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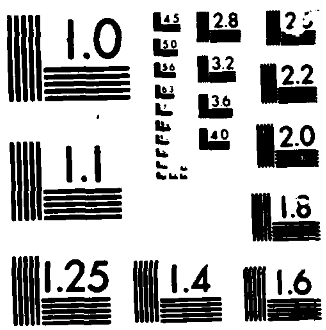

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TECHNICAL REPORT RD-85-9

THE SHIFTED LAPLACE AND Z-TRANSFORMS - REVISITED

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SEPTEMBER 1985

U.S. ARMY MISSILE COMMAND
Redstone Arsenal, Alabama 35898-5000

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Advances (left shifting) and delays (right shifting) are shown to be inverses for the Laplace and z-transforms. This is true for both bilateral and unilateral Laplace and z-transforms. This result is not in agreement with some literature on these subjects, and the oversight made in that literature is pointed out.

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I. INTRODUCTION

"Human mind like parachute, works best when open."
C. Chan

Advances (left shifting) and delays (right shifting) are shown to be inverses for Laplace and z-transforms. This is true for both bilateral and unilateral Laplace and z-transforms.

This result is not in agreement with some of the Laplace and z-transform literature, and the oversight made in that literature is pointed out.

II. THE PROBLEM

The delay or right shifting theorems of the Laplace and z-transforms are

$$\mathcal{L}[f(t - t_0)] = e^{-st_0}\ \mathcal{L}[f(t)] \tag{1}$$

and

$$Z[f(t - kT)] = z^{-k}\ Z[f(t)], \tag{2}$$

respectively, where

$$z \equiv e^{sT}. \tag{3}$$

Some literature also gives advance or left-shifted transforms [1,2,3] as

$$e^{st_0}\left[\mathcal{L}[f(t)] - \int_0^{t_0} f(t)\ e^{-st}\ dt\right] \tag{4}$$

and

$$z^k\left[Z[f(t)] - \sum_{n=0}^{k-1} f(nT)\ z^{-n}\right], \tag{5}$$

respectively. Note that these "advances" are not inverses of the delays.

It is assumed that for the function being transformed,

$$f(t) = 0,\ t < 0. \tag{6}$$

For developing the left-shifting theorems,

$$f(t - kT) = 0,\ t < kT \tag{7}$$

while for the right-shifting theorems [1,2,3] some assumed

$$f(t + kT) = 0,\ t < 0. \tag{8}$$

It would seem that

$$f(t + kT) = 0,\ t < -kT \tag{9}$$

would be more appropriate. The question is simply this, does equation (6) mean f(t) is zero when the argument is less than zero, or when time is less than zero?

III. DISCUSSION

A. The Shifted Laplace Transform

The bilateral Laplace transform [4] is

$$L[f(t)] \equiv \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt \tag{10}$$

providing there is a zone of convergence.

To form the unilateral Laplace transform, let

$$f(t) = g(t)\; u(t) \tag{11}$$

where

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases} \tag{12}$$

is the Heaviside unit step. That is the unilateral Laplace transform is

$$L\left[g(t)\; u(t)\right] = \int_{-\infty}^{\infty} g(t)\; u(t)\; e^{-st}\, dt \quad . \tag{13}$$

The advantage of the bilateral Laplace transform formulation is that a change of the integration variable does not affect the integration limits since they are both infinite. The unilateral representation using the unit step allows this property to be retained.

For a shift in time, t_0,

$$\tau = t \mp t_0 \tag{14}$$

it follows that

$$t = \tau \pm t_0 \tag{15}$$

and

$$dt = d\tau. \tag{16}$$

The bilateral Laplace transform becomes under this translation in time,

$$\int_{-\infty}^{\infty} f(t)\; e^{-st}\, dt = \int_{-\infty}^{\infty} f(\tau \pm t_0)\; e^{-s(\tau \pm t_0)}\, d\tau \tag{17}$$

and the unilateral,

$$\int_{-\infty}^{\infty} g(t)\ u(t)\ e^{-st}\ dt = \int_{-\infty}^{\infty} g(\tau \pm t_0)\ u(\tau \pm t_0)\ e^{-s(\tau \pm t_0)}\ d\tau\ . \tag{18}$$

One may "start" at anytime and the Laplace transform is the same, i.e., there is no absolute zero for time in Laplace transforms.

What if everything under consideration does not "start" at the same time, i.e., have the same time zero? Factoring out $e^{\mp st_0}$ on the left hand side of equations (17 and 18) yields

$$\int_{-\infty}^{\infty} f(t)\ e^{-st}\ dt = e^{\mp st_0} \int_{-\infty}^{\infty} f(\tau \pm t_0)\ e^{-s}\ d\tau \tag{19}$$

and

$$\int_{-\infty}^{\infty} g(t)\ u(t)\ e^{-st}\ dt = e^{\mp st_0} \int_{-\infty}^{\infty} g(\tau \pm t_0)\ u(\tau \pm t_0)\ e^{-s\tau}\ d\tau, \tag{20}$$

respectively.

From the definition(s) of the Laplace transform(s), Equations (10 and 13)

$$L\left[f(t)\right] = e^{\mp st_0}\ L\left[f(t \pm t_0)\right] \tag{21}$$

and

$$L\left[g(t)\ u(t)\right] = e^{\mp st_0}\ L\left[g(t \pm t_0)\ u(t \pm t_0)\right], \tag{22}$$

respectively.

Multiplying both sides by $e^{\pm st_0}$ yields the shifting theorems,

$$L\left[f(t \pm t_0)\right] = e^{\pm st_0}\ L\left[f(t)\right] \tag{23}$$

and

$$L\left[g(t \pm t_0)\ u(t \pm t_0)\right] = e^{\pm st_0}\ L\left[g(t)\ u(t)\right], \tag{24}$$

respectively.

B. The Shifted Z-Transform

To develop the shifted z-transform, the z-transform (a discrete Laplace transform) must be developed first.

The Laplace transform is made discrete by the sifting property [5,6],

$$\int_{-\infty}^{\infty} f(t)\ \delta(t - t_0)\ dt = f(t_0), \tag{25}$$

of a sequence of Dirac deltas,

$$\sum_{n=-\infty}^{\infty} \delta(t - nT), \tag{26}$$

that is

$$Z\left[f(t)\right] = \int_{-\infty}^{\infty} f(t) \left(\sum_{n=-\infty}^{\infty} \delta(t - nT)\right) e^{-st}\ dt\ . \tag{27}$$

Interchanging integration and summation

$$Z\left[f(t)\right] = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\ \delta(t - nT)\ e^{-st}\ dt \tag{28}$$

and applying the sifting property

$$Z\left[f(t)\right] = \sum_{n=-\infty}^{\infty} f(nT)\ e^{-snT} \tag{29}$$

or

$$Z\left[f(t)\right] = \sum_{n=-\infty}^{\infty} f(nT)\ z^{-n}\ . \tag{30}$$

Of course, for the unilateral z-transform

$$Z\left[g(t)\ u(t)\right] = \sum_{n=-\infty}^{\infty} g(nT)\ u(nT)\ z^{-n}\ . \tag{31}$$

As with the Laplace transform, the advantage of this formulation is that a change of the summation variable does not effect the summation limits since they are both infinite.

For

$$\ell = n \mp k, \tag{32}$$

then

$$n = \ell \pm k, \tag{33}$$

and

$$\sum_{n=-\infty}^{\infty} f(nT)\ z^{-n} = \sum_{\ell=-\infty}^{\infty} f(\ell T \pm kT)\ z^{-(\ell \pm k)}\ , \tag{34}$$

$$\sum_{n=-\infty}^{\infty} g(nT)\ u(nT)\ z^{-n} = \sum_{\ell=-\infty}^{\infty} g(\ell T \pm kT)\ u(\ell T \pm kT)\ z^{-(\ell \pm k)}. \tag{35}$$

As with the Laplace transform, there is no absolute zero for time in z-transforms. Factoring out $z^{\mp k}$ from the right hand side

$$\sum_{n=-\infty}^{\infty} f(nT)\ z^{-n} = z^{\mp k} \sum_{\ell=-\infty}^{\infty} f(\ell T \pm kT)\ z^{-\ell T} \tag{36}$$

and

$$\sum_{n=-\infty}^{\infty} g(nT)\ u(nT)\ z^{-n} = z^{\mp k} \sum_{n=-\infty}^{\infty} g(\ell T \pm kT)\ u(\ell T \pm kT)\ z^{-\ell T}\ . \tag{37}$$

From the definition(s) of the z-transform(s), Equations (30 and 31),

$$Z\left[f(t)\right] = z^{\mp k} Z\left[f(t \pm kT)\right] \tag{38}$$

and

$$Z\left[g(t)\ u(t)\right] = z^{\mp k} Z\left[g(t \pm kT)\ u(t \pm kT)\right] . \tag{39}$$

Multiplying both sides by $z^{\pm k}$ yields the shifting theorems,

$$Z\left[f(t \pm kT)\right] = z^{\pm k} Z\left[f(t)\right] \tag{40}$$

and

$$Z\left[g(t \pm kT)\ u(t \pm kT)\right] = z^{\pm k} Z\left[g(t)\ u(t)\right] . \tag{41}$$

IV. CONCLUSIONS

For the bilateral and unilateral Laplace and z-transforms, advances and delays have the same form. That advances are the inverse of delays fits the physical intuition of time shifting in that it is all relative. What is an advance relative to one, would appear to be a delay relative to the other.

That the bilateral and unilateral have the same form is quite interesting since while many relations are the same for both, some are not, for example, the transform of derivatives.

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LIST OF SYMBOLS

$\delta(t)$	Dirac delta distribution
$f(t)$	A proper function
$g(t)$	A proper function with at least one continuous derivative
k	An integer
ℓ	An integer
$L[\ldots]$	Laplace transform of...
n	An integer
T	Time step or interval
t	Time variable
τ	Time variable
$u(t)$	Heaviside unit step
z	Shifting operator
$Z[\ldots]$	Z-transform of...

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