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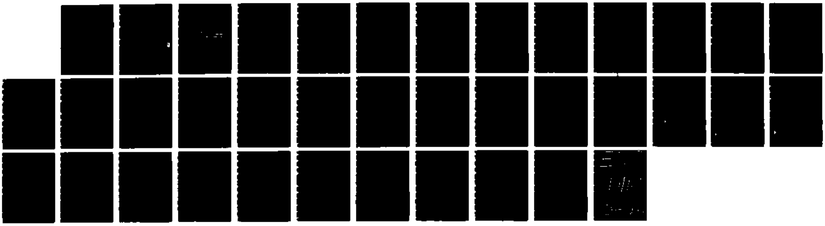

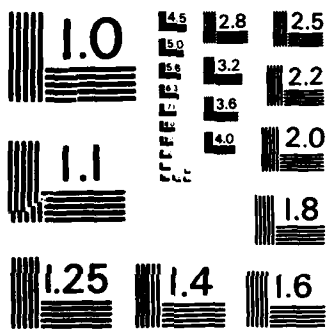

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The p-version of the finite element method is a new, important, computationally efficient, approach to finite element analysis. It is more robust than the conventional h-version and its rate of convergence, for domains with corners and for other singularity problems, is twice that of the h-version.

Hierarchic elements which implement the p-version efficiently have been formulated so as to enforce C^0 or C^1 continuity in the planar case, and so as to enforce C^0 continuity in three dimensions.

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Annual ~~and Final~~ Scientific Report

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p-version of the Finite Element Method

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1. INTRODUCTION

There are now three basic approaches to finite element analysis. In all approaches the domain Ω is divided into simple convex subdomains (usually triangles or rectangles in two dimensions, and tetrahedra or bricks in three dimensions) and over each subdomain the unknown is approximated by a (local) basis function (usually a polynomial of degree $\leq p$). Basis functions are required to meet continuously at boundaries of subdomains in the case of planar or 3 dimensional elasticity, or smoothly in the case of plate bending. The approaches are:

1. The h-version of the finite element method. In this approach the degree p of the approximating polynomial is kept fixed, usually at some low number such as 2 or 3. Convergence is achieved by allowing h, the maximum diameter of the convex subdomains, to go to zero. Estimates for the error in energy have long been known [1, 2, 3]. In all of these estimates p is assumed to be fixed and the error estimate is asymptotic in h, as h goes to zero.
2. The p-version of the finite element method. In this approach the subdivision of the domain Ω is kept fixed but p is allowed to increase until a desired accuracy is attained. The p-version is reminiscent of the Ritz method for solving partial differential equations but with a crucial distinction between the two methods. In the Ritz method a single polynomial approximation is used over the entire domain Ω (Ω, in general, is not convex). In the p-version of the finite element method polynomials are used as approximations over convex subdomains. This critical difference gives the p-version a more rapid rate of convergence than either the Ritz method or the h-version.
3. The h-p version of the finite element method. In this approach both the degree p of the approximating polynomial and the maximum diameter h of the convex subdomains are allowed to change.

The p-version of the finite element method requires families of polynomials of arbitrary degree p defined over different geometric shapes. Polynomials defined over neighboring elements join either continuously (are in C^0) for planar or three dimensional elasticity, and smoothly (are in C^1) for plate bending. In order to implement the p-version efficiently on the computer, these families should have the property that computations performed for an approximation of degree p are re-usable for computations performed for the next approximation of degree p + 1. We call families possessing this property <u>hierarchic</u> families of finite elements.

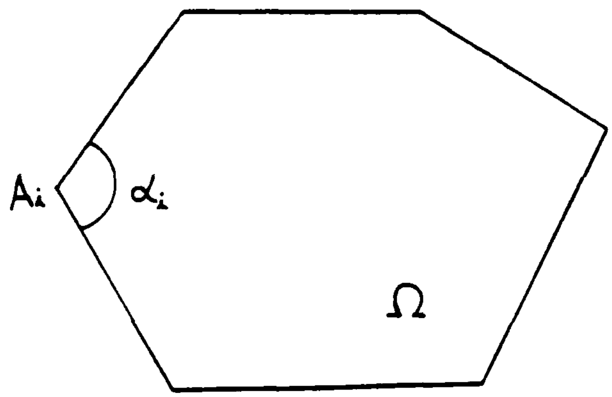
The h-version of the finite element method has been the subject of intensive study since the early 1950's and perhaps even earlier. Study of the p-version of the finite element method, on the other hand, began at Washington University in St. Louis in the early 1970's and led to a more recent study of the h-p version. Research in the p-version (formerly called The Constraint Method) has been supported in part of the Air Force Office of Scientific Research since 1976.

2. RESEARCH ACCOMPLISHMENTS

2.1. Rate of convergence of the p-version

Extensive computational experiments have long furnished empirical evidence that the rate of convergence of the p-version is significantly higher than that of the h-version (See for example [4, 5, 6])

The following two theorems provide a mathematical explanation for the efficiency of the p-version. In both theorems Ω is a bounded polygonal domain in the plane. In Theorem 1, a model problem for the C^0 case is considered, and in Theorem 2, a model problem for the C^1 case is considered. In both cases, the problems are singularity problems, that is the smoothness of the solutions are governed by the local behavior at the vertices A_i of the polygons. Suppose that α_i is a angle at vertex A_i, and that polar coordinates (r_i, ϕ_i) are used at A_i. The solution in the neighborhood of A_i is of the form



$$\rho_i(r_i)\ \theta_i(\phi_i)$$

$$\rho_i(r_i) = r_i^{\gamma_i}\, g_i(|\log r_i|),\quad \theta_i(\phi_i) \text{ is very smooth.}$$

γ_i depends upon α_i.

The domain Ω is assumed to be triangulated in such a way that A_i coincides with a vertex of a triangle. The exact assumptions on g_i and θ_i are given in [5] and [6].

<u>Theorem 1</u> <u>(model problem for C^0 Case)</u>

$$-\Delta u + u = f \quad \text{in } \Omega \tag{1}$$

$$u = 0 \quad \text{on } \partial\Omega$$

Let u be the solution to (1) in the weak sense, and let u_p be the finite element approximation to u, using polynomials of degree p with the triangulation S fixed (i.e. u_p is the solution to (1) using the p-version of the finite element method). If $u \in H^k(\Omega)$, with $k > 1$, then

$$||u - u_p||_{2,\Omega} \leq C\, p^{-\mu+\varepsilon}\, ||u||_k, \quad \mu = \min_i\,(k-1,\ 2\gamma_i), \quad \gamma_i = \frac{\pi}{\alpha_i} \tag{2}$$

where $\varepsilon > 0$ is arbitrary.

<u>Theorem 2</u> <u>(model problem for C^1 case)</u>

$$\Delta^2 w = f \quad \text{in } \Omega \tag{3}$$

$$w = \frac{\partial w}{\partial n} = 0 \quad \text{on } \partial\Omega \quad \text{(clamped edge)}$$

Let w be the solution to (2) in the weak sense, and let w_p be the solution to (3) using the p-version of the finite element method. If $w \in H^k(\Omega)$ with $k > 2$, then

$$||w - w_p||_{2,\Omega} \leq c\, p^{-\mu+\varepsilon}\, ||w||_k, \quad \mu = \min(k-2,\ 2(\gamma_i - 1)) \tag{4}$$

$$\gamma_i(\alpha_i) = \text{smallest positive root of the equation}$$

$$\sin^2(\gamma_i - 1)\alpha_i - (\gamma_i - 1)^2 \sin^2 \alpha_i = 0$$

The error estimates in (2) and (4) can be compared with analogous estimates for u_h and w_h, the solutions by the h-version of the finite element method. Assume, for convenience that in (2) $k - 1 \geq 2\gamma_i$, and in (4) $k - 2 \geq 2(\gamma_i - 1)$, that is convergence is determined by the nature of the singularities at corners. Then the analogous estimates are

$$||u - u_h||_{1,\Omega} \leq ch^{\gamma} ||u||_k, \ \gamma = \min \gamma_1 \qquad (2')$$

$$||w - w_h||_{2,\Omega} \leq ch^{(\gamma-1)} ||w||_k, \ \gamma = \min \gamma_i \ . \qquad (4')$$

If N is the number of degrees of freedom then $p \sim N^{1/2}$, $h \sim N^{-1/2}$ and

$$||u - u_p||_{1,\Omega} = O(N^{-\gamma}), \ ||u - u_h||_{1,\Omega} = O(N^{-\gamma/2})$$

$$||w - w_p||_{2,\Omega} = O(N^{-(\gamma-1)}), \ ||w - w_h||_{2,\Omega} = O(N^{-1/2(\gamma-1)})$$

Therefore, if the criterion used to compare methods is the number of degrees of freedom required to achieve a given error in energy, then <u>the rate of convergence of the p-version is twice that of the h-version</u>. This result provides a rigorous proof for the extensive computational evidence that has been gathered, and explains in part the efficiency of the p-version.

Also, it has been shown in [7, 8] that for the h-p version under the assumptions in Theorems 1 and 2 if optimal mesh refinement (not necessarily quasi uniform) is combined with optimal p-distribution then the estimate becomes (in the C^0 case)

$$||u - u_{p,h}||_{1,\Omega} \leq C e^{-\beta N^{\theta}}$$

where C, β are constants and $\theta \geq 1/3$. Therefore, for the combined h-p version the asymptotic rate of convergence becomes exponential. It is demonstrated in [7] that an exponential rate of convergence is achieved for the p-version in the case of problems with smooth solutions and that problems with stress singularities can be made to behave almost as problems with smooth solutions provided that properly graded meshes are used for the p-version.

Figure 1 illustrates graphically the rates of convergence of the h-, p- and h-p versions.

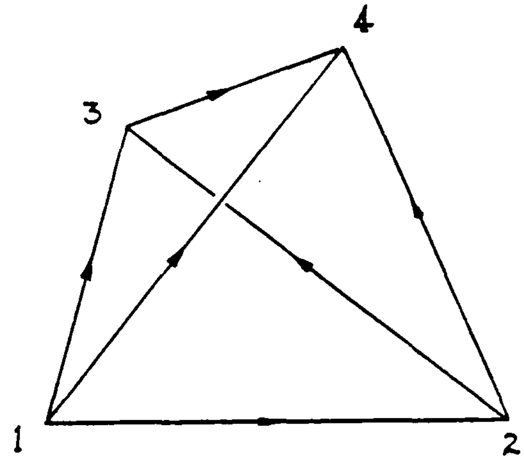
2.2. Hierarchic Families of Solid Finite Elements

In order to implement the p-version efficiently, families of finite elements are needed with the hierarchic property: computations performed for an approximation of order p should be re-usable when raising the order to p + 1. More specificially, the stiffness matrix corresponding to the polynomial approximation of degree p should be a submatrix of the polynomial approximation of degree p + 1. In terms of basis functions, this implies that the basis functions for a pth order approximation should be a subset of the basis functions for a (p + 1)st order approximation.

Hierarchic families for triangles both in the C^0 case and in the C^1 case are described in detail in [9, 10, 11, 12, 13] We now briefly describe some hierarchic families of polynomials for various three dimensional shapes. All of these families are globally in C^0.

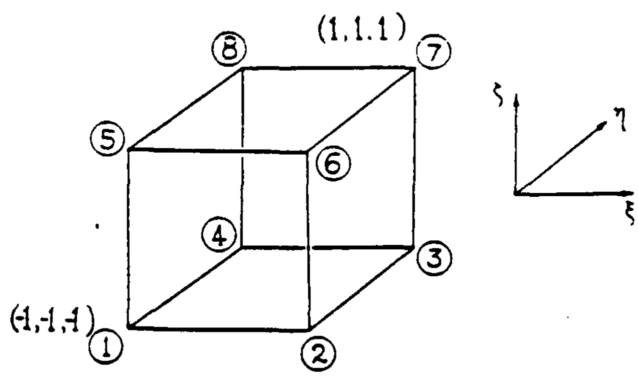
tetrahedron. A hierarchic family for the tetrahedron can be constructed from the hierarchic family for the triangle by using natural coordinates. The linear element is:

	nodal variable	shape function
vertex nodes	u(1)	L_1
	u(2)	L_2
	u(3)	L_3
	u(4)	L_4



The hierarchic family for a tetrahedron up to element of degree 4 is given in detail in [14].

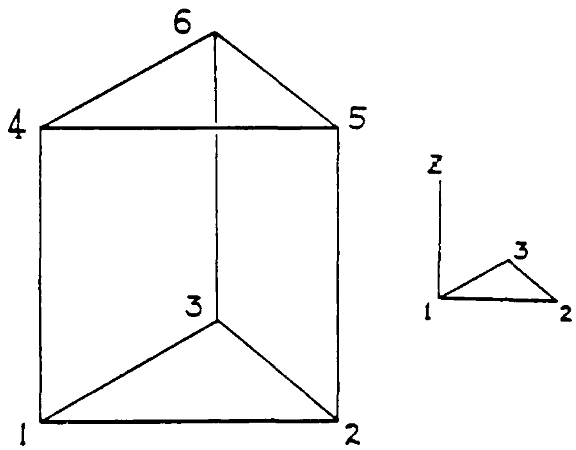
brick. A hierarchic family for the brick can be constructed from the hierarchic family for the rectangle. The linear element is shown below. Higher degree elements odd edge modes, face modes and internal modes. Details are given in [14, 15].



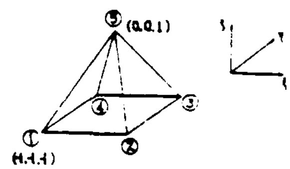
	nodal variable	shape function
vertex nodes	u(1)	$\frac{1}{8}(1-\xi)(1-\eta)(1-\zeta)$
	u(2)	$\frac{1}{8}(1+\xi)(1-\eta)(1-\zeta)$
	u(3)	$\frac{1}{8}(1+\xi)(1+\eta)(1-\zeta)$
	u(4)	$\frac{1}{8}(1-\xi)(1+\eta)(1-\zeta)$
	u(5)	$\frac{1}{8}(1-\xi)(1-\eta)(1+\zeta)$
	u(6)	$\frac{1}{8}(1+\xi)(1-\eta)(1+\zeta)$
	u(7)	$\frac{1}{8}(1+\xi)(1+\eta)(1+\zeta)$
	u(8)	$\frac{1}{8}(1-\xi)(1+\eta)(1+\zeta)$

<u>triangular prism</u> A hierarchic family for the triangular prism can be constructed by combining the hierarchic families for the triangle (in natural coordinates) and for the square (in rectangular coordinates).

Details are given in [14] and [15].

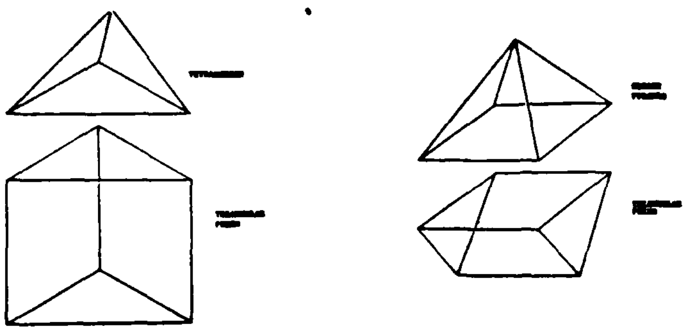


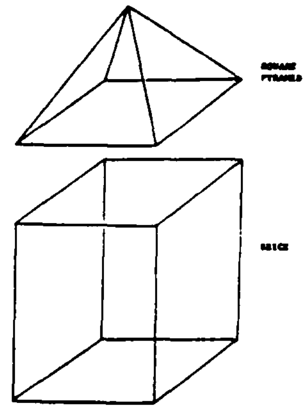
<u>square pyramid</u>. It can be shown [15] that no hierarchic family exists which consists of polynomials alone. However it is possible to supplement certain rational functions in such a way that the element of degree p contains a complete polynomial of degree p and additional rational functions. Furthermore, because of the special form of these rational functions, integration of all shape functions which appear in the elemental stiffness matrix can be performed in closed form. No numerical quadrature is required. See [15] for further details.



<u>Combinations</u>

Hierarchic families of the different shapes shown here have been constructed so that they join together continuously. Thus geometries such as the ones shown below can be modeled using as few elements as possible. In this way the subdivision of the polyhedral domain Ω can be made very coarse and accuracy can be obtained by increasing p, without using added degrees of freedom to describe the geometry. As has been shown (at least for two dimensional problems) the rate of convergence in the p-version is twice that obtained for the h-version. Therefore subdividing





in this way leads to computational efficiency, since it makes maximum use of the p-version. It also requires significantly simpler input.

2.3 Quantities of Special Interest - Extraction Techniques

Often the main purpose of a finite element analysis is to obtain values of a few important quantities with a high accuracy. In structural mechanics, for example, the values of displacements or stresses in a small number of designated areas, or the stress intensity factor at a small number of points is of critical importance for design. Both the h- and p- versions of the finite element method give approximations for these values. However, it is much more efficient to use a post-processing technique which uses weighted averages of values taken directly from the finite element approximation. The post-processor determines these quantities of special interest much more accurately. In particular, when stresses are computed pointwise as derivatives of displacements in the p-version, they may exhibit (some times severe) oscillatory behavior. In the case of the centrally cracked panel shown in Figure 2(h), for example the normal stresses σ_y along the x-axis computed for polynomial orders ranging from 1 to 7 are shown. The oscillatory behavior near the crack tip singularity is evident. Two techniques are being developed for the postprocessing of quantities of special interest. Both techniques are very accurate, yielding approximations that are of the same order of accuracy as the strain energy. (This is the square of the error in energy norm.)

2.3.1 Use of Functional Forms

This approach is based on the idea that the functional forms of the quantities of interest and generally known. In the case of a centrally cracked panel, for example, the displacements u(z), v(z), and the stresses $\sigma_x(z)$, $\tau_{xy}(z)$ and $\sigma_y(z)$ $(z = x + iy)$ are given in terms of two functions.

$$\Phi(z) = \phi_o(z) + \frac{\phi_1(z)}{\sqrt{z}} = \sum_{j=0}^{\infty} b_j z^j + \sum_{j=0}^{\infty} a_j z^{j-1/2}$$

$$\Psi(z) = -\phi_o(z) + \frac{\phi_1(z)}{\sqrt{z}} = -\sum^{\infty} b_j z^j + \sum_{j=0}^{\infty} a_j z^{j-1/2}$$

where

$$\phi_o(z) = \sum_{j=0}^{\infty} b_j z^j \qquad \phi_1(z) = \sum_{j=0}^{\infty} a_j z^j$$

are holomorphic functions. Approximate values for the coefficients a_j, b_j are determined using the displacements $u_p(z)$, $v_p(z)$ computed by the p-version of the finite element method.

This technique was used in [16] to obtain improved values of σ_y along the x-axis in the case of a centrally cracked panel.

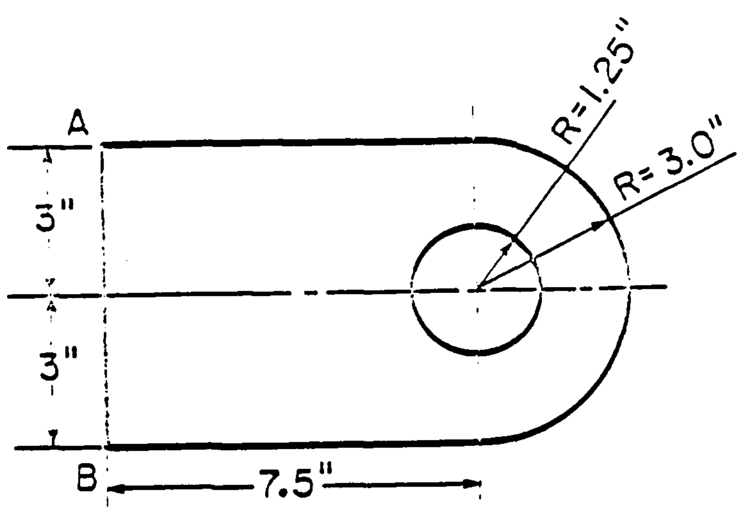
2.3.2. Extraction Techniques

A general form for post processing calculations is given in [17, 18]. Green's theorem and generalized influence functions are used together with smooth cut-off functions and blending functions in order to calculate higher derivatives of the unknown function and also stress intensity factors.

These techniques have been applied to post processing of the u_p and v_p displacement fields obtained from the p-version. Figure 3 shows two meshes for the p-version finite element analysis of a cracked plate. Figures 4 and 5 compare the absolute error in σ_y computed directly from the p-version data with that obtained by the extraction method. The stress itself is infinite at the crack tip (although the strain energy is finite). The extraction method shown in B, however, surpresses the spurious behavior of σ_y away from the crack tip.

2.4 A Sample Problem: Analysis of an Attachment Lug

An example of the use of the p-version for estimating and controlling the error in the approximation is the analysis of an attachment lug, shown below. This problem was considered in detail in [19, 20] and is of considerable interest in the aerospace industry since similar structural details are used in some critical locations of aircraft structures.

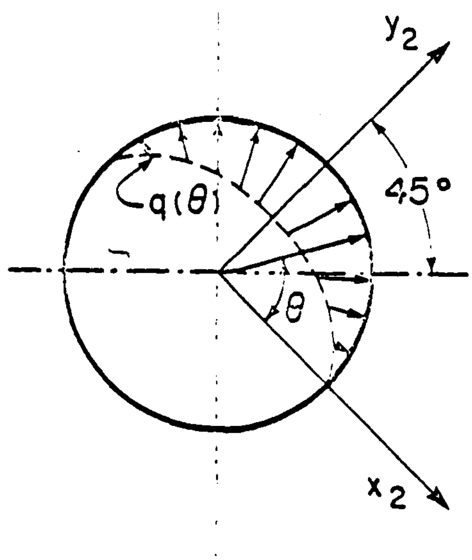


Attachment lug

The dimensions shown are in inches. The thickness of the lug is 0.5 inches (12.7mm). The material is assumed to be isotropic, elastic, with modulus of elasticity of 30,000 ksi (207000 MPa) and Poisson's ratio of 0.3. Plane stress conditions are assumed. The lug is fixed along A-B and is loaded by a tightly fitting circular pin through the hole. The pin imposes a load of 10.0 kips (44.5 kN). The line of action of the applied load passes through the center of the circular hole and is inclined to the horizontal by 45 degrees. This force is to be represented by a sinusoidal normal pressure distribution acting on the inside surface of the hole. The pressure distribution is to act within the range of +90.0 degrees to -90.0 degrees from the direction of the applied load, as shown below. The line force q is therefore:

$$q = Q\sin\theta \qquad 0 \le \theta \le \pi$$

Where Q = 5.093 kips/in (891 kN/m) and θ is measured from the x_2 axis counter-clockwise as shown below. The goal of the computation is to determine the maximum principal stress in the neighborhood of the hole.



Sinusoidal loading

In the p-version of the finite element method the mesh is normally chosen to represent the plane elastic body with a few elements as possible. An important exception is the neighborhood of singular points which should be isolated by one or more layers of small elements. In this problem points A and B are singular points. Under the assumptions of linear elasticity, the stress is infinity at these points. According to our problem statement, points A and B are outside of the region of the interest, however. Otherwise it would have been necessary to define either the goal of the computation or the support conditions differently. The purpose of the small elements A and B is to reduce the spreading of approximation error caused by the singular points. The spreading of error caused by singular points is called 'pollution'. Our mesh design is shown in Fig. 6. The mesh used in [20] when the h-version is employed is shown in Figure 7. In [19] techniques which assess the quality of finite element solutions are performed through the following tests:

(a) Estimate the relative error in energy norm;

(b) Perform an overall equilibrium test;

(c) Perform an element by element equilibrium test;

(d) Perform a convergence test on the magnitude and orientation of principal stress components of node 18;

(e) Perform a convergence test on the location, magnitude and orientation of the maximum principal stress.

Here we give the results for only (a) and (e).

(a) Estimated relative error in energy norm.

The number of degrees of freedom, N, the computed strain energy, $U(\underline{u}_{FE})$, the estimated exact strain energy, the rate of convergence β and the estimated relative error are shown in Table 1.

Table 1

Estimate of error in energy norm based on p-extension

p	N	Computed Strain Energy*	Estimated Exact Strain Energy*	[illegible]	Percent Rel. Error Est'd
1	36	0.173592D-01	-	-	-
2	100	0.276935D-01	-	-	-
3	170	0.283543D-01	0.285992D-01	2.47	9.25
4	266	0.285719D-01	0.287343D-01	1.90	7.52
5	388	0.286491D-01	0.287139D-01	2.08	4.75
6	536	0.286839D-01	0.287277D-01	1.81	3.91
7	710	0.286906D-01	0.286929D-01	4.90	0.89
8	910	0.286928D-01	0.286942D-01	3.74	0.71

* inch-kip units

It is seen that the relative error in energy norm is estimated to be under one percent for p = 7 and 8.

(e) The maximum principal stress

In elastic stress analysis the goal of finite element computation is usually to find the location and magnitude of the maximum principal stress, or the maximum shear stress. In the p-version this is done by computing the stress components on a fine grid and sorting them either according to the maximum principal stress or the maximum shear stress. The grids are imposed on the standard elements and mapped onto the 'real' element by the mapping function of the element.

In the case of the lug problem we chose a 6 x 6 grid per element and searched for the maximum principal stress on elements 3 to 10. The results are shown in Table 2. Convergence is clearly visible. To find a sharper definition of the maximum principal stress we also specified a 10 x 10 grid on element 4. The maximum principal stress on this grid was found to be 24.91 ksi, its orientation 41.1 degrees from the x_1 axis and its location: x_1 = -0.812 in; y_1 = 0.951 in.

Table 2

Location and Magnitude of the Maximum
Principal Stress (6 x 6 grid per element)

P	Element No.	Location x_1	y_1	Principal Stresses (ksi) σ_1	σ_2	ϕ*
1	4	-0.884	0.884	21.67	3.71	43.9
2	4	-0.625	1.083	20.88	1.99	35.0
3	4	-0.625	1.083	23.73	1.84	35.0
4	4	-0.625	1.083	23.64	0.58	35.2
5	4	-0.761	0.992	23.52	-0.58	39.5
6	4	-0.761	0.992	24.46	-0.77	39.2
7	4	-0.761	0.992	24.78	-0.80	38.7
8	4	-0.761	0.992	24.87	-0.04	38.4

* Orientation of first principal axis relative to axis x_1, counterclockwise in degrees.

2.5 Computer Implementations

Our earlier work on research and develonment of the p-version have resulted in several computer implementations. As our research progresses, new procedures and ideas are continually incorporated into these codes making them more efficient. Two of the codes (FIESTA/3D and PROBE) are in commercial use.

2.5.1 COMET-X

COMET-X is an experimental computer code which implements the p-version of the finite element method by using the hierarchic families which have been constructed. COMET-X is maintained by the Center for Computational Mechanics at Washington University. COMET-X can be used as a code to implement the h-version as well simply by fixing the polynomial order p and refining the mesh.

COMET-X currently has the following capabilities:

A. Element types: Stiffeners, triangular elements, triangular elements with one side curved, rectangular elements, solid elements of the shapes described earlier.

B. Types of Analysis: Laplace and Poisson equations, plane elasticity, temperature distribution in 3-dimensions.

C. Special Capabilities: non-uniform p-distribution, elastic fracture mechanics computations in two dimensions, nearly incompressible solids, linear boundary layer problems.

D. Pre- and Post processing capabilities including graphics and visual displays.

The capabilities and usage of COMET-X are described in detail in [21].

2.5.2 FIESTA/3D

FIESTA/3D is a software system for static analysis of solid structures based on the p-version of the finite element method. It is marketed by McDonnell-Douglas Automation Company. Some of the advanced features available on FIESTA/3D are:

- controllability of the quality of the solution
- efficient modeling
- treatment of stress singularities
- a method for surface identification

2.5.3 PROBE

PROBE is an advanced computer implementation of the p-version for 2-dimensional analysis currently under development at NOETIC Technologies in St. Louis, Missouri. Some of PROBE's unique features are:

- automatic error estimators. These estimators provide initial feedback on solution quality by computing element-by-element and edge-by-edge equilibrium checks.
- interactive error estimators. These estimates offer immediate feedback on solution quality by computing action/reaction checks overall equilibrium checks and a convergence trajectory of stresses and strains at user-selected points.

- load tracking, for extracting free-body diagrams
- pinpoint solutions which give specific results anywhere in the model
- Mode I (Symmetric) and Mode II (antisymmetric) Stress Intensity Factors for Fracture Mechanics
- Precise curve definitions which eliminate mesh refinement near cutouts and provide more precise solutions in critical regions.

3. FIGURES

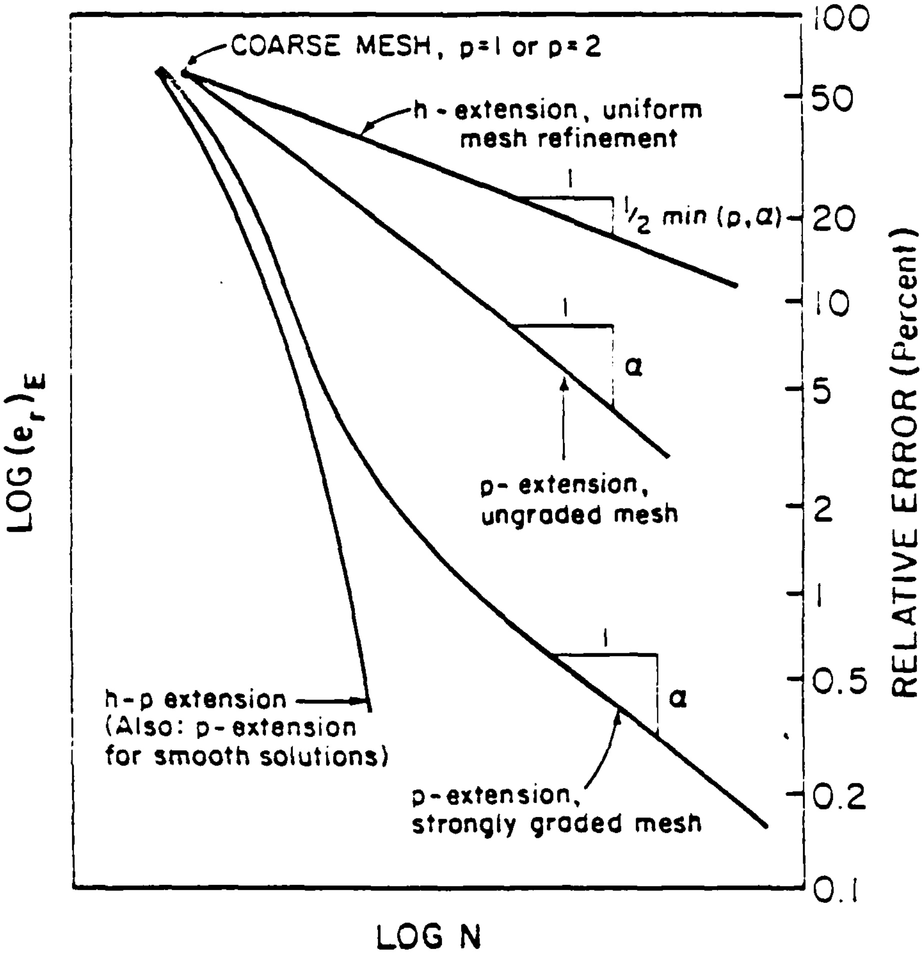


Fig. 1

Performance of the h-, p- and h-p extension processes.
(The relative error values shown are typical for certain engineering problems).

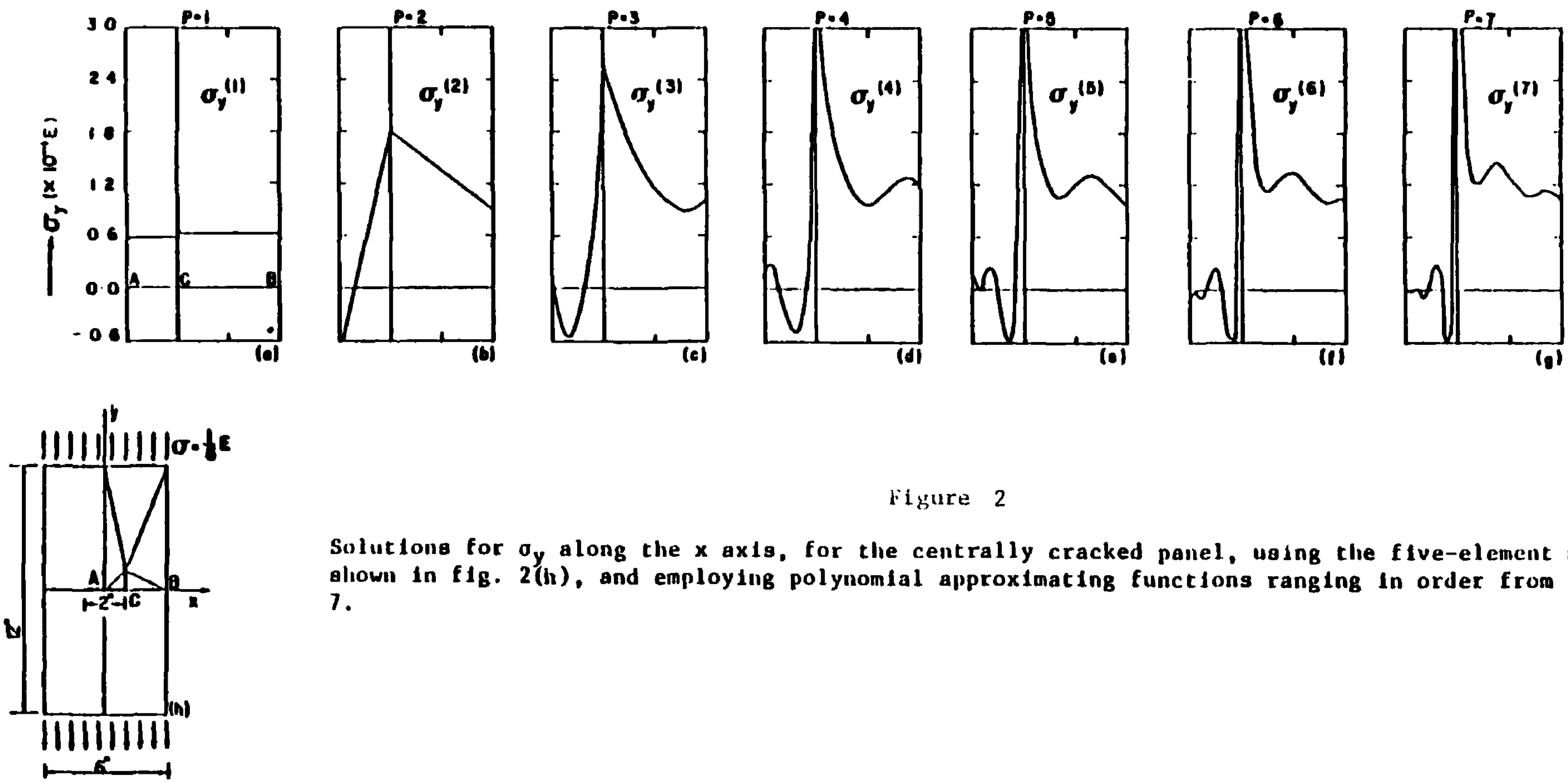


Figure 2

Solutions for σ_y along the x axis, for the centrally cracked panel, using the five-element mesh shown in fig. 2(h), and employing polynomial approximating functions ranging in order from 1 to 7.

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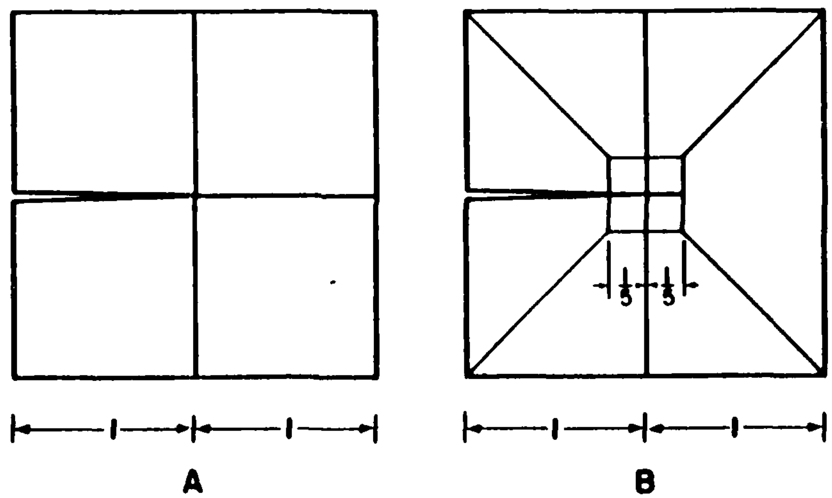
The P-Version Of The Finite Element Method
MODEL PROBLEM: A Cracked Plate



The Meshes For The P-Version, A: Mesh 1, B: Mesh 2

Figure 3

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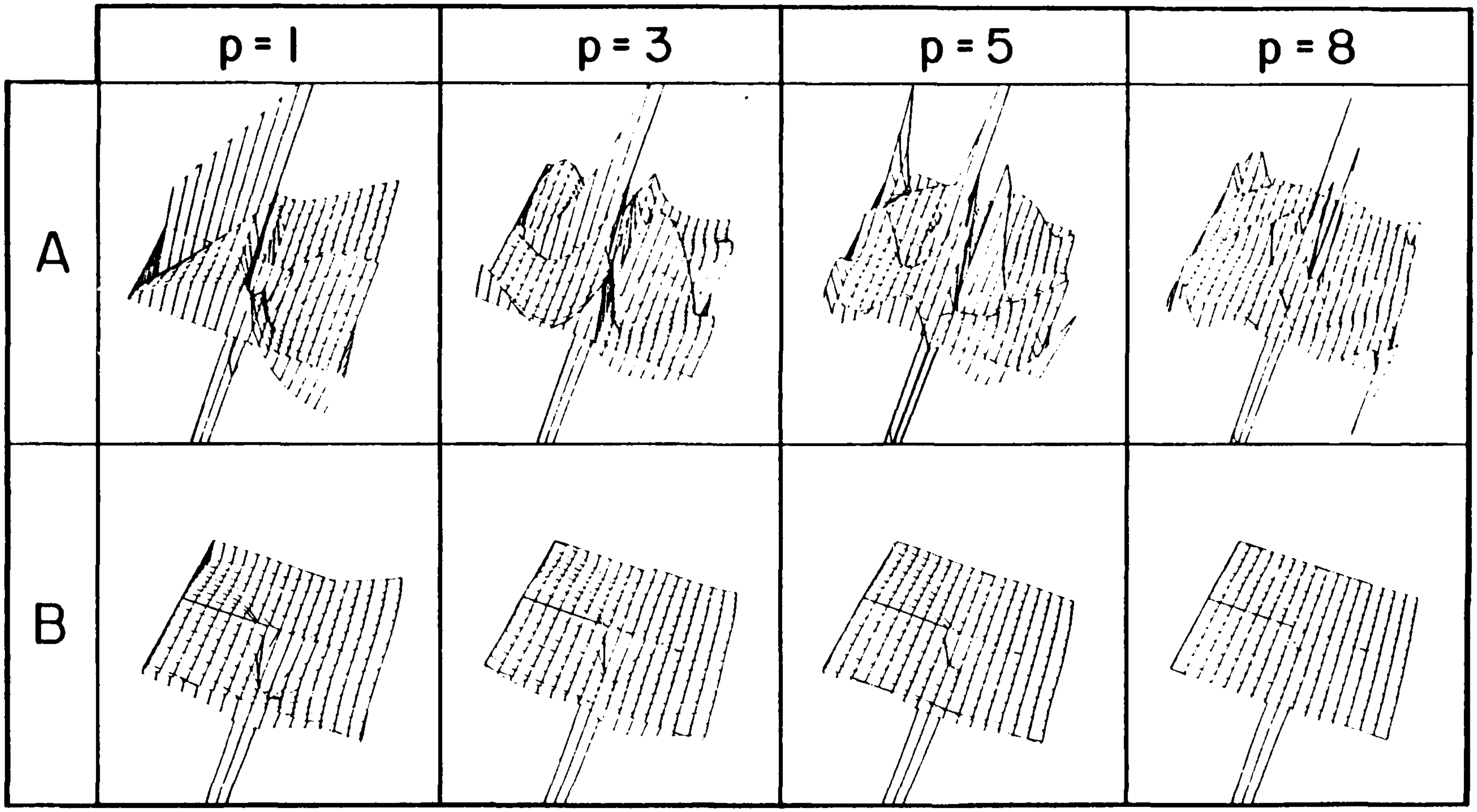
THE P-VERSION OF THE FINITE ELEMENT METHOD

Error Control In Calculation Of Stresses In A Cracked Plate



ABSOLUTE ERROR IN σ_y, MESH I

A: Computed By The Conventional Method

B: Computed By The Extraction Method

Figure 4

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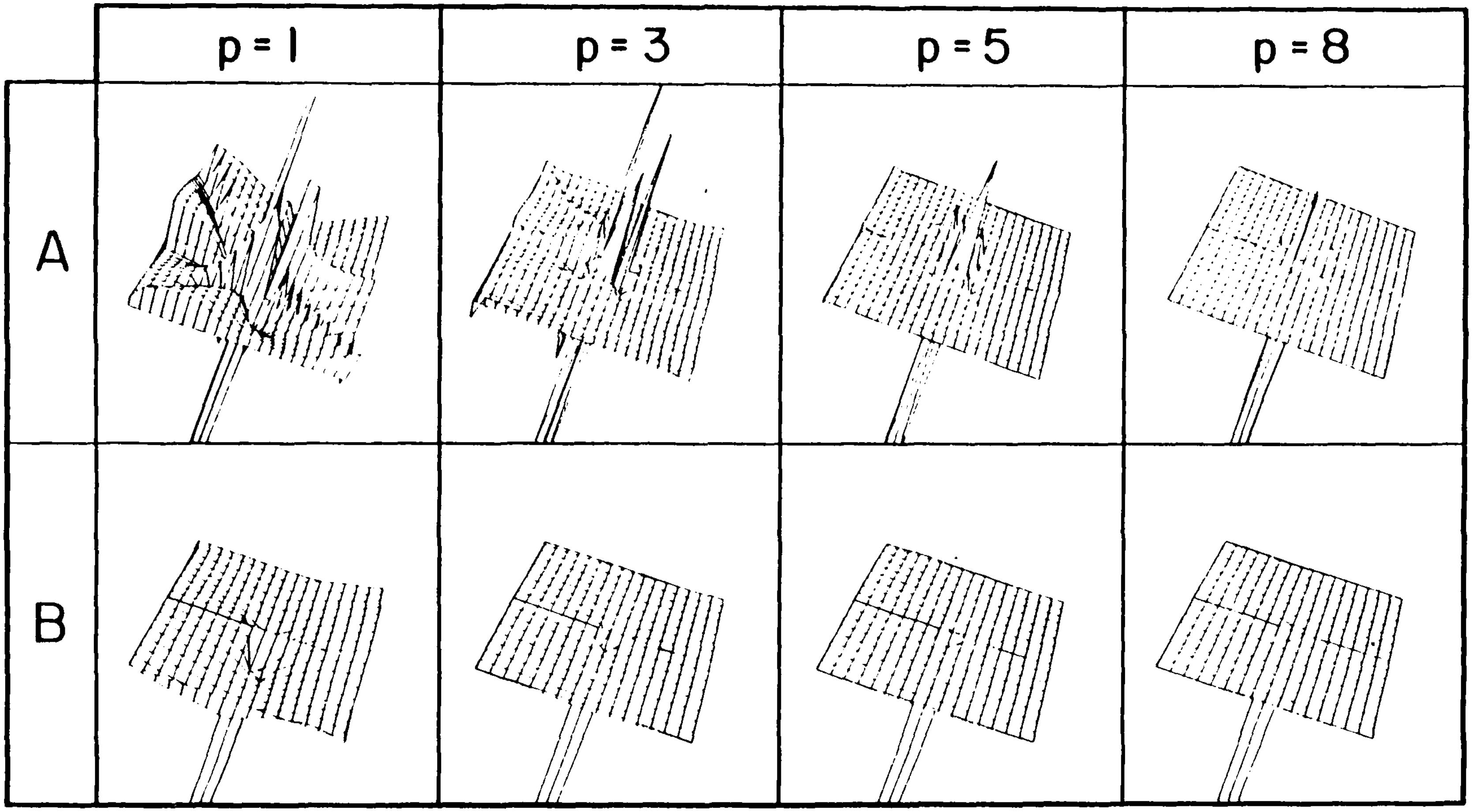
THE P-VERSION OF THE FINITE ELEMENT METHOD

Error Control In Calculation Of Stresses In A Cracked Plate



ABSOLUTE ERROR IN σ_y, MESH 2

A: Computed By The Conventional Method

B: Computed By The Extraction Method

Figure 5

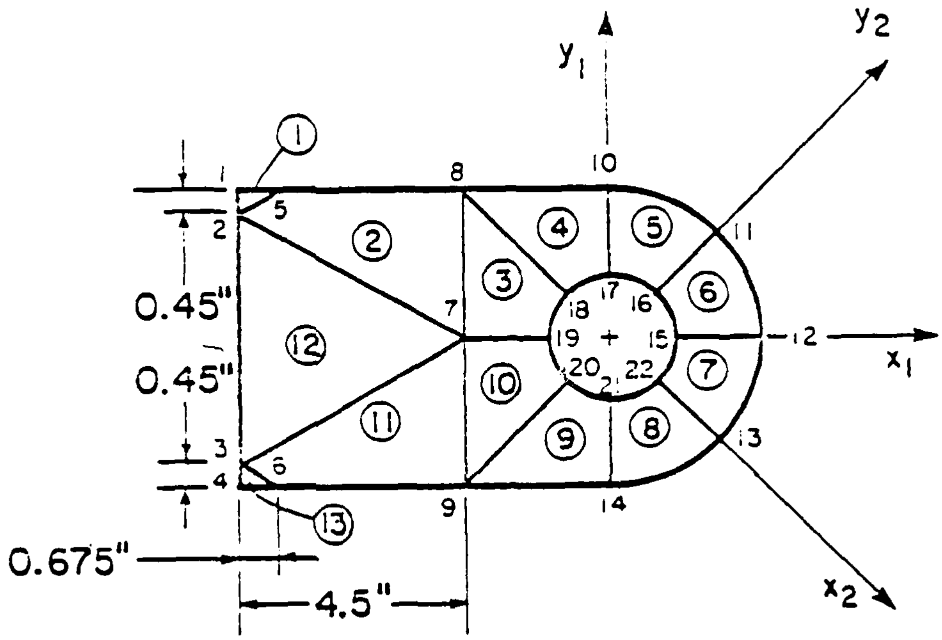


Figure 6
Mesh Design
in an attachment lug

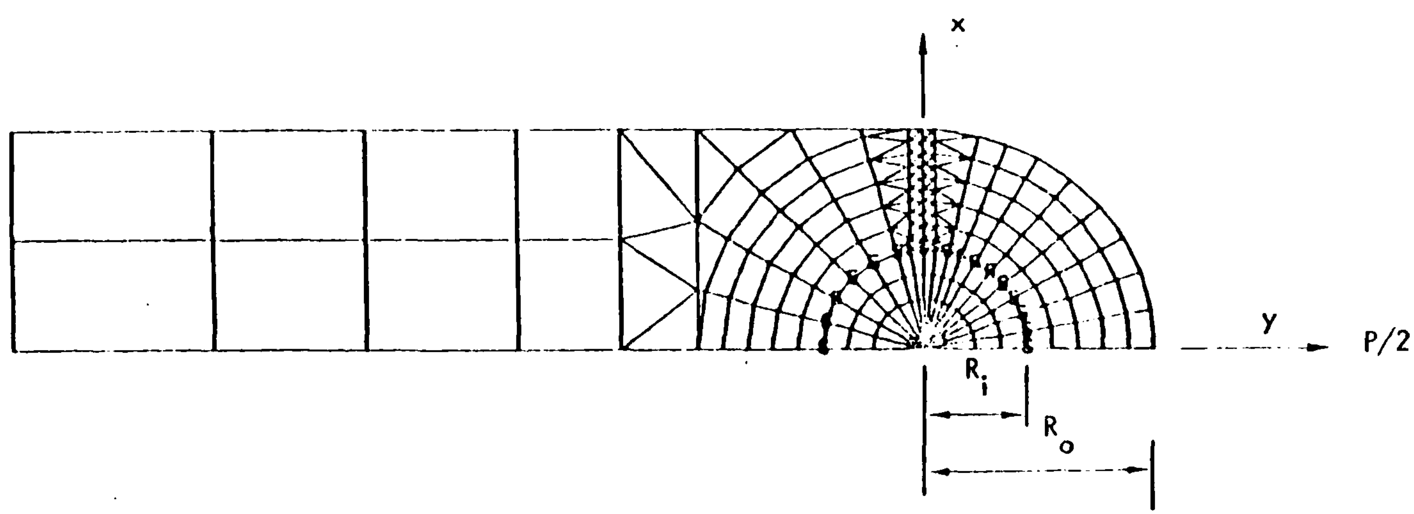


Figure 7. A Typical H-Version Finite Element Model for Unflawed Stress Analysis

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5. PAPERS PUBLISHED AND PRESENTED SINCE THE START OF THE PROJECT (1977)

5.1 Published Papers:

1. "Hierarchal Finite Elements and Precomputed Arrays", by Mark P. Rossow and I. Norman Katz, Int. J. for Num. Method in Engr., Vol. 13, No. 6 (1978) pp. 977-999.

2. "Nodal Variables for Complete Conforming Finite Elements of Arbitrary Polynomial Order", by I. Norman Katz, A. G. Peano, and Mark P. Rossow, Computers and Mathematics with Applications, Vol. 4, No. 2, (1978), pp. 85-112.

3. "Hierarchic Solid Elements for the p-version of the Finite Element Method", by I. Norman Katz, B. A. Szabo and A. G. Peano (in preparation).

4. "P-convergence Finite Element Approximations in Linear Elastic Fracture Mechanics", by Anil K. Mehta (doctoral dissertation), Department of civil Engineering, Washington University (1978).

5. "An Improved p-version Finite Element Algorithm and a Convergence Result for the p-version" by Anthony G. Kassos, Jr. (doctoral dissertation) Department of Systems Science and Mathematics, Washington University, (August, 1979).

6. "Hierarchic Families for the p-version of the Finite Element Method", I. Babuska, I. N. Katz and B. A. Szabo, invited paper presented at the Third IMACS International Symposium on Computer Methods for Partial Differential Equations, published in Advances in Computer Methods for Partial Differential Equations - III (1979) pp. 278-286.

7. "The p-version of the Finite Element Method", I. Babuska, B. A. Szabo, and I. N. Katz, SIAM J. of Numerical Analysis Vol. 18, No. 3, June 1981 pp. 515-545.

8. "Hierarchic Triangular Elements with one Curved Side for the p-version of the Finite Element Method", by I. Norman Katz (in preparation).

9. "The p-version of the Finite Element Method for Problems Requiring C^1-Continuity", by Douglas W. Wang (doctoral dissertation), Department of Systems Science and Mathematics, Washington University August 1982.

10. "Implementation of a C^1 Triangular Element based on the p-version of the Finite Element Method", by I. Norman Katz, D. W. Wang and B. Szabo, Proceedings of the Symposium in Advances and Tends in Structural and Solid Mechanics, October 4-7, 1982, Washington, D.C.

11. "The p-version of the Finite Element Method for Problems Requiring C^1-Continuity", by I. Norman Katz and Douglas W. Wang, SIAM J. of Numerical Analysis, Vol. 22, No. 6, December 1985, pp. 1082-1106.

12. "Implementation of a C^1-triangular Element based on the p-version of the Finite Element Method", by Douglas W. Wang, I. Norman Katz and Barna A. Szabo, Computers and Structures, Vol. 19, No. 3, pp. 381-392 (1984).

13. "H- and p- version analyses of a Rhombic Plate", by Douglas W. Wang, I. Norman Katz and Barna A. Szabo, International Journal for Numerical Methods in Engineering, Vol. 20, pp. 1399-1405 (1984).

14. "Implementation of a Finite Element Software System with H and P Extension Capabilities," by Barna A. Szabo, to appear in the Proceedings of the 8th Invitational Symposium on the Unification of Finite Element-Finite Differences and the Calculus of Variations.

15. "Computation of Stress Field Parameters in Areas of Steep Stress Gradients," by Barna A. Szabo, to appear in Communications in Applied Numerical Methods.

16. "On Stress Analysis with Large Length Ratios," by Barna A. Szabo, submitted to AIAA Journal.

5.2 Presented Papers:

1. "Hierarchical Approximation in Finite Element Analysis", by I. Norman Katz, International Symposim on Innovative Numerical Analysis in Applied Engineering Science, Versailles, France, May 23-27, 1977.

2. "Efficient Generation of Hierarchal Finite Elements Through the Use of Precomputed Arrays", by M. P. Rossow and I. N. Katz, Second Annual ASCE Engineering Mechanics Division Speciality Conference, North Carolina State University, Raleigh, NC, May 23-25, 1977.

3. "C^1 Triangular Elements of Arbitrary Polynomial Order Containing Corrective Rational Functions", by I. Norman Katz, SIAM 1977 National Meeting, Philadelphia, PA, June 13-15, 1977.

4. "Hierarchical Complete Conforming Tetrahedral Elements of Arbitrary Polynomial Order", by I. Norman Katz, presented at SIAM 1977 Fall Meeting, Albuquerque, NM, October 31- November 2, 1977.

5. "A Hierarchical Family of Complete Conforming Prismatic Finite Elements of Arbitrary Polynomial Order", by I. Norman Katz, presented at SIAM 1978 National Meeting, Madison, WI, May 24-26, 1978.

6. "Comparative Rates of h- and p- Convergence in the Finite Element Analysis of a Model Bar Problem", by I. Norman Katz, presented at the SIAM 1978 Fall Meeting, Knoxville, Tennessee, October 20- November 1, 1978.

7. "Smooth Approximation to a Function in $H_0^2(D)$ by Modified Bernstein Polynomials over Trangles" by A. G. Kassos, Jr. and I. N. Katz, presented at the SIAM 1979 Fall Meeting, Denver, Colorado, November 12-14, 1979.

8. "Triangles with one Curved Side for the p-version of the Finite Element Method" by I. Norman Katz, presented at the SIAM 1980 Spring Meeting, Alexandra, VA, J-ne 5-7, 1980.

9. "Hierarchic Square Pyramidal Elements for the p-version of the Finite Element Method" by I. Norman Katz, presented at the SIAM 1980 Fall Meeting, Houston, TX, November 6-8, 1980.

10. "The Rate of Convergence of the p-version of the Finite Element Method for Plate Bending Problems", by Douglas W. Wang and I. Norman Katz, presented at SIAM 1981 Fall Meeting, October 6-8, 1981, Cincinatti, Ohio.

11. "The p-version of the Finite Element Method", by I. Norman Katz, 1982 Meeting of the Illinois Section of the Mathematical Association of America, Southern Illinois University at Edwardsville, April 30-May 1, 1982.

12. "Computer Implementation of a C^1 Triangular Element based on the p-version of the Finite Element Method", by Douglas W. Wang and I. Norman Katz, SIAM 30th Anniversary Meeting, July 19-23, 1982, Stanford, California.

13. "Implementation of a C^1 Triangular Element Based on the p-version of the Finite Element Method", Symposium on Advances and Trends in Structural and Solid Mechanics, October 4-7, 1982, Washington, D.C.

14. "P-Convergent Polynomial Approximations in $H_0^2(\Omega)$" by Douglas W. Wang and I. Norman Katz, Fourth Texas Symposium on Approximation Theory. Department of Mathematics, Texas A & M Universtiy, College Station Texas 77843, January 17-21, 1983.

15. "Design Aspects of Adaptive Finite Element Codes" by D. W. Wang, I. N. Katz and M. Z. Qian, ASCE-EMD (American Society of Civil Engineers-Engineering Mechanics Division) Speciality Conference, Purdue University, May 25-28, 1983.

16. "Smoothing Stresses Computed Pointwise by the p-version of the Finite Element Method" by I. Norman Katz and Xing-ren Ying, SIAM 1983 National Meeting, Denver, Colorado, June 6-8, 1983.

17. "The Use of High Order Polynomials in the Numerical Solution of Partial Differential Equations", a Mini Symposium. I. Norman Katz, Organizer and Chairman; "The h-p version of the Finite Element Method", I. Babuska, B. Szabo, K. Izadpanah, W. Gui, and B. Guo; "A Pseudospectral Legendre Method for Hyperbolic Equations", D. Gottlieb and H. Tal-Ezer; "The Approximation Theory for the p-version of the Finite Element Method", Milo Dorr; "On the Robustness of Higher Order Elements", M. Vogelius; SIAM Summer Meeting, University of Washington, Seattle, Washington, July 16-20, 1984.

18. "Implementation of a Finite Element Software System with h- and p-extension capabilities", by Barna A. Szabo, to be presented at the 8th Invitational Symposium on the Unification of Finite Element-Finite Differences and the Calculus of Variations, University of Connecticut, Storrs, Connecticut, May 3, 1985.

19. "Stress Singularities at Angular Corners of Composite Plates", Xing-Ren Ying and I. Norman Katz, SIAM Spring Meeting, Pittsburgh, PA, June 1985.

20. "An Overview of the p-version of the Finite Element Method," by I. Norman Katz, invited colloquium presentation at the Department of Mathematics, West Virginia University, Morgantown, West Virginia, June 27, 1985.

5.3 Visits and Seminars at Government Laboratories

1. "Advanced Stress Analysis Technology" BY B. A. Szabo and I. N. Katz, presented on September 8, 1977 at the Air Force Flight Dynamic Laboratory, Wright-Patterson Air Force Base.

abstract

With one exception, all finite element software systems have element libraries in which the approximation properties of elements are frozen. The user controls only the number and distribution of finite elements. The exception is an experimental software system, developed at Washington University. This system, called COMET-X, employs conforming elements based on complete polynomials of arbitrary order. The elements are hierarchic, i.e. the stiffness matrix of each element is embedded in the stiffness matrices of all higher order elements of the same kind. The user controls not only the number and distribution of finite elements but their approximation properties as well. Thus convergence can be achieved on fixed mesh. This provides for very efficient and highly accurate approximation and a new method for computing stress intensity factors in linear elastic fracture mechanics. The theoretical developments are outlined, numerical examples are given and the concept of an advanced self-adaptive finite element software system is presented.

2. "The Constraint Method for Finite Element Stress Analysis", by I. N. Katz, presented at the National Bureau of Standards, Applied Mathematics Divison on October 19, 1977.

abstract

In conventional approaches to finite element stress analysis accuracy is obtained by fixing the degree p of the approximating polynomial and by allowing the maximum diameter h of elements in the triangulation to approach zero. An alternate approach is to fix the triangulation and to increase the degrees of approximating polynomials in those elements where more accuracy is required. In order to implement the second approach efficiently it is necessary to have a family of finite elements of arbitrary polynomial degree p with the property that as much information as possible can be retained from the pth degree approximation when computing the (p+1)st degree approximation. Such a HIERARCHIC family has been formulated with $p \geq 2$ for problems in plane stress analysis and with $p \geq 5$ for problems in plate bending. The family is described and numerical examples are presented which illustrate the efficiency of the new method.

3. "The p-version of the Finite Element Method", by I. N. Katz and B. A. Szabo, to be presented at Air-Force Flight Dynamics Laboratory Wright-Patterson Air Force Base on April 23, 1981 (tentative date).

abstract

The theoretical basis of the p-version of the finite element method has been established only quite recently. Nevertheless, the p-version is already

seen to be the most promising approach for implementing adaptivity in practical computations. The main theorems establishing asymptotic rates of convergence for the p-version, some aspects of the algorithmic structure of p-version computer codes, numerical experience and a posteriori error estimation will be discussed from the mathematical and engineering points of view.

4. Attended the meeting sponsored by AFOSR on "The Impact of Large Scale Computing on Air Force Research and Development", at Kirtland Air Force Base, Albuquerque, New Mexico, April 4-6, 1984.

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