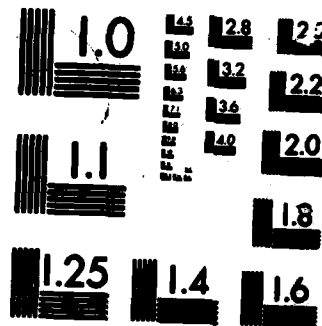


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A COMPUTER AIDED DESIGN PROCEDURE FOR GENERATING GEAR TEETH

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ABSTRACT

A procedure for computer aided design (CAD) of gear teeth is presented. It is developed for generated teeth fabricated by a hob cutter or a shaper. It provides a means for analytically and numerically determining the tooth profile, given the cutter profile. An illustrative example with involute tooth profiles is given. Application with non-standard profiles and with bevel, spiral bevel, and hypoid gears is discussed.

NOMENCLATURE

- Keyword: Curvature*
- C A plane curve, center of curvature
 - E Envelope of a family of curves
 - i, j Unit vectors parallel to $\hat{x}$ and $\hat{y}$
 - L Step side line segment
 - N Unit vector normal to E
 - O Wheel center and origin of X-Y
 - P Typical point on L, E
 - p Position vector OP
 - p_c Position vector OC
 - r Circle radius, wheel radius
 - T Unit vector tangent to E
 - t Parameter for a family of curves
 - X-Y Horizontal and vertical coordinate axes
 - X-Y Coordinate axes fixed in W
 - x, y Cartesian coordinates of P
 - W Rolling plastic wheel
 - α Roll angle
 - ρ Radius of curvature
 - ϕ Inclination angle
 - ψ Pressure angle

INTRODUCTION

Recent advances in computer graphics and computer aided design (CAD) present an opportunity for developing new procedures for optimizing gear tooth geometry. In this paper we present a basis for these procedures. The focus is upon spur gear teeth, but the same approach is applicable with helical, bevel, spiral bevel, and hypoid gear teeth.

The basic concepts underlying the method are readily seen by considering involute geometry of spur gear teeth. A widely used process for fabricating spur gear teeth is to use a rotating hob cutter. This process is based upon the concept of a reciprocating rack cutter with straight teeth moving across a gear blank as depicted in Figure 1. Geometrically this

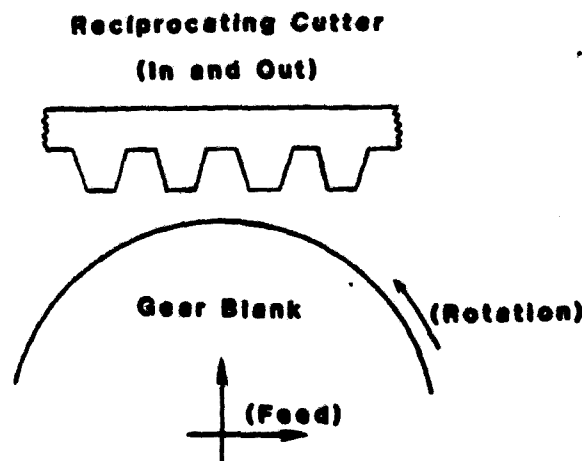


Fig. 1 Reciprocating Rack Cutter and Gear Blank.

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process may be viewed as a series of inclined line segments intersecting the circular gear blank, with the envelope of the line segments forming the tooth profile as shown in Fig. 2.

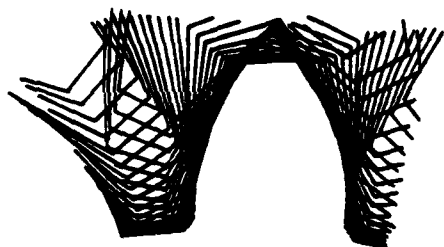


Fig. 2 Generation of an Involute Tooth Profile by a Rack Cutter.

A second way of viewing this process is to imagine a perfectly plastic wheel rolling over a "step" or obstacle in the form of a rack tooth as shown in Fig. 3. The impression (or "footprint") forms



Fig. 3 Wheel Rolling Over a Rack Tooth Forming the Tooth Profile.

the gear tooth profile. It is well known that this tooth profile is an involute of a circle. That is, the envelope of the line segments on the gear blank is an involute of a circle.

Recall that the involute of a circle may be viewed as the locus of the end of a cord being unwrapped around a circle as shown in Fig. 4. If the

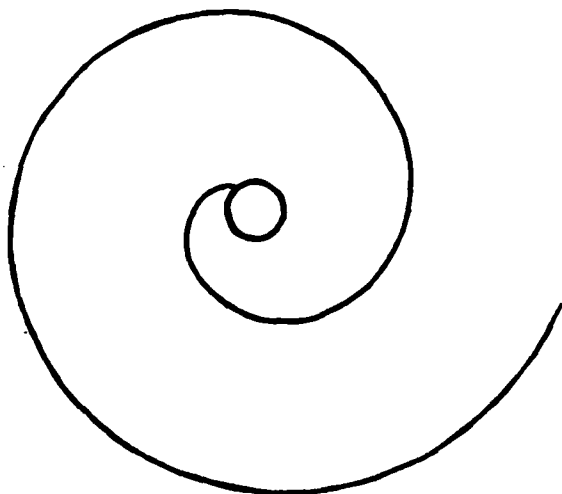


Fig. 4 Involute of a Circle.

circle has infinite radius, the involute will be straight, as with the rack tooth. While it may be intuitively clear that a tooth profile generated by the above process is an involute of a circle, the proof may be a bit more elusive. We present a computer graphic proof in the sequel.

ENVELOPE OF A FAMILY OF CURVES

Consider a plane curve C as shown in Fig. 5.

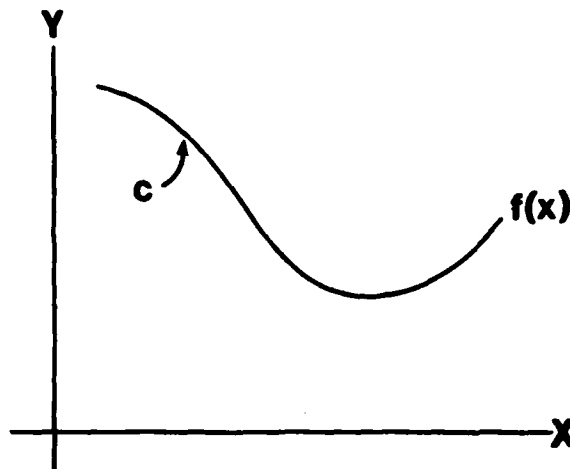


Fig. 5 A Plane Curve.

Analytically C may be represented by an equation of the form $y = f(x)$. Suppose in this functional description we introduce a parameter t defining a family of similar curves. Suppose further that as t changes the orientation of the curves change and that they intersect each other as in Fig. 6. The curve E , tangent to the intersection curves is then the envelope of the family.

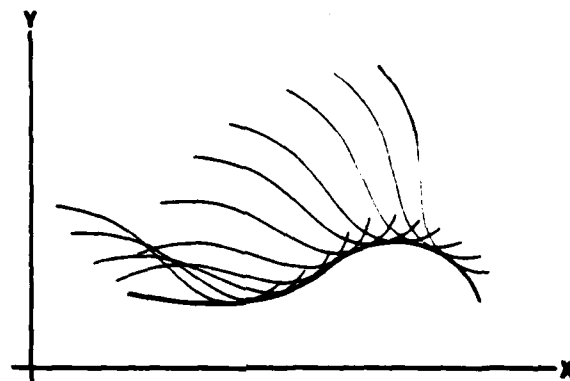


Fig. 6 A Family of Intersecting Curves.

It is relatively easy to obtain an analytical expression for the envelope. To see this, let the representation of the family of curves be $y = f(x, t)$ or $F(x, y, t) = 0$. If t is replaced by $t + \Delta t$ where Δt is a small increment in t , the expression $F(x, y, t + \Delta t) = 0$ also represents a member of the family of curves. Hence, the "difference quotient": $[F(x, y, t + \Delta t) - F(x, y, t)] / \Delta t = 0$ is a member of the

family as well. Therefore, by a limiting process, a second expression for members of the family of curves is $\partial F(x,y,t)/\partial t = 0$. By eliminating t between F and $\partial F/\partial t$ we obtain an equation of the form $G(x,y) = 0$. $G(x,y)$ thus represents the intersection of F and $\partial F/\partial t$. That is, $G(x,y)$ represents the points on the envelope E and is thus the desired analytical representation of E . (See Reference [1]* for additional details.)

To illustrate these ideas, consider the envelope of a family of lines, each a distance r from a fixed point O as depicted in Fig. 7. Let ϕ be the inclination angle of a typical member L of the family and let

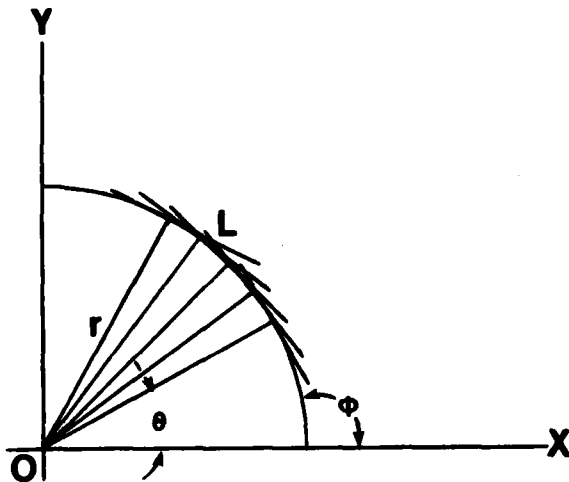


Fig. 7 A Family of Lines Equidistant From a Point.

be the inclination of the line normal to L and passing through O as shown. The equation of L might be written as

$$y - y_p = m(x - x_p) \quad (1)$$

where m is the slope of L and (x_p, y_p) are the coordinates of P , the point of intersection of L and its normal line through O . But, $m = \tan \phi$ and $\tan \phi = -\cot \theta$. Also, x_p and y_p may be expressed as $r \cos \theta$ and $r \sin \theta$. Hence, the equation of L might be rewritten as

$$y - r \sin \theta = (-\cot \theta)(x - r \cos \theta) \quad (2)$$

or as:

$$y \sin \theta + x \cos \theta - r = 0 = F(x, y, \theta) \quad (3)$$

Equation (3) may be considered as defining the family of lines with θ being the family parameter. By differentiating with respect to θ , we have

$$\partial F / \partial \theta = y \cos \theta - x \sin \theta = 0 \quad (4)$$

Finally, the equation of the envelope may be obtained by solving Equations (3) and (4) for x and y , leading to the expressions:

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta \quad (5)$$

or, by eliminating θ as:

$$x^2 + y^2 = r^2 \quad (6)$$

*Numbers in brackets refer to References.

The envelope, as expected, is a circle, with the family of lines being tangent to the circle.

DEVELOPMENT OF INVOLUTE SPUR GEAR TEETH

A similar procedure can be used to examine a spur gear tooth profile. Consider again Fig. 3 where the involute profile is generated by the step's impression on the plastic wheel. To describe the impression we need to find the envelope, in the wheel, of the line segments representing the sides of the step. To this end, consider Fig. 8 where L is a step

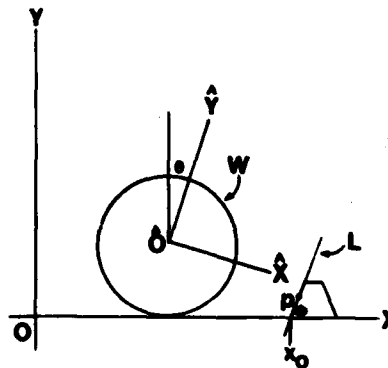


Fig. 8 Representation of a Perfectly Plastic Wheel Rolling Over a Step.

side, line segment. L is inclined at an angle ϕ to the X -axis and it intersects the X -axis at a distance x_0 from the origin. The wheel W has a radius r , center O' , and roll angle θ . $\hat{X}$ and $\hat{Y}$ are coordinate axes fixed in W with origin at O' . The objective is then to express the envelope of L in the $\hat{X}$ - $\hat{Y}$ system.

Let (x, y) and $(\hat{x}, \hat{y})$ be coordinates of a typical point P on L , relative to the X - Y and $\hat{X}$ - $\hat{Y}$ systems. Then it is easily shown that $\hat{x}$ and $\hat{y}$ are related to x and y by the expressions:

$$x = r\theta + \hat{x} \cos \theta + \hat{y} \sin \theta \quad (7)$$

and

$$y = r - \hat{x} \sin \theta + \hat{y} \cos \theta \quad (8)$$

The equation of L is

$$y = (x - x_0) \tan \phi \quad (9)$$

Using Equations (7) and (8), L may be described in terms of $\hat{x}$ and $\hat{y}$ as:

$$\begin{aligned} \hat{y}(\cos \theta - \tan \phi \sin \theta) - \hat{x}(\sin \theta + \tan \phi \cos \theta) \\ + r + (x_0 - r\theta) \tan \phi = 0 = F(\hat{x}, \hat{y}, \theta) \end{aligned} \quad (10)$$

Equation (10), like Equation (3) can be considered as describing a family of lines relative to $\hat{X}$ - $\hat{Y}$ with θ being the parameter. Hence, by differentiating with respect to θ we have

$$\begin{aligned} \partial F / \partial \theta = 0 = \hat{y}(\sin \theta + \tan \phi \cos \theta) \\ + \hat{x}(\cos \theta - \tan \phi \sin \theta) + r \tan \phi \end{aligned} \quad (11)$$

By solving Equations (10) and (11) for $\hat{x}$ and $\hat{y}$, we obtain:

$$\begin{aligned}\hat{x} &= r \sin \theta + (x_0 - r \theta) \sin \phi \cos \phi (\sin \theta + \tan \phi \cos \theta) \\ \text{and} \\ \hat{y} &= -r \cos \theta - (x_0 - r \theta) \sin \phi \cos \phi (\cos \theta - \tan \phi \sin \theta)\end{aligned}\quad (12)$$

Equations (12) are a pair of parametric equations representing the envelope of L relative to W. Therefore, Equations (12) describe the tooth profile impression created by the cutter step. To show this numerically, the line segments generating the envelope were plotted with a computer for $r=1$, $\phi=70^\circ$, $x_0=0.70021$, and $0 \leq \theta \leq 270^\circ$. Fig. 9 shows the

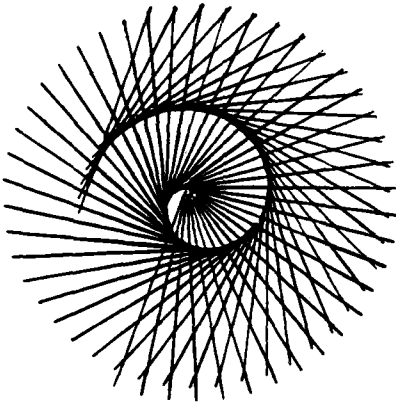


Fig. 9 Computer Drawn Cutting Lines at 5° Intervals for $\phi=70^\circ$, $r=1$, $x_0=0.70021$, and $0 \leq \theta \leq 270^\circ$.

results for 5° increments in θ . Fig. 10 shows a computer drawn graph of Equations (12). The "natural"

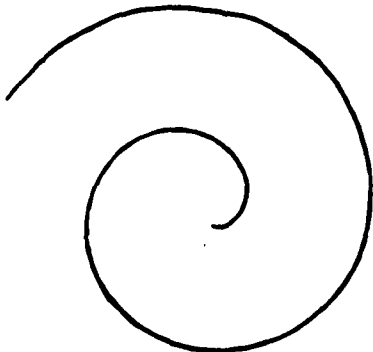


Fig. 10 Graph of Equations (12).

envelope in Fig. 9 is thus seen to be the same as the "analytical" envelope of Fig. 10.

* When $\phi=0$, the coordinate axes $\hat{X}-\hat{Y}$ and $X-Y$ are parallel and $\hat{O}$ is on the Y -axis. Also, x_0 has the value $r(1-\cos \phi)/\sin \phi$ so that L is tangent to W when $\theta=0$.

What remains to be shown is that the envelope is indeed an involute of a circle. A comparison of Fig. 4 and Fig. 10 suggests that it is. However, to show this analytically, let P be a typical point of the envelope E . The radius of curvature ρ of E at P may be expressed as [2,3]:

$$\rho = \frac{(\hat{x}_\theta^2 + \hat{y}_\theta^2)^{3/2}}{\hat{y}_{\theta\theta}\hat{x}_\theta - \hat{x}_{\theta\theta}\hat{y}_\theta} \quad (13)$$

where the subscript θ indicates partial differentiation with respect to θ . By substituting from Equations (12) into Equation (13) and after reduction, ρ takes the relatively simple form:

$$\rho = 1/[r \cos \phi + (x_0 - r \theta) \sin \phi] \quad (14)$$

The center of curvature C of E at P is located on a line perpendicular to E at a distance ρ from E . Hence, C may be located relative to the wheel center $\hat{O}$ by the vector $\hat{p} + \rho \hat{N}$, where $\hat{p}$ locates P relative to $\hat{O}$ and $\hat{N}$ is a unit vector perpendicular to E as shown in Fig. 11. Recall that a unit vector $\hat{T}$ tangent to E

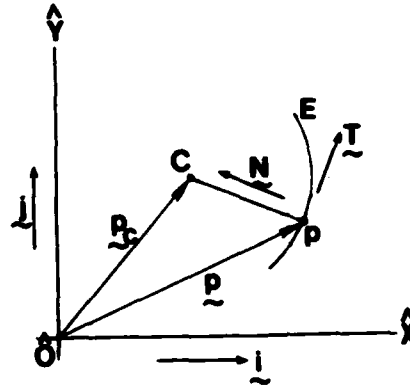


Fig. 11 The Center of Curvature C at a Typical Point P of the Envelope.

at P may be expressed as [3]:

$$\hat{T} = (\hat{x}_\theta \hat{i} + \hat{y}_\theta \hat{j}) / (\hat{x}_\theta^2 + \hat{y}_\theta^2)^{1/2} \quad (15)$$

Hence, $\hat{N}$ may be written as:

$$\hat{N} = (-\hat{y}_\theta \hat{i} + \hat{x}_\theta \hat{j}) / (\hat{x}_\theta^2 + \hat{y}_\theta^2)^{1/2} \quad (16)$$

Let $\hat{p}_c$ be the vector from $\hat{O}$ to C . Then, using Equation (13), $\hat{p}_c$ may be written as:

$$\begin{aligned}\hat{p}_c &= \hat{p} + \rho \hat{N} = \hat{x}_c \hat{i} + \hat{y}_c \hat{j} \\ &= \hat{x} \hat{i} + \hat{y} \hat{j} + (-\hat{y}_\theta \hat{i} + \hat{x}_\theta \hat{j}) \frac{(\hat{x}_\theta^2 + \hat{y}_\theta^2)^{3/2}}{(\hat{x}_\theta \hat{y}_{\theta\theta} - \hat{y}_\theta \hat{x}_{\theta\theta})}\end{aligned} \quad (17)$$

where $(\hat{x}_c, \hat{y}_c)$ are the coordinates of C relative to the $\hat{X}-\hat{Y}$ system, fixed in W . By substituting from Equations (12) and by performing the indicated differentiations, it is seen that the ratio $(\hat{x}_\theta^2 + \hat{y}_\theta^2)^{3/2} / (\hat{x}_\theta \hat{y}_{\theta\theta} - \hat{y}_\theta \hat{x}_{\theta\theta})$ is unity and that $\hat{x}_c$ and $\hat{y}_c$ are:

$$\hat{x}_c = \hat{x} - \hat{y} / \tan \phi = -r(\cos \theta - \tan \phi \sin \theta) \sin \phi \cos \phi$$

and

$$\dot{y}_c = \dot{y} + 3\dot{x}/\phi = -r(\sin\theta + \tan\phi \cos\theta)\sin\phi \cos\phi \quad (18)$$

The locus of the centers of curvature can now be seen to be a circle. That is, from Equations (18), we find that

$$x_c^2 + y_c^2 = r^2 \sin^2 \phi \quad (19)$$

By recalling the construction of an involute as the locus of the end points of an unwrapping cord around a circle, we see that the cord is perpendicular to the involute and its unwrapped length is the radius of curvature of the involute. Hence, the centers of curvature of the envelope are located on the generating circle (the "evolute"). Therefore, the envelope E above is the involute of the circle of Equation (19).

DISCUSSION AND APPLICATION

Recall that for a spur gear the circle generating the involute tooth profile is called the base circle. Recall also that if meshing spur gears are viewed as rolling cylinders, the cylinder cross sections define the pitch circles. Then, if the pressure angle is defined as the angle between the radial line and the tooth profile at the pitch circle, it is readily seen [4] that the ratio of the radii of the base and pitch circle is

$$R_b/R_p = \cos \phi \quad (20)$$

In the above analysis, the wheel profile is the pitch circle, the pressure angle is the complement of ϕ (that is, $\cos\psi = \sin\phi$), and the generating circle is the locus of the centers of curvature of the involute (the evolute). Then, from Equation (19), the base circle radius is $r\sin\phi$ ($= r\cos\psi$) where r is the pitch circle radius. Therefore, the ratio of the radii of the base and pitch circles is simply $\sin\phi$, or $\cos\psi$, -- a result consistent with Equation (20). Fig. 12 shows a computer generated drawing of the base circle, the pitch circle, and an involute curve forming a portion of a tooth profile.

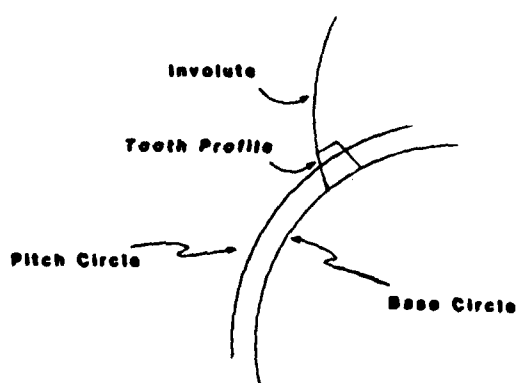


Fig. 12 Computer Drawing of Base Circle, Pitch Circle, and Involute.

A major application of the procedures of this paper is with the computer aided design and analysis of nonstandard tooth forms. For example, suppose that the cutter profile is not straight, but has a profile defined by the expression $y = f(x)$ [instead of

$y = (x - x_0)\tan\phi$ as in Equation (9)]. Then, in terms of the $X-Y$ coordinate system of the gear blank, the resulting tooth profile is determined from the expressions:

$$r - x\sin\phi + y\cos\phi - f = 0 \quad (21)$$

and

$$-x\cos\phi - y\sin\phi - (r - x\sin\phi + y\cos\phi)f' = 0 \quad (22)$$

where from Equation (11) the argument of f is $(r + x\cos\phi + y\sin\phi)$ and where f' is the derivative of f with respect to its argument. When Equations (21) and (22) are solved for x and y in terms of ϕ , they form a pair of parametric equations for the envelope of the cutter profile, which is the tooth profile.

A second major application of the procedures is with the design and analysis of bevel, spiral bevel, and hypoid gears. In this case, the procedures are generalized to three dimensions, and the cutter profile creates a family of surfaces whose envelope in the gear blank is the tooth surface. When these procedures are developed numerically, the resulting representation of the tooth surface appears in a form suitable for kinematic, stress, and life analyses.

ACKNOWLEDGEMENT

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