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EXPONENTIAL BOUNDS OF MEAN ERROR FOR THE NEAREST
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FOR MULTIVARIATE ANALYSIS L C ZHAO NOV 85 TR-85-44

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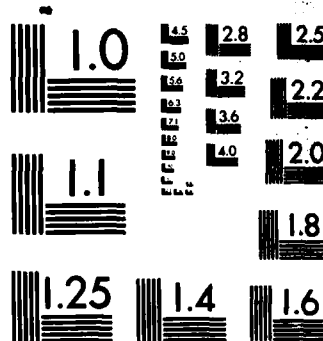
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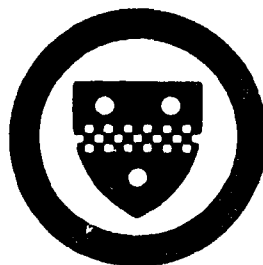
EXPONENTIAL BOUNDS OF MEAN ERROR FOR THE
NEAREST NEIGHBOR ESTIMATES OF
REGRESSION FUNCTIONS*

L. C. Zhao

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EXponential BOUNDS OF MEAN ERROR
FOR THE NEAREST NEIGHBOR ESTIMATES
OF REGRESSION FUNCTIONS

L. C. Zhao

ABSTRACT

Let $(X, Y), (X_1, Y_1), \dots, (X_n, Y_n)$ be i.i.d. $R^r \times R$ -valued random vectors with $E|Y| < \infty$, and let $m_n(x)$ be a nearest neighbor estimate of the regression function $m(x) = E(Y|X=x)$. In this paper, we establish an exponential bound of the mean deviation between $m_n(x)$ and $m(x)$ given the training sample $Z_n = (X_1, Y_1), \dots, (X_n, Y_n)$, under the conditions as weak as possible. This is a substantial improvement on Beck's result.

Key words. Regression function, nearest neighbor estimate, exponential bound, mean error, training sample.

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1. INTRODUCTION

Let $(X, Y), (X_1, Y_1), \dots, (X_n, Y_n)$ be i.i.d. $R^d \times R$ -valued random vectors with $E|Y| < \infty$. To estimate $m(x) = E(Y|X=x)$, the regression function of Y with respect to X , Stone (1977) and others proposed the so-called weight estimation

$$(1) \quad m_n(x) = \sum_{j=1}^n W_{nj}(x) Y_j,$$

where $W_{nj}(x) = W_{nj}(x, X_1, \dots, X_n)$ is a Borel-measurable function of its arguments.

Let V_{nj} , $j = 1, \dots, n$, be non-negative real number such that $\sum_{j=1}^n V_{nj} = 1$. For suitable-chosen metric $\|a-b\|$ on R^d (such as L_2 or L_∞), rearrange X_j , $j = 1, \dots, n$:

$$(2) \quad \|X_1^X - x\| \leq \|X_2^X - x\| \leq \dots \leq \|X_n^X - x\|$$

(ties are broken by comparing indices), and set

$$(3) \quad m_n(x) = \sum_{j=1}^n V_{nj} Y_j^X.$$

Then we obtain the nearest neighbor (NN) estimates of $m(x)$.

Many scholars studied convergence problem of these estimates from different points of view. (For the universal consistency, one can refer to, for example, Stone (1977). For the pointwise moment-consistency, see Devroye (1981). For the pointwise a.s. consistency, see Devroye (1981), Zhao and Bai (1984)). In this paper, we study another convergency of these estimates.

Write $X^n = (X_1, \dots, X_n)$, $Y^n = (Y_1, \dots, Y_n)$ and $Z^n = (X^n, Y^n)$. Let $g_n = g_n(x, Z^n)$ be an estimate of $m(x)$. In some problems, we are interested in the following mean deviation of g_n given the training sample Z^n :

$$(4) \quad \begin{aligned} D(g_n) &= E\{|g_n(X, Z^n) - m(x)| | Z^n\} \\ &= \int_{R^d} |g_n(x, Z^n) - m(x)| Q(dx), \end{aligned}$$

where Q denotes the distribution of X .

Take $k=k_n \leq n$, and put

$$(5) \quad \tilde{m}_n(x) = \frac{1}{k} \sum_{j=1}^k y_j^x.$$

For this class of estimates, Beck (1979) established the following theorem:

Suppose that the following conditions are satisfied:

- (6) (i) Y is bounded.
 (ii) $m(x)$ is continuous on R^d .
 (iii) Q has a continuous density f .
 (iv) $k \rightarrow \infty$ and $k/n \rightarrow 0$ as $n \rightarrow \infty$.

Then, for any given $\varepsilon > 0$,

$$P\{D(\tilde{m}_n) \geq \varepsilon\} \leq e^{-cn}$$

where $C > 0$ is a constant independent of n .

This theorem deals only with a special case of NN estimates, and the assumptions are rather restrictive. Recently, we substantially improved this result. We established the following:

Theorem 1. Let $m_n(x)$ be a NN estimate of $m(x)$ defined by (2) and (3).

Suppose that the following conditions are satisfied:

- (7) (i) Y is bounded.
 (ii) Q has a density f .
 (iii) There exists a sequence of integers $k = k_n$ such that

$$k \rightarrow \infty, \quad k/n \rightarrow 0, \\ \sup_n \{k \max_{1 \leq j \leq k} V_{nj}\} < \infty \quad \text{and} \quad \sum_{j=k+1}^n V_{nj} \rightarrow 0.$$

Then for any given $\varepsilon > 0$, we have

$$P\{D(m_n) \geq \varepsilon\} \leq e^{-cn},$$

where $C > 0$ is a constant independent of n .

Note that the special case considered by Beck is included in this theorem. Besides, this theorem gives a substantial improvement of Beck's result, by getting rid of the continuity requirement of $m(x)$ and $f(x)$, the density of Q .

2. SOME LEMMAS.

Theorem 1 is valid for the L_2 norm or L_∞ norm on R^d , here we only give the proof for L_∞ norm. For simplicity, we make the following convention: $\varepsilon, \varepsilon_1, \varepsilon_2, \dots, C, C_0, C_1, \dots, \alpha, \beta_1, \beta_2, \delta$, etc., are all constants independent of n . I_A or $I(A)$ denotes the indicator of a set A . $\#(A)$ denotes the cardinal of set A . $S_{x,\rho} = \{u \in R^d: ||u-x|| \leq \rho\}$. Q^* and λ^* denote the outer measure generated by Q and the Lebesgue measure λ (on R^d), respectively. We need the following lemmas in the sequel.

Lemma 1 (Besicovitch Covering Lemma). Let E be bounded subset of R^d , and let K be a family of cubes covering E which contains a cube D_x with center x for each $x \in E$. Then there exist points $\{x_k\}$ in E such that

$$(i) \quad E \subset \bigcup D_{x_k}.$$

$$(ii) \quad \text{there exists a constant } \sigma \text{ depending only on } d \text{ such that } \sum_k I(D_{x_k}) \leq \sigma.$$

Refer to Wheeden and Zygmund (1977), pp. 185-187.

Let Q_n be the empirical measure of $X_1, \dots, X_n$, and $T > 0$ be a given constant. Fix $\delta \in (0, 1/2\sigma)$ and assume that $h = h_n \in (0, 1)$. Set

$$(8) \quad G_n^* = \{x \in S_{0,T}: Q_n(S_{x,h}) < \delta Q(S_{x,h})\}.$$

and

$$(9) \quad E^* = \{x \in S_{0,T}: \beta_1(2\rho)^d \leq Q(S_{x,h}) \leq \beta_2(2\rho)^d$$

for any $\rho \in (0, 1)\}$,

Where $\beta_1 > 0$ and $\beta_2 > 0$ are constants to be chosen later.

LEMMA 2. Suppose that Q has a density f . Then for any $\epsilon > 0$, we can choose β_1 small enough and β_2 large enough such that $Q^*(S_{0,T}-E^*) < \epsilon$.

Note that for any Borel-measurable set $E \subseteq E^*$, we have

$$\beta_1 \leq f(x) \leq \beta_2, \text{ for almost all } x \in E(\lambda).$$

LEMMA 3. Suppose that Q has a density f , $h = h_n \in (0,1)$ and $nh^d \rightarrow \infty$. Then for any given $\epsilon > 0$, we have

$$P\{Q^*(G_n^*) \geq \epsilon\} < e^{-C_1 n}.$$

Lemmas 2 and 3 can be deduced from Lemma 1. For the proof, see Zhao (1985).

Lemma 4. Suppose that $\int_{\mathbb{R}^d} |g(x)|^p F(dx) < \infty$ for some $p > 0$, then

$$\lim_{h \rightarrow \infty} \int_{S_{x,h}} |g(u) - g(x)|^p F(du) / F(S_{x,h}) = 0$$

for almost all $x(F)$.

Refer to Wheeden and Zygmund (1977), p. 191, example 20.

3. Proof of Theorem 1

Suppose that $|Y| \leq M$. Then

$$\int \left| \sum_{j>k} V_{nj} (Y_j^x - m(x)) \right| Q(dx) \leq 2M \sum_{j>k} V_{nj} \rightarrow 0$$

as $n \rightarrow \infty$. Without loss of generality, we can assume $\sum_{j>k} V_{nj} = 0$ for any n . It is enough to prove that for each fixed $T > 0$,

$$(10) \quad P\left\{ \int_{S_{0,T/2}} |m_n(x) - m(x)| Q(dx) \geq \epsilon \right\} < e^{-cn}.$$

By Lemma 2, there exists $\beta_i = \beta_i(\epsilon)$, $i=1,2$, and a compact set $E \subseteq E^*$ such that

$$(11) \quad Q(S_{0,T}-E) < \epsilon/8M,$$

where E^* is defined by (9).

Fix $\delta \in (0, \frac{1}{2}\sigma)$, and take $\alpha \geq (2^d \beta_1 \delta)^{-1}$. Set

$$h = h_n = (\alpha k/n)^{1/d},$$

then $h \rightarrow 0$ and $nh^d \rightarrow \infty$ as $n \rightarrow \infty$.

By Lemma 3, there exists a compact set H_n such that with h as above

$$(12) \quad H_n \subset \{x \in S_{0,T} : Q_n(S_{x,h}) \geq \delta Q(S_{x,h})\}$$

and

$$(13) \quad P\{Q(S_{0,T}-H_n) \geq \epsilon/8M\} < e^{-C_1 n}.$$

For $x \in H_n \cap E$, $Q_n(S_{x,h}) \geq \delta Q(S_{x,h}) \geq \beta_1 \delta \lambda(S_{x,h}) = \beta_1 \delta 2^d \alpha k/n \geq k/n$, so that $x_1^x, x_2^x, \dots, x_k^x$ all fall into $S_{x,h}$.

Partition R^d into sets with the form $\prod_{j=1}^d [(i_j-1)h, i_j h)$, where $i_1, \dots, i_d = 0, \pm 1, \dots$. Call the partition Ψ . Set $\Psi' = \{B \in \Psi, B \subset S_{0,T}\}$. For $B \in \Psi'$, put

$$\tilde{W}(B) = \{B' \in \Psi, \rho(B, B') < 3h\}, \quad W(B) = \bigcup_{B' \in \tilde{W}(B)} B',$$

where $\rho(B, B') = \inf\{\|x-x'\| : x \in B, x' \in B'\}$. Then there exists a constant C_d such that for any $B \in \Psi'$ we have $\#\tilde{W}(B) \leq C_d$. It is easy to show by induction that, Ψ' can be divided into $C_2 (\leq C_d^2)$ disjoint subsets Ψ_i , $i=1, \dots, C_2$, such that for any two sets B_1, B_2 in the same Ψ_i , we have

$$W(B_1) \cap W(B_2) = \emptyset.$$

Denote by $B(x)$ the cube $B \in \Psi$ which contains x . If $x \in H_n \cap E$ and $B(x) \in \Psi'$, then for any $u \in B(x)$, we have $S_{x,h} \subset S_{u,2h} \subset W(B(x))$, so that, from $Q_n(S_{x,h}) \geq k/n$ it follows that $x_1^u, \dots, x_k^u$ are also contained in $W(B(x))$. If we write

$$A_n = \{B \in \Psi^1: B \cap H_n \cap E \neq \emptyset\}$$

then, as mentioned above, for any $B \in A_n$, $W(B)$ contains the k nearest neighbors of each $x \in B$. Further, we set $H_i = A_n \cap \Psi_i$, $i=1,2,\dots,C_2$. It is easy to see that

$$\int_{S_{0,T/2}} |m_n(x) - m(x)| Q(dx) \leq \int_{S_{0,T-E}} + \int_{S_{0,T-H_n}} + \int_{H_n \cap E \cap S_{0,T/2}}.$$

By (11), we have

$$\int_{S_{0,T-E}} |m_n(x) - m(x)| Q(dx) \leq 2MQ(S_{0,T-E}) < \epsilon/4.$$

By (13),

$$\begin{aligned} P\left(\int_{S_{0,T-H_n}} |m_n(x) - m(x)| Q(dx) \geq \epsilon/4\right) \\ \leq P\{Q(S_{0,T-H_n}) \geq \epsilon/8M\} < e^{-c_1 n}. \end{aligned}$$

Hence to prove (10), it is enough to prove that

$$(14) \quad P\left\{\int_{H_n \cap E \cap S_{0,T/2}} |m_n(x) - m(x)| Q(dx) \geq \epsilon/2\right\} < e^{-c_3 n}.$$

For large n ,

$$\begin{aligned} & \int_{H_n \cap E \cap S_{0,T/2}} |m_n(x) - m(x)| Q(dx) \\ & \leq \sum_{B \in A_n} \int_{B \cap E} |m_n(x) - m(x)| Q(dx) \\ & \leq \sum_{i=1}^{C_2} \sum_{B \in H_i} \int_{B \cap E} |m_n(x) - m(x)| Q(dx). \end{aligned}$$

Put

$$\tilde{m}_n(x) = \sum_{j=1}^k v_{nj} m(x_j^x),$$

$$I_{ni} = \sum_{B \in H_i} \int_{B \cap E} |m_n(x) - \tilde{m}_n(x)| Q(dx),$$

$$(15) \quad J_{ni} = \sum_{B \in H_i} \int_{B \cap E} |\tilde{m}_n(x) - m(x)| Q(dx), \quad i=1, \dots, C_2.$$

$$\phi(B) = \int_{B \cap E} \left| \sum_{j=1}^k V_{nj} (Y_j^X - m(X_j^X)) \right| Q(dx) / Q(B \cap E),$$

$$d_{ni} = \#\{B \in H_i, \phi(B) \geq \epsilon / (8C_2)\}, \quad i=1, \dots, C_2.$$

To prove (14), it is enough to show that, for each i , $1 \leq i \leq C_2$, we have

$$(16) \quad P\{I_{ni} \geq \epsilon / (4C_2)\} < e^{-C_4 n}$$

$$(17) \quad P\{J_{ni} \geq \epsilon / (4C_2)\} < e^{-C_5 n}.$$

For almost all $x \in B \cap E(\lambda)$, $f(x) \leq \beta_2$. Hence,

$$I_{ni} \leq \epsilon / (8C_2) + 2M d_{ni} \beta_2^{\alpha k} / n.$$

Write $C_6 = \epsilon(16M C_2^{\alpha \beta_2})^{-1}$, then

$$(18) \quad P\{I_{ni} \geq \epsilon / (4C_2)\} \leq P\{d_{ni} \geq C_6 n / k\}.$$

Now we proceed to prove that, for any $B \in H_i$,

$$(19) \quad P\{\phi(B) \geq \epsilon / 8C_2 | X^n\} < e^{-C_7 k},$$

where $X^n = (X_1, \dots, X_n)$ is defined as before.

For any $\epsilon_1 > 0$ and $s > 0$, by Jensen's inequality we have

$$(20) \quad P\{\phi(B) \geq \epsilon_1 | X^n\} \leq e^{-s\epsilon_1} E\{\exp(s\phi(B)) | X^n\} \\ \leq e^{-s\epsilon_1} \int_{B \cap E} E\{\exp(s \left| \sum_{j=1}^k V_{nj} [Y_j^X - m(X_j^X)] \right| | X^n\} Q(dx) / Q(B \cap E).$$

When $\{X_j^X, j \leq k\}$ is given, $Y_1^X, \dots, Y_k^X$ are independent. From this and the inequality $|e^t - 1 - t| \leq \frac{1}{2}t^2 e^{|t|}$ for any real t , it follows that,

$$\begin{aligned} & E\{\exp(s \sum_{j=1}^k V_{nj} [Y_j^X - m(X_j^X)]) | X^n\} \\ &= \prod_{j=1}^k E\{\exp(s V_{nj} [Y_j^X - m(X_j^X)]) | X_j^X\} \\ &\leq \prod_{j=1}^k \{1 + s^2 C_8^2 k^{-2} \exp(2s C_8 k^{-1})\} \\ &\leq \exp\{s^2 C_8^2 k^{-1} \exp(2s C_8 k^{-1})\}. \end{aligned}$$

Here we have written $C_9 = \sup_n \{k \max_{j \leq k} V_{nj}\}$ and $C_8 = C_9 M$. In the same way,

$$\begin{aligned} & E\{\exp(s \sum_{j=1}^k V_{nj} [m(X_j^X) - Y_j^X]) | X^n\} \\ &\leq \exp\{s^2 C_8^2 k^{-1} \exp(2s C_8 k^{-1})\}. \end{aligned}$$

In view of (20), we get

$$P\{\phi(B) \geq \epsilon_1 | X^n\} \leq 2 \exp\{-s\epsilon_1 + s^2 C_8^2 k^{-1} \exp(2s C_8 k^{-1})\}$$

Take $s = \mu k$ with μ being small enough, we have

$$P\{\phi(B) \geq \epsilon_1 | X^n\} < e^{-C_{10} k}.$$

This is just (19).

Since for each $B \in H_i$, $W(B)$ contains the k nearest neighbors of each $x \in B$, and $W(B_1) \cap W(B_2) = \emptyset$ for any $B_1, B_2 \in H_i$, we see that when $X^n = (X_1, \dots, X_n)$ is given, $\{\phi(B), B \in H_i\}$ is a group of conditionally independent variables. Put $G(B) = \{\phi(B) \geq \epsilon_1\}$. Then by (19) and $\#(H_i) \leq \#(\Psi') \leq C_{11} n/k$, we have

$$\begin{aligned}
& P\{d_{ni} \geq C_6 n/k | X^n\} \\
& \leq P\{U_H \subset H_i, \#(H) \geq C_6 n/k \cap \bigcap_{B \in H} G(B) | X^n\} \\
& \leq \sum_{H \subset H_i, \#(H) \geq C_6 n/k} P(\bigcap_{B \in H} G(B) | X^n) \\
& = \sum_{H \subset H_i, \#(H) \geq C_6 n/k} \prod_{B \in H} P(G(B) | X^n) \\
(21) \quad & \leq \sum_{C_6 n/k \leq j \leq \#(H_i)} \binom{\#(H_i)}{j} (e^{-C_7 k})^j \\
& \leq e^{-C_6 C_7 n} \frac{\#(H_i)}{2} \leq 2^{-C_6 C_7 n} C_{11}^{n/k} \leq e^{-C_{12} n}.
\end{aligned}$$

From (18) and (21) it follows (16) is valid.

Now we proceed to prove (17). As mentioned above, for each $B \in H_i$, $X_1^X, \dots, X_k^X$ all fall into $W(B)$. Noticing the conditions imposed on V_{nj} 's, we see that

$$\begin{aligned}
(22) \quad J_{ni} &= \sum_{B \in H_i} \int_{B \cap E} \left| \sum_{j=1}^k V_{nj} (m(X_j^X) - m(x)) \right| Q(dx) \\
&\leq C_9 k^{-1} \sum_{B \in \Psi_i} \sum_{j=1}^n I_{W(B)}(X_j) \int_{B \cap E} |m(X_j) - m(x)| Q(dx) \\
&= C_9 k^{-1} \sum_{B \in \Psi_i} \sum_{j=1}^n I_{W(B)}(X_j) Z_B(X_j),
\end{aligned}$$

where

$$(23) \quad Z_B(u) = \int_{B \cap E} |m(u) - m(x)| Q(dx) \leq 2M\beta_2 \alpha k/n.$$

Here, the following facts are used: $|m(x)| \leq M$, $f(x) \leq \beta_2$ for $x \in B \cap E$ and, $\lambda(B) \leq h^d = \alpha k/n$.

Put $\varepsilon_2 = \varepsilon(8C_2C_9)^{-1}$. To prove (17), it suffices to prove that

$$(24) \quad P\left\{ \sum_{B \in \Psi_i} \sum_{j=1}^n I_{W(B)}(X_j) Z_B(X_j) \geq 2k\varepsilon_2 \right\} < e^{-C_{13} n}.$$

Let N be a Poisson random variable with parameter n , which is independent of $X_1, X_2, \dots$. If $|N-n| < n\epsilon_3 = n\epsilon_2/(2M\beta_2\alpha)$, then by (23)

$$\begin{aligned} & \left| \sum_{B \in \Psi_i} \left(\sum_{j=1}^n I_{W(B)}(X_j) Z_B(X_j) - \sum_{j=1}^N I_{W(B)}(X_j) Z_B(X_j) \right) \right| \\ & \leq |N-n| 2M\beta_2\alpha k/n < \epsilon_2 k. \end{aligned}$$

It follows that

$$\begin{aligned} (25) \quad & P\left\{ \sum_{B \in \Psi_i} \sum_{j=1}^n I_{W(B)}(X_j) Z_B(X_j) \geq 2k\epsilon_2 \right\} \\ & \leq P\{|N-n| \geq n\epsilon_3\} + P\left\{ \sum_{B \in \Psi_i} \sum_{j=1}^N I_{W(B)}(X_j) Z_B(X_j) > k\epsilon_2 \right\} \end{aligned}$$

It is easy to show that

$$(26) \quad P\{|N-n| \geq n\epsilon_3\} < e^{-C_{14}n}.$$

Since $W(B)$, $B \in \Psi_i$, are disjoint, we see that for $t > 0$,

$$\begin{aligned} (27) \quad & P\left\{ \sum_{B \in \Psi_i} \sum_{j=1}^N I_{W(B)}(X_j) Z_B(X_j) > k\epsilon_2 \right\} \\ & \leq e^{-t\epsilon_2 k} \sum_{\ell=0}^{\infty} \frac{e^{-n} n^{\ell}}{\ell!} (E\{\exp(t \sum_{B \in \Psi_i} I_{W(B)}(X_1) Z_B(X_1))\})^{\ell} \\ & = e^{-t\epsilon_2 k} e^{-n} \sum_{\ell=0}^{\infty} \frac{n^{\ell}}{\ell!} \left(\sum_{B \in \Psi_i} \int_{W(B)} e^{tZ_B(u)} Q(du) + 1 - Q\left(\bigcup_{B \in \Psi_i} W(B)\right) \right)^{\ell} \\ & = \exp\{-t\epsilon_2 k + n \sum_{B \in \Psi_i} \int_{W(B)} (e^{tZ_B(u)} - 1) Q(du)\} \end{aligned}$$

Now we proceed to show that

$$(28) \quad \limsup_{n \rightarrow \infty} \sum_{B \in \Psi_i} \int_{W(B)} [\exp(\frac{n}{k} Z_B(u)) - 1] Q(du) = 0.$$

By (23), there exist constants C_{15} , C_{16} such that

$$\frac{n}{k} Z_B(u) \leq C_{15}$$

and

$$\exp\left(\frac{n}{k} Z_B(u)\right) - 1 \leq C_{16} \frac{n}{k} Z_B(u).$$

To prove (28), it suffices to show that

$$(29) \quad \limsup_{n \rightarrow \infty} \frac{n}{k} \sum_{B \in \Psi_i} \int_{B \cap E} Q(dx) \int_{W(B)} |m(u) - m(x)| Q(du) = 0.$$

Assume that $B \in \Psi_i$, $B \cap E \neq \emptyset$ and $x \in B \cap E$, then $W(B) \subset S_{x,5h}$, where $h = (\alpha k/n)^{1/d}$. By Lemma 2,

$$Q(S_{x,5h}) \leq \beta_2 (10h)^d = 10^d \beta_2 \alpha k/n.$$

Put $C_{17} = 10^d \beta_2 \alpha$, then

$$\begin{aligned} (30) \quad & \frac{n}{k} \sum_{B \in \Psi_i} \int_{B \cap E} Q(dx) \int_{W(B)} |m(u) - m(x)| Q(du) \\ & \leq C_{17} \sum_{B \in \Psi_i} \int_{B \cap E} Q(dx) \left\{ \int_{S_{x,5h}} |m(u) - m(x)| Q(du) / Q(S_{x,5h}) \right\} \\ & \leq C_{17} \int Q(dx) \left\{ \int_{S_{x,5h}} |m(u) - m(x)| Q(du) / Q(S_{x,5h}) \right\} \end{aligned}$$

By Lemma 4, for almost all $x(Q)$,

$$\lim_{n \rightarrow \infty} \int_{S_{x,5h}} |m(u) - m(x)| Q(du) / Q(S_{x,5h}) = 0.$$

Further, for $x \in S(Q)$, the support of Q , we have

$$\int_{S_{x,5h}} |m(u) - m(x)| Q(du) / Q(S_{x,5h}) \leq 2M$$

Hence, by the dominated convergence theorem, (29) is valid. Thus (28) is proved.

Take $t = n/k$ in (27), we have

$$\begin{aligned}
 (31) \quad & P\left\{\sum_{B \in \Psi_i} \sum_{j=1}^N I_{W(B)}(X_j) Z_B(X_j) > k\epsilon_2\right\} \\
 & \leq \exp\{-\epsilon_2 n + o(n)\} \leq e^{-C_{18} n}.
 \end{aligned}$$

From (25), (26) and (31), it follows that (24) holds, and (17) is valid. From (16) and (17), Theorem 1 is proved.

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