleq q_n(t) e^{2\pi j \tilde{\phi}(t)} \quad (\text{A-9})$$

and

$$\begin{aligned} \tilde{\phi}(t) = & (-f_n - b_n \tau(t) / 2T) \tau(t) \\ & + (\bar{f}_n + f_n + \bar{b}_n c_n / T + b_n \tau(t) / T)(t - n\Delta) \\ & + (-\bar{b}_n / 2T - b_n / 2T)(t - n\Delta)^2. \end{aligned} \quad (\text{A-10})$$

The orthogonality follows from the assumption that the product $q_n(t)q_m(t)$ is identically zero for all t when $n \neq m$. From (A-7), $s:t \rightarrow s(t)$ can also be defined by

$$s(t) = \sum_{n=-N}^N \gamma_n u_n(\tau)(t) \quad (A-11)$$

where

$$\gamma_n = A_n e^{2\pi j(-\bar{f}_n c_n - \bar{b}_n c_n^2/2T)}. \quad (A-12)$$

From (A-11), (A-12) and the definition of S_τ it is clear that $\{u_n(\tau)\}_n$ spans S_τ .

Then it follows that $s(\hat{\alpha}_\tau, \tau)$, the solution of the linear subproblem (3.2-3) is defined by

$$s(\hat{\alpha}_\tau, \tau)(t) = \sum_{n=-N}^N \hat{\gamma}_n u_n(\tau)(t) \quad (A-13)$$

where

$$\begin{aligned} \hat{\gamma}_n &= \langle r, u_n(\tau) \rangle / \|u_n(\tau)\|^2 \\ &= \hat{A}_n e^{2\pi j(-\bar{f}_n c_n - \bar{b}_n c_n^2/2T)}. \end{aligned} \quad (A-14)$$

We may formally pass from a space of continuous time functions to E^K by 'sampling' the values of each continuous function at times t_k for $k = 1, 2, \dots, K$. Although in general some caution is required, in this trivial case sampling will not destroy the orthogonality of the functions $u_n(\tau)$. The definition of inner product required in (A-14) becomes

$$\langle r, u_n(\tau) \rangle \triangleq \sum_{k=1}^K r(t_k) u_n^*(\tau)(t_k) \quad (\text{A-15})$$

and, of course,

$$\|u_n(\tau)\|^2 \triangleq \langle u_n(\tau), u_n(\tau) \rangle. \quad (\text{A-16})$$

Therefore, in E^K the closed form solution of the linear subproblem is given by (A-13) and (A-14) using the definitions (A-15) and (A-16).

As discussed in Section 3, $s(\hat{\alpha}_\tau, \tau)$ is required to solve the alternate problem (3.2-5) or the final problem (3.2-10) by numerical optimization over T . Solutions of either of these problems are shown in Section 3 to be solutions of the principal problem (3.2-1). With the definition of inner product and norm given by (A-15) and (A-16), the solution of the principal problem is called the least squares solution because of the particular form assumed by the norm of the error $r - s(\alpha, \tau)$.

In order to estimate the number of real floating point operations per unit time required to approximate the solution of the final problem (3.2-10) define

- s = number of complex receiver samples per pulse, and
- p = number of pulses.

The objective is to maximize $\|s(\hat{\alpha}_\tau, \tau)\|$, or equivalently $\|s(\hat{\alpha}_\tau, \tau)\|^2$, over T .

Suppose that sine and cosine evaluations do not require floating point operations and that $s > 10$ (so that per-sample operations dominate per-pulse operations). Then in E^k the computation of $\|s(\hat{\alpha}_\tau, \tau)\|^2$ requires about $(16 + 2M)sp$ floating point operations, more or less.

The choice of a numerical optimization algorithm to maximize $\|s(\hat{\alpha}_\tau, \tau)\|^2$ over T is beyond the scope of this report. Just for purposes of estimating the number of floating point operations required per unit time, assume that the algorithm is similar to the Algol procedure presented by Brent in Ref. [3]. This algorithm optimizes by successive quadratic approximation. Brent's experience suggests that approximately $3(M + 1)^2$ evaluations of $\|s(\hat{\alpha}_\tau, \tau)\|^2$ are required to compute a quadratic approximation. Because the transmitted signal $x:t \rightarrow x(t)$ and the propagation delay $\tau:t \rightarrow \tau(t)$ are chosen to make $\|s(\hat{\alpha}_\tau, \tau)\|^2$ a well behaved function of τ_m , three successive quadratic approximations should adequately approximate the maximum of $\|s(\hat{\alpha}_\tau, \tau)\|^2$. The total computation cost becomes about $9(16 + 2M)(M + 1)^2sp$ floating point operations and Δp time units are available for real time computation. Thus the required rate R is given by

$$R = \frac{9(16 + 2M)(M + 1)^2sp}{\Delta p} = \frac{9(16 + 2M)(M + 1)^2S}{\Delta} \quad (A-17)$$

floating point operations per unit time.

Suppose a quadratic target delay was to be estimated in a system using 16 samples per pulse and .030 second interpulse periods. Then

$$R = .86(10^6)$$

(A-18)

floating point operations per second. Such rates are easily achieved in modern commercial hardware.

We briefly note that from (A-17) the rate is independent of the value of p , the number of pulses. Thus to reduce the variance of estimated quantities it is possible to increase the value of p (assuming other factors remain unchanged) until the computing system storage is fully utilized. As a practical matter, a large enough increase in the value of p finally requires that the value of M also be increased, indirectly relating the computation rate R to the number of pulses p .

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) The range estimation problem is generally solved by assuming a low order target mass center motion description (such as constant velocity or constant acceleration) or by postulating a well defined maneuver. Assumptions are often made that require the receiver signal associated with a well tracked target to have a narrow bandwidth. These assumptions are unreasonable for certain range estimation problems. An approach general enough for use with virtually any pulsed narrowband transmitter waveform and a variety of finite parameter descriptions of time varying target range and cross section is developed. The associated best approximation problem is nonlinear but has a special structure which permits a computable solution in applications of interest involving thousands of unknowns. An Appendix provides an example of estimating a polynomial propagation delay from observations of the radar receiver signal.								

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