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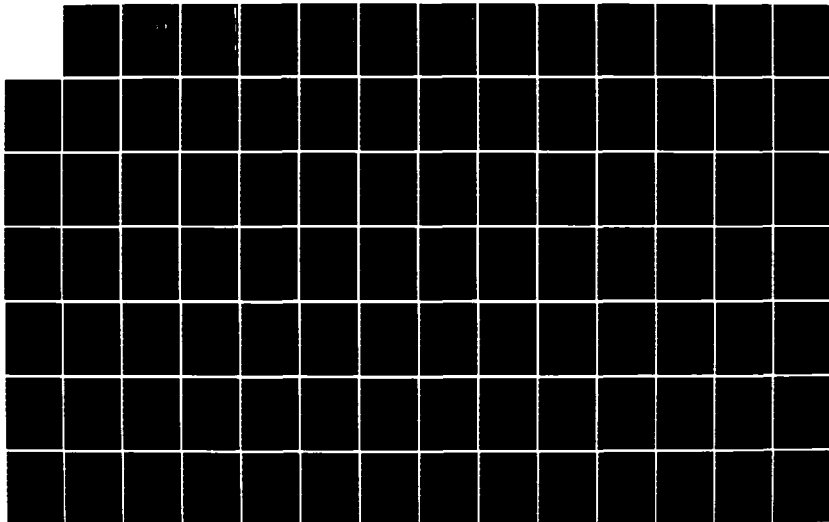
COMPUTER PROGRAMS FOR THE DETERMINATION OF STRESSES AND  
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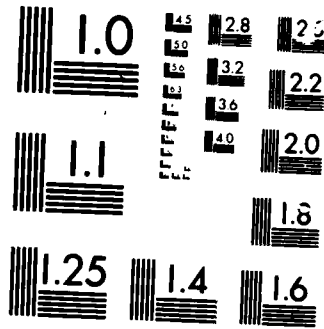
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COMPUTER PROGRAMS

FOR THE DETERMINATION OF

STRESSES AND DISPLACEMENTS

IN FOUR LAYERED SYSTEMS WITH FIXED BOTTOM

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By Dr. Ir. F. Van Cauwelaert, Head of the Department of Civil Engineering  
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CENTRE DE RECHERCHES  
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COMPUTER PROGRAMS  
FOR THE DETERMINATION OF  
STRESSES AND DISPLACEMENTS  
IN FOUR LAYERED SYSTEMS WITH FIXED BOTTOM

By Dr. Ir. F. Van Cauwelaert, Head of the Department of Civil Engineering  
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STRESSES AND DISPLACEMENTS  
IN A FOUR LAYERED SYSTEM  
WITH FIXED BOTTOM

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Introduction

↳ This report deals with the mathematical aspects required for the establishment of a computer program able to calculate all stresses and displacements in a four layered system.

The materials of the different layers may be isotropic or cross-anisotropic.

The interface conditions cover all the cases from full friction to full slip included partial friction.

The bottom of the fourth layer is considered to be fixed (no vertical deflections).

The report is based on

(2) *previous research on*

existing material: ~~isotropic multilayer theory (BURMISTER, 1943)~~  
~~and anisotropic multilayer theory (VAN CAUWELAERT, 1983).~~

*and (2)*

↳ original research <sup>on</sup> interface conditions (fixed bottom, partial friction) and satisfactory convergency, thus complete accuracy, at the surface and in the first layer of the system.

This report contains ~~three parts~~ (1)

Part 1: a theoretical outline where<sup>e</sup> in the basic equations are given, the specific boundary conditions discussed and the particular numerical problems related to the accuracy at the surface and in the first layer <sup>are</sup> solved. (2)

Part 2: the general mathematical analysis of the chosen numerical solution and a description of the programs and their utilization. *and (3-)*

Part 3: (appendices) the detailed mathematical and algebra<sup>a</sup>l analyses, ~~esses~~ for the different considered cases, isotropic or anisotropic, full slip or not.

The programs are written in FORTRAN 77 and run on IBM PC or ~~all~~ other compatible equipment. *Keywords: Pavement design.*

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## SUMMARY

### Introduction

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- 3 : Analysis of the anisotropic case with full and partial friction
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- 5 : Explanatory notice.

PART 1. THEORETICAL OUTLINE

1.1 THE BASIC EQUATIONS AND FUNDAMENTAL HYPOTHESES

The stresses and displacements in a layer of a multilayered system submitted to a vertical flexible load are obtained for an isotropic body from following stressfunction (BURMISTER, 1943) :

$$\phi = pa \int_0^{\infty} \frac{J_0(mr) \cdot J_1(ma)}{m} [A_i e^{mz} - B_i e^{-mz} + zC_i e^{mz} - zD_i e^{-mz}] dm$$

and are given by

$$\sigma_z = pa \int_0^{\infty} J_0(mr) \cdot J_1(ma) [A_i m^2 e^{mz} + B_i m^2 e^{-mz} - C_i m(1 - 2\nu_i - mz)e^{mz} + D_i m(1 - 2\nu_i + mz)e^{-mz}] dm$$

$$\sigma_r = -pa \int_0^{\infty} J_0(mr) \cdot J_1(ma) [A_i m^2 e^{mz} + B_i m^2 e^{-mz} + C_i m(1 + 2\nu_i + mz)e^{mz} - D_i m(1 + 2\nu_i - mz)e^{-mz}] dm$$

$$+ pa \int_0^{\infty} \frac{J_1(mr) \cdot J_1(ma)}{mr} [A_i m^2 e^{mz} + B_i m^2 e^{-mz} + C_i m(1 + mz)e^{mz} - D_i m(1 - mz)e^{-mz}] dm$$

$$\sigma_\theta = -pa \int_0^{\infty} J_0(mr) \cdot J_1(ma) [C_i m \cdot e^{mz} - D_i m \cdot e^{-mz}] \cdot 2\nu_i \cdot dm$$

$$- pa \int_0^{\infty} \frac{J_1(mr) \cdot J_1(ma)}{mr} [A_i m^2 e^{mz} + B_i m^2 e^{-mz} + C_i m(1 + mz)e^{mz} - D_i m(1 - mz)e^{-mz}] dm$$

$$\tau_{rz} = -pa \int_0^{\infty} J_1(mr) \cdot J_1(ma) [A_i m^2 e^{mz} - B_i m^2 e^{-mz} + C_i m(2\nu_i + mz)e^{mz} - D_i m(2\nu_i - mz)e^{-mz}] dm$$

$$w = \frac{1+\mu_i}{E_i} p a \int_0^{\infty} \frac{J_0(mr) \cdot J_1(ma)}{m} [A_i m^2 e^{mz} - B_i m^2 e^{-mz} - C_i m (2 - 4\mu_i - mz) e^{mz} - D_i m (2 - 4\mu_i + mz) e^{-mz}] dm$$

$$u = -\frac{1+\mu_i}{E_i} p a \int_0^{\infty} \frac{J_1(mr) \cdot J_1(ma)}{m} [A_i m^2 e^{mz} + B_i m^2 e^{-mz} + C_i m (1 + mz) e^{mz} - D_i m (1 - mz) e^{-mz}] dm$$

where

- $a$  is the radius of a circular uniformly distributed load
- $p$  is the value of the vertical pressure
- $r$  is the horizontal distance from the axis in a cylindrical coordinate system
- $z$  is the depth
- $\sigma_z$  is the vertical stress
- $\sigma_r$  is the horizontal radial stress,  $\sigma_\theta$  is the circumferential stress
- $\tau_{rz}$  is the shearstress
- $w$  is the vertical deflection
- $u$  is the radial (horizontal) displacement
- $E_i$  is the Young modulus of the concerned layer
- $\mu_i$  is the Poisson's ratio of the concerned layer
- $A_i, D_i$  are unknown parameters to be determined by the boundary conditions
- $J_0$  is the Besselfunction of the first kind of order zero
- $J_1$  is the Besselfunction of the first kind of order one
- $m$  is an integrating parameter

In the case of a cross anisotropic body, they are obtained from  
(VAN CAUWELAERT, 1983) :

$$\phi = p a \int_0^{\infty} \frac{J_0(mr) \cdot J_1(ma)}{m} [A_i e^{mz} - B_i e^{-mz} + C_i e^{m s_i z} - D_i e^{-m s_i z}] dm$$

This stress function differs fundamentally from the preceding one : indeed in putting  $s_i = 1$  in it, we do not obtain the stress function for the isotropic case. We conclude that the two cases must be handled in a separate way.

The stresses and displacements are given by

$$\sigma_z = pa \int_0^{\infty} J_0(mr) \cdot J_1(ma) [n_i(1 + \mu_i) (A_i m^2 e^{mz} + B_i m^2 e^{-mz}) + n_i(n_i + \mu_i) (C_i \delta_i m^2 e^{\delta_i m z} + D_i \delta_i m^2 e^{-\delta_i m z})] dm$$

$$\sigma_r = -pa \int_0^{\infty} J_0(mr) \cdot J_1(ma) [n_i(1 + \mu_i) (A_i m^2 e^{mz} + B_i m^2 e^{-mz}) + \frac{n_i(n_i - \mu_i^2)}{n_i - \mu_i} (C_i \delta_i m^2 e^{\delta_i m z} + D_i \delta_i m^2 e^{-\delta_i m z})] dm + pa \int_0^{\infty} \frac{J_1(mr) \cdot J_1(ma)}{mr} n_i(1 + \mu_i) [A_i m^2 e^{mz} + B_i m^2 e^{-mz} + C_i \delta_i m^2 e^{\delta_i m z} + D_i \delta_i m^2 e^{-\delta_i m z}] dm$$

$$\sigma_\theta = -pa \int_0^{\infty} J_0(mr) \cdot J_1(ma) [C_i \delta_i m^2 e^{\delta_i m z} + D_i \delta_i m^2 e^{-\delta_i m z}] \frac{n_i \mu_i (1 - n_i)}{n_i - \mu_i} - pa \int_0^{\infty} \frac{J_1(mr) \cdot J_1(ma)}{mr} [A_i m^2 e^{mz} + B_i m^2 e^{-mz} + C_i \delta_i m^2 e^{\delta_i m z} + D_i \delta_i m^2 e^{-\delta_i m z}] n_i(1 + \mu_i) dm$$

$$\tau_{rz} = -pa \int_0^{\infty} J_1(mr) \cdot J_1(ma) [n_i(1 + \mu_i) (A_i m^2 e^{mz} - B_i m^2 e^{-mz}) + n_i \delta_i (n_i + \mu_i) (C_i \delta_i m^2 e^{\delta_i m z} - D_i \delta_i m^2 e^{-\delta_i m z})] dm$$

$$w = \frac{1 + \mu_i}{E_i} pa \int_0^{\infty} \frac{J_0(mr) \cdot J_1(ma)}{m} n_i(1 + \mu_i) (A_i m^2 e^{mz} - B_i m^2 e^{-mz}) + \frac{n_i \delta_i (n_i + \mu_i)^2}{(1 + \mu_i)} (C_i \delta_i m^2 e^{\delta_i m z} - D_i \delta_i m^2 e^{-\delta_i m z})] dm$$

$$u = \frac{(1 + \mu_i) n_i (n_i + \mu_i)}{E_i} p a \int_0^{\infty} \frac{J_1(ma) \cdot J_1(mz)}{m} [A_i m^2 e^{mz} + B_i m^2 e^{-mz} + C_i s_i m^2 e^{s_i m z} + D_i s_i m^2 e^{-s_i m z}] dm$$

where

$n_i = E_{vi} / E_{hi}$  is the degree of anisotropy, the ratio between the vertical and the horizontal Young moduli of the concerned layer ;

$\mu_i$  is Poisson's ratio expressing a strain in the horizontal plane induced by a stress in the vertical direction ;

$s_i = \sqrt{\frac{n_i - \mu_i^2}{n_i - \mu_i^2}}$  is the index of anisotropy.

The anisotropic relations are established in the assumption that

- the shear modulus in the vertical plane,  $G_{hz}$ , is related to the other elastic constants by (BARDEN, 1963 ; VAN CAUWELAERT, 1983) :

$$\frac{1}{G_{hzi}} = \frac{1 + n_i + 2\mu_i}{E_i}$$

- Poisson's ratio in the horizontal plane,  $\nu$ , is related to Poisson's ratio in the vertical plane by (FTIMIE, 1973) :

$$\nu_i = \mu_i / n_i$$

## 1.2. THE BOUNDARY CONDITIONS

### 1.2.1. The partial friction condition

Let us consider an layered system, consisting in  $(n - 1)$  layers of a finite thickness built on a semi-infinite body.

For each layer exists a stress function  $\phi_i (A_i B_i C_i D_i)$  with 4 unknown parameters : the total of unknown parameters is  $4n$ .

Two parameters depend on the shape of the load at the surface

$$\sigma_z = f(p) \quad \text{for } z \leq a$$

$$\tau_{xz} = 0$$

At infinite depth stresses and displacements must vanish and thus  $A_n$  and  $C_n = 0$ .

We remain with  $4(n - 1)$  parameters to be determined with 4 conditions at each interface.

The hypothesis is introduced at this stage that under effect of the load, the layers remain individually fully in contact, what we express by imposing that at the bottom of each layer and at the surface of next layer vertical stresses ( $\sigma_z$ ), shearstresses ( $\tau_{xz}$ ) and vertical displacements ( $w$ ) are identic. The fourth interface condition depends on the relative adhesion in the horizontal plane between the considered layers.

The two extremes are

- full continuity, expressed by setting that the horizontal displacement ( $u$ ) are identic ;
- frictionless interface by considering the interface as a principal plane and thus by setting the shearstresses equal zero.

Partial adhesion has been temptatively introduced by several authors, utilizing, in the same way as WESTERGAARD (1926), a relation between horizontal displacements and shearstress :

$$K (u_i - u_{i+1}) = \tau_{xz i}$$

where  $u_i$  is the horizontal displacement at the bottom of the  $i$ -th layer and  $u_{i+1}$  that at the surface of the  $(i+1)$ -th layer.

We shall prove that such a relation cannot be correct in the case of a multilayer.

One has, for an isotropic body, following relations between displacements, shearstrains and shearstresses :

$$\frac{\partial u_i}{\partial z} + \frac{\partial w_i}{\partial h} = \gamma_{hz_i} = [ 2 (1 + \nu_i) / E_i ] \tau_{hz_i}$$

$$\frac{\partial u_{i+1}}{\partial z} + \frac{\partial w_{i+1}}{\partial h} = \gamma_{hz_{i+1}} = [ 2 (1 + \nu_{i+1}) / E_{i+1} ] \tau_{hz_{i+1}}$$

We know from the interface boundary conditions that

$$w_i = w_{i+1} \quad \tau_{hz_i} = \tau_{hz_{i+1}}$$

and thus that

$$\frac{\partial}{\partial z} (u_i - u_{i+1}) = 2 \left[ \frac{1+\nu_i}{E_i} - \frac{1+\nu_{i+1}}{E_{i+1}} \right] \tau_{hz_i}$$

$$(u_i - u_{i+1}) = 2 \left[ \frac{1+\nu_i}{E_i} - \frac{1+\nu_{i+1}}{E_{i+1}} \right] \int \tau_{hz_i} \cdot dz$$

On the other hand

$$K (u_i - u_{i+1}) = \tau_{hz_i}$$

so that

$$2 K \left[ \frac{1+\nu_i}{E_i} - \frac{1+\nu_{i+1}}{E_{i+1}} \right] \int \tau_{hz_i} \cdot dz = \tau_{hz_i}$$

which solution is

$$\tau_{hz_i} = \exp \left[ 2K \cdot z \left( \frac{1+\nu_i}{E_i} - \frac{1+\nu_{i+1}}{E_{i+1}} \right) \right] \cdot f(h)$$

Comparing this solution with the relation above for the shearstress, we conclude that the obtained expression cannot be deduced from a stress-function that is still solution of the compatibility equations.

Compatibility is thus not respected and relation  $K(u_i - u_{i+1}) = \tau_{hz_i}$  cannot be accepted.

Our meaning is that the only way to express partial adhesion consists in writing

$$u_i = Ku_{i+1}$$

with  $K \in ]0, \infty[$

When  $K = 1$ , one has full continuity

$K \neq 1$ , one has partial continuity (normally  $K > 1$ )

Full slip is then a completely other case, for which another program has to be written, what the before mentioned authors tried to avoid by introducing their particular relation.

### 1.2.2. The fixed bottom condition

The boundary conditions discussed in previous alinea implicate that the last layer of the multilayer is considered as a semi-infinite body. One can also consider the case of a multilayer built on an undeformable body, that thus any vertical displacement vanishes at the contact face with the undeformable body : we shall call this a fixed bottom condition. This condition can be introduced in several manners and thus demand a detailed analysis.

A vertical displacement is obtained by integration of the vertical strain :

$$w = \int \epsilon_z \cdot dz$$

It would not be correct to resort to an integration between limits, such as

$$w = \int_0^H \epsilon_z \cdot dz$$

where  $H$  could, for example, be the depth at which we want the bottom to be fixed. In doing so we would ignore the contribution (zero or not) to the vertical deflection (or displacement) due to other parts of the body that we neglect by integrating between specified limits.

The correct way consists in writing (TIMOSHENKO, 1970) :

$$w = \int \epsilon_z \cdot dz + f(x)$$

where  $f(x)$  is a function of  $x$  only, and thus a constant regarding  $z$ , so that by differentiating we obtain again

$$\frac{\partial w}{\partial z} = \epsilon_z$$

By choosing an appropriate expression for  $f(\kappa)$ , we obtain the bottom fixed at the desired depth : by doing this, we introduce in fact a geometrical condition fixing the reference level for the vertical deflections at the chosen depth.

But with the general solutions of the compatibility equations in multi-layer theory, we can do it also in another way by expressing that  $w = 0$  at the desired depth and determining the corresponding values of the parameters  $A_n, B_n, C_n, D_n$ .

Since there are thus several possibilities to express a same boundary condition, it is necessary to compare the results obtained and retain the one that seems the most appropriate.

To do this, we shall consider the most simple case, that of the semi-infinite body : the one-layer case.

#### a) Basic equations

Let us consider a semi-infinite anisotropic body submitted to a uniform distributed vertical pressure at its surface.

The stressfunction is

$$\phi = p\alpha \int_0^{\infty} \frac{J_0(mr) \cdot J_1(ma)}{m} [Ae^{mz} - Be^{-mz} + Ce^{msz} - De^{-msz}] dm$$

The surface boundary conditions ( $\sigma_z = p, \tau_{rz} = 0, z = 0$ ) are induced from the relations given in § 1.1. for the stresses

$$n(1+\mu)(Am^2 + Bm^2) + n(n+\mu)(Csm^2 + Dsm^2) = 1$$

$$n(1+\mu)(Am^2 - Bm^2) + ns(n+\mu)(Csm^2 - Dsm^2) = 0$$

Solving this system for  $B$  and  $D$ , one obtains :

$$n(1-s)(1+\mu)Bm^2 = -s + n(1+s)(1+\mu)Am^2 + 2ns(n+\mu)Csm^2$$

$$n(1-s)(n+\mu)Dsm^2 = 1 - 2n(1+\mu)Am^2 - n(1+s)(n+\mu)Csm^2$$

The next step depends on the boundary condition that fixes the deflection at a depth  $H$ .

b) Fixed bottom expressed by zero deflection

Referring to the stress and displacements equations given in paragraph 1.1 the condition  $w = 0$  at a depth  $H$  is written

$$n(1+\mu) (Am^2 e^{mH} - Bm^2 e^{-mH}) + \frac{n\delta(n+\mu)^2}{(1+\mu)} (C\delta m^2 e^{\delta mH} - D\delta m^2 e^{-\delta mH}) = 0$$

Replacing  $B$  and  $D$  by their values

$$\begin{aligned} Am^2 \left[ n(1+\mu)e^{mH} - \frac{n(1+\mu)(1+\mu)}{(1-\delta)} e^{-mH} + \frac{2n\delta(n+\mu)}{(1-\delta)} e^{-\delta mH} \right] \\ + C\delta m^2 \left[ -\frac{n\delta(n+\mu)^2}{(1+\mu)} e^{\delta mH} - \frac{2n\delta(n+\mu)}{(1-\delta)} e^{-mH} + \frac{n\delta(1+\delta)(n+\mu)^2}{(1+\mu)(1-\delta)} e^{-\delta mH} \right] \\ = \frac{\delta(n+\mu)}{(1+\mu)} \frac{1}{1-\delta} e^{-\delta mH} - \frac{\delta}{1-\delta} e^{-mH} \end{aligned}$$

During the integration process, the value of  $m$  tends to infinity, so that the limit expression of the equation becomes

$$\lim_{m \rightarrow \infty} \left[ Am^2(1+\mu)^2 e^{mH} + C\delta m^2 \delta(n+\mu)^2 e^{\delta mH} \right] = 0$$

This relation can only be satisfied by setting

$$C = 0 \quad \text{when} \quad \delta > 1$$

$$A = 0 \quad \text{when} \quad \delta < 1$$

It is easily shown that, in the case of an isotropic body, one must always have  $C = 0$ , because of the factor  $z$  multiplying  $C$  in the stressfunction. Taking  $\delta < 1$ ,  $C$  becomes

$$n\delta(n+\mu) C\delta m^2 = \frac{e^{-\delta mH} [(1+\mu)e^{-mH} - (n+\mu)e^{-\delta mH}]}{2(1+\mu)e^{-mH} e^{-\delta mH} - (n+\mu)(1-\delta) - (1+\delta)(n+\mu)e^{-2\delta mH}}$$

The expression of the deflection at the surface and in the axis of the load [ $J_0(mr) = 1$  for  $r = 0$ ] is

$$w = \frac{1+\mu}{E} \cdot p\alpha \int_0^\infty \frac{J_1(ma)}{m} [1 - 2n\delta(n+\mu) C\delta m^2] dm$$

c) Fixed bottom by an appropriate function

The vertical deflection at a depth  $z$  is given by

$$w = \frac{1+\mu}{E} \cdot pa \int_0^{\infty} \frac{J_0(mr) \cdot J_1(ma)}{m} [ n(1+\mu) (Am^2 e^{mz} - Bm^2 e^{-mz}) + \frac{n\delta(n+\mu)^2}{(1+\mu)} (C\delta m^2 e^{\delta mz} - D\delta m^2 e^{-\delta mz}) ] dm$$

We now choose an appropriate function  $f(\lambda)$  so that  $w$  becomes zero for  $z = H$

$$f(\lambda) = - \frac{1+\mu}{E} \cdot pa \int_0^{\infty} \frac{J_0(mr) \cdot J_1(ma)}{m} [ n(1+\mu) (Am^2 e^{mH} - Bm^2 e^{-mH}) + \frac{n\delta(n+\mu)^2}{(1+\mu)} (C\delta m^2 e^{\delta mH} - D\delta m^2 e^{-\delta mH}) ] dm$$

The final expression for the deflection is then

$$w = w_1 + f(\lambda)$$

One verifies that  $f(\lambda)$  is indeed only a function of  $\lambda$  and that

$$w = 0 \quad \text{for } z = H.$$

For the reasons developed before, one of the parameters  $A$  or  $C$  must be zero.

The other parameter is obtained by a supplementary boundary condition (a mechanical condition) :

$$- \tau_{\lambda z} = 0 \quad \text{at the depth } H$$

$$- u = 0 \quad \text{at the depth } H$$

If we still suppose  $\delta < 1$  and thus  $A = 0$ , one obtains in the case that  $\tau_{\lambda z} = 0$

$$n\delta(n+\mu) C\delta m^2 = \frac{e^{-Hm} \cdot e^{-\delta Hm} - e^{-2\delta Hm}}{2e^{-Hm} \cdot e^{-\delta Hm} - (1-\delta) - (1+\delta)e^{-2\delta Hm}}$$

The deflection at the surface is then

$$\begin{aligned}
 w = & pa \frac{\delta(1-n)}{E(1-\delta)} \int_0^{\infty} \frac{J_1(am)}{m} dm \\
 & + pa \frac{1+\mu}{E} \int_0^{\infty} \frac{J_1(am)}{m} \left\{ \frac{\delta}{(1-\delta)} \frac{(n+\mu)}{(1+\mu)} e^{-\delta llm} - \frac{\delta}{1-\delta} e^{-mll} \right. \\
 & + ns(n+\mu) Csm^2 \left[ \frac{2\delta(n-1)}{(1-\delta)(1+\mu)} + \frac{2\delta}{1-\delta} e^{-mll} \right. \\
 & \left. \left. - \frac{n+\mu}{1+\mu} \delta e^{\delta llm} - \frac{n+\mu}{1+\mu} \delta \frac{1+\delta}{1-\delta} e^{-\delta mll} \right] \right\} dm
 \end{aligned}$$

In the case  $\mu = 0$

$$ns(n+\mu) Csm^2 = \frac{e^{-\delta llm} [(n+\mu)\delta e^{-llm} - (1+\mu)e^{-\delta llm}]}{2\delta(n+\mu)e^{-llm} e^{-\delta llm} + (1+\mu)(1-\delta) - (1+\mu)(1+\delta)e^{-2\delta llm}}$$

The deflection at the surface is given by the same relation as above with the appropriate value for C.

One sees that in the 3 cases, the deflection is composed of a first term

$$\begin{aligned}
 w_- &= pa \frac{\delta(1-n)}{E(1-\delta)} \int_0^{\infty} \frac{J_1(am)}{m} dm \\
 &= pa \frac{\delta(1-n)}{E(1-\delta)}
 \end{aligned}$$

This term is the deflection on top of a semi-infinite body.

In the case of an isotropic body, one has ( $n = 1$ ):

$$w_- = -pa \frac{2(1-\mu^2)}{E}$$

The second term  $w_h$  depends on the chosen boundary condition ; but in this 3 cases it reduces the value of  $w_-$  because of the fixed bottom.

We have computed the values of  $w_-$  and  $w_h$  for different values of  $ll/a$ .

The results are given below, at a factor  $(1+\mu)/E$ , for  $\delta = 0.5$  and for the isotropic case.

$$\delta = 0.5 \quad \omega_{-} = 1.025$$

$H/a$	$w_n(\tau_{xz} = 0)$	$w_n(u = 0)$	$w_n(w = 0)$
1	0.339	0.729	0.153
2	0.120	0.295	0.043
3	0.058	0.147	0.019
4	0.034	0.086	0.011
5	0.022	0.057	0.007
6	0.015	0.040	0.005
7	0.011	0.029	0.004
8	0.009	0.023	0.003
9	0.007	0.018	0.002
10	0.006	0.015	0.002

$$n = 1 \quad \omega_{-} = 1.000$$

1	0.402	0.934	0.276
2	0.143	0.442	0.086
3	0.069	0.231	0.040
4	0.040	0.138	0.023
5	0.026	0.091	0.015
6	0.018	0.064	0.010
7	0.013	0.048	0.008
8	0.010	0.037	0.006
9	0.008	0.029	0.005
10	0.007	0.024	0.004

We conclude that from a depth of about  $H/a = 5$ , the absolute influence of the fixed boundary is negligible. This influence will still be much lesser in the case of a roadstructure where the E-moduli of the layers are sensitively higher than the modulus of the subground.

We conclude also that the relative influence on the deflection is much more important when we fix the horizontal displacements ( $u = 0$ ). This should be the case in a laboratory testpit with lateral walls, but less in the case of a real road where lateral movements are not restricted.

The relative influence of the condition  $\tau_{xz} = 0$  is also more important than that of the condition  $w = 0$ , although less important than the condition  $u = 0$ . It seems nevertheless very unlikely that there would be no friction between the subground and the last layer.

The easiest way to fix the bottom from a mathematical point of view is on the other hand the condition  $w = 0$ .

Taking then into account the little influence of the chosen condition on the deflection at the surface and the fact that conditions  $u = 0$  and  $\tau_{xz} = 0$  have less physical sense, we retain the condition  $w = 0$  as the most indicated fixed bottom condition.

### 1.3. SOLUTION OF PARTICULAR NUMERICAL PROBLEMS

#### 1.3.1. The full slip interface condition

The value of any stress or displacement is obtained from one of the above mentioned relations.

Let us consider, for example, the vertical stress in the  $i$ -th layer of an isotropic layer :

$$\sigma_z = \rho a \int_0^{\infty} J_0(mr) \cdot J_1(ma) [A_i m^2 e^{mz} + B_i m^2 e^{-mz} - C_i m(1-2\nu_i - mz)e^{mz} + D_i m(1-2\nu_i + mz)e^{-mz}] dm$$

The integration can only be performed numerically.

One must thus calculate the value of the stress for a set of values of  $m$  growing from 0 to a value high enough to ensure convergency.

For each value of  $m$ , those of the parameters  $A_i$ ,  $B_i$ ,  $C_i$  and  $D_i$  must be determined out of the set of boundary conditions, a system of  $(4n - 1)$  equations with  $(4n - 1)$  unknowns in the case of a fixed bottom and  $n$  layers above it.

The first programs solved this problem by inverting the matrix of the  $(4n - 1)$  unknowns. Nevertheless the inversion procedure leads in some cases to unsoluble difficulties because of the presence of the negative exponents tending to zero in the determinant of the denominator.

Other programs have tried to avoid the inversion procedure as follows : one chooses appropriate values for  $B_n$  and  $D_n$ , goes through the whole set of equations and verifies in how far the surface conditions are met.

One then chooses another pair of values for  $B_n$  and  $D_n$  and follows the same procedure. Since the whole process is linear, the correct values for  $B_n$  and  $D_n$  can finally be obtained by linear interpolation after two runs. The difficulty lies in the appropriate choice of the values of  $B_n$  and  $D_n$  to ensure a numerically correct interpolation.

However, even those programs are not entirely appropriate for the cases with frictionless conditions at some interfaces.

We shall show this with the most simple case, that of a two layer.

Writing  $A_i$ ,  $B_i$ ,  $C_i$ ,  $D_i$  in stead of  $A_i m^2$ ,  $B_i m^2$ ,  $C_i m$ ,  $D_i m$ , the boundary conditions are, in the case of two isotropic layers, with the origin ( $z = 0$ ) at the interface, the thickness of the first layer being  $H$  and the second layer semi-infinite :

At the surface ( $z = -H$ ) :

$$A_1 e^{-mH} + B_1 e^{mH} - C_1 (1-2\mu_1 + mH) e^{-mH} + D_1 (1-2\mu_1 - mH) e^{mH} = 1$$

$$A_1 e^{-mH} - B_1 e^{mH} + C_1 (2\mu_1 - mH) e^{-mH} + D_1 (2\mu_1 + mH) e^{mH} = 0$$

At the frictionless interface ( $z = 0$ ) :

$$A_1 + B_1 - C_1 (1-2\mu_1) + D_1 (1-2\mu_1) = B_2 + D_2 (1-2\mu_2)$$

$$A_1 - B_1 + 2\mu_1 C_1 + 2\mu_1 D_1 = 0$$

$$-B_2 + 2\mu_2 D_2 = 0$$

$$\frac{1+\mu_1}{E_1} [A_1 - B_1 - 2C_1(1-2\mu_1) - 2D_1(1-2\mu_1)] = \frac{1+\mu_2}{E_2} [-B_2 - 2D_2(1-2\mu_2)]$$

Solving the system for  $C_1$  and  $D_1$ , one obtains (BURMISTER, 1943)

$$C_1 = \left[ (1-F+mH) e^{mH} - (1-F) e^{-mH} \right] \cdot \frac{1}{\nabla}$$

$$D_1 = \left[ F e^{mH} - (F-mH) e^{-mH} \right] \cdot \frac{1}{\nabla}$$

where

$$\nabla = F e^{2mH} + (2F-1) \cdot 2mH - (1+2m^2 H^2) + (1-F) e^{-2mH}$$

$$F = \frac{(1-\mu_2) + n(1-\mu_1)}{2(1-\mu_2)}$$

$$n = \frac{E_2}{E_1} \cdot \frac{(1+\mu_1)}{(1+\mu_2)}$$

$$A_1 = C_1(F-2\mu_1) - D_1(1-F)$$

$$B_1 = C_1 F - D_1(1-2\mu_1-F)$$

The vertical deflection at the surface is

$$w = pa \frac{1+\mu_1}{E_1} \int_0^{\infty} \frac{J_0(mr) \cdot J_1(ma)}{m} [A_1 m^2 e^{-mH} - B_1 m^2 e^{mH} - (2-4\mu_1+mH) C_1 m e^{-mH} - (2-4\mu_1-mH) D_1 m e^{mH}] dm$$

Replacing  $A_1$ ,  $B_1$ ,  $C_1$  and  $D_1$  by their values, the deflection becomes

$$w = -pa \frac{2(1-\mu_1^2)}{E_1} \int_0^{\infty} \frac{J_0(mr) \cdot J_1(ma)}{m} \left[ \frac{F e^{2mH} - (2F-1-2mH) - (1-F)e^{-2mH}}{F e^{2mH} + (2F-1)2mH - (1+2m^2H^2) + (1-F)e^{-2mH}} \right] dm$$

At the origin of the integration ( $m = 0$ ), the term in between brackets becomes indefinite :  $0/0$ .

This has no influence when computing stresses, because the Bessel functions products occurring here are also zero at the origin :  $J_0(mr) \cdot J_1(ma) = 0$  for  $m = 0$ .

But in the case of the deflection  $\lim_{m \rightarrow 0} \frac{J_0(mr) \cdot J_1(ma)}{m} = \frac{a}{2}$

It is therefore absolutely necessary to have the term in between brackets in a sufficient closed form to be able to determine its value for  $m = 0$ .

The importance of the first term of the series is not negligible :

for  $m = 0$  the term in between brackets is equal to  $E_1(1-\mu_2^2)/[E_2(1-\mu_1^2)]$ .

If  $h$  is the interval chosen for the numerical integration, one can then write

$$w = -pa \frac{2(1-\mu_1^2)}{E_1} \left( \frac{1}{2} \frac{E_1(1-\mu_2^2)}{E_2(1-\mu_1^2)} \frac{h}{3} + \int_h^{\infty} \frac{J_0(mr) \cdot J_1(ma)}{m} [ ] dm \right)$$

and, if we make a semi-infinite body from the two layer ( $E_1 = E_2$ ,  $\mu_1 = \mu_2$ ) :

$$w = -pa \frac{2(1-\mu^2)}{E} \left( \frac{h}{6} + \int_0^{\infty} \frac{J_0(mr) \cdot J_1(ma)}{m} [ ] dm \right)$$

Comparing this expression with that for the deflection at the surface of a semi-infinite body

$$w_s = - pa.2(1-\mu^2) / E$$

One concludes that the contribution of the first term  $h/6$  is indeed not negligible, especially when we have in mind that the only practical measurement that can be performed on a real roadstructure is the vertical deflection at the surface.

### 1.3.2. Over and underflow problems

During the integration procedure  $m$  varies from 0 to a value high enough to ensure convergency. We mean by this that the integration procedure can be stopped from the moment on that the terms of the series become so small that they have no more influence on the final result and can thus be neglected.

Practically, however this means that  $m$  can reach quite high values such as 20 or 30 for example.

To illustrate the influence of this, let us go back to the two-layer developed in the preceeding paragraph.

The values of  $C_1$  and  $D_1$ , from which the values of all the other parameters can be deduced, are

$$C_1 = \frac{[(1-F+mH)e^{mH} - (1-F)e^{-mH}]}{Fe^{2mH} + (2F-1).2mH - (1+2m^2H^2) + (1-F)e^{-2mH}}$$

$$D_1 = \frac{[Fe^{mH} - (F-mH)e^{-mH}]}{Fe^{2mH} + (2F-1).2mH - (1+2m^2H^2) + (1-F)e^{-2mH}}$$

The geometrical units are generally expressed in function of  $a$ , the radius of the load.

Let us consider  $H/a = 5$ .

One immediately sees that no computer can handle exponents as  $e^{mH/a}$  and  $e^{2mH/a}$  without overflow occuring for values of  $m$  above 10.

However this problem can easily be solved by dividing both numerator and denominator by  $e^{2mH}$  :

$$C_1 = \frac{[(1-F+mH)e^{-mH} - (1-F)e^{-3mH}]}{F + [(2F-1).2mH - (1+2m^2H^2)]e^{-2mH} + (1-F)e^{-4mH}}$$

$$D_1 = \frac{[Fe^{-mH} - (F-mH)e^{-3mH}]}{F + [(2F-1).2mH - (1+2m^2H^2)]e^{-2mH} + (1-F)e^{-4mH}}$$

Underflow will obviously occur now, but most of the computers have a routine that sets variables subjected to underflow equal to zero.

If such a routine does not exist, it is very easy to build it into the program.

But more interesting is the fact that, having transformed the relations for  $C_1$  and  $D_1$ , convergency will occur quite quickly and in a complete safe way : the numerators tend both to zero, while the denominator tends to a constant  $F$ .

This can be obtained automatically in writing the boundary conditions at the surface ( $z = -H$ ) as follows :

$$A_1 e^{-3mH} + B_1 e^{-mH} - C_1 (1 - 2\mu_1 + mH) e^{-3mH} + D_1 (1 - 2\mu_1 - mH) e^{-mH} = e^{-2mH}$$

$$A_1 e^{-3mH} - B_1 e^{-mH} + C_1 (2\mu_1 - mH) e^{-3mH} + D_1 (2\mu_1 + mH) e^{-mH} = 0$$

However this is only true in the case of a two-layer system.

In a three-layer with  $H_1$ , the thickness of the first layer, and  $H_2$ , the thickness of the second layer, occur exponents such as

$$e^{2mH_1 \cdot 2mH_2} \quad e^{-2mH_1 \cdot 2mH_2}$$

but they eliminate when writing the denominator in close forme so that dividing the expressions by the largest out of  $e^{2mH_1}$  and  $e^{2mH_2}$  is enough. If one should divide by  $e^{2mH_1 \cdot 2mH_2}$ , the denominator would also tend to zero, which should stop the program because of dividing by zero. Here is another reason for writing the equations in closeform for three- and more-layers.

### 1.3.3. The vertical deflection at the surface

When one computes the deflections at the surface, convergency is obtained only very slowly.

To illustrate this, let us look at the expression of the deflection at the surface developed in § 1.3.1.

$$w = -pa \frac{2(1-\nu_1^2)}{E_1} \int_0^{\infty} \frac{J_0(mr) \cdot J_1(ma)}{m} \left( \frac{Fe^{2mH} - (2F-1-2mH) - (1-F)e^{-2mH}}{Fe^{2mH} + (2F-1)2mH - (1+2m^2H^2) + (1-F)e^{-2mH}} \right) dm$$

To avoid overflow problems we divide numerator and denominator by  $e^{2mH}$

$$w = -pa \frac{2(1-\nu_1^2)}{E_1} \int_0^{\infty} \frac{J_0(mr) \cdot J_1(ma)}{m} \left( \frac{F - (2F-1-2mH)e^{-2mH} - (1-F)e^{-4mH}}{F + [(2F-1)2mH - (1+2m^2H^2)]e^{-2mH} + (1-F)e^{-4mH}} \right) dm$$

For large values of  $m$ , numerator and denominator tend both to  $F$ , so that for, let us say  $m = m_L$ , the expression above could be written as follows :

$$w = -pa \frac{2(1-\nu_1^2)}{E_1} \int_0^{m_L} \frac{J_0(mr) \cdot J_1(ma)}{m} \left( \frac{F - (2F-1-2mH)e^{-2mH} - (1-F)e^{-4mH}}{F + [(2F-1)2mH - (1+2m^2H^2)]e^{-2mH} + (1-F)e^{-4mH}} \right) dm - pa \frac{2(1-\nu_1^2)}{E_1} \int_{m_L}^{\infty} \frac{J_0(mr) \cdot J_1(ma)}{m} dm \quad (1)$$

The first integral converges fast, the second converges proportionally to  $1/m$ . This means that if one should want a result correct at  $10^{-5}$ , one has to perform the numerical integration of the second integral until values of  $m$  above 100,000 !

This is practically impossible.

One could of course be tempted to interrupt the integration procedure when the first integral has converged to a satisfactory level, "hoping" that the second integral can be neglected at that moment.

To illustrate the danger of such an approach, we return to the semi-infinite body.

In the case of an isotropic body, the deflection at the surface and the vertical stress at a depth  $z$  are given in the axis of the load by

$$w = -pa \frac{2(1-\nu^2)}{E} \int_0^{\infty} \frac{J_1(ma)}{m} dm \quad (2)$$

$$\sigma_z = pa \int_0^{\infty} J_1(ma) (1-mz) e^{-mz} dm \quad (3)$$

Those integrals can of course be solved analytically

$$w = -pa \frac{2(1-\nu^2)}{E} \quad (4)$$

$$\sigma_z = p \left[ 1 - \frac{z^3}{(a^2 + z^2)^{3/2}} \right] \quad (5)$$

We can compare (2) with the second integral of (1) and (3) with the first integral of (1).

We perform then a numerical integration of (2) and (3) and stop the procedure when (3) has converged to a satisfactory level, which is easily checked by comparing the obtained result with the correct one given by (5). The difference between the numerical result for  $w$  obtained at that moment by integration of (2) with the analytical result given by (4) will give us an illustration of the possible error when integrating (1) and stopping the process when its first integral has converged.

This difference is illustrated on figure 1.

In absciss, we have the convergency level adopted for the vertical stress and in ordinate the error, expressed in %, on the values of the vertical stress and the vertical deflection.

One sees that, for even such low levels as  $10^{-3}$ , the value of the vertical stress is absolutely correct, while the error on the deflection varies between - 5% and + 8%, depending on the chosen convergency level and the relative depth at which the vertical stress is computed; the worst of all is that we have no means to predict either the direction either the amplitude of the error on  $w$ .

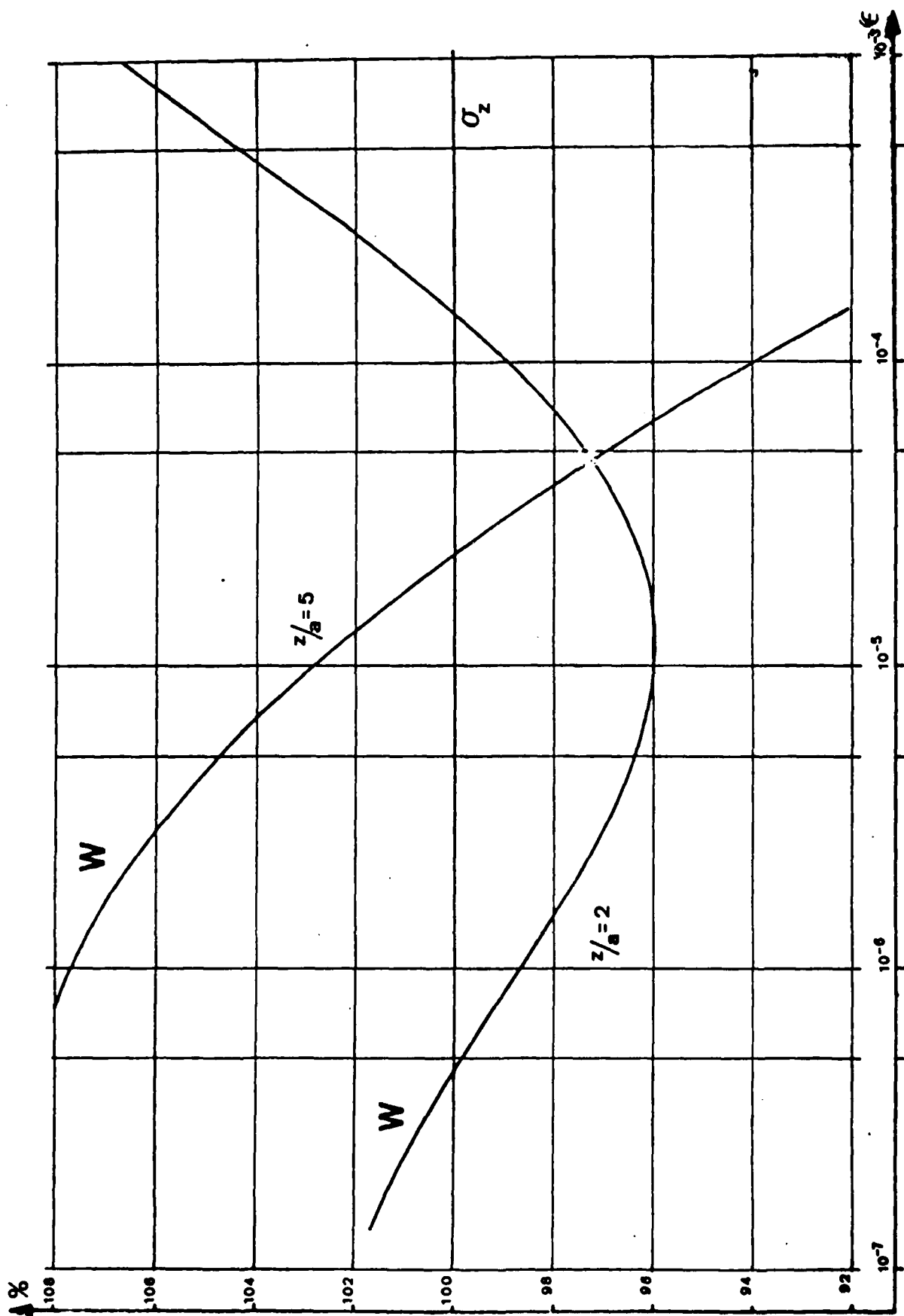


FIGURE 1.

The only way to solve the problem satisfactory is to split the expression of the deflection in another way than the one we had done.

We write first the expression of the deflection with negative exponents only :

$$w = -pa \frac{2(1-\nu_1^2)}{E_1} \int_0^\infty \frac{J_0(mr) \cdot J_1(ma)}{m} \left[ \frac{F - (2F-1-2mH)e^{-2mH} - (1-F)e^{-4mH}}{+ [(2F-1)2mH - (1+2m^2H^2)] e^{-2mH} + (1-F)e^{-4mH}} \right] dm$$

We divide then the numerator of the term in between brackets by the denominator :

$$w = -pa \frac{2(1-\nu_1^2)}{E} \int_0^\infty \frac{J_0(mr) \cdot J_1(ma)}{m} \left( 1 + 2 \frac{[(1-F)(1+mH) + m^2H^2] e^{-2mH} - (1-F)e^{-4mH}}{F + [(2F-1)2mH - (1+2m^2H^2)] e^{-2mH} + (1-F)e^{-4mH}} \right) dm$$

and split the integral into two parts from which the first is integrable analytically and the second converges in the usual way.

$$\text{For } r = 0, \text{ one has } \int_0^\infty \frac{J_1(ma)}{m} dm = 1$$

$$\text{For } r = a, \text{ one has } \int_0^\infty \frac{J_0(mr) \cdot J_1(ma)}{m} dm = 2/\pi$$

$$\text{For } r < a, \text{ one has } \int_0^\infty \frac{J_0(mr) \cdot J_1(ma)}{m} dm = F(1/2, -1/2; 1; \frac{r^2}{a^2})$$

where  $F$  is the hypergeometric function of GAUSS :

$$F(a, b; e; z) = \sum_0^\infty \frac{(a)_n (b)_n}{n! (c)_n} z^n$$

$$(a)_n = a(a+1)(a+2) \dots (a+n-1)$$

$$(a)_0 = 1$$

For  $\kappa > a$ , one has 
$$\int_0^\infty \frac{J_0(m\kappa) \cdot J_1(ma)}{m} dm = \frac{a}{\kappa^2} F\left(1/2, 1/2; 2; \frac{a^2}{\kappa^2}\right)$$

The obtained result will now be correct, while convergency is reached as fast as for the other equations for stresses.

But again, if we will be able to compute as indicated, we must dispose over the equations in closeform, although in such a form that the integral can be split.

### 1.3.4. Accuracy in the first layer

As for the deflection at the surface, the numerical computation of the stresses in the first layer, in fact nearby the surface, converges also very slowly.

To illustrate this let us look at the relation for the vertical stress in the first layer (the equation is given in § 1.1) of a two layer :

$$\sigma_z = pa \int_0^{\infty} J_0(mr) \cdot J_1(ma) [ A_1 m^2 e^{mz} + B_1 m^2 e^{-mz} - C_1 m(1-2\nu_1 - mz) e^{mz} + D_1 m(1-2\nu_1 + mz) e^{-mz} ] dm$$

We replace  $A_1, B_1, C_1, D_1$  by their values obtained in § 1.3.1.

$$\sigma_z = pa \int_0^{\infty} J_0(mr) \cdot J_1(ma) \left\{ F [1 + m(H+z)] e^{-mH} \cdot e^{-mz} + [ ] e^{-mH} \cdot e^{mz} + [ ] e^{-3mH} \cdot e^{mz} + [ ] e^{-3mH} \cdot e^{-mz} \right\} \frac{dm}{\nu}$$

The values of  $z$  are negative in the first layer ( $z = 0$  at the interface). The term  $F [1 + m(H+z)] e^{-mH} \cdot e^{-mz} \cdot \frac{1}{\nu}$  converges very slowly for values of  $-z$  nearly equal to  $H$ , the other terms converge normally.

To solve the problem created by the first term we divide again numerator by denominator :

$$F [1 + m(H+z)] e^{-mH} \cdot e^{-mz} \cdot \frac{1}{\nu} = [1 + m(H+z)] e^{-mH} \cdot e^{-mz} \frac{[1 + m(H+z)] e^{-mH} \cdot e^{-mz} [ [(2F-1)2mH - (1+2m^2H^2)] e^{-2mH} + (1-F)e^{-4mH} ]}{F + [(2F-1)2mH - (1+2m^2H^2)] e^{-2mH} + (1-F)e^{-4mH}}$$

The second term of the second member converges normally so that we can again split the integral in several parts from which the one that converges slowly is

$$pa \int_0^{\infty} J_0(mr) \cdot J_1(ma) [1 + m(H+z)] e^{-mH} \cdot e^{-mz} dm$$

This integral is known as a LIPSCHITZ-HANKEL integral, but only some particular cases are integrable analytically. To solve the problem for all cases we have to make a detour through the analysis of stresses and displacements in a semi-infinite body submitted to an isolated local force P.

### 1.3.5. Stresses and displacements under an isolated load

The HANKEL transform in the case of a uniform distributed load is

$$\Psi(m) = pa \int_0^{\infty} \frac{J_1(am)}{m} \phi \, dm$$

We consider the resulting load  $P = \pi pa^2$  acting on a surface whose area reduces to zero.

$$\begin{aligned} \Psi(m) &= \lim_{a \rightarrow 0} \frac{P}{\pi} \int_0^{\infty} \frac{J_1(am)}{am} \phi \, dm \\ &= \frac{P}{2\pi} \int_0^{\infty} \phi \, dm \end{aligned}$$

The relations for the stresses and the displacements are then given, in the case of an isotropic body, by :

$$\begin{aligned} \sigma_r &= \frac{P}{2\pi} \left[ \int_0^{\infty} m J_0(mr) e^{-mz} \, dm - z \int_0^{\infty} m^2 J_0(mr) e^{-mz} \, dm \right. \\ &\quad \left. - (1-2\mu) \int_0^{\infty} \frac{J_1(mr)}{r} e^{-mz} \, dm + z \int_0^{\infty} m \frac{J_1(mr)}{r} e^{-mz} \, dm \right] \\ &= \frac{P}{2\pi} \left[ \frac{z}{(z^2+r^2)^{3/2}} - \left[ \frac{2z}{(z^2+r^2)^{3/2}} - \frac{3zr^2}{(z+r)^{5/2}} \right] \right. \\ &\quad \left. - (1-2\mu) \frac{(r^2+z^2)^{1/2} z}{r^2 (r^2+z^2)^{1/2}} + \frac{z}{(z^2+r^2)^{3/2}} \right] \end{aligned}$$

$$\begin{aligned} \sigma_z &= \frac{P}{2\pi} \left[ \int_0^{\infty} m J_0(mr) e^{-mz} \, dm + z \int_0^{\infty} m^2 J_0(mr) e^{-mz} \, dm \right] \\ &= \frac{P}{2\pi} \left[ \frac{z}{(z^2+r^2)^{3/2}} + \left[ \frac{2z}{(z^2+r^2)^{3/2}} - \frac{3zr^2}{(z^2+r^2)^{5/2}} \right] \right] \end{aligned}$$

$$\tau_{rz} = \frac{Pz}{2\pi} \int_0^{\infty} m^2 J_1(mr) e^{-mz} dm = \frac{P}{2\pi} \frac{3z^2 r}{(r^2 + z^2)^{3/2}}$$

$$\begin{aligned} w &= -\frac{P}{2\pi} \frac{2(1-\mu^2)}{E} \int_0^{\infty} J_0(mr) e^{-mz} dm - \frac{Pz}{2\pi} \frac{(1+\mu)}{E} \int_0^{\infty} m J_0(mr) e^{-mz} dm \\ &= -\frac{P}{2\pi} \frac{2(1-\mu^2)}{E} \frac{1}{(z^2 + r^2)^{1/2}} - \frac{P}{2\pi} \frac{(1+\mu)}{E} \frac{z}{(z^2 + r^2)^{3/2}} \end{aligned}$$

$$\begin{aligned} u &= \frac{P}{2\pi} \frac{(1+\mu)(1-2\mu)}{E} \int_0^{\infty} J_1(mr) e^{-mz} dm - \frac{Pz}{2\pi} \frac{(1+\mu)}{E} \int_0^{\infty} m J_1(mr) e^{-mz} dm \\ &= \frac{P}{2\pi} \frac{(1+\mu)(1-2\mu)}{E} \cdot \frac{(r^2 + z^2)^{1/2} z}{r (r^2 + z^2)^{1/2}} - \frac{P}{2\pi} \frac{(1+\mu)}{E} \cdot \frac{zr}{(z^2 + r^2)^{3/2}} \end{aligned}$$

### 1.3.6. Stresses and displacements under a uniform distributed load

The stresses and displacements under a uniform distributed load can be obtained by integrating the relations under an isolated load over the concerned area

$$\sigma_D = \int_0^{2\pi} \int_0^a \sigma_I \rho d\rho d\theta$$

where  $\sigma_I$  is given by one of the relations of § 1.3.5. wherein the distance  $r$  must be replaced by  $(r^2 + \rho^2 - 2r\rho \cos \theta)^{1/2}$

a) Relations for the stress  $\sigma_r$

The relation for  $\sigma_r$  under a distributed load is given by

$$\begin{aligned} \sigma_r = & pa \int_0^{\infty} J_0(mr) \cdot J_1(ma) e^{-mz} dm - pa \int_0^{\infty} J_0(mr) \cdot J_1(ma) mz e^{-mz} dm \\ & - pa (1-2\mu) \int_0^{\infty} \frac{J_1(mr)}{mr} J_1(ma) e^{-mz} dm + pa \int_0^{\infty} \frac{J_1(mr)}{r} J_1(ma) z e^{-mz} dm \end{aligned}$$

so that comparing with the expression for the stress under an isolated load we can conclude that :

$$I_1 : pa \int_0^{\infty} J_0(mr) \cdot J_1(ma) e^{-mz} dm = \frac{p}{2\pi} \int_0^{2\pi} \int_0^a \frac{z \rho}{(z^2 + r^2 + \rho^2 - 2r\rho \cos \theta)^{3/2}} d\rho d\theta$$

$$\begin{aligned} I_2 : & pa \int_0^{\infty} J_0(mr) \cdot J_1(ma) mz e^{-mz} dm \\ & = \frac{p}{2\pi} \int_0^{2\pi} \int_0^a \left[ \frac{2z\rho}{(z^2 + r^2 + \rho^2 - 2r\rho \cos \theta)^{3/2}} - \frac{3z(r^2 + \rho^2 - 2r\rho \cos \theta)\rho}{(z^2 + r^2 + \rho^2 - 2r\rho \cos \theta)^{5/2}} \right] d\rho d\theta \end{aligned}$$

$$\begin{aligned} I_3 : & pa \int_0^{\infty} \frac{J_1(mr)}{mr} J_1(ma) e^{-mz} dm = \\ & = \frac{p}{2\pi} \int_0^{2\pi} \int_0^a \frac{(z^2 + r^2 + \rho^2 - 2r\rho \cos \theta)^{1/2} - z}{(r^2 + \rho^2 - 2r\rho \cos \theta) (z^2 + r^2 + \rho^2 - 2r\rho \cos \theta)^{1/2}} \rho d\rho d\theta \end{aligned}$$

$$I_4 : pa \int_0^{\infty} \frac{J_1(mr)}{r} \cdot J_1(ma) z e^{-mz} dm = \frac{p}{2} \int_0^{2\pi} \int_0^a \frac{z \rho d\rho d\theta}{(z^2 + r^2 + \rho^2 - 2r\rho \cos \theta)^{3/2}}$$

b) Relations for the stress  $\sigma_z$

These relations can be deduced from those established for the stress  $\sigma_r$ .

c) Relation for the stress  $\tau_{rz}$

$$\tau_{rz} = pa \int_0^{\infty} J_1(mr) \cdot J_1(ma) m z e^{-mz} dm$$

$$I_5 : pa \int_0^{\infty} J_1(mr) \cdot J_1(ma) m z e^{-mz} dm = \frac{p}{2\pi} \int_0^{2\pi} \int_0^a \frac{3z^2 (\kappa^2 + \rho^2 - 2\kappa\rho \cos \theta)^{1/2}}{(z^2 + \kappa^2 + \rho^2 - 2\kappa\rho \cos \theta)^{5/2}} \rho d\rho d\theta$$

d) Relations for the vertical displacement  $w$

$$w = -pa \frac{2(1-\nu^2)}{E} \int_0^{\infty} \frac{J_0(mr) \cdot J_1(ma)}{m} e^{-mz} dm$$

$$- pa \frac{(1+\nu)}{E} \int_0^{\infty} J_0(mr) \cdot J_1(ma) z e^{-mz} dm$$

$$I_6 : pa \int_0^{\infty} \frac{J_0(mr) \cdot J_1(ma)}{m} e^{-mz} dm = \frac{p}{2\pi} \int_0^{2\pi} \int_0^a \frac{\rho d\rho d\theta}{(z^2 + \kappa^2 + \rho^2 - 2\kappa\rho \cos \theta)^{1/2}}$$

$$I_7 : pa \int_0^{\infty} J_0(mr) \cdot J_1(ma) z e^{-mz} dm = \frac{p}{2\pi} \int_0^{2\pi} \int_0^a \frac{z^2 \rho d\rho \cdot d\theta}{(z^2 + \kappa^2 + \rho^2 - 2\kappa\rho \cos \theta)^{3/2}}$$

e) Relations for the horizontal displacement  $u$

$$u = pa \frac{(1+\nu)(1-2\nu)}{E} \int_0^{\infty} \frac{J_1(mr) \cdot J_1(ma)}{m} e^{-mz} dm$$

$$- pa \frac{(1+\nu)}{E} \int_0^{\infty} J_1(mr) \cdot J_1(ma) z e^{-mz} dm$$

$$I_8 : pa \int_0^{\infty} \frac{J_1(mr) \cdot J_1(ma)}{m} e^{-mz} dm$$

$$= \frac{p}{2\pi} \int_0^{2\pi} \int_0^a \frac{[(z^2 + \kappa^2 + \rho^2 - 2\kappa\rho \cos \theta)^{1/2} - z] \rho d\rho d\theta}{(\kappa^2 + \rho^2 - 2\kappa\rho \cos \theta)^{1/2} (z^2 + \kappa^2 + \rho^2 - 2\kappa\rho \cos \theta)^{1/2}}$$

$$I_3 : pa \int J_1(mr) \cdot J_1(ma) z e^{-mz} dm = \frac{p}{2\pi} \int_0^{2\pi} \int_0^a \frac{z(\lambda^2 + \rho^2 - 2\lambda\rho \cos \theta)^{1/2} \rho d\rho d\theta}{(z^2 + \lambda^2 + \rho^2 - 2\lambda\rho \cos \theta)^{3/2}}$$

### 1.3.7. Resolution of the double integrals

The integrals of § 1.3.6 are most easily solved in transforming the variable  $\theta$  and  $\rho$  by setting :  $x = \rho \cos \theta$   $y = \rho \sin \theta$   $\rho d\rho d\theta = dx dy$

One obtains

$$I_1 = \frac{p}{2\pi} \int_{-a}^a \frac{2z(a^2 - x^2)^{1/2}}{(z^2 + x^2 + \lambda^2 - 2\lambda x)(z^2 + \lambda^2 + a^2 - 2\lambda x)^{1/2}} dx$$

that can easily be computed numerically.

$$I_2 = \frac{p}{2\pi} \int_{-a}^a \frac{2z(\lambda^2 + x^2 - 2\lambda x)(a^2 - x^2)^{1/2}}{(z^2 + x^2 + \lambda^2 - 2\lambda x)(z^2 + \lambda^2 + a^2 - 2\lambda x)^{1/2}} \left[ \frac{1}{(\lambda^2 + a^2 - 2\lambda x + z^2)} + \frac{2}{(z^2 + x^2 + \lambda^2 - 2\lambda x)} \right] dx$$

$$+ \frac{p}{2\pi} \int_{-a}^a \frac{2z(a^2 - x^2)^{1/2}}{(z^2 + \lambda^2 + a^2 - 2\lambda x)^{3/2}} dx$$

$$+ \frac{p}{2\pi} \int_{-a}^a \frac{2z(a^2 - x^2)^{1/2}}{(z^2 + x^2 + \lambda^2 - 2\lambda x)(z^2 + \lambda^2 + a^2 - 2\lambda x)^{1/2}} dx$$

$I_3$ ,  $I_4$ , and  $I_5$  are particular LIPSCHITZ-HANKEL integrals that we shall solve in next paragraph.

$$I_6 = \frac{p}{2\pi} \int_{-a}^a \ln \frac{(\lambda^2 - 2\lambda x + a^2 + z^2)^{1/2} + (a^2 - x^2)^{1/2}}{(\lambda^2 - 2\lambda x + a^2 + z^2)^{1/2} - (a^2 - x^2)^{1/2}} dx$$

$I_7$  can be deduced from  $I_1$ ,

$I_8$  and  $I_9$  are particular LIPSCHITZ-HANKEL integrals.

### 1.3.8. Resolution of the LIPSCHITZ-HANKEL integrals

The solution of the Lipschitz-Hankel integrals is given by WATSON (1960).

$$\int_0^{\infty} e^{-at} J_{\nu}(bt) \cdot J_{\nu}(ct) t^{\mu-1} dt =$$

$$\frac{(bc)^{\nu} \Gamma(\mu+2\nu)}{\pi a^{\mu+2\nu} \Gamma(2\nu+1)} \int_0^{\pi} F\left(\frac{\mu+2\nu}{2}, \frac{\mu+2\nu+1}{2}; \nu+1; -\frac{w^2}{a^2}\right) \sin^{2\nu} \phi d\phi$$

where

$$w^2 = b^2 + c^2 - 2bc \cos \phi$$

$$I_3 = \frac{pa}{\lambda} \int_0^{\infty} \frac{J_1(m\lambda) \cdot J_1(ma)}{m} e^{-mz} dm$$

$$= p \frac{a^2}{z^2} \cdot \frac{1}{2\pi} \int_0^{\pi} F\left(1, 3/2; 2; -\frac{w^2}{z^2}\right) \sin^2 \phi d\phi$$

with (WAYLAND, 1970)

$$F\left(1, 3/2; 2; -\frac{w^2}{z^2}\right) = \frac{z^2}{w^2} \left[ 1 - \left(1 + \frac{w^2}{z^2}\right)^{-1/2} \right]$$

$$\text{and } w^2 = a^2 + \lambda^2 - 2a\lambda \cos \phi$$

The resulting integral  $I_3 = p \frac{a^2}{\pi} \int_0^{\pi} \frac{1}{w^2} \left[ 1 - \frac{z}{(z^2+w^2)^{1/2}} \right] \sin \phi d\phi$  can easily be solved numerically.

$$I_4 = p \frac{az}{\lambda} \int_0^{\infty} J_1(m\lambda) \cdot J_1(ma) e^{-mz} dm$$

$$= p \frac{a^2}{z^2} \cdot \frac{1}{\pi} \int_0^{\pi} F\left(3/2, 2; 2; -\frac{w^2}{z^2}\right) \sin^2 \phi d\phi$$

$$F\left(3/2, 2; 2; -\frac{w^2}{z^2}\right) = \left(1 + \frac{w^2}{z^2}\right)^{-3/2}$$

$$I_4 = p \frac{a^2 z}{\pi} \int_0^{\pi} \frac{\sin^2 \phi d\phi}{(z^2 + w^2)^{3/2}}$$

$$I_5 = p a z \int_0^{\infty} J_1(m\lambda) \cdot J_1(ma) m e^{-mz} dm$$

$$= \frac{3pa^2 \lambda}{\pi z^3} \int_0^{\pi} F\left(2, 5/2; 2; -\frac{w^2}{z^2}\right) \sin^2 \phi d\phi$$

$$F\left(2, 5/2; 2; -\frac{w^2}{z^2}\right) = \left(1 + \frac{w^2}{z^2}\right)^{-5/2}$$

$$I_5 = \frac{3pa^2 \lambda z^2}{\pi} \int_0^{\pi} \frac{\sin^2 \phi d\phi}{(z^2 + w^2)^{5/2}}$$

$I_8$  can be deduced from  $I_3$ .

$I_9$  can be deduced from  $I_4$ .

### 1.3.9. Expressions for computation in the axis of the load

When stresses and displacements are computed in the axis of the load, one has :

$$I_1 : pa \int_0^{\infty} J_1(ma) e^{-mz} dm = \frac{\sqrt{a^2+z^2}-z}{a \sqrt{a^2+z^2}} \cdot pa$$

$$I_2 : pa \int_0^{\infty} J_1(ma) m z e^{-mz} dm = \frac{a z}{(a^2+z^2)^{3/2}} \cdot pa$$

$$I_3 : pa \int_0^{\infty} \frac{J_1(mr) \cdot J_1(ma)}{mr} e^{-mz} dm = \frac{1}{2} pa \int_0^{\infty} J_1(ma) e^{-mz} dm = \frac{I_1}{2}$$

$$I_4 : pa \int_0^{\infty} \frac{J_1(ma)}{r} \cdot J_1(ma) z e^{-mz} dm = \frac{1}{2} pa \int_0^{\infty} J_1(ma) m z e^{-mz} dm = \frac{I_2}{2}$$

$$I_5 : pa \int_0^{\infty} J_1(mr) \cdot J_1(ma) m z e^{-mz} dm = 0$$

$$I_6 : pa \int_0^{\infty} \frac{J_1(ma)}{m} e^{-mz} dm = \frac{(a^2+z^2)^{1/2}-z}{a} \cdot pa$$

$$I_7 : pa \int_0^{\infty} J_1(ma) z e^{-mz} dm = z \cdot I_1$$

$$I_8 : pa \int_0^{\infty} \frac{J_1(mr) \cdot J_1(ma)}{m} e^{-mz} dm = 0$$

$$I_9 : pa \int_0^{\infty} J_1(ma) \cdot J_1(ma) z e^{-mz} dm = 0$$

To be applicable, all the developments of paragraph 1.3.4., again require all the equations to be available in closeform.

## PART 2 : NUMERICAL RESOLUTION

### 2.1. THE MATHEMATICAL ANALYSIS

We give here only the main principles of the analysis. The detailed analyses are given in appendix.

#### 2.1.1. Full or partial friction at the interfaces

Next system of boundary conditions has to be solved

- At the surface

$$\sigma_z = p$$

$$\tau_{rz} = 0$$

- At each interface

$$\sigma_{zi} = \sigma_{zi+1}$$

$$\tau_{rzi} = \tau_{rzi+1}$$

$$w_i = w_{i+1}$$

$$u_i = \lambda_i u_{i+1}$$

- At the bottom

$$w = 0$$

$$A_4 \text{ or } C_4 = 0$$

This is a system of 16 equations with 16 unknowns ( $A_1, B_1, \dots, C_4, D_4$ ).

It is very difficult to solve this system in a complete analytical way, as BURMISTER did, for example for the two and the three layered structures. Indeed it is nearly impossible in doing so to avoid mathematical errors in the analysis because of the extension taken by the different expressions in the elimination procedure of the unknowns.

Thus it is necessary to reduce the analysis to a more comprehensive model but in such a way that all the numerical problems detailed in part 1 can still be solved completely accurately.

The main objective is to obtain an expression for each unknown parameter consisting in a numerator containing negative exponents only and a denominator containing a constant term and negative exponents.

During the integration procedure, when the variable tends to infinity, the numerator will then tend to zero and the denominator to a constant value.

It is thus absolutely necessary that all the exponents appear in closeform in the analysis. The factors multiplying the exponents may then be expressed in a more comprehensive form.

The mathematical analysis can then be resumed as follows.

### First step

Replace in the boundary equations of the third interface the parameters  $A_4$  and  $C_4$  by their values obtained from the fixed bottom condition.

### Second step

Write all the interface conditions in matrixform.

- At the surface  $MI (A_1 B_1 C_1 D_1)^T = (1 \ 0)^T$

where  $MI$  is a  $2 \times 4$  matrix

- At the first interface  $M_1 (A_1 B_1 C_1 D_1)^T = M_2 (A_2 B_2 C_2 D_2)^T$

where  $M_1$  and  $M_2$  are  $4 \times 4$  matrices

- At the second interface  $M_3 (A_2 B_2 C_2 D_2)^T = M_4 (A_3 B_3 C_3 D_3)^T$

where  $M_3$  and  $M_4$  are  $4 \times 4$  matrices

- At the third interface  $M_5 (A_3 B_3 C_3 D_3)^T = M_6 (B_4 D_4)^T$

where  $M_5$  is a  $4 \times 4$  matrix

and  $M_6$  is a  $4 \times 2$  matrix.

### Third step

Invert the matrices  $M_1$ ,  $M_3$  and  $M_5$

The system becomes

$$MI (A_1 B_1 C_1 D_1)^T = (1 \ 0)^T$$

$$(A_1 B_1 C_1 D_1)^T = M_1^{-1} M_2 (A_2 B_2 C_2 D_2)^T$$

$$(A_2 B_2 C_2 D_2)^T = M_3^{-1} M_4 (A_3 B_3 C_3 D_3)^T$$

$$(A_3 \ B_3 \ C_3 \ D_3)^T = M_5^{-1} M_6 (B_4 \ D_4)^T$$

and finally

$$M_1 M_1^{-1} M_2 M_3^{-1} M_4 M_5^{-1} M_6 (B_4 \ D_4)^T = (1 \ 0)^T$$

The product  $M_1 M_1^{-1} M_2 M_3^{-1} M_4 M_5^{-1} M_6$  is a  $2 \times 2$  matrix, so that we can write

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} B_4 \\ D_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and solve the resulting system :

$$B_4 = \frac{a_{22}}{a_{11} a_{22} - a_{21} a_{12}}$$

$$D_4 = \frac{-a_{21}}{a_{11} a_{22} - a_{21} a_{12}}$$

Utilizing the different matrix equations, we express then the other parameters in function of  $B_4$  and  $D_4$ .

To obtain the exponents in closeform, all the matrices are split in such a way that the exponents can be put out of the brackets.

For example, in the isotropic case :

$$M_1^{-1} = -\frac{1}{4(1-\mu_1)} [ e^{-x} M_{11} + e^x M_{12} ]$$

$$M_2 = [ e^x M_{21} + e^{-x} M_{22} ]$$

$$M_3^{-1} = -\frac{1}{4(1-\mu_2)} [ e^{-y} M_{31} + e^y M_{32} ]$$

$$M_4 = [ e^y M_{41} + e^{-y} M_{42} ]$$

$$M_5^{-1} = -\frac{1}{4(1-\mu_3)} [ e^{-x} M_{51} + e^x M_{52} ]$$

$$M_6 = [ e^{-u} M_{61} + e^{-z} M_{62} ]$$

where  $x = mh_1$

$$y = m(h_1 + h_2)$$

$$z = m(h_1 + h_2 + h_3)$$

$$u = m(h_1 + h_2 + h_3 + 2h_4)$$

$h_i$  being the thickness of layer  $i$ .

The terms of all the matrices are of course known in closeform, so that they can be introduced as input values for the numerical procedure. The terms of the resulting products are not expressed in closeform : it is sufficient to know which coloms or rows of each matrix contain only zeros. Indeed in the final product intermediate matrices disappear because they are identic zero. Practically, those matrices vanish, which are preceeded by positive exponents.

The values of all the unknown parameters can then be expressed in the desired way, a numerator with only negatve exponents and a denominator with a constant term and negative exponents, excepted for the parameters  $B_1$  and  $D_1$  at the surface.

Indeed at the surface the values of the parameters  $B_1$  and  $D_1$  contain in both numerator and denominator a constant term, followed by negative exponents, so that satisfactory convergency cannot be obtained.

The parameters  $B_1$  and  $D_1$  are then expressed, utilizing the surface conditions, in function of the parameters  $A_1$  and  $C_1$ .

The resulting expression for a given stress or displacement becomes then

$$\sigma = pa \int_0^{\infty} J_0(mx).J_1(ma) [ K + f_1(A_1) + f_2(C_1) ] dm$$

where  $K$  is a constant term and  $f_1(A_1)$  and  $f_2(C_1)$  are functions of the parameters  $A_1$  and  $C_1$  which converge normally.

The expression of the stress can then he split into two parts

$$\sigma_1 = pa \int_0^{\infty} J_0(mx).J_1(ma). K dm$$

which can be solved analytically

$$\sigma_2 = pa \int_0^{\infty} J_0(mx).J_1(ma) [ f_1(A_1) + f_2(C_1) ] dm$$

which can be solved numerically in an accurate way.

The same difficulty arises in the first layer near the surface. Although all the parameters converge, again the parameters  $B_1$  and  $D_1$  converge very slowly so that complete accuracy is also here hard to insure.

Therefore, again utilizing the surface conditions, we split the expressions for stresses and displacements into two parts.

The first part, known as a LIPSCHITZ-HANKEL integral (paragraph 1.3.4), is solved analytically; the second part is solved numerically.

### 2.1.2. Full slip at the first two interfaces

The procedure described in previous paragraph cannot be applied here.

The boundary conditions are :

- At the surface

$$\sigma_x = p$$

$$\tau_{xz} = 0$$

- At the first two interfaces :

$$\sigma_{zi} = \sigma_{zi+1} \quad (1)$$

$$\tau_{nzi} = 0 \quad (2)$$

$$\tau_{nzi+1} = 0 \quad (3)$$

$$w_i = w_{i+1} \quad (4)$$

- At the third interface :

$$\sigma_{zi} = \sigma_{zi+1}$$

$$\tau_{nzi} = \tau_{nzi+1}$$

$$w_i = w_{i+1}$$

$$u_i = u_{i+1}$$

- At the bottom :

$$w = 0$$

$$A_4 \text{ or } C_4 = 0$$

The systems of equations at the first two interfaces cannot be expressed in matrixform, because two of the four equations are homogeneous.

The mathematical analysis is here :

#### First step

Replace in the boundary equations of the third interface the parameters  $A_4$  and  $C_4$  by their values obtained from the fixed bottom condition.

#### Second step

Write the boundary equations at the third interface (friction) in matrix

form  $(A_3 \ B_3 \ C_3 \ D_3)^T = M_5^{-1} \cdot M_6 (B_4 \ D_4)^T$

Third step

Using the surface conditions, express  $A_1$  and  $B_1$  in function of  $C_1$  and  $D_1$

$$(A_1 \ B_1)^T = M_0(C_1 \ D_1)^T$$

Replace in conditions (1) and (2) at the first interface  $A_1$  and  $B_1$  by their values and solve the system by expressing  $C_1$  and  $D_1$  in function of  $A_2$ ,  $B_2$ ,  $C_2$ ,  $D_2$ .

$$(C_1 \ D_1)^T = M_1(A_2 \ B_2 \ C_2 \ D_2)^T$$

Replace in condition (4) at the first interface  $A_1$ ,  $B_1$ ,  $C_1$  and  $D_1$  by their values in function of  $A_2$ ,  $B_2$ ,  $C_2$  and  $D_2$ .

Using then conditions (3) and (4) at the first interface, express  $A_2$  and  $B_2$  in function of  $C_2$  and  $D_2$

$$(A_2 \ B_2)^T = M_2(C_2 \ D_2)^T$$

Replace in condition (1) and (2) at the second interface  $A_2$  and  $B_2$  by their values and solve the system by expressing  $C_2$  and  $D_2$  in function of  $A_3$ ,  $B_3$ ,  $C_3$ ,  $D_3$ .

$$(C_2 \ D_2)^T = M_3(A_3 \ B_3 \ C_3 \ D_3)^T$$

Replace in condition (4) at the second interface  $A_2$ ,  $B_2$ ,  $C_2$ ,  $D_2$  by their values in function of  $A_3$ ,  $B_3$ ,  $C_3$ ,  $D_3$

Equations (3) and (4) at the second interface reduce then to following system

$$M_4(A_3 \ B_3 \ C_3 \ D_3)^T = (K \ 0)^T$$

where  $M_4$  is a  $2 \times 4$  matrix and  $K$  a function of the integration variable. One obtains finally

$$M_4 \cdot M_5^{-1} \cdot M_6(B_4 \ D_4)^T = (K \ 0)^T$$

a system which can be solved in the same way as in previous paragraph.

The same procedure is also utilized to express all the unknown parameters in function of  $B_4$  and  $D_4$ .

Nevertheless a supplementary difficulty arises in the full slip case, regarding the expression of the vertical deflection :

$$w = pa \frac{1+\nu_i}{E_i} \int_0^{\infty} \frac{J_0(mr) \cdot J_1(ma)}{m} f_i(A_i B_i C_i D_i) dm$$

When integrating numerically, this expression becomes, for  $m = 0$

$$\lim_{m \rightarrow 0} \frac{J_0(mr) \cdot J_1(ma)}{m} = \frac{a}{2}$$

$$\lim_{m \rightarrow 0} f_i(A_i B_i C_i D_i) = \frac{0}{0}$$

so that at the origin ( $m=0$ ), the expression for the vertical deflection is undefined.

But, thanks of the fixed bottom condition, we know that at the bottom  $w=0$  for all values of the integrating parameter  $m_i$  thus also for  $m=0$ . Using the boundary conditions we can then write :

$$\text{At the bottom } f_4(A_4 B_4 C_4 D_4) = 0$$

$$\text{At the third interface } f_3(A_3 B_3 C_3 D_3) = f_4(A_4 B_4 C_4 D_4)$$

$$\text{At the second interface } f_2(A_2 B_2 C_2 D_2) = f_3(A_3 B_3 C_3 D_3)$$

$$\text{and, at the first interface } f_1(A_1 B_1 C_1 D_1) = f_2(A_2 B_2 C_2 D_2)$$

$$\text{Thus, in all layers, we have, for } m = 0 : f_i(A_i B_i C_i D_i) = 0$$

### 2.1.3. The detailed mathematical analysis

The completely detailed analysis is given

in appendix 1 for the isotropic case with full and partial friction

in appendix 2 for the isotropic case with full slip at the first  
two interfaces

in appendix 3 for the anisotropic case with full and partial friction

in appendix 4 for the anisotropic case with full slip at the first  
two interfaces.

## 2.2. THE NUMERICAL PROCEDURE

The stresses and displacements are obtained by numerical integration of expressions such as

$$\sigma = pa \int_0^{\infty} J_0(mr) \cdot J_1(ma) f(m) dm$$

Therefore we use Simpson's method.

$$I(a,b) = \frac{h}{3} [f_0 + 4f_1 + 2f_2 + \dots + 4f_{2n-1} + f_{2n}]$$

where  $[a,b]$  is the interval of integration of  $I(a,b)$  subdivided in  $2n$  equal segments of length  $h$ .

Here the interval of integration goes until infinity.

The numerical computation is interrupted when the value of the function  $f(m,z)$  (in fact the values of all the parameters  $A_i, B_i, C_i, D_i$ ) becomes smaller than the imposed convergency level :

$$f_i(m,z) < \epsilon$$

The function  $f_i(mz)$  varies quickly for small values of the integrating parameter  $m$  and much slower for high values of  $m$ .

Thus it seems appropriate to increase the value of  $h$  for higher values of  $m$ .

The numerical integral can then be written :

$$\begin{aligned} I[0, \infty] &= \frac{h_1}{3} [f_0 + 4f_1 + 2f_2 + 4f_3 + f_4] \quad \text{for } m \leq L_1 \\ &+ \frac{h_2}{3} [f_4 + 4f_5 + 2f_6 + 4f_7 + f_8] \quad \text{for } L_1 < m \leq L_2 \\ &+ \frac{h_3}{3} [f_8 + 4f_9 + 2f_{10} + 4f_{11} + f_{12}] \quad \text{for } L_2 < m \\ &+ \dots \end{aligned}$$

For practical reasons we take

$$h_2 = 2h_1, \quad h_3 = 2h_2, \dots$$

Unless in the case of the vertical deflection, for which the function is computed in a separated way, the first term  $f_0 = 0$ ; the last one  $f_{2n}$  (here  $f_{12}$ ) can be neglected.

The integral becomes then :

$$\begin{aligned}
 I [ \cdot, \cdot ] &= \frac{h}{3} [ 4f_1 + 2f_2 + 4f_3 + 2f_4 ] + \frac{h}{3} f_4 \\
 &+ \frac{2h}{3} [ 4f_5 + 2f_6 + 4f_7 + 2f_8 ] + \frac{2h}{3} f_8 \\
 &+ \frac{4h}{3} [ 4f_9 + 2f_{10} + 4f_{11} + 2f_{12} ]
 \end{aligned}$$

The main computation routine is

$$I [ L_i, L_i ] = \frac{2^i h}{3} [ 4f_{i+1} + 2f_{i+2} + \dots + 4f_{i+2n-1} + 2f_{i+2n} ]$$

When modifying the length of the integration segment, a secondary routine increases the already obtained with a value  $\frac{2^i h}{3} \cdot f_{i+2n}$

The initial length  $h$  of the integration segment is chosen by the user. It is multiplied by a factor 2 when the values of  $f(mz)$  become smaller than  $10^{-2}$ ,  $10^{-3}$ ,  $10^{-4}$  and  $10^{-5}$ . The final convergency level is  $10^{-6}$ .

### 2.3. RESUMED FLOW SHEET

The main steps of the program can be resumed as follows.

#### 2.3.1. INPUT procedures

- Input of the data (by display or file) regarding loads and structure
- Input of the coordinates (by display or file) of the points where stresses and displacements are to be computed.
- Choice of the length of the integration segment.

#### 2.3.2. Computation

- Computation of the value of the vertical deflection at the surface and in the first layer, for  $m = 0$ .
- Computation of the parameters  $A_i, B_i, C_i, D_i$  for each value of  $m$ .
- Computation of the values of the Bessel functions for each value of  $m$ .
- Computation of the stresses and displacements in cylindrical coordinates for each value of  $m$ .
- Computation of the stresses and displacements in cartesian coordinates for each value of  $m$ .
- Upbuilding of the vectors containing the results using Simpson's rule.
- Convergency test for each  $f_{2k}$  function.
- When the final convergency is reached, computation of the additional values at the surface and in the first layer (LIPSCHITZ-HANKEL Integrals).

### 2.3.3. OUTPUT Procedures

- Creating of a file containing the results :  
 stresses  $\sigma_x$   $\sigma_y$   $\sigma_z$   $\tau_{xy}$   $\tau_{xz}$   $\tau_{yz}$   
 displacements  $u_x$   $v_y$   $w_z$
- Computation of  
 principal stresses  $\sigma_1$   $\sigma_2$   $\sigma_3$   
 principal strains  $\epsilon_1$   $\epsilon_2$   $\epsilon_3$   
 linear strains  $\epsilon_x$   $\epsilon_y$   $\epsilon_z$
- Printing (if desired) of the results.

### 2.3.4. Capabilities of the program

The number of circular loads, with different radii and contact pressures, is limited to 20.

Stresses and displacements can be computed at 30 places in the horizontal plane and at each place at 30 depths (included 8 values at the surface and the interfaces).

### 2.3.5. Used language

The language in which the program is written is FORTRAN 77.

## 2.4. PARTICULAR FEATURES

### 2.4.1. Verification of the accuracy of the results

Stresses and displacements are computed at each interface with the appropriate stress function. This means that, at each interface, the computations are performed at the bottom of layer  $i$ , using parameters  $A_i, B_i, C_i, D_i$ , and, at the surface of layer  $i+1$ , using parameters  $A_{i+1}, B_{i+1}, C_{i+1}, D_{i+1}$ .

Thus stresses and displacements are computed at a same point with two different functions. This procedure allows to verify the results (mainly to insure that the chosen length of the integration segment is small enough) taking into account that the boundary conditions have to be

satisfied :

- In the friction case :  $\sigma_{zi} = \sigma_{zi+1}$

$$\tau_{rzi} = \tau_{rzi+1}$$

$$w_i = w_{i+1}$$

$$u_i = \lambda_i u_{i+1}$$

- In the full slip case :  $\sigma_{zi} = \sigma_{zi+1}$

$$\tau_{rzi} = \tau_{rzi+1} = 0$$

$$w_i = w_{i+1}$$

This is of particular importance at the first interface where the stresses at the bottom of the first layer are computed in a complete different way (part of it analytically).

In the full slip case, one must be careful by choosing the length of the integration segment.

Indeed by choosing it too small (0,02 for example when a good average value is 0,1) there is a risk that the values of the function  $f(mz)$ , undefined for  $m = 0$ , result in complete abnormal figures for values of  $m$  near to 0 : for example tensile stresses for  $\sigma_z$ .

Beside the preceding procedure, the best way of verifying the results is to compare them with existing tables such as these of JONES (1962) established for a three layered system with full friction at the interfaces. It is very easy to reduce the four layered system to a three layered one by setting the Young's moduli and Poisson's ratio of two successive layers identical. That creates no numerical difficulties.

On the other hand it is necessary, because of the fixed bottom, to give an important thickness to the last layer in order to assimilate it to a semi-infinite body, as considered by Jones.

Of course the comparison of the results can only be done at the two interfaces, but there also the values of Jones are completely accurate.

### 2.4.2. Convergency of the Bessel functions

The Bessel functions are computed by series expansion such as

$$J_0(x) = \sum_0^{\infty} (-)^k \frac{(x/2)^{2k}}{k! k!}$$

For high values of  $x$  this expansion leads to overflow problems, because of the factorials in the denominator.

For values of  $x$  higher than 16, the Bessel functions are approximated by their asymptotic values given by

$$J_0(x) \approx \sqrt{\frac{2}{\pi x}} \cos(x - \pi/4)$$

$$J_1(x) \approx \sqrt{\frac{2}{\pi x}} \sin(x - \pi/4)$$

The introduced error is less than  $10^{-4}$

### 2.4.3. Underflow problems

FORTRAN 77 does not have a routine that sets variables subjected to underflow equal to zero. Thus, this routine had to be built in the program. The limit value for underflow on the IBM PC is  $\exp(-88)$

Maximum there occur in the computation 4 products of exponentials. So in limiting the value of each exponent to (-20) normally there should occur no underflow.

We have that  $\exp(-20) \approx 10^{-8}$

Normally one computation requires about 60 loops, let us thus say a maximum of 100.

The maximal error resulting in the limitation of the values of negative exponents is then

$$100 \times 10^{-8} = 10^{-6}$$

which is the value of the convergency level of the whole procedure.

In doing so, we insure thus accuracy of the results at a level of  $10^{-6}$ .

#### 2.4.4. The computation of the principal stresses

The principal stresses are solution of following relations (TIMOSHENKO and GOODIER, 1961)

$$S^3 - i_1.S^2 + i_2.S - i_3 = 0$$

where the stress invariants are

$$i_1 = \sigma_x + \sigma_y + \sigma_z$$

$$i_2 = \sigma_x.\sigma_y + \sigma_x.\sigma_z + \sigma_z.\sigma_y - \tau_{xy}^2 - \tau_{xz}^2 - \tau_{yx}^2$$

$$i_3 = \sigma_x.\sigma_y.\sigma_z - 2\tau_{xy}.\tau_{yz}.\tau_{xz} - \sigma_y.\tau_{xz}^2 - \sigma_x.\tau_{yz}^2 - \sigma_z.\tau_{xy}^2$$

The equation is transformed into

$$t^3 + at + b = 0$$

$$\text{with } t = S + i_1/3$$

$$a = \frac{1}{3}(3i_2 - i_1^2)$$

$$b = \frac{1}{27}(-2i_1^3 + 9i_1.i_2 - 27i_3)$$

Knowing the trigonometric identity

$$4\cos^3\theta - 3\cos\theta - \cos(3\theta) \equiv 0$$

we may write, with

$$t = m.\cos\theta$$

$$m^3\cos^3\theta + am\cos\theta + b \equiv 4\cos^3\theta - 3\cos\theta - \cos(3\theta)$$

$$\text{so that } \cos(3\theta) = \frac{3b}{am} \quad \text{and} \quad m = \left(-\frac{4}{3}a\right)^{1/2}$$

The solution is then given by

$$\begin{aligned} \theta_1 &= \frac{1}{3} \arccos\left(\frac{3b}{a}\right) & t_1 &= \cos \theta_1 & S_1 &= t_1 - i_1/3 \\ \theta_2 &= \frac{1}{3} \arccos\left(\frac{3b}{a}\right) + \frac{2\pi}{3} & t_2 &= \cos \theta_2 & S_2 &= t_2 - i_1/3 \\ \theta_3 &= \frac{1}{3} \arccos\left(\frac{3b}{a}\right) + \frac{4\pi}{3} & t_3 &= \cos \theta_3 & S_3 &= t_3 - i_1/3 \end{aligned}$$

When  $a = 0$ , the value of  $\cos(3\theta)$  seems to become infinite.

Nevertheless, this is not true.

We have always

$$(\sigma_x - \sigma_y)^2 > 0$$

$$(\sigma_y - \sigma_z)^2 > 0$$

$$(\sigma_z - \sigma_x)^2 > 0$$

and thus

$$\sigma_x^2 + \sigma_y^2 > 2\sigma_x \cdot \sigma_y$$

$$\sigma_y^2 + \sigma_z^2 > 2\sigma_y \cdot \sigma_z$$

$$\sigma_z^2 + \sigma_x^2 > 2\sigma_z \cdot \sigma_x$$

$$\text{By adding } \sigma_x^2 + \sigma_y^2 + \sigma_z^2 > \sigma_x \cdot \sigma_y + \sigma_y \cdot \sigma_z + \sigma_z \cdot \sigma_x$$

Expanding parameter  $a$ , we have

$$\begin{aligned} a &= \frac{1}{3} [3\sigma_x \cdot \sigma_y + 3\sigma_y \cdot \sigma_z + 3\sigma_z \cdot \sigma_x - 3\tau_{xy}^2 - 3\tau_{yz}^2 - 3\tau_{zx}^2 \\ &\quad - \sigma_x^2 - \sigma_y^2 - \sigma_z^2 - 2\sigma_x \cdot \sigma_y - 2\sigma_y \cdot \sigma_z - 2\sigma_z \cdot \sigma_x] \\ &= \frac{1}{3} [\sigma_x \cdot \sigma_y + \sigma_y \cdot \sigma_z + \sigma_z \cdot \sigma_x - 3\tau_{xy}^2 - 3\tau_{yz}^2 - 3\tau_{zx}^2 \\ &\quad - \sigma_x^2 - \sigma_y^2 - \sigma_z^2] \end{aligned}$$

$$\text{Since } \sigma_x^2 + \sigma_y^2 + \sigma_z^2 > \sigma_x \cdot \sigma_y + \sigma_y \cdot \sigma_z + \sigma_z \cdot \sigma_x$$

we have that always  $a < 0$ .

The parameter  $a$  can only be equal to zero when

$$\sigma_x^2 + \sigma_y^2 + \sigma_z^2 + 3\tau_{xy}^2 + 3\tau_{yz}^2 + 3\tau_{zx}^2 = \sigma_x \cdot \sigma_y + \sigma_y \cdot \sigma_z + \sigma_z \cdot \sigma_x$$

This is only possible when  $\sigma_x = \sigma_y = \sigma_z = \sigma$

$$\tau_{xy} = \tau_{yz} = \tau_{zx} = 0$$

Thus when  $\sigma_x$ ,  $\sigma_y$  and  $\sigma_z$  are the principal stresses in a spherical stress situation.

The value of  $b$  is then

$$b = \frac{1}{27} [54 \sigma^3 - 27 \sigma^3 - 27 \sigma^3] = 0$$

and the value of  $m$  is undefined.

We conclude that, when  $\sigma_x = \sigma_y = \sigma_z$ , the principal stresses are immediately obtained by

$$\sigma_1 = \sigma_2 = \sigma_3 = \sigma_x = \sigma_y = \sigma_z$$

## 2.5. USE OF THE PROGRAMS

### 2.5.1. Generalities

The programs are available in two versions :

- EXECUTABLE version, in which the executable program is made up of only one block with automatic loading, when DOS is ready.
- SOURCE version, in which all controls, data and modules are in separated files.

Each program is written on a separate floppy disk.

The names of the programs are :

FLIP : Four layered isotropic system with partial friction  
at the interfaces.

FLIS : Four layered isotropic system with full slip  
at the first two interfaces.

FLAP : Four layered anisotropic system with partial friction  
at the interfaces.

FLAS : Four layered anisotropic system with full slip  
at the first two interfaces.

Each diskette contains an explanatory notice which can be called by  
next instructions :

FLIP NO.TXT	for FLIP
FLIS NO.TXT	for FLIS
FLAP NO.TXT	for FLAP
FLAS NO.TXT	for FLAS

The notice FLIP NO.TXT is given in appendix 5.

### 2.5.2. Source version

For this detailed analysis of the source version, we refer to the notice of program FLIP in appendix 5.

Each program is built up in 9 modules.

#### Module 1 (FLIP1.FOR)

Main module controlling the whole computation.

#### Module 2 (FLIP2.FOR)

Data input module with following subroutines :

DOMEC : Mechanical data of the structure  
 DOTRA : Traffic (load) data  
 POCAL : Stress coordinates in the  $xy$  plane  
 POINT : Depth coordinates  
 PAS : Length of the integration segment  
 CHECK : Desired geometric scale  
 ERROR : Error routines on input procedures.

#### Module 3 (FLIP3.FOR)

Initialization module

ECHDE : Scaling of the geometric parameters  
 VINIT : Initialization of the integration parameters  
 ZERO : Initialization of the results vectors  
 FINIT : Computation of the values of the vertical deflections at the surface and in the first layer for  $m = 0$

#### Module 4 (FLIP4.FOR)

Computation of the parameters  $A_i B_i C_i D_i$  for each value of  $m$

P4442 : Product of a matrix (4,4) with a matrix (4,2)  
 PCT22 : Product of a constant with a matrix (2,2)  
 PCT42 : Product of a constant with a matrix (4,2)  
 SOM42 : Sum of two matrices (4,2)  
 SOM22 : Sum of two matrices (2,2)  
 P4444 : Product of two matrices (4,4)  
 P2442 : Product of a matrix (2,4) with a matrix (4,2)  
 CONST : Computation of the parameters  $A_i C_i B_i D_i$

Module 5 (FLIP 5.FOR)

Computation of the stresses for each value of  $m$

BESJ1 : Computation of  $J_1(ma)$   
 BJOJ2 : Computation of  $J_0(mr)$  and  $J_1(mr)$   
 SURFA : Computation of the stresses at the surface  
 COUC 1 : Computation of the stresses in the first layer  
 COUC 2 : Computation of the stresses in the second layer  
 COUC 3 : Computation of the stresses in the third layer  
 COUC 4 : Computation of the stresses in the fourth layer.

Module 6 (FLIP 6.FOR)

Integration procedure

TITRE : Display control  
 MODIF : Alteration of the length of the integration segment  
 SIMPS : Routine for Simpson's rule and up building of the results vectors.

Module 7 (FLIP 7.FOR)

Definite values of stresses and displacements

COMSU : Computation of the additionnal analytical values at the surface  
 FONC : Computation of the Gauss function for the vertical deflection  
 at the surface  
 CCOU 1 : Computation of the additionnal analytical values in the first layer  
 FONC 1 : Computation of the LIPSCHITZ-HANKEL integral  $I_1$  (appendix 1)  
 FONC 2 : Computation of the LIPSCHITZ-HANKEL integral  $I_2$   
 FONC 3 : Computation of the LIPSCHITZ-HANKEL integral  $I_3$   
 FONC 4 : Computation of the LIPSCHITZ-HANKEL integral  $I_4$   
 FONC 5 : Computation of the LIPSCHITZ-HANKEL integral  $I_5$   
 FONC 6 : Computation of the LIPSCHITZ-HANKEL integral  $I_6$   
 ECHEF : Rescaling of the displacements.

Module 8 (FLIP 8.FOR)

Printing of the results

IMDON : File storing and printing of the data  
 IMRES : File storing and printing of the results.

Module 9 (FLIP 9.FOR)

Display instructions.

### 2.5.3. Running of the program

We refer again to the notice in appendix 5, on page 5, from where on the complete running procedure is explained.

### 2.5.4. Miscellaneous

It is important to notice that the mechanical input data (load pressures and Young's moduli) must be expressed with the same units and that also the geometrical data (radii of the loads, thickness of the layers, coordinates of the different locations) must be expressed with the same units.

Nevertheless it is not necessary that mechanical data and geometrical data are expressed in coherent units : for example, mechanical data may be expressed in psi and geometrical data in meters.

The stresses will be expressed with the same units as the mechanical data and the displacements with the same units as the geometrical data.

#### - Accuracy of anisotropic results

It is not possible to compare the results of the anisotropic programs with those of the isotropic ones in setting the degree  $n$  of anisotropy equal to one. Indeed, in doing so, all the relations become indefinite.

Approximate checking is possible by setting for the stresses  $n = 1.05$  and for the displacements  $n = 1.20$ .

#### - Scale

Routine ECHDE of module 3 scales the geometric parameters of the structure. This routine has been introduced to reduce the problems regarding accuracy : in using the same scale one can use the same length for the integration segment.

Convergency problems arise essentially at the surface and in the first layer. The best scale factor is therefore the thickness of the first layer. Another appropriate scale factor is the radius of the loads, obviously when all loads have the same radius. The computation of the Besselfunction  $J_1(ma)$  is then the same for all loads. But this scale factor should only be utilized when its order of magnitude is the same as that of the thickness of the first layer.

Of course, at the end of the whole procedure, the values of the displacements have to be rescaled in their original values (routine ECHEF of module 7).

### - Poisson's ratio

The Young's moduli of the materials are known. This is not always the case with the Poisson's ratio. Therefore the program asks if Poisson's ratio is known. If not, a value of 0.5 is taken in the isotropic case. In the anisotropic case the procedure is as follows :

- First question : "Do you know the anisotropic Poisson's ratio ? "  
 IF YES : Input the value  
 IF NO : Second question.
- Second question : "Do you know the Poisson's ratio when considering the material as isotropic ? "  
 IF NO : The program utilizes the maximum tolerated value : 0.5 for an isotropic material  

$$\frac{1.5\nu n}{2+n}$$
 for an anisotropic material  
 (LEKHNITSKII, 1963).

IF YES : Input the isotropic value  $u_i$

The program computes then an adapted anisotropic

value 
$$u_a = \frac{u_i}{0.5} \times \frac{1.5\nu n}{2+n} = u_i \cdot \frac{3\nu n}{2+n}$$

### - Principal strains

The program computes the principal stresses in routine IMRES of module 8. It also computes, in the isotropic case, the principal strains by applying HOOKE's law.

It does not compute the principal strains in the anisotropic case.

There indeed, the values of the Young moduli varie with the considered direction. Thus to apply HOOKE's law one must know the principal directions to be able to compute the values of the Young moduli. This computation requires the resolution of a system of three trigonometric equations with three unknowns. Although this could theoretically be done, accuracy of th results can practically not be insured.

### - Intermediate printing

During the computation, the successive values of the integrating parameter  $m$  are displayed. This is only done to show that the program is running and to protect the user from despair when utilizing the model with an important number of loads.

"remain patient" on the display means that the program is computing the parameters  $A_i$   $B_i$   $C_i$   $D_i$  and the stresses (modules 4, 5 and 6).

"remain more patient" means that the program is computing the LIPSCHITZ-HANKEL integrals (module 7), which can need a certain amount of time with several 10 ds out of the vertical axis where stresses are computed.

#### 2.5.5. Values for n and $l_a$ .

##### - The degree of anisotropy n.

Granular materials and soils can be considered as anisotropic due to compaction or settlement. In most of the cases the value of n, the degree of anisotropy, is higher than 1, which leads to the phenomenon of vertical stress concentration; the value of n can be related to Frohlich's stress concentration factor k, approximatively by

$$n = \frac{2k - 1 - \sqrt{4k - 3}}{2}$$

The value of k varies from 2 to 5 (3 = isotropic), thus the value of n from 0.4 to 2.4.

For superconsolidated clays, n is lower than 1.

##### - The partial friction ratio $l_a$ .

Full friction is introduced by setting  $l_a = 1$ .

When  $l_a$  tends to infinity, the shearstresses tend to zero.

Thus partial partial friction is obtained for values of  $l_a$  between 1 and infinity.

The practical value depends on the value of the maximum shearstress (located near the edge of the load) tolerated at the interface.

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APPENDIX 1

Algebraical Analysis of a four-layer isotropic System  
with fixed bottom and partial or full friction inter-  
face conditions.

## APPENDIX 1

Algebraical Analysis of a four-layer isotropic System with fixed bottom and partial friction interface conditions.

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### 1. Boundary conditions.

We write

$$A_i = A_i m^2 \quad B_i = B_i m^2 \quad C_i = C_i m \quad D_i = D_i m$$

$$F_w = \frac{E_1(1+\mu_2)}{E_2(1+\mu_1)} \quad F_u = \lambda_1 \frac{E_1(1+\mu_2)}{E_2(1+\mu_1)}$$

$$K_w = \frac{E_2(1+\mu_3)}{E_3(1+\mu_2)} \quad K_u = \lambda_2 \frac{E_2(1+\mu_3)}{E_3(1+\mu_2)}$$

$$L_w = \frac{E_3(1+\mu_4)}{E_4(1+\mu_3)} \quad L_u = \lambda_3 \frac{E_3(1+\mu_4)}{E_4(1+\mu_3)}$$

$$x = m H_1$$

$$y = m (H_1 + H_2)$$

$$z = m (H_1 + H_2 + H_3)$$

$$t = m (H_1 + H_2 + H_3 + H_4)$$

$$u = -2t + z$$

where  $H_1, H_2, H_3$  and  $H_4$  are the thicknesses of the four layers. The index 1 applies to the surface layer.

Boundary conditions at the surface ( $z = 0$ ):

$$\sigma_z = p \quad A_1 + B_1 - C_1(1-2\mu_1) + D_1(1-2\mu_1) = 1$$

$$\tau_{xz} = 0 \quad A_1 - B_1 + C_1 \cdot 2\mu_1 + D_1 \cdot 2\mu_1 = 0$$

Boundary conditions at the first interface ( $z = H_1$ ):

$$\tau_{xz}: A_1 e^x + B_1 e^{-x} - C_1(1-2\mu_1-x)e^x + D_1(1-2\mu_1+x)e^{-x} = A_2 e^x + B_2 e^{-x} - C_2(1-2\mu_2-x)e^x + D_2(1-2\mu_2+x)e^{-x}$$

$$\tau_{yz}: A_1 e^x - B_1 e^{-x} + C_1(2\mu_1+x)e^x + D_1(2\mu_1-x)e^{-x} =$$

$$A_2 e^x - B_2 e^{-x} + C_2(2\mu_2+x)e^x + D_2(2\mu_2-x)e^{-x}$$

$$w: A_1 e^x - B_1 e^{-x} - C_1 (2-4\mu_1-x) e^x - D_1 (2-4\mu_1+x) e^{-x} =$$

$$F_w [A_2 e^x - B_2 e^{-x} - C_2 (2-4\mu_2-x) e^x - D_2 (2-4\mu_2+x) e^{-x}]$$

$$u: A_1 e^x + B_1 e^{-x} + C_1 (1+x) e^x - D_1 (1-x) e^{-x} =$$

$$F_u [A_2 e^x + B_2 e^{-x} + C_2 (1+x) e^x - D_2 (1-x) e^{-x}]$$

Boundary conditions at the second interface ( $z = H_1 + H_2$ ):

$$\sigma_z: A_2 e^\gamma + B_2 e^{-\gamma} - C_2 (1-2\mu_2-\gamma) e^\gamma + D_2 (1-2\mu_2+\gamma) e^{-\gamma} =$$

$$A_3 e^\gamma + B_3 e^{-\gamma} - C_3 (1-2\mu_3-\gamma) e^\gamma + D_3 (1-2\mu_3+\gamma) e^{-\gamma}$$

$$\tau_{rz}: A_2 e^\gamma - B_2 e^{-\gamma} + C_2 (2\mu_2+\gamma) e^\gamma + D_2 (2\mu_2-\gamma) e^{-\gamma} =$$

$$A_3 e^\gamma - B_3 e^{-\gamma} + C_3 (2\mu_3+\gamma) e^\gamma + D_3 (2\mu_3-\gamma) e^{-\gamma}$$

$$w: A_2 e^\gamma - B_2 e^{-\gamma} - C_2 (2-4\mu_2-\gamma) e^\gamma - D_2 (2-4\mu_2+\gamma) e^{-\gamma} =$$

$$k_w [A_3 e^\gamma - B_3 e^{-\gamma} - C_3 (2-4\mu_3-\gamma) e^\gamma - D_3 (2-4\mu_3+\gamma) e^{-\gamma}]$$

$$u: A_2 e^\gamma + B_2 e^{-\gamma} + C_2 (1+\gamma) e^\gamma - D_2 (1-\gamma) e^{-\gamma} =$$

$$k_u [A_3 e^\gamma + B_3 e^{-\gamma} + C_3 (1+\gamma) e^\gamma - D_3 (1-\gamma) e^{-\gamma}]$$

Boundary conditions at the third interface ( $z = H_1 + H_2 + H_3$ ):

$$\sigma_z: A_3 e^z + B_3 e^{-z} - C_3 (1-2\mu_3-z) e^z + D_3 (1-2\mu_3+z) e^{-z} =$$

$$A_4 e^z + B_4 e^{-z} - C_4 (1-2\mu_4-z) e^z + D_4 (1-2\mu_4+z) e^{-z}$$

$$\tau_{rz}: A_3 e^z - B_3 e^{-z} + C_3 (2\mu_3+z) e^z + D_3 (2\mu_3-z) e^{-z} =$$

$$A_4 e^z - B_4 e^{-z} + C_4 (2\mu_4+z) e^z + D_4 (2\mu_4-z) e^{-z}$$

$$w: A_3 e^z - B_3 e^{-z} - C_3 (2-4\mu_3-z) e^z - D_3 (2-4\mu_3+z) e^{-z} =$$

$$L_w [A_4 e^z - B_4 e^{-z} - C_4 (2-4\mu_4-z) e^z - D_4 (2-4\mu_4+z) e^{-z}]$$

$$u: A_3 e^z + B_3 e^{-z} + C_3 (1+z) e^z - D_3 (1-z) e^{-z} =$$

$$L_u [A_4 e^z + B_4 e^{-z} + C_4 (1+z) e^z - D_4 (1-z) e^{-z}]$$

Boundary conditions at the bottom ( $z = H_1 + H_2 + H_3 + H_4$ ):

$$w: A_4 e^t - B_4 e^{-t} - C_4 (2-4\mu_4-t) e^t - D_4 (2-4\mu_4+t) e^{-t} = 0$$

$$C_4 = 0$$

## 2. Resolution of the system of 16 equations.

In the equations of the conditions at the third interface,  $A_4$  is replaced by its value taken from the fixed bottom condition:

$$A_4 e^t = B_4 e^{-t} + D_4 (2 - 4\mu_4 + t) e^{-t}$$

We write the conditions at the third interface in matrixform

$$M_5 (A_3 B_3 C_3 D_3)^T = M_6 (B_4 D_4)^T$$

We invert  $M_5$

$$(A_3 B_3 C_3 D_3)^T = M_5^{-1} \cdot M_6 (B_4 D_4)^T$$

where

$$M_5^{-1} = -\frac{1}{4(1-\mu_3)} e^{-2z} \begin{pmatrix} 7(1+z) & -(2-4\mu_3+z) & -(2\mu_3+z) & -(1-2\mu_3-z) \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= -\frac{1}{4(1-\mu_3)} e^z \begin{pmatrix} 0 & 0 & 0 & 0 \\ -(1-z) & (2-4\mu_3+z) & (2\mu_3-z) & -(1-2\mu_3+z) \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 \end{pmatrix}$$

$$M_6 = e^{-u} \begin{pmatrix} 1 & (2-4\mu_4+t) \\ 1 & (2-4\mu_4+t) \\ Lw & Lw(2-4\mu_4+t) \\ Lu & Lu(2-4\mu_4+t) \end{pmatrix} + e^{-z} \begin{pmatrix} 1 & (1-2\mu_4+z) \\ -1 & (2\mu_4-z) \\ -Lw & -Lw(2-4\mu_4+z) \\ Lu & -Lu(1-z) \end{pmatrix}$$

$$(A_3 B_3 C_3 D_3)^T = -\frac{1}{4(1-\mu_3)} \left[ e^{-z} \cdot M_{51} + e^z \cdot M_{52} \right]$$

$$\left[ e^{-u} \cdot M_{61} + e^{-z} \cdot M_{62} \right] \cdot (B_4 D_4)^T$$

$$= -\frac{1}{4(1-\mu_3)} \left[ e^{-2t} \cdot M_{5161} + e^{-2z} \cdot M_{5162} + e^{-2(t-z)} \cdot M_{5261} + MA \right] \cdot (B_4 D_4)^T$$

$$M_{51} \cdot M_{61} = \begin{pmatrix} + & + \\ 0 & 0 \\ + & + \\ 0 & 0 \end{pmatrix} = M_{5161}$$

$$M_{51} \cdot M_{62} = \begin{pmatrix} + & + \\ 0 & 0 \\ + & + \\ 0 & 0 \end{pmatrix} = M_{5162}$$

$$M_{52} \cdot M_{61} = \begin{pmatrix} 0 & 0 \\ + & + \\ 0 & 0 \\ + & + \end{pmatrix} = M_{5261}$$

$$M_{52} \cdot M_{62} = \begin{pmatrix} 0 & 0 \\ L_1 & L_4 \\ 0 & 0 \\ -L_3 & L_2 \end{pmatrix} + 2 \begin{pmatrix} 0 & 0 \\ L_3 & (L_1 - L_2) \\ 0 & 0 \\ 0 & -L_3 \end{pmatrix} + 2^2 \begin{pmatrix} 0 & 0 \\ 0 & L_3 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= MA$$

$$L_1 = -3 + 4\mu_3 - L_u - 2\mu_3(L_w - L_u)$$

$$L_2 = -1 - 2L_w + 4L_w \cdot \mu_4 - L_u$$

$$L_3 = L_w - L_u$$

$$L_4 = -1 + 6\mu_4 - 8\mu_3\mu_4 - 4\mu_3 \cdot L_w + 8\mu_3\mu_4 L_w + L_u - 2\mu_3 L_u$$

We write the conditions at the second interface in matrix form

$$M_3 (A_2 B_2 C_2 D_2)^T = M_4 (A_3 B_3 C_3 D_3)^T$$

$$(A_2 B_2 C_2 D_2)^T = M_3^{-1} \cdot M_4 (A_3 B_3 C_3 D_3)^T$$

$$M_3^{-1} = -\frac{1}{4(1-\mu_2)} \cdot e^{-y} \begin{pmatrix} -(1+y) & -(2-4\mu_2-y) & -(2\mu_2+y) & -(1-2\mu_2-y) \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$-\frac{1}{4(1-\mu_2)} \cdot e^y \begin{pmatrix} 0 & 0 & 0 & 0 \\ -(1-y) & (2-4\mu_2+y) & (2\mu_2-y) & -(1-2\mu_2+y) \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 \end{pmatrix}$$

$$M_4 = e^y \begin{pmatrix} 1 & 0 & -(1-2\mu_3-y) & 0 \\ 1 & 0 & (2\mu_3+y) & 0 \\ k_w & 0 & -k_w(2-4\mu_3-y) & 0 \\ k_u & 0 & k_u(1+y) & 0 \end{pmatrix}$$

$$+ e^{-y} \begin{pmatrix} 0 & 1 & 0 & (1-2\mu_3+y) \\ 0 & -1 & 0 & (2\mu_3-y) \\ 0 & -k_w & 0 & -k_w(2-4\mu_3+y) \\ 0 & k_u & 0 & -k_u(1-y) \end{pmatrix}$$

$$(A_2 B_2 C_2 D_2)^T = \frac{1}{16(1-\mu_2)(1-\mu_3)} \cdot [M_{31} \cdot e^{-y} + M_{32} \cdot e^y]$$

$$[M_{41} \cdot e^y + M_{42} \cdot e^{-y}] (A_3 B_3 C_3 D_3)^T$$

$$(A_2 B_2 C_2 D_2)^T = \frac{1}{16(1-\mu_1)(1-\mu_3)} \left[ M_{31} \cdot M_{41} + M_{32} \cdot M_{42} + M_{31} \cdot M_{42} e^{-2\gamma} + M_{32} \cdot M_{41} e^{2\gamma} \right]$$

$$\left[ M_{5161} \cdot e^{-2\alpha} + M_{5162} e^{-2\alpha} + M_{5261} e^{-2(\Gamma-\alpha)} + MA \right] (B_4 D_4)^T$$

$$M_{31} \cdot M_{42} = \begin{pmatrix} 0 & + & 0 & + \\ 0 & 0 & 0 & 0 \\ 0 & + & 0 & + \\ 0 & 0 & 0 & 0 \end{pmatrix} = M_{3142}$$

$$M_{32} \cdot M_{41} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ + & 0 & + & 0 \\ 0 & 0 & 0 & 0 \\ + & 0 & + & 0 \end{pmatrix} = M_{3241}$$

$$M_{31} \cdot M_{41} + M_{32} \cdot M_{42} = \begin{pmatrix} k_1 & 0 & -k_4 & 0 \\ 0 & k_1 & 0 & k_4 \\ k_3 & 0 & k_2 & 0 \\ 0 & -k_3 & 0 & k_2 \end{pmatrix} + \gamma \begin{pmatrix} -k_3 & 0 & (k_1 - k_2) & 0 \\ 0 & k_3 & 0 & (k_1 - k_2) \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & -k_3 \end{pmatrix}$$

$$+ \gamma^2 \begin{pmatrix} 0 & 0 & -k_3 & 0 \\ 0 & 0 & 0 & k_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = MB$$

$$k_1 = -3 + 4\mu_2 - k_u - 2\mu_2(k_w - k_u)$$

$$k_2 = -1 - 2k_w + 4k_w \cdot \mu_3 - k_u$$

$$k_3 = k_w - k_u$$

$$k_4 = -1 + 6\mu_3 - 8\mu_2\mu_3 - 4\mu_2 k_w + 8\mu_2\mu_3 k_w + k_u - 2\mu_2 k_u$$

$$M_B \cdot M_A = \begin{pmatrix} 0 & 0 \\ + & + \\ 0 & 0 \\ + & + \end{pmatrix} = N_{11}$$

$$M_{3142} \cdot M_{5261} = \begin{pmatrix} + & + \\ 0 & 0 \\ + & + \\ 0 & 0 \end{pmatrix} = N_{21}$$

$$M_{3241} \cdot M_{5161} = \begin{pmatrix} 0 & 0 \\ + & + \\ 0 & 0 \\ + & + \end{pmatrix} = N_{12}$$

$$M_{3142} \cdot M_A = \begin{pmatrix} + & + \\ 0 & 0 \\ + & + \\ 0 & 0 \end{pmatrix} = N_{22}$$

$$M_{3241} \cdot M_{5162} = \begin{pmatrix} 0 & 0 \\ + & + \\ 0 & 0 \\ + & + \end{pmatrix} = N_{13}$$

$$M_B \cdot M_{5161} = \begin{pmatrix} + & + \\ 0 & 0 \\ + & + \\ 0 & 0 \end{pmatrix} = N_{23}$$

$$M_B \cdot M_{5261} = \begin{pmatrix} 0 & 0 \\ + & + \\ 0 & 0 \\ + & + \end{pmatrix} = N_{14}$$

$$M_B \cdot M_{5162} = \begin{pmatrix} + & + \\ 0 & 0 \\ + & + \\ 0 & 0 \end{pmatrix} = N_{24}$$

$$M_{3142} \cdot M_{5161} = M_{3142} \cdot M_{5162} = M_{3241} \cdot M_{5261} = M_{3241} \cdot M_A = 0$$

$$(A_2 B_2 C_2 D_2)^T = \frac{1}{16(1-\mu_2)(1-\mu_3)} \cdot \left[ N_{11} + e^{-2(t-y)} N_{12} + e^{-2(x-y)} N_{13} \right. \\ \left. + e^{-2(t-x)} N_{14} + e^{-2y-2(t-x)} N_{21} + e^{-2y} N_{22} + e^{-2t} N_{23} + e^{-2x} N_{24} \right] \cdot (B_4 D_4)^T$$

We write the conditions at the first interface in matrixform

$$M_1 (A_1 B_1 C_1 D_1)^T = M_2 (A_2 B_2 C_2 D_2)^T$$

$$(A_1 B_1 C_1 D_1)^T = M_1^{-1} \cdot M_2 (A_2 B_2 C_2 D_2)^T$$

$$M_1^{-1} = -\frac{1}{4(1-\mu_1)} \cdot e^{-x} \begin{pmatrix} -(1+x) & -(2-4\mu_1-x) & -(2\mu_1+x) & -(1-2\mu_1-x) \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$-\frac{1}{4(1-\mu_1)} e^x \begin{pmatrix} 0 & 0 & 0 & 0 \\ -(1-x) & (2-4\mu_1+x) & (2\mu_1-x) & -(1-2\mu_1+x) \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 \end{pmatrix}$$

$$M_2 = e^x \begin{pmatrix} 1 & 0 & -(1-2\mu_2-x) & 0 \\ 1 & 0 & (2\mu_2+x) & 0 \\ F_w & 0 & -F_w(2-4\mu_2-x) & 0 \\ F_u & 0 & F_u(1+x) & 0 \end{pmatrix}$$

$$+ e^{-x} \begin{pmatrix} 0 & 1 & 0 & (1-2\mu_2+x) \\ 0 & -1 & 0 & (2\mu_2-x) \\ 0 & -F_w & 0 & -F_w(2-4\mu_2+x) \\ 0 & F_u & 0 & -F_u(1-x) \end{pmatrix}$$

$$(A_1 B_1 C_1 D_1)^T = -\frac{1}{64(1-\mu_1)(1-\mu_2)(1-\mu_3)} \cdot \left[ M_{11} \cdot e^{-x} + M_{12} \cdot e^x \right]$$

$$\left[ M_{21} \cdot e^x + M_{22} \cdot e^{-x} \right] (A_2 B_2 C_2 D_2)^T$$

$$(A_1 B_1 C_1 D_1)^T_2 = \frac{1}{64(1-\mu_1)(1-\mu_2)(1-\mu_3)} \left[ M_{11} \cdot M_{21} + e^{2x} M_{11} \cdot M_{22} \right.$$

$$\left. + e^{2x} M_{12} \cdot M_{21} + M_{12} \cdot M_{22} \right] \cdot \left[ N_{11} + e^{-2(t-y)} N_{12} + e^{-2(x-y)} N_{13} \right.$$

$$\left. + e^{-2(t-z)} N_{14} + e^{-2y} \cdot e^{-2(t-z)} N_{21} + e^{-2y} N_{22} + e^{-2t} N_{23} + e^{-2z} N_{24} \right] \cdot (B_4 D_4)^T$$

$$M_{11} \cdot M_{21} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ + & 0 & + & 0 \\ 0 & 0 & 0 & 0 \\ + & 0 & + & 0 \end{pmatrix} = M_{1221}$$

$$M_{11} \cdot M_{22} = \begin{pmatrix} 0 & + & 0 & + \\ 0 & 0 & 0 & 0 \\ 0 & + & 0 & + \\ 0 & 0 & 0 & 0 \end{pmatrix} = M_{1122}$$

$$M_{11} \cdot M_{21} + M_{12} \cdot M_{22} = \begin{pmatrix} F_1 & 0 & -F_4 & 0 \\ 0 & F_1 & 0 & F_4 \\ F_3 & 0 & F_2 & 0 \\ 0 & -F_3 & 0 & F_2 \end{pmatrix}$$

$$+ X \begin{pmatrix} -F_3 & 0 & F_1 - F_2 & 0 \\ 0 & F_3 & 0 & F_1 - F_2 \\ 0 & 0 & F_3 & 0 \\ 0 & 0 & 0 & -F_3 \end{pmatrix} + X^2 \begin{pmatrix} 0 & 0 & -F_3 & 0 \\ 0 & 0 & 0 & F_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = MC$$

$$F_1 = -3 + 4\mu_1 - F_u - 2\mu_1 (F_w - F_u)$$

$$F_2 = -1 - 2F_w + 4F_w \cdot \mu_2 - F_u$$

$$F_3 = F_w - F_u$$

$$F_4 = -1 + 6\mu_2 - 8\mu_1 \mu_2 - 4\mu_1 F_w + 8\mu_1 \mu_2 F_w + F_u - 2\mu_1 F_u$$

$$MC \cdot N_{1i} = \begin{pmatrix} 0 & 0 \\ + & + \\ 0 & 0 \\ + & + \end{pmatrix} = T_{1i}$$

$$MC \cdot N_{2i} = \begin{pmatrix} + & + \\ 0 & 0 \\ + & + \\ 0 & 0 \end{pmatrix} = T_{2i}$$

$$M_{1122} \cdot N_{1i} = \begin{pmatrix} + & + \\ 0 & 0 \\ + & + \\ 0 & 0 \end{pmatrix} = T_{3i}$$

$$M_{1221} \cdot N_{2i} = \begin{pmatrix} 0 & 0 \\ + & + \\ 0 & 0 \\ + & + \end{pmatrix} = T_{4i}$$

$$M_{1122} \cdot N_{2i} = M_{1221} \cdot N_{1i} = 0$$

$$\begin{aligned}
 (A, B, C, D)^T = & - \frac{1}{64(1-\mu_1)(1-\mu_2)(1-\mu_3)} \cdot \left[ T_{11} + T_{12} e^{-2(t-y)} + T_{13} e^{-2(z-y)} \right. \\
 & + T_{14} e^{-2(t-z)} + T_{21} e^{-2y} e^{-2(t-x)} + T_{22} e^{-2y} + T_{23} e^{-2t} + T_{24} e^{-2z} \\
 & + T_{31} e^{-2x} + T_{32} e^{-2x} e^{-2(t-y)} + T_{33} e^{-2x} e^{-2(z-y)} + T_{34} e^{-2x} e^{-2(t-z)} \\
 & \left. + T_{41} e^{-2(y-x)} e^{-2(t-z)} + T_{42} e^{-2(y-x)} + T_{43} e^{-2(t-x)} + T_{44} e^{-2(z-x)} \right] \cdot (B_2, D_4)^T
 \end{aligned}$$

We write the conditions at the surface in matrixform:

$$\begin{pmatrix} 1 & 1 & -(1-2\mu_1) & (1-2\mu_1) \\ 1 & -1 & 2\mu_1 & 2\mu_1 \end{pmatrix} (A, B, C, D)^T = (1 \ 0)^T$$

$$\text{MI} = \begin{pmatrix} 1 & 1 & -(1-2\mu_1) & (1-2\mu_1) \\ 1 & -1 & 2\mu_1 & 2\mu_1 \end{pmatrix}$$

$$\text{MI} \cdot T_{ij} = U_{ij} = \begin{matrix} + & + \\ + & + \end{matrix}$$

$$\begin{aligned}
 & \left[ U_{11} + U_{12} e^{-2(t-y)} + U_{13} e^{-2(z-y)} + U_{14} e^{-2(t-z)} + U_{21} e^{-2y} e^{-2(t-x)} \right. \\
 & + U_{22} e^{-2y} + U_{23} e^{-2t} + U_{24} e^{-2z} + U_{31} e^{-2x} + U_{32} e^{-2x} e^{-2(t-y)} \\
 & + U_{33} e^{-2x} e^{-2(z-y)} + U_{34} e^{-2x} e^{-2(t-z)} + U_{41} e^{-2(y-x)} e^{-2(t-z)} \\
 & \left. + U_{42} e^{-2(y-x)} + U_{43} e^{-2(t-x)} + U_{44} e^{-2(z-x)} \right] (B_2, D_4)^T =
 \end{aligned}$$

$$= -64(1-\mu_1)(1-\mu_2)(1-\mu_3) (1 \ 0)^T$$

where the matrices  $U_{ij}$  are all  $(2 \times 2)$  square matrices.

We write the terms of matrix  $U_{11}$

$$U_{11} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

and the sum of the terms of all other matrices

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

We notice that the terms  $a_{ij}$  contain constants and linear functions of the variables  $x, y, z$  and their products.

The terms  $c_{ij}$  contain all negative exponential functions of the variables  $x, y, z$  and  $t$ . For high values of the negative exponent they all tend to zero.

The matricial equation of  $B_4$  and  $D_4$  can be written in the following way:

$$(a_{11} + c_{11}) B_4 + (a_{12} + c_{12}) D_4 = -64(1-\mu_1)(1-\mu_2)(1-\mu_3)$$

$$(a_{21} + c_{21}) B_4 + (a_{22} + c_{22}) D_4 = 0$$

The solution of this system is:

$$B_4 = - \frac{64(1-\mu_1)(1-\mu_2)(1-\mu_3)}{\Delta} [a_{22} + c_{22}]$$

$$D_4 = \frac{64(1-\mu_1)(1-\mu_2)(1-\mu_3)}{\Delta} [a_{21} + c_{21}]$$

$$\Delta = a_{11} \cdot a_{22} - a_{21} \cdot a_{12} + (a_{11} + c_{11}) c_{22} - (a_{21} + c_{21}) \cdot c_{12} + c_{11} \cdot a_{22} - c_{21} \cdot a_{12}$$

One verifies that

$$a_{11} \cdot a_{22} - a_{21} \cdot a_{12} = (F_1 F_2 + F_3 F_4)(k_1 k_2 + k_3 k_4)(L_1 L_2 + L_3 L_4)$$

All the linear functions of the variables  $x$ ,  $y$  and  $z$  have disappeared in this relation, so that for high values of the variables, the value of the denominator  $\nabla$  tends to a constant

$$FKL = (F_1 F_2 + F_3 F_4)(k_1 k_2 + k_3 k_4)(L_1 L_2 + L_3 L_4)$$

### 3. Values of the parameters $A_i, D_i$ .

#### 3.1. Values of the parameters $A_1, C_1$ .

At the bottom of the first layer, parameters  $A_1$  and  $C_1$  are factors of the positive exponent  $e^x$ . Thus when computing stresses in the first layer we have to input at least two positive exponents which necessarily will lead to overflow problems. Therefore we shall express next modified values of the parameters  $A_1$  and  $C_1$ :  $A_1 \cdot e^x$  and  $C_1 \cdot e^x$ .

Regarding the parameters  $B_1$  and  $D_1$ , we shall replace them in the equations for the first layer in function of  $A_1$  and  $C_1$ , using the boundary conditions at the surface:

$$A_1 + B_1 - C_1(1 - 2\mu_1) + D_1(1 - 2\mu_1) = 1$$

$$A_1 - B_1 + C_1 \cdot 2\mu_1 + D_1 \cdot 2\mu_1 = 0$$

The resolution of this system leads to

$$B_1 = 2\mu_1 + A_1(1 - 4\mu_1) + 4\mu_1 C_1(1 - 2\mu_1)$$

$$D_1 = 1 - 2A_1 + C_1(1 - 4\mu_1)$$

The values of  $A_1, B_1, C_1$  and  $D_1$  are given by

$$(A_1, B_1, C_1, D_1)^T = - \frac{1}{64(1-\mu_1)(1-\mu_2)(1-\mu_3)} \left[ \begin{aligned} & T_{11} + T_{12} \cdot e^{-2(t-y)} + T_{13} \cdot e^{-2(z-y)} \\ & + T_{14} \cdot e^{-2(t-z)} + T_{21} \cdot e^{-2y} \cdot e^{-2(t-x)} + T_{22} \cdot e^{-2y} + T_{23} \cdot e^{-2t} + T_{24} \cdot e^{-2z} \\ & + T_{31} \cdot e^{-2x} + T_{32} \cdot e^{-2x} \cdot e^{-2(t-y)} + T_{33} \cdot e^{-2x} \cdot e^{-2(z-y)} + T_{34} \cdot e^{-2x} \cdot e^{-2(t-x)} \\ & + T_{41} \cdot e^{-2(y-x)} \cdot e^{-2(t-x)} + T_{42} \cdot e^{-2(y-x)} + T_{43} \cdot e^{-2(t-x)} + T_{44} \cdot e^{-2(z-x)} \end{aligned} \right] (B_4, D_4)^T$$

We notice that the matrices  $T_{1i}$  and  $T_{4i}$  contain nothing but zeros in their first and third rows and that the matrices  $T_{2i}$  and  $T_{3i}$  contain nothing but zeros in their second and fourth rows, so that the values of  $A_1$  and  $C_1$  depend only on the matrices  $T_{2i}$  and  $T_{3i}$ .

We obtain then

$$e^x \cdot (A_1 \ 0 \ C_1 \ 0)^T = - \frac{1}{64(1-\mu_1)(1-\mu_2)(1-\mu_3)} \cdot$$

$$\left[ T_{21} \cdot e^{-(2\gamma-x)} \cdot e^{-2(t-z)} + T_{22} \cdot e^{-(2\gamma-x)} + T_{23} \cdot e^{-(2t-x)} + T_{24} \cdot e^{-(2z-x)} + T_{31} \cdot e^{-x} \right.$$

$$\left. + T_{32} \cdot e^{-x} \cdot e^{-2(t-\gamma)} + T_{33} \cdot e^{-x} \cdot e^{-2(z-\gamma)} + T_{34} \cdot e^{-x} \cdot e^{-2(t-z)} \right] \cdot (B_4 \ D_4)^T$$

We notice that all the terms contain a negative exponential function. For high values of the variables, the numerators tend thus to zero and the denominator to the constant FKL.

### 3.2. Values of the parameters $A_2, B_2, C_2, D_2$ .

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For the reasons explained in previous paragraph we express next modified values of the parameters:  $A_2 e^y, B_2 e^{-x}, C_2 e^y, D_2 e^{-x}$ .

The values of the parameters are given by

$$(A_2 \ B_2 \ C_2 \ D_2)^T = \frac{1}{16(1-\mu_2)(1-\mu_3)} \cdot \left[ N_{11} + N_{12} \cdot e^{-2(t-\gamma)} \right.$$

$$\left. + N_{13} \cdot e^{-2(z-\gamma)} + N_{14} \cdot e^{-2(t-z)} + N_{21} \cdot e^{-2\gamma} \cdot e^{-2(t-z)} + N_{22} \cdot e^{-2\gamma} \right.$$

$$\left. + N_{23} \cdot e^{-2t} + N_{24} \cdot e^{-2z} \right] \cdot (B_4 \ D_4)^T$$

We split this relation in

$$e^y (A_2 \ 0 \ C_2 \ 0)^T = \frac{1}{16(1-\mu_2)(1-\mu_3)} \left[ N_{21} \cdot e^{-y} \cdot e^{-2(t-z)} + \right.$$

$$\left. + N_{22} \cdot e^{-y} + N_{23} \cdot e^{-(2t-\gamma)} + N_{24} \cdot e^{-(2z-\gamma)} \right] \cdot (B_4 \ D_4)^T$$

$$e^{-x} (0 \ B_2 \ 0 \ D_2)^T = \frac{1}{16(1-\mu_2)(1-\mu_3)} \left[ N_{11} \cdot e^{-x} + \right.$$

$$\left. + N_{12} \cdot e^{-x} \cdot e^{-2(t-\gamma)} + N_{13} \cdot e^{-x} \cdot e^{-2(z-\gamma)} + N_{14} \cdot e^{-x} \cdot e^{-2(t-z)} \right] \cdot (B_4 \ D_4)^T$$

### 3.3. Values of the parameters $A_3, B_3, C_3, D_3$ :

The values of  $A_3, B_3, C_3$  and  $D_3$  are given by

$$(A_3 B_3 C_3 D_3)^T = -\frac{1}{4(1-\mu_3)} \left[ M_{5161} \cdot e^{-2t} + M_{5162} \cdot e^{-2x} \right. \\ \left. + M_{5261} \cdot e^{-2(t-x)} + MA \right] (B_4 D_4)^T$$

We split this relation to obtain

$$e^x (A_3 \ 0 \ C_3 \ 0)^T = -\frac{1}{4(1-\mu_3)} \left[ M_{5161} e^{-(2t-x)} + M_{5162} \cdot e^{-x} \right] (B_4 \ D_4)^T$$

$$e^{-y} (0 \ B_3 \ 0 \ D_3)^T = -\frac{1}{4(1-\mu_3)} \left[ M_{5261} \cdot e^{-y} \cdot e^{-2(t-x)} + MA \cdot e^{-y} \right] (B_4 \ D_4)^T$$

### 3.4. Values of the parameters $B_4, D_4$ :

Following the same procedure, we obtain

$$B_4 \cdot e^{-x} = -\frac{64(1-\mu_1)(1-\mu_2)(1-\mu_3)}{\nabla} [a_{22} + c_{22}] \cdot e^{-x}$$

$$D_4 \cdot e^{-x} = \frac{64(1-\mu_1)(1-\mu_2)(1-\mu_3)}{\nabla} [a_{21} + c_{21}] \cdot e^{-x}$$

#### 4. Determination of the stresses and the displacements.

##### 4.1. Mathematical procedure.

The program calculates the stresses and the displacements using following relations:

$$\sigma_z = pa \int_0^a J_0(mr) \cdot J_1(ma) \left[ A_1 m^2 e^{mz} + B_1 m^2 e^{-mz} - C_1 m(1-2\nu_1-mz) e^{mz} + D_1 m(1-2\nu_1+mz) e^{-mz} \right] dm$$

$$\frac{\sigma_r + \sigma_\theta}{2} = -\frac{pa}{2} \int_0^a J_0(mr) \cdot J_1(ma) \left[ A_1 m^2 e^{mz} + B_1 m^2 e^{-mz} + C_1 m(1+4\nu_1+mz) e^{mz} - D_1 m(1+4\nu_1-mz) e^{-mz} \right] dm$$

$$\frac{\sigma_r - \sigma_\theta}{2} = -\frac{pa}{2} \int_0^a J_0(mr) \cdot J_1(ma) \left[ A_1 m^2 e^{mz} + B_1 m^2 e^{-mz} + C_1 m(1+mz) e^{mz} - D_1 m(1-mz) e^{-mz} \right] dm$$

$$+ pa \int_0^a \frac{J_1(mr) \cdot J_1(ma)}{mr} \left[ A_1 m^2 e^{mz} + B_1 m^2 e^{-mz} + C_1 m(1+mz) e^{mz} - D_1 m(1-mz) e^{-mz} \right] dm$$

$$\tau_{rz} = -pa \int_0^a J_1(mr) \cdot J_1(ma) \left[ A_1 m^2 e^{mz} - B_1 m^2 e^{-mz} + C_1 m(2\nu_1+mz) e^{mz} + D_1 m(2\nu_1-mz) e^{-mz} \right] dm$$

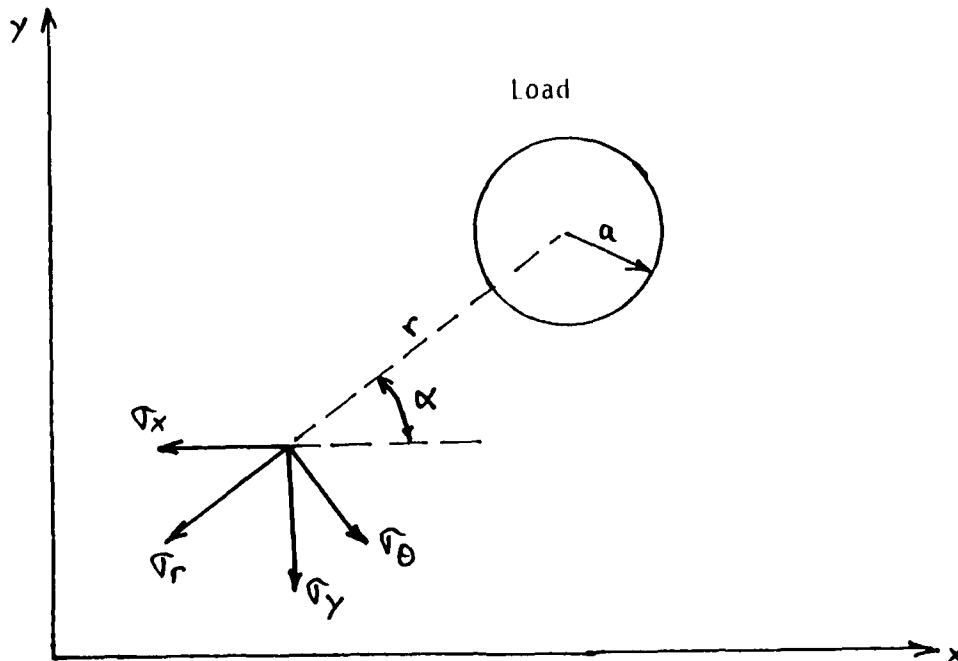
$$w = +\frac{1+\nu_1}{E_1} pa \int_0^a \frac{J_0(mr) \cdot J_1(ma)}{m} \left[ A_1 m^2 e^{mz} - B_1 m^2 e^{-mz} - C_1 m(2-4\nu_1-mz) e^{mz} - D_1 m(2-4\nu_1+mz) e^{-mz} \right] dm$$

$$u = -\frac{1+\nu_1}{E_1} pa \int_0^a \frac{J_1(mr) \cdot J_1(ma)}{m} \left[ A_1 m^2 e^{mz} + B_1 m^2 e^{-mz} + C_1 m(1+mz) e^{mz} - D_1 m(1-mz) e^{-mz} \right] dm$$

Stresses and displacements are calculated in a system of cylindrical coordinates. The stresses due to several loads are to be added together. Therefore we must express them in cartesian coordinates using following relations:

$$\begin{aligned}\sigma_x &= \sigma_r \cdot \cos^2 \alpha + \sigma_\theta \cdot \sin^2 \alpha = \frac{\sigma_r + \sigma_\theta}{2} + \frac{\sigma_r - \sigma_\theta}{2} \cos 2\alpha \\ \sigma_y &= \sigma_r \cdot \sin^2 \alpha + \sigma_\theta \cdot \cos^2 \alpha = \frac{\sigma_r + \sigma_\theta}{2} - \frac{\sigma_r - \sigma_\theta}{2} \cos 2\alpha \\ \sigma_z &= \sigma_z \\ \tau_{yz} &= \tau_{rz} \cdot \sin \alpha \\ \tau_{xz} &= \tau_{rz} \cdot \cos \alpha \\ \tau_{xy} &= (\sigma_r - \sigma_\theta) \cos \alpha \cdot \sin \alpha = \frac{\sigma_r - \sigma_\theta}{2} \cdot \sin 2\alpha \\ u_x &= u_r \cdot \cos \alpha \\ v_y &= u_r \cdot \sin \alpha \\ w &= w\end{aligned}$$

wherein the signification of the angle  $\alpha$  is illustrated below.



4.2. Stresses and displacements at the surface ( $z = 0$ )

$$\begin{aligned}
 -\sigma_z &= p \quad (r < a) \\
 &= p/2 \quad (r = a) \\
 &= 0 \quad (r > a)
 \end{aligned}$$

$$\begin{aligned}
 -\frac{\sigma_r + \sigma_\theta}{2} &= -\frac{pa}{2} \int_0^\infty J_0(mr) \cdot J_1(ma) [A_1 + B_1 + (1+4\mu_1)C_1 - (1+4\mu_1)D_1] dm \\
 &= -\frac{pa}{2} \int_0^\infty J_0(mr) \cdot J_1(ma) [-(1+2\mu_1) + 4(1+\mu_1)A_1 + 8\mu_1(1+\mu_1)C_1] dm \\
 &= \frac{pa}{2} \int_0^\infty J_0(mr) \cdot J_1(ma) (1+2\mu_1) dm \\
 &\quad - 2pa(1+\mu_1) \int_0^\infty J_0(mr) J_1(ma) [A_1 e^{-x} + 2\mu_1(C_1 e^{-x})] e^{-x} dm
 \end{aligned}$$

with

$$\frac{pa}{2} (1+2\mu_1) \int_0^\infty J_0(mr) \cdot J_1(ma) dm = \begin{cases} (1+2\mu_1) \frac{p}{2} & (r < a) \\ (1+2\mu_1) \frac{p}{4} & (r = a) \\ 0 & (r > a) \end{cases}$$

$$\begin{aligned}
 -\frac{\sigma_r - \sigma_\theta}{2} &= -\frac{pa}{2} \int_0^\infty J_0(mr) J_1(ma) [A_1 + B_1 + C_1 - D_1] dm \\
 &\quad + pa \int_0^\infty \frac{J_1(mr) \cdot J_1(ma)}{mr} [A_1 + B_1 + C_1 - D_1] dm \\
 &= \frac{pa}{2} (1-2\mu_1) \int_0^\infty J_0(mr) J_1(ma) dm \\
 &\quad - pa(1-2\mu_1) \int_0^\infty \frac{J_1(mr) \cdot J_1(ma)}{mr} dm \\
 &\quad - 2pa(1-\mu_1) \int_0^\infty \left[ J_0(mr) - \frac{2J_1(mr)}{mr} \right] J_1(ma) [A_1 e^{-x} + 2\mu_1(C_1 e^{-x})] e^{-x} dm
 \end{aligned}$$

with

$$\frac{pa}{2}(1-2\mu_1) \int_0^\infty J_0(mr) \cdot J_1(ma) dm = \begin{cases} (1-2\mu_1) \frac{p}{2} & (r < a) \\ (1-2\mu_1) \frac{p}{4} & (r = a) \\ 0 & (r > a) \end{cases}$$

$$-pa(1-2\mu_1) \int_0^\infty \frac{J_1(mr) \cdot J_1(ma)}{mr} dm = \begin{cases} -(1-2\mu_1) \frac{p}{2} & (r < a) \\ -(1-2\mu_1) \frac{pa^2}{2r^2} & (r > a) \end{cases}$$

$$-r\tau_z = -pa \int_0^\infty J_1(mr) \cdot J_1(ma) [A_1 - B_1 + 2\mu_1 C_1 + 2\mu_1 D_1] dm$$

$$= 0$$

$$-w = + \frac{1+\mu_1}{E_1} pa \int_0^\infty \frac{J_0(mr) \cdot J_1(ma)}{m} [A_1 - B_1 - (2-4\mu_1)C_1 - (2-4\mu_1)D_1] dm$$

$$= + \frac{2(1-\mu_1^2)}{E_1} pa \int_0^\infty \frac{J_0(mr) \cdot J_1(ma)}{m} dm$$

$$- \frac{4(1-\mu_1^2)}{E_1} pa \int_0^\infty \frac{J_0(mr) \cdot J_1(ma)}{m} [A_1 e^{-x} - (1-2\mu_1)(C_1 e^{-x})] e^{-x} dm$$

with

$$\int_0^\infty \frac{J_0(mr) \cdot J_1(ma)}{m} dm = \begin{cases} 1 & (r=0) \\ F\left(\frac{1}{2}, -\frac{1}{2}; 1; r^2/a^2\right) & (r < a) \\ 2/\pi & (r=a) \\ \frac{a}{2r} F\left(\frac{1}{2}, \frac{1}{2}; 2; \frac{a^2}{r^2}\right) & (r > a) \end{cases}$$

$$-u = - \frac{1+\mu_1}{E_1} pa \int_0^\infty \frac{J_1(mr) \cdot J_0(ma)}{m} [A_1 + B_1 + C_1 - D_1] dm$$

$$u = \frac{1+\nu_1}{E_1} (1-2\nu_1) \cdot pa \int_0^a \frac{J_1(mr) \cdot J_1(ma)}{m} dm$$

$$- \frac{4(1-\nu_1^2)}{E_1} \cdot pa \int_0^\infty \frac{J_1(mr) \cdot J_1(ma)}{m} \left[ (A_1 e^x) + 2\nu_1 (C_1 e^x) \right] e^{-x} dm.$$

with

$$\int_0^a \frac{J_1(mr) \cdot J_1(ma)}{m} dm = \begin{cases} \frac{r}{2a} & (r < a) \\ \frac{a}{2r} & (r > a) \end{cases}$$

#### 4.3. Stresses and displacements in the first layer ( $0 < h < H_1$ ):

$$-r_z = pa \int_0^\infty J_0(mr) \cdot J_1(ma) \left[ A_1 e^{mh} + B_1 e^{-mh} - C_1 (1-2\nu_1 - mh) e^{mh} \right. \\ \left. + D_1 (1-2\nu_1 + mh) e^{-mh} \right] dm$$

$$= pa \int_0^a J_0(mr) \cdot J_1(ma) e^{-mh} dm + pa \int_a^\infty J_0(mr) \cdot J_1(ma) m h e^{-mh} \\ + pa \int_0^a J_0(mr) \cdot J_1(ma) \left[ (A_1 e^x) \cdot e^{-(x-mh)} - (C_1 e^x) (1-2\nu_1 - mh) e^{-(x-mh)} \right. \\ \left. - (A_1 e^x) (1+2mh) e^{-(x+mh)} + (C_1 e^x) (1-2\nu_1 + mh - 4\nu_1 mh) e^{-(x+mh)} \right] dm$$

with

$$pa \int_0^a J_0(mr) J_1(ma) e^{-mh} dm = p \cdot \frac{\sqrt{a^2 + h^2} - h}{\sqrt{a^2 + h^2}} \quad (r=0)$$

$$= \frac{p}{2\pi} \int_{-a}^a \frac{2h(a^2 - x^2)^{1/2}}{(h^2 + x^2 + r^2 - 2xr)(h^2 + r^2 + a^2 - 2xr)^{1/2}} dx \quad (r \neq 0)$$

$$pa \int_0^a J_0(mr) \cdot J_1(ma) m h e^{-mh} dm = p \cdot \frac{a^2 x}{(a^2 + x^2)^{3/2}} \quad (r=0)$$

$$= \frac{p}{2\pi} \int_{-a}^a \frac{2h(a^2 - x^2)^{1/2}}{(h^2 + x^2 + r^2 - 2rx)(h^2 + r^2 + a^2 - 2rx)^{1/2}} dx$$

$$+ \frac{p}{2\pi} \int_{-a}^a \frac{2h(a^2 - x^2)^{1/2}}{(h^2 + r^2 + a^2 - 2rx)^{3/2}} dx$$

$$- \frac{p}{2\pi} \int_{-a}^a \frac{2h(r^2 + x^2 - 2rx)(a^2 - x^2)^{1/2}}{(h^2 + x^2 + r^2 - 2rx)(h^2 + r^2 + a^2 - 2rx)^{3/2}} \left[ \frac{1}{(r^2 + a^2 - 2rx + h^2)} + \frac{2}{(h^2 + x^2 + r^2 - 2rx)} \right] dx$$

$$- \frac{v_r + v_\theta}{r} = - \frac{pa}{2} \int_0^a J_0(mr) \cdot J_1(ma) \left[ A_1 e^{mh} + B_1 e^{-mh} + C_1 (1 + 4\mu_1 + mh) e^{mh} - D_1 (1 + 4\mu_1 - mh) e^{-mh} \right] dm$$

$$= pa \cdot \frac{(1+4\mu_1)}{2} \int_0^a J_0(mr) J_1(ma) e^{-mh} dm - \frac{pa}{2} \int_0^a J_0(mr) J_1(ma) m h e^{-mh} dm$$

$$- \frac{pa}{2} \int_0^a J_0(mr) \cdot J_1(ma) \left[ (A_1 e^x) e^{-(x-mh)} + (C_1 e^x) (1 + 4\mu_1 + mh) e^{-(x-mh)} + (A_1 e^x) (3 + 4\mu_1 - 2mh) e^{-(x+mh)} - (C_1 e^x) (1 - 4\mu_1 - 8\mu_1^2 - mh + 4\mu_1 mh) e^{-(x+mh)} \right] dm$$

$$- \frac{v_r - v_\theta}{2} = - \frac{pa}{2} \int_0^a J_0(mr) J_1(ma) \left[ A_1 e^{mh} + B_1 e^{-mh} + C_1 (1 + mh) e^{mh} - D_1 (1 - mh) e^{-mh} \right] dm$$

$$+ pa \int_0^a \frac{J_1(mr) J_1(ma)}{mr} \left[ A_1 e^{mh} + B_1 e^{-mh} + C_1 (1 + mh) e^{mh} - D_1 (1 - mh) e^{-mh} \right] dm$$

$$\begin{aligned}
\frac{\sigma_r - \sigma_\theta}{2} &= p a \frac{(1-2\mu_1)}{2} \int_0^\infty J_0(mr) \cdot J_1(ma) e^{-mh} dm - \frac{p a}{2} \int_0^\infty J_0(mr) \cdot J_1(ma) m h e^{-mh} dm \\
&\quad - p a (1-2\mu_1) \int_0^\infty \frac{J_1(mr) \cdot J_1(ma) e^{-mh}}{mr} dm + p a \int_0^\infty \frac{J_1(mr) \cdot J_1(ma)}{mr} m h e^{-mh} dm \\
&\quad - p a \int_0^\infty \left[ \frac{J_0(mr)}{2} - \frac{J_1(mr)}{mr} \right] \cdot J_1(ma) \left[ (A_1 e^x) e^{-(x-mh)} \right. \\
&\quad \left. + (C_1 e^x) (1+mh) e^{-(x-mh)} + (A_1 e^x) (3-4\mu_1 - 2mh) e^{-(x+mh)} \right. \\
&\quad \left. - (C_1 e^x) (1-2\mu_1 + 2\mu_1^2 - mh + 4\mu_1 mh) e^{-(x+mh)} \right] dm
\end{aligned}$$

with

$$p a \int_0^\infty \frac{J_1(mr) \cdot J_1(ma)}{mr} e^{-mh} dm = \frac{p}{2} \cdot \frac{\sqrt{a^2+h^2} - h}{\sqrt{a^2+h^2}} \quad (r=0)$$

$$= p \cdot \frac{a^2}{\pi} \int_0^\pi \frac{1}{\omega^2} \left[ 1 - \frac{h}{(h^2+\omega^2)^{1/2}} \right] \sin^2 \phi d\phi \quad (r \neq 0)$$

$$\omega^2 = a^2 + r^2 - 2ar \cos \phi$$

$$p a \int_0^\infty \frac{J_1(mr) \cdot J_1(ma)}{mr} m h e^{-mh} dm = \frac{p}{2} \frac{a^2 h}{(a^2+h^2)^{3/2}} \quad (r=0)$$

$$= p \frac{a^2}{\pi} \int_0^\pi \frac{h}{(h^2+\omega^2)^{3/2}} \sin^2 \phi d\phi \quad (r \neq 0)$$

$$\begin{aligned}
-\bar{\tau}_{rz} &= -p a \int_0^\infty J_1(mr) \cdot J_1(ma) \left[ A_1 e^{mh} - B_1 e^{-mh} + C_1 (2\mu_1 + mh) e^{mh} \right. \\
&\quad \left. + D_1 (2\mu_1 - mh) e^{-mh} \right] dm.
\end{aligned}$$

$$\begin{aligned} \tau_{rz} = & \rho a \int_0^a J_1(mr) J_1(ma) m h e^{-mh} dm \\ & - \rho a \int_0^a J_1(mr) J_1(ma) \left[ (A, e^x) e^{-(x-mh)} + (C, e^x) (2\mu_1 + mh) e^{-(x-mh)} \right. \\ & \left. - (A, e^x) (1-2mh) e^{-(x+mh)} - (C, e^x) (2\mu_1 + mh - 4\mu_1 mh) e^{-(x+mh)} \right] dm \end{aligned}$$

with

$$\rho a \int_0^a J_1(mr) J_1(ma) m h e^{-mh} dm = 0 \quad (\sigma = 0)$$

$$= \frac{3\rho a^2 r}{\pi} \int_0^\pi \frac{h^2}{(h^2 + \omega^2)^{5/2}} \cdot \sin^2 \phi d\phi \quad (r \neq 0)$$

$$-w = + \frac{1+\mu_1}{E_1} \rho a \int_0^a \frac{J_0(mr) J_1(ma)}{m} \left[ A, e^{mh} - B, e^{-mh} - C, (2-4\mu_1 - mh) e^{mh} \right. \\ \left. - D, (2-4\mu_1 + mh) e^{-mh} \right] dm$$

$$= + \frac{2(1+\mu_1^2)}{E_1} \rho a \int_0^a \frac{J_0(mr) J_1(ma)}{m} e^{-mh} dm - \frac{(1+\mu_1)}{E_1} \rho a \int_0^a \frac{J_0(mr) J_1(ma)}{m} m h e^{-mh} dm$$

$$- \frac{(1+\mu_1)}{E_1} \rho a \int_0^a \frac{J_0(mr) J_1(ma)}{m} \left[ (A, e^x) e^{-(x-mh)} - (C, e^x) (2-4\mu_1 - mh) e^{-(x-mh)} \right. \\ \left. + (A, e^x) (3-4\mu_1 + 2mh) e^{-(x+mh)} - (C, e^x) (2-8\mu_1 + 8\mu_1^2 + mh - 4\mu_1 mh) e^{-(x+mh)} \right] dm$$

with

$$\rho a \int_0^a \frac{J_0(mr) J_1(ma)}{m} e^{-mh} dm = p \left[ (a^2 + h^2)^{1/2} - h \right] \quad (r=0)$$

$$= \frac{1}{2\pi} \int_{-a}^a \ln \frac{(r^2 - 2xr + a^2 + h^2)^{1/2} + (a^2 - x^2)^{1/2}}{(r^2 - 2xr + a^2 + h^2)^{1/2} - (a^2 - x^2)^{1/2}} dx \quad (r \neq 0)$$

$$-u = - \frac{1+\mu_1}{E_1} \rho a \int_0^a \frac{J_1(mr) J_1(ma)}{m} \left[ A, e^{mh} + B, e^{-mh} + C, (1+mh) e^{mh} - D, (1-mh) e^{-mh} \right] dm$$

$$\begin{aligned}
u = & \frac{(1+\mu_1)(1-2\mu_1)}{E_1} p a \int_0^{\infty} \frac{J_1(mr) \cdot J_1(ma)}{m} e^{-mh} dm \\
& - \frac{(1+\mu_1)}{E_1} p a \int_0^{\infty} \frac{J_1(mr) \cdot J_1(ma)}{m} \cdot mh e^{-mh} dm \\
& - \frac{(1+\mu_1)}{E_1} p a \int_0^{\infty} \frac{J_1(mr) \cdot J_1(ma)}{m} \left[ (A_1 e^x) e^{-(x-mh)} + (C_1 e^x) (1+\mu_1) e^{-(x-mh)} \right. \\
& \left. + (A_1 e^x) (3-4\mu_1 - 2mh) e^{-(x+mh)} - (C_1 e^x) (1-8\mu_1 + 8\mu_1^2 - mh + 4\mu_1 mh) e^{-(x+mh)} \right] dm
\end{aligned}$$

#### 4.4. Stresses and displacements in the other layers.

The relations for the stresses and displacements are given in paragraph 4.1. The terms  $A_i m^2 e^{mz}$ ,  $B_i m^2 e^{-mz}$ ,  $C_i m e^{mz}$  and  $D_i m e^{-mz}$  are to be replaced

In the second layer ( $H_1 < h < H_1 + H_2$ ) by

$$(A_2 e^y) \cdot e^{-(y-mh)} \quad (B_2 e^{-x}) e^{-(mh-x)} \quad (C_2 e^y) e^{-(y-mh)} \quad (D_2 e^{-x}) e^{-(mh-x)}$$

In the third layer ( $H_1 + H_2 < h < H_1 + H_2 + H_3$ ) by

$$(A_3 e^z) \cdot e^{-(z-mh)} \quad (B_3 e^{-y}) e^{-(mh-y)} \quad (C_3 e^z) e^{-(z-mh)} \quad (D_3 e^{-y}) e^{-(mh-y)}$$

In the fourth layer ( $H_1 + H_2 + H_3 < h < H_1 + H_2 + H_3 + H_4$ ) by

$$\begin{aligned}
B_4 e^{-mh} & \longrightarrow (B_4 e^{-z}) \cdot e^{-(mh-z)} \\
D_4 e^{-mh} & \longrightarrow (D_4 e^{-z}) \cdot e^{-(mh-z)} \\
A_4 e^{mh} & = (A_4 e^t) e^{-(t-mh)} \\
& = (D_4 e^{-z}) e^{-(t-z)} \cdot e^{-(t-mh)} \\
& \quad + (2-4\mu_4 + t) (D_4 e^{-z}) \cdot e^{-(t-z)} \cdot e^{-(t-mh)}
\end{aligned}$$


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APPENDIX 2

Algebraical Analysis of a four-layer isotropic System  
with fixed bottom, full slip condition at the first  
and second interface, full friction at the third  
interface.

APPENDIX 2

Algebraical Analysis of a four-layer isotropic System with fixed bottom and full slip condition at the first and second interfaces.

---

1. Boundary conditions.

We write

$$A_i = A_i m^2 \quad B_i = B_i m^2 \quad C_i = C_i m \quad D_i = D_i m$$

$$F = \frac{E_1(1+\mu_2)}{E_2(1+\mu_1)} \quad K = \frac{E_2(1+\mu_3)}{E_3(1+\mu_1)} \quad L = \frac{E_3(1+\mu_4)}{E_4(1+\mu_3)}$$

$$x = mH_1$$

$$y = m(H_1 + H_2)$$

$$z = m(H_1 + H_2 + H_3)$$

$$t = m(H_1 + H_2 + H_3 + H_4)$$

$$u = -2t + z$$

where  $H_1, H_2, H_3$  and  $H_4$  are the thicknesses of the four layers. The index 1 applies to the first layer.

Boundary conditions at the surface ( $z = 0$ ):

$$\begin{aligned} \sigma_z = p & \quad A_1 + B_1 - C_1(1-2\mu_1) + D_1(1-2\mu_1) = 1 \\ \tau_{rz} = 0 & \quad A_1 - B_1 + C_1 \cdot 2\mu_1 + D_1 \cdot 2\mu_1 = 0 \end{aligned}$$

Boundary conditions at the first interface ( $z = H_1$ ):

$$\begin{aligned} \sigma_z &: A_1 e^x + B_1 e^{-x} - C_1(1-2\mu_1-x)e^x + D_1(1-2\mu_1+x)e^{-x} = \\ & \quad A_2 e^x + B_2 e^{-x} - C_2(1-2\mu_2-x)e^x + D_2(1-2\mu_2+x)e^{-x} \\ \tau_{rz} = 0 &: A_1 e^x - B_1 e^{-x} + C_1(2\mu_1+x)e^x + D_1(2\mu_1-x)e^{-x} = 0 \\ \tau_{rz} = 0 &: A_2 e^x - B_2 e^{-x} + C_2(2\mu_2+x)e^x + D_2(2\mu_2-x)e^{-x} = 0 \\ w &: A_1 e^x - B_1 e^{-x} - C_1(2-4\mu_1-x)e^x - D_1(2-4\mu_1+x)e^{-x} = \\ & \quad F[A_2 e^x - B_2 e^{-x} - C_2(2-4\mu_2-x)e^x - D_2(2-4\mu_2+x)e^{-x}] \end{aligned}$$

Boundary conditions at the second interface ( $z = H_1 + H_2$ ):

$$\sigma_z: A_2 e^\gamma + B_2 e^{-\gamma} - C_2 (1 - 2\mu_2 - \gamma) e^\gamma + D_2 (1 - 2\mu_2 + \gamma) e^{-\gamma} =$$

$$A_3 e^\gamma + B_3 e^{-\gamma} - C_3 (1 - 2\mu_3 - \gamma) e^\gamma + D_3 (1 - 2\mu_3 + \gamma) e^{-\gamma}$$

$$\tau_{rz}=0: A_2 \gamma - B_2 \gamma + C_2 (2\mu_2 + \gamma) e^\gamma + D_2 (2\mu_2 - \gamma) e^{-\gamma} = 0$$

$$\tau_{rz}=0: A_3 \gamma - B_3 \gamma + C_3 (2\mu_3 + \gamma) e^\gamma + D_3 (2\mu_3 - \gamma) e^{-\gamma} = 0$$

$$w: A_2 e^\gamma - B_2 e^{-\gamma} - C_2 (2 - 4\mu_2 - \gamma) e^\gamma - D_2 (2 - 4\mu_2 + \gamma) e^{-\gamma} =$$

$$\kappa [A_3 e^\gamma - B_3 e^{-\gamma} - C_3 (2 - 4\mu_3 - \gamma) e^\gamma - D_3 (2 - 4\mu_3 + \gamma) e^{-\gamma}]$$

Boundary conditions at the third interface ( $z = H_1 + H_2 + H_3$ ):

$$\sigma_z: A_3 e^z + B_3 e^{-z} - C_3 (1 - 2\mu_3 - z) e^z + D_3 (1 - 2\mu_3 + z) e^{-z} =$$

$$A_4 e^z + B_4 e^{-z} - C_4 (1 - 2\mu_4 - z) e^z + D_4 (1 - 2\mu_4 + z) e^{-z}$$

$$\tau_{rz}: A_3 e^z - B_3 e^{-z} + C_3 (2\mu_3 + z) e^z + D_3 (2\mu_3 - z) e^{-z} =$$

$$A_4 e^z - B_4 e^{-z} + C_4 (2\mu_4 + z) e^z + D_4 (2\mu_4 - z) e^{-z}$$

$$w: A_3 e^z - B_3 e^{-z} - C_3 (2 - 4\mu_3 - z) e^z - D_3 (2 - 4\mu_3 + z) e^{-z} =$$

$$L [A_4 e^z - B_4 e^{-z} - C_4 (2 - 4\mu_4 - z) e^z - D_4 (2 - 4\mu_4 + z) e^{-z}]$$

$$u: A_3 e^z + B_3 e^{-z} + C_3 (1 + z) e^z - D_3 (1 - z) e^{-z} =$$

$$L [A_4 e^z + B_4 e^{-z} + C_4 (1 + z) e^z - D_4 (1 - z) e^{-z}]$$

Boundary conditions at the bottom ( $z = H_1 + H_2 + H_3 + H_4$ ):

$$w: A_4 e^t - B_4 e^{-t} - C_4 (2 - 4\mu_4 - t) e^t - D_4 (2 - 4\mu_4 + t) e^{-t} = 0$$

$$C_4 = 0$$

## 2. Resolution of the system of 16 equations.

In the equations of the conditions at the third interface,  $A_4$  is replaced by its value taken from the fixed bottom condition:

$$A_4 e^t = B_4 e^{-t} + D_4 (2 - 4\mu_4 + t) e^{-t}$$

We write the conditions at the third interface in matrixform

$$M_5 (A_3 B_3 C_3 D_3)^T = M_6 (B_4 D_4)^T$$

We invert  $M_5$

$$(A_3 B_3 C_3 D_3)^T = M_5^{-1} \cdot M_6 (B_4 D_4)^T$$

where

$$M_5^{-1} = -\frac{1}{4(1-\mu_3)} \cdot e^{-z} \begin{pmatrix} -(1+z) & -(2-4\mu_3-z) & -(2\mu_3+z) & -(1-2\mu_3-z) \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$-\frac{1}{4(1-\mu_3)} \cdot e^z \begin{pmatrix} 0 & 0 & 0 & 0 \\ -(1-z) & (2-4\mu_3+z) & (2\mu_3-z) & -(1-2\mu_3+z) \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 \end{pmatrix}$$

$$M_6 = e^{-u} \begin{pmatrix} 1 & (2-4\mu_4+t) \\ 1 & (2-4\mu_4+t) \\ L & L(2-4\mu_4+t) \\ L & L(2-4\mu_4+t) \end{pmatrix} + e^{-z} \begin{pmatrix} 1 & (1-2\mu_4+2) \\ -1 & (2\mu_4-2) \\ -L & -L(2-4\mu_4+2) \\ L & -L(1-\mu_4) \end{pmatrix}$$

$$\begin{aligned} (A_3 B_3 C_3 D_3)^T &= -\frac{1}{4(1-\mu_3)} \left[ e^{-z} \cdot M_{51} + e^z \cdot M_{52} \right] \cdot \left[ e^{-u} \cdot M_{61} + e^{-z} \cdot M_{62} \right] (B_4 D_4)^T \\ &= -\frac{1}{4(1-\mu_3)} \left[ e^{-2t} \cdot M_{5161} + e^{-2z} \cdot M_{5162} + e^{-2(t-z)} \cdot M_{5261} \right. \\ &\quad \left. + M_{5262} \right] \cdot (B_4 D_4)^T \end{aligned}$$

$$M_{51} \cdot M_{61} = \begin{pmatrix} + & + \\ 0 & 0 \\ + & + \\ 0 & 0 \end{pmatrix} = M_{5161}$$

$$M_{51} \cdot M_{62} = \begin{pmatrix} + & + \\ 0 & 0 \\ + & + \\ 0 & 0 \end{pmatrix} = M_{5162}$$

$$M_{52} \cdot M_{61} = \begin{pmatrix} 0 & 0 \\ + & + \\ 0 & 0 \\ + & + \end{pmatrix} = M_{5261}$$

$$M_{52} \cdot M_{62} = \begin{pmatrix} 0 & 0 \\ L_1 & L_3 \\ 0 & 0 \\ 0 & L_2 \end{pmatrix} + 2(L_1 - L_2) \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = M_{5262}$$

$$L_1 = -2(1-2\mu_3) - (1+L)$$

$$L_2 = -2L(1-2\mu_4) - (1+L)$$

$$L_3 = \frac{1}{2} [L_1(1-4\mu_4) - L_2(1-4\mu_3)]$$

We consider now the boundary conditions at the surface.  
Adding and subtracting the surface conditions, we obtain

$$\begin{aligned} 2A_1 &= 1 + C_1(1-4\mu_1) - D_1 \\ 2B_1 &= 1 + C_1 - D_1(1-4\mu_1) \end{aligned}$$

Adding and subtracting then the first two conditions at the first interface, we obtain

$$\begin{aligned} 2A_1 e^x - C_1(1-4\mu_1 - 2x)e^x + D_1 e^{-x} &= [A_2] \\ 2B_1 e^{-x} - C_1 e^x + D_1(1-4\mu_1 + 2x)e^{-x} &= [A_2] \end{aligned}$$

where

$$[A_2] = A_2 e^x + B_2 e^{-x} - C_2(1-2\mu_2 - x)e^x + D_2(1-2\mu_2 + x)e^{-x}$$

We replace  $A_1$  and  $B_1$  by their values in function of  $C_1$  and  $D_1$  and solve the system

$$\begin{aligned} C_1 e^x &= \frac{[A_2] [(2x-1)e^{-2x} + 1] - (1+2x)e^{-x} + e^{-3x}}{2(1+2x^2)e^{-2x} - e^{-4x} - 1} \\ D_1 e^{-x} &= \frac{[A_2] [(1+2x)e^{-2x} - e^{-4x}] + (1-2x)e^{-3x} - e^{-x}}{2(1+2x^2)e^{-2x} - e^{-4x} - 1} \end{aligned}$$

We combine the  $\tau_{rz}$ -conditions with the  $w$ -condition so that

$$2(1-\mu_1)C_1 e^x + 2(1-\mu_1)D_1 e^{-x} = F [2(1-\mu_2)C_2 e^x + 2(1-\mu_2)D_2 e^{-x}]$$

and with  $F_1 = \frac{F(1-\mu_2)}{(1-\mu_1)}$

$$C_1 e^x + D_1 e^{-x} = F_1 [C_2 e^x + D_2 e^{-x}]$$

We replace  $C_1$  and  $D_1$  by their values in function of  $A_2$ ,  $B_2$ ,  $C_2$  and  $D_2$

$$\frac{[A_2] [1+4xe^{-2x} - e^{-4x}] - 2(1+x)e^{-x} + 2(1-x)e^{-3x}}{2(1+2x^2)e^{-2x} - e^{-4x} - 1} = F_1 [C_2 e^x + D_2 e^{-x}]$$

Together with the  $\tau_{r2}$ -condition, we obtain then the system

$$A_2 e^x - B_2 e^{-x} + C_2 (2\mu_2 + x) e^x + D_2 (2\mu_2 - x) e^{-x} = 0$$

$$A_2 e^x + B_2 e^{-x} - C_2 (1 - 2\mu_2 - x + R_3) e^x + D_2 (1 - 2\mu_2 + x - R_3) e^{-x} = -\frac{R_2}{R_1}$$

where

$$R_1 = 1 + 4x e^{-2x} - e^{-4x}$$

$$R_2 = 2(1-x)e^{-3x} - 2(1+x)e^{-x}$$

$$\nabla_1 = 2(1+2x^2)e^{-2x} - e^{-4x} - 1$$

$$R_3 = \frac{F_1 \cdot \nabla_1}{R_1}$$

We notice that for  $m = \infty$

$$R_1 = 1 \quad R_2 = 0 \quad \nabla_1 = -1 \quad R_3 = -F_1$$

We solve the system for  $A_2$  and  $B_2$

$$2A_2 = -\frac{R_2}{R_1} e^{-x} + C_2 (1 - 4\mu_2 - 2x + R_3) - D_2 (1 - R_3) e^{-2x}$$

$$2B_2 = -\frac{R_2}{R_1} e^x + C_2 (1 + R_3) e^{2x} - D_2 (1 - 4\mu_2 + 2x - R_3)$$

Adding and subtracting now the first two conditions at the second interface, we obtain

$$2A_2 e^y - C_2 (1 - 4\mu_2 - 2y) e^y + D_2 e^{-y} = [A_3]$$

$$2B_2 e^{-y} - C_2 e^y + D_2 (1 - 4\mu_2 + 2y) e^{-y} = [A_3]$$

where

$$[A_3] = A_3 e^y + B_3 e^{-y} - C_3 (1 - 2\mu_3 - y) e^y + D_3 (1 - 2\mu_3 + y) e^{-y}$$

We replace  $A_2$  and  $B_2$  by their values in function of  $C_2$  and  $D_2$  and solve the system.

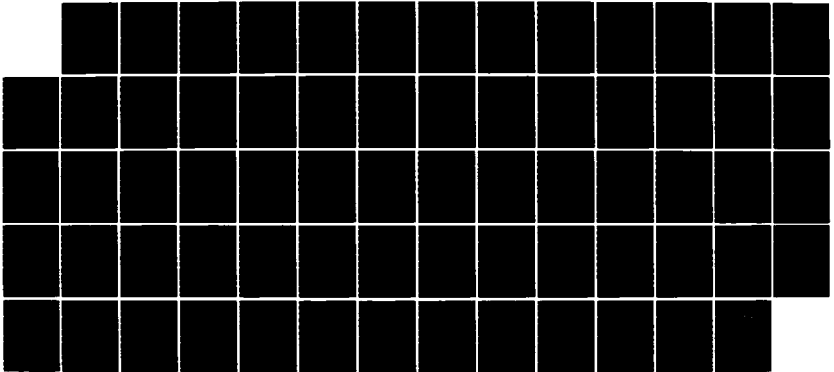
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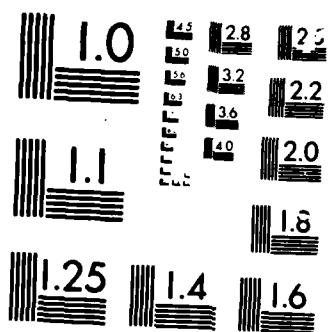
COMPUTER PROGRAMS FOR THE DETERMINATION OF STRESSES AND  
DISPLACEMENTS IN. (U) CENTRE DE RECHERCHES DE L'INST  
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MICROCOPY RESOLUTION TEST CHART

$$C_2 e^y = \frac{[A_3][(2y-2x+R_3-1)e^{-2(y-x)} + (1-R_3)] + \frac{R_2}{R_1}[(1+2y-2x)e^{-(y-x)} - e^{-3(y-x)}]}{2[1+2(y-x)^2]e^{-2(y-x)} + 4R_3(y-x)e^{-2(y-x)} - (1+R_3)e^{-4(y-x)} - (1-R_3)}$$

$$D_2 e^{-y} = \frac{[A_3][(1+2y-2x+R_3)e^{-2(y-x)} - (1+R_3)e^{-4(y-x)}] + \frac{R_2}{R_1}[e^{-(y-x)} - (1-2y+2x)e^{-3(y-x)}]}{2[1+2(y-x)^2]e^{-2(y-x)} + 4R_3(y-x)e^{-2(y-x)} - (1+R_3)e^{-4(y-x)} - (1-R_3)}$$

We combine the  $\tau_{rz}$ -conditions with the w-condition so that

$$2(1-\mu_1)C_2 e^y + 2(1-\mu_1)D_2 e^{-y} = k [2(1-\mu_3)C_3 e^y + 2(1-\mu_3)D_3 e^{-y}]$$

and with

$$k_1 = \frac{k(1-\mu_3)}{(1-\mu_1)}$$

$$C_2 e^y + D_2 e^{-y} = k_1 [C_3 e^y + D_3 e^{-y}]$$

We replace  $C_2$  and  $D_2$  by their values in function of  $A_3$ ,  $B_3$ ,  $C_3$  and  $D_3$

$$\frac{[A_3][2(2y-2x+R_3)e^{-2(y-x)} + (1-R_3) - (1+R_3)e^{-4(y-x)}] + \frac{2R_2}{R_1}[(1+y-x)e^{-(y-x)} - (1-y+x)e^{-3(y-x)}]}{2[1+2(y-x)^2]e^{-2(y-x)} + 4R_3(y-x)e^{-2(y-x)} - (1+R_3)e^{-4(y-x)} - (1-R_3)}$$

$$= k_1 [C_3 e^y + D_3 e^{-y}]$$

Together with the  $\tau_{rz}$ -condition, we obtain then the system

$$A_3 e^y - B_3 e^{-y} + C_3(2\mu_3+y)e^y + D_3(2\mu_3-y)e^{-y} = 0$$

$$A_3 e^y + B_3 e^{-y} - C_3(1-2\mu_3-y+Q_3)e^y + D_3(1-2\mu_3+y-Q_3)e^{-y} = -\frac{Q_2}{Q_1}$$

where

$$Q_1 = 2(2y-2x+R_3)e^{-2(y-x)} + (1-R_3) - (1+R_3)e^{-4(y-x)}$$

$$Q_2 = \frac{2R_2}{R_1} [(1+y-x)e^{-(y-x)} - (1-y+x)e^{-3(y-x)}]$$

$$\nabla_2 = 2 \left[ 1 + 2(\gamma - x)^2 \right] e^{-2(\gamma - x)} + 4R_3(\gamma - x) e^{-2(\gamma - x)} - (1 + R_3) e^{-4(\gamma - x)} - (1 - R_3)$$

$$Q_3 = \frac{\kappa_1 \cdot \nabla_2}{Q_1}$$

We notice that for  $m = \infty$

$$Q_1 = 1 - R_3 \quad Q_2 = 0 \quad \nabla_2 = -(1 - R_3) \quad Q_3 = -\kappa_1$$

We write the system in matrix form

$$\left[ \begin{array}{c|cccc} e^{\gamma} & 1 & 0 & -(1 - 2\mu_3 - \gamma + Q_3) & 0 \\ & 1 & 0 & 2\mu_3 + \gamma & 0 \\ + e^{-\gamma} & 0 & 1 & (1 - 2\mu_3 + \gamma - Q_3) & \\ & 0 & -1 & 2\mu_3 - \gamma & \end{array} \right] \begin{bmatrix} A_3 \\ B_3 \\ C_3 \\ D_3 \end{bmatrix} = \begin{bmatrix} -\frac{Q_2}{Q_1} \\ 0 \end{bmatrix}$$

or

$$\left[ e^{\gamma} \cdot M_{41} + e^{-\gamma} M_{42} \right] (A_3 \ B_3 \ C_3 \ D_3)^T = \left( -\frac{Q_2}{Q_1} \ 0 \right)^T$$

We replace the matrix  $(A_3 \ B_3 \ C_3 \ D_3)^T$  in function of  $B_4$  and  $D_4$

$$\left[ e^{\gamma} \cdot M_{41} + e^{-\gamma} M_{42} \right] \left[ e^{-2t} M_{5161} + e^{-2x} M_{5162} + e^{-2(t-x)} M_{5261} + M_{5262} \right] (B_4 \ D_4)^T \\ = 4(1 - \mu_3) \left( \frac{Q_2}{Q_1} \ 0 \right)^T$$

We notice that  $M_{41} \cdot M_{5261} = 0$

$$M_{41} \cdot M_{5262} = 0$$

$$M_{42} \cdot M_{5161} = 0$$

$$M_{42} \cdot M_{5162} = 0$$

We call

$$M_{41} \cdot M_{5161} = N_1$$

$$M_{41} \cdot M_{5162} = N_2$$

$$M_{42} \cdot M_{5261} = N_3$$

$$M_{42} \cdot M_{5262} = N_4$$

and write

$$\begin{aligned} & \left[ e^{-2(r-\gamma)} N_1 + e^{-2(z-\gamma)} N_2 + e^{-2(r-z)} e^{-\gamma} N_3 + e^{-\gamma} N_4 \right] (D_4 \ D_4)^T \\ & = 4(1-\mu_3) \left( \frac{Q_2}{Q_1} \ 0 \right)^T \end{aligned}$$

that we transform into

$$\begin{aligned} & \left[ e^{-2(r-\gamma)} N_1 + e^{-2(z-\gamma)} N_2 + e^{-2(r-z)} N_3 + N_4 \right] (B_4 e^{-\gamma} \ D_4 e^{-\gamma})^T \\ & = 4(1-\mu_3) \left( \frac{Q_2}{Q_1} \ 0 \right)^T \end{aligned}$$

We write

$$e^{-2(r-\gamma)} N_1 + e^{-2(z-\gamma)} N_2 + e^{-2(r-z)} N_3 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$N_4 = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$$

where all the terms  $a_{ij}$  converge when  $m = \infty$

We develop the matrix equation

$$(a_{11} + b_{11}) B_4 e^{-\gamma} + (a_{12} + b_{12}) D_4 e^{-\gamma} = 4(1-\mu_3) \frac{Q_2}{Q_1}$$

$$(a_{21} + b_{21}) B_4 e^{-\gamma} + (a_{22} + b_{22}) D_4 e^{-\gamma} = 0$$

We solve the system

$$B_4 e^{-y} = 4(1-\tau_3) \frac{Q_2}{Q_1} \frac{a_{22} + b_{22}}{V_3}$$

$$D_4 e^{-y} = -4(1-\tau_3) \frac{Q_2}{Q_1} \frac{a_{21} + b_{21}}{V_3}$$

where

$$\begin{aligned} V_3 &= (a_{11} + b_{11})(a_{22} + b_{22}) - (a_{12} + b_{12})(a_{21} + b_{21}) \\ &= (a_{11} + b_{11})a_{22} - (a_{12} + b_{12})a_{21} + b_{11} \cdot b_{22} - b_{12} \cdot b_{21} + a_{11} \cdot b_{22} - a_{12} \cdot b_{21} \end{aligned}$$

The term  $b_{11} \cdot b_{22} - b_{12} \cdot b_{21}$  contains linear functions of the variables and has to be developed in close form

$$\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 & 1-2\tau_3+\gamma-Q_3 \\ 0 & -1 & 0 & 2\tau_3-\gamma \end{vmatrix} \cdot \begin{vmatrix} 0 & 0 \\ L_1 & L_3+2(L_1-L_2) \\ 0 & 0 \\ 0 & L_2 \end{vmatrix}$$

$$b_{11} = L_1$$

$$b_{12} = L_3 + 2(L_1 - L_2) + L_2(1 - 2\tau_3 + \gamma - Q_3)$$

$$b_{21} = -L_1$$

$$b_{22} = -L_3 - 2(L_1 - L_2) + L_2(2\tau_3 - \gamma)$$

$$\begin{aligned} b_{11} \cdot b_{22} - b_{12} \cdot b_{21} &= L_1 \left[ -L_3 - 2(L_1 - L_2) + L_2(2\tau_3 - \gamma) \right. \\ &\quad \left. + L_3 + 2(L_1 - L_2) + L_2(1 - 2\tau_3 + \gamma - Q_3) \right] \\ &= L_1 L_2 (1 - Q_3) \end{aligned}$$

The linear functions of the variables have disappeared so that for  $m = \infty$  the numerators of  $B_4 \cdot e^{-y}$  and  $D_4 \cdot e^{-y}$  tend both to zero, because of the factor  $Q_2$ , and the denominator tends to a constant:

$$\lim_{m \rightarrow \infty} Q_1 L_1 L_2 (1 - Q_3) = (1 + F_1)(1 + k_1) L_1 L_2$$

and finally

$$B_4 e^{-z} = B_4 e^{-y} \cdot e^{-(z-y)} \quad D_4 e^{-z} = D_4 e^{-y} \cdot e^{-(z-y)}$$

### 3. Values of the parameters $A_i, D_i$ .

We express the values of the parameters  $A_i, D_i$  in the same way as explained in appendix 1.

#### 3.1. Values of the parameters $A_3, B_3, C_3, D_3$

The values of the parameters  $A_3, B_3, C_3$  and  $D_3$  are obtained from the relation

$$(A_3 \ B_3 \ C_3 \ D_3)^T = -\frac{1}{4(1-\mu_3)} \left[ e^{-2t} M_{5161} + e^{-2z} M_{5162} + e^{-2(t-z)} M_{5261} + M_{5262} \right] (B_4 \ D_4)^T$$

The matrices  $M_{5261}$  and  $M_{5262}$  contain nothing but zeros in their first and third rows, so that we can write

$$(A_3 \ 0 \ C_3 \ 0)^T = -\frac{1}{4(1-\mu_3)} \left[ e^{-2t} M_{5161} + e^{-2z} M_{5162} \right] (B_4 \ D_4)^T$$

$$(A_3 e^z \ 0 \ C_3 e^z \ 0)^T = -\frac{1}{4(1-\mu_3)} \left[ e^{-(t-z)} e^{-(t-z)} M_{5161} + e^{-(z-z)} M_{5162} \right] (B_4 e^{-y} \ D_4 e^{-y})^T$$

The matrices  $M_{5161}$  and  $M_{5162}$  contain nothing but zeros in their second and fourth rows, so that we can write

$$(0 \ B_3 \ 0 \ D_3)^T = -\frac{1}{4(1-\mu_3)} \left[ e^{-2(t-z)} M_{5261} + M_{5262} \right] (B_4 \ D_4)^T$$

$$(0 \ B_3 e^{-y} \ 0 \ D_3 e^{-y})^T = -\frac{1}{4(1-\mu_3)} \left[ e^{-2(t-z)} M_{5261} + M_{5262} \right] (B_4 e^{-y} \ D_4 e^{-y})^T$$

### 3.2. Values of the parameters $A_2, B_2, C_2, D_2$ .

We have

$$C_2 e^y = \frac{[A_3] [(2y - 2x + R_3 - 1) e^{-2(y-x)} + (1 - R_3)] + \frac{R_2}{R_1} [(1 + 2y - 2x) e^{-(y-x)} - e^{-3(y-x)}]}{\Delta_2}$$

$$D_2 e^{-y} = \frac{[A_3] [(1 + 2y - 2x + R_3) e^{-2(y-x)} - (1 + R_3) e^{-4(y-x)}] + \frac{R_2}{R_1} [e^{-(y-x)} - (1 - 2y + 2x) e^{-3(y-x)}]}{\Delta_2}$$

so that

$$\begin{aligned} D_2 e^{-x} &= D_2 e^{-y} \cdot e^{(y-x)} \\ &= \frac{[A_3] [(1 + 2y - 2x + R_3) e^{-(y-x)} - (1 + R_3) e^{-3(y-x)}] + \frac{R_2}{R_1} [1 - (1 - 2y + 2x) e^{-2(y-x)}]}{\Delta_2} \end{aligned}$$

where

$$\begin{aligned} [A_3] &= A_3 e^y + B_3 e^{-y} - C_3 (1 - 4\mu_3 - y) e^y + D_3 (1 - 4\mu_3 + y) e^{-y} \\ &= (A_3 e^2) \cdot e^{(2-y)} + B_3 e^{-y} - (C_3 e^2) \cdot (1 - 4\mu_3 - y) e^{(2-y)} + D_3 (1 - 4\mu_3 + y) e^{-y} \end{aligned}$$

We have also that

$$2A_2 e^y - C_2 (1 - 4\mu_2 - 2y) e^y + D_2 e^{-y} = [A_3]$$

so that

$$A_2 e^y = \frac{1}{2} \left\{ [A_3] + C_2 (1 - 4\mu_2 - 2y) e^y - (D_2 e^{-x}) e^{(y-x)} \right\}$$

and finally we have that

$$2B_2 = -\frac{R_2}{R_1} e^x + C_2 (1 + R_3) e^x - D_2 (1 - 4\mu_2 + 2x - R_3)$$

so that

$$B_2 e^{-x} = \frac{1}{2} \left[ -\frac{R_2}{R_1} + (C_2 e^y) (1 + R_3) e^{(y-x)} - (D_2 e^{-x}) \cdot (1 - 4\mu_2 + 2x - R_3) \right]$$

### 3.3. Values of the parameters $A_1$ and $C_1$ .

We have that

$$C_1 e^x = \frac{[A_2] \cdot [(2x-1)e^{-2x} + 1] - (1+2x)e^{-x} + e^{-3x}}{V_1}$$

$$D_1 e^{-x} = \frac{[A_2] [(1+2x)e^{-2x} - e^{-4x}] + (1-2x)e^{-3x} - e^{-x}}{V_1}$$

where

$$\begin{aligned} [A_2] &= A_2 e^x + B_2 e^{-x} - C_2 (1-2\mu_2-x)e^x + D_2 (1-2\mu_2+x)e^{-x} \\ &= (A_2 e^y) \cdot e^{-(y-x)} + B_2 e^{-x} - (C_2 e^y) \cdot (1-2\mu_2-x)e^{-(y-x)} + D_2 e^{-x} \cdot (1-2\mu_2+x) \end{aligned}$$

We have also that

$$2A_1 e^x - C_1 (1-4\mu_1-2x)e^x + D_1 e^{-x} = [A_2]$$

so that

$$A_1 e^x = \frac{1}{2} \left\{ [A_2] + C_1 (1-4\mu_1-2x)e^x - D_1 e^{-x} \right\}$$

The values of  $B_1$  and  $D_1$  are obtained from the surface conditions

$$A_1 + B_1 - C_1 (1-2\mu_1) + D_1 (1-2\mu_1) = 1$$

$$A_1 - B_1 + C_1 2\mu_1 + D_1 2\mu_1 = 0$$

so that

$$B_1 = 2\mu_1 + A_1 (1-4\mu_1) + 4\mu_1 C_1 (1-2\mu_1)$$

$$= 2\mu_1 + (A_1 e^x) (1-4\mu_1) e^{-x} + 4\mu_1 (C_1 e^x) (1-2\mu_1) e^{-x}$$

$$D_1 = 1 - 2A_1 + C_1 (1-4\mu_1)$$

$$= 1 - 2(A_1 e^x) e^{-x} + (C_1 e^x) (1-4\mu_1) e^{-x}$$

#### 4. Relations for the stresses and the displacements.

The relations for the stresses and displacements are completely the same as those developed in appendix 1, by replacing the parameters  $A_i, D_i$  by their adequate values.

Nevertheless, there is a problem in the computation of the vertical displacement: its value at the origin ( $m = 0$ ) is undetermined. The relation for the vertical deflection at the surface is given by

$$w = p\alpha \frac{1+\mu_1}{E_1} \int_0^{\infty} \frac{J_0(mr) J_1(ma)}{m} [A_1 - B_1 - C_1 (2-4\mu_1) - D_1 (2-4\mu_1)]$$

which, to avoid convergency problems, is transformed into

$$w = -p\alpha \frac{2(1-\mu_1^2)}{E_1} \int_0^{\infty} \frac{J_0(mr) J_1(ma)}{m} \left\{ 1 - 2 [A_1 - (1-2\mu_1)C_1] \right\} dm$$

The numerators and denominator of  $A_1$  and  $C_1$  are both zero for  $m = 0$ . To eliminate the indetermination we should develop  $A_1$  and  $C_1$  in a Taylor series. Although this is theoretically possible, the required computation is very long and the risks of introducing errors, in doing so, are enormous. Fortunately, we dispose over the fixed bottom condition, which, for  $m = 0$ , transforms into

$$A_4 - B_4 - C_4 (2-4\mu_4) - D_4 (2-4\mu_4) = 0$$

The w-conditions at the other interfaces transform into

$$A_3 - B_3 - C_3 (2-4\mu_3) - D_3 (2-4\mu_3) = L [A_4 - B_4 - C_4 (2-4\mu_4) - D_4 (2-4\mu_4)]$$

$$A_2 - B_2 - C_2 (2-4\mu_2) - D_2 (2-4\mu_2) = K [A_3 - B_3 - C_3 (2-4\mu_3) - D_3 (2-4\mu_3)]$$

$$A_1 - B_1 - C_1 (2-4\mu_1) - D_1 (2-4\mu_1) = F [A_2 - B_2 - C_2 (2-4\mu_2) - D_2 (2-4\mu_2)]$$

so that, for  $m = 0$ ,

$$A_1 - B_1 - C_1 [2-4\mu_1] - D_1 [2-4\mu_1] = 0$$

and thus

$$[A_1 - (1-2\mu_1)C_1] = \frac{1}{2}$$

so that the problem is solved without any difficulty.

APPENDIX 3

Algebraical Analysis of a four-layer anisotropic System  
with fixed bottom and partial or full friction interface  
conditions.

### APPENDIX 3

Algebraical analysis of an anisotropic four layered structure with fixed bottom and partial friction interface conditions.

#### 1. Boundary conditions.

Because of the presence of 4 different exponential functions in the expressions for stresses and displacements, the algebraical analysis is more laborious than in the isotropic case (appendices 1 and 2) and has to be developed in a more detailed way.

To reduce the number of exponentials we write

$$\begin{aligned} A_i m^2 n_i (1 + \mu_i) e^{m(H_1 + H_2 + \dots + H_i)} &= A_i \\ B_i m^2 n_i (1 + \mu_i) e^{-m(H_1 + H_2 + \dots + H_{i-1})} &= B_i \\ C_i m^2 n_i s_i (n_i + \mu_i) e^{m s_i (H_1 + H_2 + \dots + H_i)} &= C_i \\ D_i m^2 n_i s_i (n_i + \mu_i) e^{-m s_i (H_1 + H_2 + \dots + H_{i-1})} &= D_i \end{aligned}$$

Further we also write

$$\begin{aligned} F_1 &= \frac{E_1}{E_2} & F_2 &= \lambda_1 \frac{E_1}{E_2} \\ K_1 &= \frac{E_2}{E_3} & K_2 &= \lambda_2 \frac{E_2}{E_3} \\ L_1 &= \frac{E_3}{E_4} & L_2 &= \lambda_3 \frac{E_3}{E_4} \end{aligned} \quad S_i = \left( \frac{n_i - \mu_i^2}{n_i^2 \mu_i^2} \right)^{\frac{1}{2}}$$

$$\begin{aligned} x &= m H_1 \\ y &= m (H_1 + H_2) \\ z &= m (H_1 + H_2 + H_3) \\ t &= m (H_1 + H_2 + H_3 + H_4) \end{aligned}$$

where  $H_1, H_2, H_3$  and  $H_4$  are the thicknesses of the four layers.

The index 1 applies to the first layer.

Boundary conditions at the surface ( $z = 0$ ):

$$\begin{aligned} \sigma_z = p &: A_1 e^{-x} + B_1 + C_1 e^{-s_1 x} + D_1 = 1 \\ \tau_{rz} = 0 &: A_1 e^{-x} - B_1 + s_1 C_1 e^{-s_1 x} - s_1 D_1 = 0 \end{aligned}$$

Boundary conditions at the first interface ( $z = H_1$ )

$$\begin{aligned} \sigma_z: A_1 + B_1 e^{-x} + C_1 + D_1 e^{-s_1 x} &= A_2 e^{-(y-x)} + B_2 + C_2 e^{-s_2(y-x)} + D_2 \\ \tau_{xz}: A_1 - B_1 e^{-x} + s_1 C_1 - s_1 D_1 e^{-s_1 x} &= A_2 e^{-(y-x)} - B_2 + s_2 C_2 e^{-s_2(y-x)} - s_2 D_2 \\ w: (1+\mu_1) A_1 - (1+\mu_1) B_1 e^{-x} + s_1(n_1+\mu_1) C_1 - s_1(n_1+\mu_1) D_1 e^{-s_1 x} &= \\ F_1 [(1+\mu_2) A_2 e^{-(y-x)} - (1+\mu_2) B_2 + s_2(n_2+\mu_2) C_2 e^{-s_2(y-x)} - s_2(n_2+\mu_2) D_2] \\ u: (n_1+\mu_1) A_1 + (n_1+\mu_1) B_1 e^{-x} + (1+\mu_1) C_1 + (1+\mu_1) D_1 e^{-s_1 x} &= \\ F_2 [(n_2+\mu_2) A_2 e^{-(y-x)} + (n_2+\mu_2) B_2 + (1+\mu_2) C_2 e^{-s_2(y-x)} + (1+\mu_2) D_2] \end{aligned}$$

Boundary conditions at the second interface ( $z = H_1 + H_2$ ):

$$\begin{aligned} \sigma_z: A_2 + B_2 e^{-(y-x)} + C_2 + D_2 e^{-s_2(y-x)} &= A_3 e^{-(z-y)} + B_3 + C_3 e^{-s_3(z-y)} + D_3 \\ \tau_{xz}: A_2 - B_2 e^{-(y-x)} + s_2 C_2 - s_2 D_2 e^{-s_2(y-x)} &= A_3 e^{-(z-y)} - B_3 + s_3 C_3 e^{-s_3(z-y)} - s_3 D_3 \\ w: (1+\mu_2) A_2 - (1+\mu_2) B_2 e^{-(y-x)} + s_2(n_2+\mu_2) C_2 - s_2(n_2+\mu_2) D_2 e^{-s_2(y-x)} &= \\ K_1 [(1+\mu_3) A_3 e^{-(z-y)} - (1+\mu_3) B_3 + s_3(n_3+\mu_3) C_3 e^{-s_3(z-y)} - s_3(n_3+\mu_3) D_3] \\ u: (n_2+\mu_2) A_2 + (n_2+\mu_2) B_2 e^{-(y-x)} + (1+\mu_2) C_2 + (1+\mu_2) D_2 e^{-s_2(y-x)} &= \\ K_2 [(n_3+\mu_3) A_3 e^{-(z-y)} + (n_3+\mu_3) B_3 + (1+\mu_3) C_3 e^{-s_3(z-y)} + (1+\mu_3) D_3] \end{aligned}$$

Boundary conditions at the third interface ( $z = H_1 + H_2 + H_3$ ):

$$\begin{aligned} \sigma_z: A_3 + B_3 e^{-(z-y)} + C_3 + D_3 e^{-s_3(z-y)} &= A_4 e^{-(t-x)} + B_4 + C_4 e^{s_4(t-x)} + D_4 \\ \tau_{xz}: A_3 - B_3 e^{-(z-y)} + s_3 C_3 - s_3 D_3 e^{-s_3(z-y)} &= A_4 e^{-(t-x)} - B_4 + s_4 C_4 e^{s_4(t-x)} - s_4 D_4 \\ w: (1+\mu_3) A_3 - (1+\mu_3) B_3 e^{-(z-y)} + s_3(n_3+\mu_3) C_3 - s_3(n_3+\mu_3) D_3 e^{-s_3(z-y)} &= \\ L_1 [(1+\mu_4) A_4 e^{-(t-x)} - (1+\mu_4) B_4 + s_4(n_4+\mu_4) C_4 e^{s_4(t-x)} - s_4(n_4+\mu_4) D_4] \\ u: (n_3+\mu_3) A_3 + (n_3+\mu_3) B_3 e^{-(z-y)} + (1+\mu_3) C_3 + (1+\mu_3) D_3 &= \\ L_2 [(n_4+\mu_4) A_4 e^{-(t-x)} + (n_4+\mu_4) B_4 + (1+\mu_4) C_4 e^{s_4(t-x)} + (1+\mu_4) D_4] \end{aligned}$$

Boundary condition at the bottom ( $z = H_1 + H_2 + H_3 + H_4$ ):

$$w: (1+\mu_4) A_4 - (1+\mu_4) B_4 e^{-(t-x)} + s_4(n_4+\mu_4) C_4 - s_4(n_4+\mu_4) D_4 e^{-s_4(t-x)} = 0$$

$$\text{If } s_4 > 1 \quad C_4 = 0 \\ A_4 e^{-(t-x)} = B_4 e^{-2(t-x)} + \frac{s_4(n_4+\mu_4)}{(1+\mu_4)} D_4 e^{-(1+s_4)(t-x)}$$

$$\text{If } s_4 < 1 \quad A_4 = 0 \\ C_4 e^{-s_4(t-x)} = \frac{(1+\mu_4)}{s_4(n_4+\mu_4)} B_4 e^{-(1+s_4)(t-x)} + D_4 e^{-2s_4(t-x)}$$

2. Expression of the boundary conditions in matrixform.

2.1. At the third interface.

In the equations at the third interface,  $A_4$ , or  $C_4$ , is replaced by its value obtained from the fixed bottom condition.

We write the conditions at the third interface in matrixform

$$M_5 (A_3 B_3 C_3 D_3)^T = M_6 (B_4 D_4)^T$$

We invert  $M_5$

$$(A_3 B_3 C_3 D_3)^T = M_5^{-1} M_6 (B_4 D_4)^T$$

$$M_5^{-1} = \frac{1}{2s_3(1+n_3)} \begin{pmatrix} s_3(1+\mu_3) & -s_3(n_3+\mu_3) & s_3 & -s_3 \\ s_3(1+\mu_3)e^{s_3(z-y)} & s_3(n_3+\mu_3)e^{s_3(z-y)} & -s_3e^{s_3(z-y)} & -s_3e^{s_3(z-y)} \\ -s_3(n_3+\mu_3) & (1+\mu_3) & -1 & s_3 \\ -s_3(n_3+\mu_3)e^{s_3(z-y)} & -(1+\mu_3)e^{s_3(z-y)} & e^{s_3(z-y)} & s_3e^{s_3(z-y)} \end{pmatrix}$$

If  $s_4 > 1$

$$M_6 = \begin{pmatrix} 1 + e^{-2(t-z)} & 1 + \frac{s_4(n_4+\mu_4)}{s_4(n_4+\mu_4)} e^{-(1+s_4)(t-z)} \\ -1 + e^{-2(t-z)} & -s_4 + \frac{(1+\mu_4)}{s_4(n_4+\mu_4)} e^{-(1+s_4)(t-z)} \\ L_1(1+\mu_4) [-1 + e^{-2(t-z)}] & L_1 s_4(n_4+\mu_4) [-1 + e^{-(1+s_4)(t-z)}] \\ L_2(n_4+\mu_4) [1 + e^{-2(t-z)}] & L_2(1+\mu_4) \left[ 1 + \frac{s_4(n_4+\mu_4)^2}{(1+\mu_4)^2} e^{-(1+s_4)(t-z)} \right] \end{pmatrix}$$

If  $s_4 < 1$

$$M_6 = \begin{pmatrix} 1 + \frac{1+\mu_4}{s_4(n_4+\mu_4)} e^{-(1+s_4)(t-z)} & 1 + e^{-2s_4(t-z)} \\ -1 + \frac{1+\mu_4}{(n_4+\mu_4)} e^{-(1+s_4)(t-z)} & -s_4 + e^{-2s_4(t-z)} \cdot s_4 \\ L_1(1+\mu_4) [-1 + e^{-(1+s_4)(t-z)}] & L_1 s_4(n_4+\mu_4) [-1 + e^{-2s_4(t-z)}] \\ L_2(n_4+\mu_4) \left[ 1 + \frac{(1+\mu_4)^2}{s_4(n_4+\mu_4)^2} e^{-(1+s_4)(t-z)} \right] & L_2(1+\mu_4) [1 + e^{-2s_4(t-z)}] \end{pmatrix}$$

We write  $A_3$ ,  $B_3$ ,  $C_3$  and  $D_3$  in function of  $B_4$  and  $D_4$ .

$$A_3 = \frac{\sum M_5(i_1) \cdot M_6(i_1) \cdot B_4 + \sum M_5(i_1) \cdot M_6(i_2) D_4}{2j_3(1-n_3)}$$

$$B_3 = \frac{\sum M_5(i_1) \cdot M_6(i_1) \cdot B_4 + \sum M_5(i_2) \cdot M_6(i_2) D_4}{2j_3(1-n_3)} \cdot e^{(2-y)}$$

$$C_3 = \frac{\sum M_5(i_1) \cdot M_6(i_1) \cdot B_4 + \sum M_5(i_2) \cdot M_6(i_2) D_4}{2j_3(1-n_3)}$$

$$D_3 = \frac{\sum M_5(i_1) \cdot M_6(i_1) \cdot B_4 + \sum M_5(i_2) \cdot M_6(i_2) D_4}{2j_3(1-n_3)} e^{j_3(2-y)}$$

where  $M_5(i,j)$  are the constants in  $M_5^{-1}$ .

We write

$$P_{j1} = \sum M_5(ji) M_6(i_1)$$

$$P_{j2} = \sum M_5(ji) M_6(i_2)$$

so that

$$\begin{pmatrix} A_3 \\ B_3 \\ C_3 \\ D_3 \end{pmatrix} = \frac{1}{2j_3(1-n_3)} \begin{pmatrix} P_{11} & P_{12} \\ P_{21}e^{(2-y)} & P_{22}e^{(2-y)} \\ P_{31} & P_{32} \\ P_{41}e^{j_3(2-y)} & P_{42}e^{j_3(2-y)} \end{pmatrix} \begin{pmatrix} B_4 \\ D_4 \end{pmatrix}$$

## 2.2. At the second interface.

We write the conditions at the second interface in matrixform.

$$M_3 (A_2 B_2 C_2 D_2)^T = M_4 (A_3 B_3 C_3 D_3)^T$$

We invert  $M_3$

$$(A_2 B_2 C_2 D_2)^T = M_3^{-1} \cdot M_4 (A_3 B_3 C_3 D_3)^T$$

$$M_5^{-1} = \frac{1}{2s_2(1-h_2)} \begin{pmatrix} s_2(1+\mu_2) & -s_2(n_2+\mu_2) & s_2 & -s_2 \\ s_2(1+\mu_2)e^{(y-x)} & s_2(n_2+\mu_2)e^{(y-x)} & -s_2 e^{(y-x)} & -s_2 e^{(y-x)} \\ -s_2(n_2+\mu_2) & (1+\mu_2) & -1 & s_2 \\ -s_2(n_2+\mu_2)e^{s_1(y-x)} & -(1+\mu_2)e^{s_1(y-x)} & e^{s_2(y-x)} & s_2 e^{s_2(y-x)} \end{pmatrix}$$

$$M_4 = \begin{pmatrix} e^{-(x-y)} & 1 & e^{-s_3(x-y)} & 1 \\ e^{-(x-y)} & -1 & s_3 e^{-s_3(x-y)} & -s_3 \\ k_1(1+\mu_3)e^{-(x-y)} & -k_1(1+\mu_3) & k_1 s_3(n_3+\mu_3)e^{-s_3(x-y)} & -k_1 s_3(n_3+\mu_3) \\ k_2(n_3+\mu_3)e^{-(x-y)} & k_2(n_3+\mu_3) & k_2(1+\mu_3)e^{-s_3(x-y)} & k_2(1+\mu_3) \end{pmatrix}$$

We write  $A_2$ ,  $B_2$ ,  $C_2$  and  $D_2$  in function of  $A_3$ ,  $B_3$ ,  $C_3$  and  $D_3$ .

$$A_2 = \frac{\sum M_3(1i) \cdot M_4(i1) e^{-(x-y)} \cdot A_3 + \sum M_3(1i) \cdot M_4(i2) \cdot B_3}{2s_2(1-h_2)} \\ + \frac{\sum M_3(1i) \cdot M_4(i3) e^{-s_3(x-y)} \cdot C_3 + \sum M_3(1i) \cdot M_4(i4) \cdot D_3}{2s_2(1-h_2)}$$

$$B_2 = \frac{\sum M_3(2i) \cdot M_4(i1) e^{-(x-y)} \cdot A_3 + \sum M_3(2i) \cdot M_4(i2) \cdot B_3}{2s_2(1-h_2)} \cdot e^{(y-x)} \\ + \frac{\sum M_3(2i) \cdot M_4(i3) e^{-s_3(x-y)} \cdot C_3 + \sum M_3(2i) \cdot M_4(i4) \cdot D_3}{2s_2(1-h_2)} \cdot e^{(y-x)}$$

$$C_2 = \frac{\sum M_3(3i) \cdot M_4(i1) e^{-(x-y)} \cdot A_3 + \sum M_3(3i) \cdot M_4(i2) \cdot B_3}{2s_2(1-h_2)} \\ + \frac{\sum M_3(3i) \cdot M_4(i3) e^{-s_3(x-y)} \cdot C_3 + \sum M_3(3i) \cdot M_4(i4) \cdot D_3}{2s_2(1-h_2)}$$

$$D_2 = \frac{\sum M_3(4i) \cdot M_4(i1) e^{-(x-y)} \cdot A_3 + \sum M_3(4i) \cdot M_4(i2) \cdot B_3}{2s_2(1-h_2)} e^{s_2(y-x)} \\ + \frac{\sum M_3(4i) \cdot M_4(i3) e^{-s_3(x-y)} \cdot C_3 + \sum M_3(4i) \cdot M_4(i4) \cdot D_3}{2s_2(1-h_2)} e^{s_2(y-x)}$$

where  $M_3(i,j)$  and  $M_4(i,j)$  are the constants in  $M_3^{-1}$  and  $M_4$ .

We write

$$Q_{jl} = \sum M_3(ji) \cdot M_4(il)$$

so that

$$\begin{pmatrix} A_2 \\ B_2 \\ C_2 \\ D_2 \end{pmatrix} = \frac{1}{2s_2(1-n_2)} \begin{pmatrix} Q_{11}e^{-(z-y)} & Q_{12} & Q_{13}e^{-s_2(z-y)} & Q_{14} \\ Q_{21}e^{-(z-y)}e^{(y-x)} & Q_{22}e^{(y-x)} & Q_{23}e^{-s_2(z-y)}e^{(y-x)} & Q_{24}e^{(y-x)} \\ Q_{31}e^{-(z-y)} & Q_{32} & Q_{33}e^{-s_2(z-y)} & Q_{34} \\ Q_{41}e^{-(z-y)}e^{s_2(y-x)} & Q_{42}e^{s_2(y-x)} & Q_{43}e^{-s_2(z-y)}e^{s_2(y-x)} & Q_{44}e^{s_2(y-x)} \end{pmatrix} \begin{pmatrix} A_3 \\ B_3 \\ C_3 \\ D_3 \end{pmatrix}$$

2.3. At the first interface.

We write the conditions at the first interface in matrixform.

$$M_1(A, B, C, D)^T = M_2(A_2 B_2 C_2 D_2)^T$$

We invert  $M_1$

$$(A, B, C, D)^T = M_1^{-1} \cdot M_2(A_2 B_2 C_2 D_2)^T$$

$$M_1^{-1} = \frac{1}{2s_1(1-n_1)} \begin{pmatrix} s_1(1+p_1) & -s_1(n_1+p_1) & s_1 & -s_1 \\ s_1(1+p_1)e^x & s_1(n_1+p_1)e^x & -s_1e^x & -s_1e^x \\ -s_1(n_1+p_1) & (1+p_1) & -1 & s_1 \\ -s_1(n_1+p_1)e^{s_1x} & -(1+p_1)e^{s_1x} & e^{s_1x} & s_1e^{s_1x} \end{pmatrix}$$

$$M_2 = \begin{pmatrix} e^{-(y-x)} & 1 & e^{-s_2(y-x)} & 1 \\ e^{-(y-x)} & -1 & s_2e^{-s_2(y-x)} & -s_2 \\ F_1(1+p_2)e^{-(y-x)} & -F_1(1+p_2) & F_1s_2(n_2+p_2)e^{-s_2(y-x)} & -F_1s_2(n_2+p_2) \\ F_2(n_2+p_2)e^{-(y-x)} & F_2(n_2+p_2) & F_2(1+p_2)e^{-s_2(y-x)} & F_2(1+p_2) \end{pmatrix}$$

We write  $A_1, B_1, C_1$  and  $D_1$  in function of  $A_2, B_2, C_2$  and  $D_2$

$$\begin{aligned}
 A_1 &= \frac{\sum M_1(i1) M_2(i1) e^{-(y-x)} A_2 + \sum M_1(i1) \cdot M_2(i2) B_2}{2s_1(1-n_1)} \\
 &+ \frac{\sum M_1(i1) M_2(i3) e^{-s_2(y-x)} C_2 + \sum M_1(i1) \cdot M_2(i4) D_2}{2s_1(1-n_1)} \\
 B_1 &= \frac{\sum M_1(2i) \cdot M_2(i1) e^{-(y-x)} A_2 + \sum M_1(2i) \cdot M_2(i2) B_2}{2s_1(1-n_1)} \cdot e^x \\
 &+ \frac{\sum M_1(2i) \cdot M_2(i3) e^{-s_2(y-x)} C_2 + \sum M_1(2i) \cdot M_2(i4) D_2}{2s_1(1-n_1)} \cdot e^x \\
 C_1 &= \frac{\sum M_1(3i) \cdot M_2(i1) e^{-(y-x)} A_2 + \sum M_1(3i) \cdot M_2(i2) B_2}{2s_1(1-n_1)} \\
 &+ \frac{\sum M_1(3i) \cdot M_2(i3) e^{-s_2(y-x)} C_2 + \sum M_1(3i) \cdot M_2(i4) D_2}{2s_1(1-n_1)} \\
 D_1 &= \frac{\sum M_1(4i) \cdot M_2(i1) e^{-(y-x)} A_2 + \sum M_1(4i) \cdot M_2(i2) B_2}{2s_1(1-n_1)} \cdot e^{s_1 x} \\
 &+ \frac{\sum M_1(4i) \cdot M_2(i3) e^{-s_2(y-x)} C_2 + \sum M_1(4i) \cdot M_2(i4) D_2}{2s_1(1-n_1)} \cdot e^{s_1 x}
 \end{aligned}$$

where  $M_1(i,j)$  and  $M_2(i,j)$  are the constants in  $M_1^{-1}$  and  $M_2$ .

We write

$$R_{jl} = M_1(ji) \cdot M_2(ij)$$

so that

$$\begin{pmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{pmatrix} = \frac{1}{2s_1(1-n_1)} \begin{pmatrix} R_{11} e^{-(y-x)} & R_{12} & R_{13} e^{-s_2(y-x)} & R_{14} \\ R_{21} e^{-(y-x)} \cdot e^x & R_{22} e^x & R_{23} e^{-s_2(y-x)} \cdot e^x & R_{24} e^x \\ R_{31} e^{-(y-x)} & R_{32} & R_{33} e^{-s_2(y-x)} & R_{34} \\ R_{41} e^{-(y-x)} \cdot e^{s_1 x} & R_{42} e^{s_1 x} & R_{43} e^{-s_2(y-x)} \cdot e^{s_1 x} & R_{44} e^{s_1 x} \end{pmatrix} \begin{pmatrix} A_2 \\ B_2 \\ C_2 \\ D_2 \end{pmatrix}$$

### 3. Resolution of the system of boundary conditions.

We write the conditions at the surface in matrixform

$$\begin{vmatrix} e^{-x} & 1 & e^{-s_1 x} & 1 \\ e^{-x} & -1 & s_1 e^{-s_1 x} & -s_1 \end{vmatrix} \cdot (A, B, C, D)_1^T = \begin{vmatrix} 1 \\ 0 \end{vmatrix}$$

We have then following system of matrix equations

$$MI \cdot (A, B, C, D)_1^T = (1 \ 0)^T$$

$$(A, B, C, D)_1^T = \frac{1}{2s_1(1-n_1)} MR (A_2 B_2 C_2 D_2)^T$$

$$(A_2 B_2 C_2 D_2)^T = \frac{1}{2s_2(1-n_2)} MQ (A_3 B_3 C_3 D_3)^T$$

$$(A_3 B_3 C_3 D_3)^T = \frac{1}{2s_3(1-n_3)} MP (B_4 D_4)^T$$

so that

$$\frac{1}{8s_1 s_2 s_3 (1-n_1)(1-n_2)(1-n_3)} \cdot MI \cdot MR \cdot MQ \cdot MP (B_4 D_4)^T = (1 \ 0)^T$$

$$MI \cdot MR = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{vmatrix}$$

$$MQ \cdot MP = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{vmatrix}$$

$$MI \cdot MR \cdot MQ \cdot MP = \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix}$$

$$B_4 = 8s_1 s_2 s_3 (1-h_1)(1-h_2)(1-h_3) \cdot \frac{C_{22}}{C_{11} \cdot C_{22} - C_{12} \cdot C_{21}}$$

$$D_4 = -8s_1 s_2 s_3 (1-h_1)(1-h_2)(1-h_3) \cdot \frac{C_{21}}{C_{11} \cdot C_{22} - C_{12} \cdot C_{21}}$$

### 3.1. Determination of $C_{11} \cdot C_{22} - C_{21} \cdot C_{12}$

The expression of the denominator  $C_{11} \cdot C_{22} - C_{21} \cdot C_{12}$  must be established in complete closeform although regarding the exponentials.

$$C_{11} = a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31} + a_{14} \cdot b_{41}$$

$$C_{12} = a_{11} \cdot b_{12} + a_{12} \cdot b_{22} + a_{13} \cdot b_{32} + a_{14} \cdot b_{42}$$

$$C_{21} = a_{21} \cdot b_{11} + a_{22} \cdot b_{21} + a_{23} \cdot b_{31} + a_{24} \cdot b_{41}$$

$$C_{22} = a_{21} \cdot b_{12} + a_{22} \cdot b_{22} + a_{23} \cdot b_{32} + a_{24} \cdot b_{42}$$

$$C_{11} \cdot C_{22} - C_{12} \cdot C_{21} =$$

$$\begin{aligned} & (a_{11} \cdot a_{22} - a_{12} \cdot a_{21}) \cdot (b_{11} \cdot b_{22} - b_{12} \cdot b_{21}) \\ & + (a_{11} \cdot a_{23} - a_{13} \cdot a_{21}) \cdot (b_{11} \cdot b_{32} - b_{12} \cdot b_{31}) \\ & + (a_{11} \cdot a_{24} - a_{14} \cdot a_{21}) \cdot (b_{11} \cdot b_{42} - b_{12} \cdot b_{41}) \\ & + (a_{12} \cdot a_{23} - a_{13} \cdot a_{22}) \cdot (b_{21} \cdot b_{32} - b_{22} \cdot b_{31}) \\ & + (a_{12} \cdot a_{24} - a_{14} \cdot a_{22}) \cdot (b_{21} \cdot b_{42} - b_{22} \cdot b_{41}) \\ & + (a_{13} \cdot a_{24} - a_{14} \cdot a_{23}) \cdot (b_{31} \cdot b_{42} - b_{32} \cdot b_{41}) \end{aligned}$$

$$a_{11} = e^{-(y-x)} [R_{11} e^{-x} + R_{21} e^x + R_{31} e^{-s_1 x} + R_{41} e^{s_1 x}]$$

$$a_{12} = [R_{12} e^{-x} + R_{22} e^x + R_{32} e^{-s_1 x} + R_{42} e^{s_1 x}]$$

$$a_{13} = e^{-s_1(y-x)} [R_{13} e^{-x} + R_{23} e^x + R_{33} e^{-s_1 x} + R_{43} e^{s_1 x}]$$

$$a_{14} = [R_{14} e^{-x} + R_{24} e^x + R_{34} e^{-s_1 x} + R_{44} e^{s_1 x}]$$

$$a_{21} = e^{-(\gamma-x)} [R_{11}e^{-x} - R_{21}e^x + s_1 R_{31}e^{-s_1 x} - s_1 R_{41}e^{s_1 x}]$$

$$a_{22} = [R_{12}e^{-x} - R_{22}e^x + s_1 R_{32}e^{-s_1 x} - s_1 R_{42}e^{s_1 x}]$$

$$a_{23} = e^{-s_2(\gamma-x)} [R_{13}e^{-x} - R_{23}e^x + s_1 R_{33}e^{-s_1 x} - s_1 R_{43}e^{s_1 x}]$$

$$a_{24} = [R_{14}e^{-x} - R_{24}e^x + s_1 R_{34}e^{-s_1 x} - s_1 R_{44}e^{s_1 x}]$$

$$a_{11} \cdot a_{22} - a_{12} \cdot a_{21} = e^{-(\gamma-x)} \cdot e^{s_1 x} \cdot e^x$$

$$R_{11} \cdot [-2R_{22}e^{-x}e^{-s_1 x} + (s_1 - 1)R_{32}e^{-2x}e^{-2s_1 x} - (s_1 + 1)R_{42}e^{-2x}]$$

$$+ R_{21} [2R_{12}e^{-x}e^{-s_1 x} + (1 + s_1)R_{32}e^{-2s_1 x} + (1 - s_1)R_{42}]$$

$$+ R_{31} [(1 - s_1)R_{12}e^{-2x}e^{-2s_1 x} - (1 + s_1)R_{22}e^{-2s_1 x} - 2s_1 R_{42}e^{-x}e^{-s_1 x}]$$

$$+ R_{41} [(1 + s_1)R_{12}e^{-2x} + (s_1 - 1)R_{22} + 2s_1 R_{32}e^{-x}e^{-s_1 x}]$$

$$a_{11} \cdot a_{23} - a_{13} \cdot a_{21} = e^{-(\gamma-x)} \cdot e^{-s_2(\gamma-x)} \cdot e^{s_1 x} \cdot e^x$$

$$R_{11} [-2R_{23}e^{-x}e^{-s_1 x} + (s_1 - 1)R_{33}e^{-2x}e^{-2s_1 x} - (s_1 + 1)R_{43}e^{-2x}]$$

$$+ R_{21} [2R_{13}e^{-x}e^{-s_1 x} + (1 + s_1)R_{33}e^{-2s_1 x} + (1 - s_1)R_{43}]$$

$$+ R_{31} [(1 - s_1)R_{13}e^{-2x}e^{-2s_1 x} - (s_1 + 1)R_{23}e^{-2s_1 x} - 2s_1 R_{43}e^{-x}e^{-s_1 x}]$$

$$+ R_{41} [(1 + s_1)R_{13}e^{-2x} + (s_1 - 1)R_{23} + 2s_1 R_{33}e^{-x}e^{-s_1 x}]$$

$$a_{11} \cdot a_{24} - a_{14} \cdot a_{21} = e^{-(\gamma-x)} \cdot e^{s_1 x} \cdot e^x$$

$$R_{11} [-2R_{24}e^{-x}e^{-s_1 x} + (s_1 - 1)R_{34}e^{-2x}e^{-2s_1 x} - (1 + s_1)R_{44}e^{-2x}]$$

$$+ R_{21} [2R_{14}e^{-x}e^{-s_1 x} + (1 + s_1)R_{34}e^{-2s_1 x} + (1 - s_1)R_{44}]$$

$$+ R_{31} [(1 - s_1)R_{14}e^{-2x}e^{-2s_1 x} - (1 + s_1)R_{24}e^{-2s_1 x} - 2s_1 R_{44}e^{-x}e^{-s_1 x}]$$

$$+ R_{41} [(1 + s_1)R_{14}e^{-2x} + (s_1 - 1)R_{24} + 2s_1 R_{34}e^{-x}e^{-s_1 x}]$$

$$a_{12} \cdot a_{23} - a_{13} \cdot a_{22} = e^{-s_2(\gamma-x)} \cdot e^{s_1 x} \cdot e^x$$

$$\begin{aligned} & R_{12} \left[ -2R_{23} e^{-x} e^{-s_1 x} + (s_1-1) R_{33} e^{-2x} e^{-2s_1 x} - (1+s_1) R_{43} e^{-2x} \right] \\ & + R_{22} \left[ 2R_{13} e^{-x} e^{-s_1 x} + (1+s_1) R_{33} e^{-2s_1 x} + (1-s_1) R_{43} \right] \\ & + R_{32} \left[ (1-s_1) R_{13} e^{-2x} e^{-2s_1 x} - (1+s_1) R_{23} e^{-2s_1 x} - 2s_1 R_{43} e^{-x} e^{-s_1 x} \right] \\ & + R_{42} \left[ (1+s_1) R_{13} e^{-2x} + (s_1-1) R_{23} + 2s_1 R_{33} e^{-x} e^{-s_1 x} \right] \end{aligned}$$

$$a_{12} \cdot a_{24} - a_{14} \cdot a_{22} = e^{s_1 x} \cdot e^x$$

$$\begin{aligned} & R_{12} \left[ -2R_{24} e^{-x} e^{-s_1 x} + (s_1-1) R_{34} e^{-2x} e^{-2s_1 x} - (1+s_1) R_{44} e^{-2x} \right] \\ & + R_{22} \left[ 2R_{14} e^{-x} e^{-s_1 x} + (1+s_1) R_{34} e^{-2s_1 x} + (1-s_1) R_{44} \right] \\ & + R_{32} \left[ (1-s_1) R_{14} e^{-2x} e^{-2s_1 x} - (1+s_1) R_{24} e^{-2s_1 x} - 2s_1 R_{44} e^{-x} e^{-s_1 x} \right] \\ & + R_{42} \left[ (1+s_1) R_{14} e^{-2x} + (s_1-1) R_{24} + 2s_1 R_{34} e^{-x} e^{-s_1 x} \right] \end{aligned}$$

$$a_{13} \cdot a_{24} - a_{14} \cdot a_{23} = e^{-s_2(\gamma-x)} \cdot e^{s_1 x} \cdot e^x$$

$$\begin{aligned} & R_{13} \left[ -2R_{24} e^{-x} e^{-s_1 x} + (s_1-1) R_{34} e^{-2x} e^{-2s_1 x} - (1+s_1) R_{44} e^{-2x} \right] \\ & + R_{23} \left[ 2R_{14} e^{-x} e^{-s_1 x} + (1+s_1) R_{34} e^{-2s_1 x} + (1-s_1) R_{44} \right] \\ & + R_{33} \left[ (1-s_1) R_{14} e^{-2x} e^{-2s_1 x} - (1+s_1) R_{24} e^{-2s_1 x} - 2s_1 R_{44} e^{-x} e^{-s_1 x} \right] \\ & + R_{43} \left[ (1+s_1) R_{14} e^{-2x} + (s_1-1) R_{24} + 2s_1 R_{34} e^{-x} e^{-s_1 x} \right] \end{aligned}$$

$$a_{11} \cdot a_{22} - a_{12} \cdot a_{21} = e^{-(\gamma-x)} \cdot e^{s_1 x} \cdot e^x \cdot A_{12}$$

$$a_{11} \cdot a_{23} - a_{13} \cdot a_{21} = e^{-(\gamma-x)} \cdot e^{-s_2(\gamma-x)} \cdot e^{s_1 x} \cdot e^x \cdot A_{13}$$

$$a_{11} \cdot a_{24} - a_{14} \cdot a_{21} = e^{-(\gamma-x)} \cdot e^{s_1 x} \cdot e^x \cdot A_{14}$$

$$a_{12} \cdot a_{23} - a_{13} \cdot a_{22} = e^{-s_2(\gamma-x)} \cdot e^{s_1 x} \cdot e^x \cdot A_{23}$$

$$a_{12} \cdot a_{24} - a_{14} \cdot a_{22} = e^{s_1 x} \cdot e^x \cdot A_{24}$$

$$a_{13} \cdot a_{24} - a_{14} \cdot a_{23} = e^{-s_2(\gamma-x)} \cdot e^{s_1 x} \cdot e^x \cdot A_{34}$$

$$\begin{aligned}
 b_{11} &= Q_{11} \cdot P_{11} \cdot e^{-(z-y)} + Q_{12} \cdot P_{21} \cdot e^{(z-y)} + Q_{13} \cdot P_{31} \cdot e^{-s_3(z-y)} + Q_{14} \cdot P_{41} \cdot e^{s_3(z-y)} \\
 b_{12} &= Q_{11} \cdot P_{12} + Q_{12} \cdot P_{22} + Q_{13} \cdot P_{32} + Q_{14} \cdot P_{42} \\
 b_{21} &= [Q_{21} \cdot P_{11} + Q_{22} \cdot P_{21} + Q_{23} \cdot P_{31} + Q_{24} \cdot P_{41}] e^{(y-x)} \\
 b_{22} &= [Q_{21} \cdot P_{12} + Q_{22} \cdot P_{22} + Q_{23} \cdot P_{32} + Q_{24} \cdot P_{42}] e^{(y-x)} \\
 b_{31} &= Q_{31} \cdot P_{11} + Q_{32} \cdot P_{21} + Q_{33} \cdot P_{31} + Q_{34} \cdot P_{41} \\
 b_{32} &= Q_{31} \cdot P_{12} + Q_{32} \cdot P_{22} + Q_{33} \cdot P_{32} + Q_{34} \cdot P_{42} \\
 b_{41} &= [Q_{41} \cdot P_{11} + Q_{42} \cdot P_{21} + Q_{43} \cdot P_{31} + Q_{44} \cdot P_{41}] e^{s_2(y-x)} \\
 b_{42} &= [Q_{41} \cdot P_{12} + Q_{42} \cdot P_{22} + Q_{43} \cdot P_{32} + Q_{44} \cdot P_{42}] e^{s_2(y-x)}
 \end{aligned}$$

$$b_{11} \cdot b_{22} - b_{12} \cdot b_{21} = e^{(y-x)} \cdot e^{(z-y)} \cdot e^{s_3(z-y)}$$

$$\left\{ \begin{aligned}
 & [Q_{11} \cdot Q_{22} - Q_{12} \cdot Q_{21}] [P_{11} \cdot P_{22} - P_{12} \cdot P_{21}] e^{-(z-y)} \cdot e^{-s_3(z-y)} \\
 & + [Q_{11} \cdot Q_{24} - Q_{14} \cdot Q_{21}] [P_{11} \cdot P_{42} - P_{12} \cdot P_{41}] e^{-2(z-y)} \\
 & + [Q_{12} \cdot Q_{23} - Q_{13} \cdot Q_{22}] [P_{21} \cdot P_{32} - P_{22} \cdot P_{31}] e^{-2s_3(z-y)} \\
 & + [Q_{13} \cdot Q_{21} - Q_{11} \cdot Q_{23}] [P_{31} \cdot P_{12} - P_{32} \cdot P_{11}] e^{-2(z-y)} \cdot e^{-2s_3(z-y)} \\
 & + [Q_{13} \cdot Q_{24} - Q_{14} \cdot Q_{23}] [P_{31} \cdot P_{42} - P_{32} \cdot P_{41}] e^{-(z-y)} \cdot e^{-s_3(z-y)} \\
 & + [Q_{14} \cdot Q_{22} - Q_{12} \cdot Q_{24}] [P_{41} \cdot P_{22} - P_{42} \cdot P_{21}]
 \end{aligned} \right\}$$

We write

$$\begin{aligned}
 p_{12} &= [P_{11} \cdot P_{22} - P_{12} \cdot P_{21}] e^{-(z-y)} \cdot e^{-s_3(z-y)} \\
 p_{14} &= [P_{11} \cdot P_{42} - P_{12} \cdot P_{41}] e^{-2(z-y)} \\
 p_{23} &= [P_{21} \cdot P_{32} - P_{22} \cdot P_{31}] e^{-2s_3(z-y)} \\
 p_{31} &= [P_{31} \cdot P_{12} - P_{32} \cdot P_{11}] e^{-2(z-y)} \cdot e^{-2s_3(z-y)} \\
 p_{34} &= [P_{31} \cdot P_{42} - P_{32} \cdot P_{41}] e^{-(z-y)} \cdot e^{-s_3(z-y)} \\
 p_{42} &= [P_{41} \cdot P_{22} - P_{42} \cdot P_{21}]
 \end{aligned}$$

$$b_{11} \cdot b_{32} - b_{12} \cdot b_{31} = e^{(z-y)} \cdot e^{s_3(z-y)}$$

$$\left\{ \begin{aligned} & [\Phi_{11} \cdot \Phi_{32} - \Phi_{12} \cdot \Phi_{31}] \cdot p_{12} + [\Phi_{11} \cdot \Phi_{34} - \Phi_{13} \cdot \Phi_{31}] \cdot p_{14} \\ & + [\Phi_{12} \cdot \Phi_{33} - \Phi_{13} \cdot \Phi_{32}] \cdot p_{23} + [\Phi_{13} \cdot \Phi_{31} - \Phi_{11} \cdot \Phi_{33}] \cdot p_{31} \\ & + [\Phi_{13} \cdot \Phi_{34} - \Phi_{14} \cdot \Phi_{33}] \cdot p_{34} + [\Phi_{14} \cdot \Phi_{32} - \Phi_{12} \cdot \Phi_{34}] \cdot p_{42} \end{aligned} \right\}$$

$$b_{11} \cdot b_{42} - b_{12} \cdot b_{41} = e^{s_2(y-x)} \cdot e^{(z-y)} \cdot e^{s_3(z-y)}$$

$$\left\{ \begin{aligned} & [\Phi_{11} \cdot \Phi_{42} - \Phi_{12} \cdot \Phi_{41}] \cdot p_{12} + [\Phi_{11} \cdot \Phi_{44} - \Phi_{14} \cdot \Phi_{41}] \cdot p_{14} \\ & + [\Phi_{12} \cdot \Phi_{43} - \Phi_{13} \cdot \Phi_{42}] \cdot p_{23} + [\Phi_{13} \cdot \Phi_{41} - \Phi_{11} \cdot \Phi_{43}] \cdot p_{31} \\ & + [\Phi_{13} \cdot \Phi_{44} - \Phi_{14} \cdot \Phi_{43}] \cdot p_{34} + [\Phi_{14} \cdot \Phi_{42} - \Phi_{12} \cdot \Phi_{44}] \cdot p_{42} \end{aligned} \right\}$$

$$b_{21} \cdot b_{32} - b_{22} \cdot b_{31} = e^{(y-x)} \cdot e^{(z-y)} \cdot e^{s_3(z-y)}$$

$$\left\{ \begin{aligned} & [\Phi_{21} \cdot \Phi_{32} - \Phi_{22} \cdot \Phi_{31}] \cdot p_{12} + [\Phi_{21} \cdot \Phi_{34} - \Phi_{24} \cdot \Phi_{31}] \cdot p_{14} \\ & + [\Phi_{22} \cdot \Phi_{33} - \Phi_{23} \cdot \Phi_{32}] \cdot p_{23} + [\Phi_{23} \cdot \Phi_{31} - \Phi_{21} \cdot \Phi_{33}] \cdot p_{31} \\ & + [\Phi_{23} \cdot \Phi_{34} - \Phi_{24} \cdot \Phi_{33}] \cdot p_{34} + [\Phi_{24} \cdot \Phi_{32} - \Phi_{22} \cdot \Phi_{34}] \cdot p_{42} \end{aligned} \right\}$$

$$b_{21} \cdot b_{42} - b_{22} \cdot b_{41} = e^{s_2(y-x)} \cdot e^{(z-y)} \cdot e^{s_3(z-y)} \cdot e^{(y-x)}$$

$$\left\{ \begin{aligned} & [\Phi_{21} \cdot \Phi_{42} - \Phi_{22} \cdot \Phi_{41}] \cdot p_{12} + [\Phi_{21} \cdot \Phi_{44} - \Phi_{24} \cdot \Phi_{41}] \cdot p_{14} \\ & + [\Phi_{22} \cdot \Phi_{43} - \Phi_{23} \cdot \Phi_{42}] \cdot p_{23} + [\Phi_{23} \cdot \Phi_{41} - \Phi_{21} \cdot \Phi_{43}] \cdot p_{31} \\ & + [\Phi_{23} \cdot \Phi_{44} - \Phi_{24} \cdot \Phi_{43}] \cdot p_{34} + [\Phi_{24} \cdot \Phi_{42} - \Phi_{22} \cdot \Phi_{44}] \cdot p_{42} \end{aligned} \right\}$$

$$b_{31} \cdot b_{42} - b_{32} \cdot b_{41} = e^{s_1(y-x)} \cdot e^{(z-y)} \cdot e^{s_3(z-y)}$$

$$\left\{ \begin{aligned} & [\Phi_{31} \cdot \Phi_{42} - \Phi_{32} \cdot \Phi_{41}] \cdot p_{12} + [\Phi_{31} \cdot \Phi_{44} - \Phi_{34} \cdot \Phi_{41}] \cdot p_{14} \\ & + [\Phi_{32} \cdot \Phi_{43} - \Phi_{33} \cdot \Phi_{42}] \cdot p_{23} + [\Phi_{33} \cdot \Phi_{41} - \Phi_{31} \cdot \Phi_{43}] \cdot p_{31} \\ & + [\Phi_{33} \cdot \Phi_{44} - \Phi_{34} \cdot \Phi_{43}] \cdot p_{34} + [\Phi_{34} \cdot \Phi_{42} - \Phi_{32} \cdot \Phi_{44}] \cdot p_{42} \end{aligned} \right\}$$

$$b_{11} \cdot b_{22} - b_{12} \cdot b_{21} = e^{(y-x)} \cdot e^{(z-y)} \cdot e^{s_3(z-y)} \cdot B_{12}$$

$$b_{11} \cdot b_{32} - b_{12} \cdot b_{31} = e^{(z-y)} \cdot e^{s_3(z-y)} \cdot B_{13}$$

$$b_{11} \cdot b_{42} - b_{12} \cdot b_{41} = e^{s_2(y-x)} \cdot e^{(z-y)} \cdot e^{s_3(z-y)} \cdot B_{14}$$

$$b_{21} \cdot b_{32} - b_{22} \cdot b_{31} = e^{(y-x)} \cdot e^{(z-y)} \cdot e^{s_3(z-y)} \cdot B_{23}$$

$$b_{21} \cdot b_{42} - b_{22} \cdot b_{41} = e^{(y-x)} \cdot e^{s_2(y-x)} \cdot e^{(z-y)} \cdot e^{s_3(z-y)} \cdot B_{24}$$

$$b_{31} \cdot b_{42} - b_{32} \cdot b_{41} = e^{s_2(y-x)} \cdot e^{(z-y)} \cdot e^{s_3(z-y)} \cdot B_{34}$$

The value of the denominator is then finally

$$c_{11} \cdot c_{22} - c_{12} \cdot c_{21} = e^x \cdot e^{s_1 x} \cdot e^{(y-x)} \cdot e^{s_2(y-x)} \cdot e^{(z-y)} \cdot e^{s_3(z-y)}$$

$$\begin{aligned} & \left[ A_{12} \cdot B_{12} \cdot e^{-(y-x)} \cdot e^{-s_2(y-x)} \right. \\ & + A_{13} \cdot B_{13} \cdot e^{-2(y-x)} \cdot e^{-2s_2(y-x)} \\ & + A_{14} \cdot B_{14} \cdot e^{-2(y-x)} \\ & + A_{23} \cdot B_{23} \cdot e^{-2s_2(y-x)} \\ & + A_{24} \cdot B_{24} \\ & \left. + A_{34} \cdot B_{34} \cdot e^{-(y-x)} \cdot e^{-s_2(y-x)} \right] \end{aligned}$$

The term  $A_{24} \cdot B_{24}$  contains the constant

$$\left[ (1-s_1) R_{22} \cdot R_{44} + (s_1-1) R_{42} \cdot R_{24} \right]$$

$$\left[ Q_{24} \cdot Q_{42} - Q_{22} \cdot Q_{44} \right] \cdot \left[ P_{41} \cdot P_{22} - P_{42} \cdot P_{21} \right]$$

We write

$$c_{11} \cdot c_{22} - c_{12} \cdot c_{21} = e^x \cdot e^{s_1 x} \cdot e^{(y-x)} \cdot e^{s_2(y-x)} \cdot e^{(z-y)} \cdot e^{s_3(z-y)} \cdot C_{12}$$

### 3.2. Determination of the parameters $B_4$ and $C_4$ .

We write

$$a'_{21} = [R_{11} \cdot e^{-2x} \cdot e^{-s_1 x} - R_{21} e^{-s_1 x} + s_1 R_{31} e^{-x} \cdot e^{-2s_1 x} - s_1 R_{41} e^{-x}]$$

$$a'_{22} = [R_{12} \cdot e^{-2x} \cdot e^{-s_1 x} - R_{22} e^{-s_1 x} + s_1 R_{32} e^{-x} \cdot e^{-2s_1 x} - s_1 R_{42} e^{-x}]$$

$$a'_{23} = [R_{13} \cdot e^{-2x} \cdot e^{-s_1 x} - R_{23} e^{-s_1 x} + s_1 R_{33} e^{-x} \cdot e^{-2s_1 x} - s_1 R_{43} e^{-x}]$$

$$a'_{24} = [R_{14} \cdot e^{-2x} \cdot e^{-s_1 x} - R_{24} e^{-s_1 x} + s_1 R_{34} e^{-x} \cdot e^{-2s_1 x} - s_1 R_{44} e^{-x}]$$

$$b'_{11} = [Q_{11} \cdot P_{11} \cdot e^{-2(z-y) - s_3(z-y)} + Q_{12} \cdot P_{21} \cdot e^{-s_3(z-y)} + Q_{13} \cdot P_{31} \cdot e^{-(2-y) - 2s_3(z-y)} + Q_{14} \cdot P_{41} \cdot e^{-(z-y)}]$$

$$b'_{12} = [Q_{11} \cdot P_{12} \quad + Q_{12} \cdot P_{22} \quad + Q_{13} \cdot P_{32} \quad + Q_{14} \cdot P_{42}]$$

$$b'_{21} = [Q_{21} \cdot P_{11} \quad + Q_{22} \cdot P_{21} \quad + Q_{23} \cdot P_{31} \quad + Q_{24} \cdot P_{41}]$$

$$b'_{22} = [Q_{21} \cdot P_{12} \quad + Q_{22} \cdot P_{22} \quad + Q_{23} \cdot P_{32} \quad + Q_{24} \cdot P_{42}]$$

$$b'_{31} = [Q_{31} \cdot P_{11} \quad + Q_{32} \cdot P_{21} \quad + Q_{33} \cdot P_{31} \quad + Q_{34} \cdot P_{41}]$$

$$b'_{32} = [Q_{31} \cdot P_{12} \quad + Q_{32} \cdot P_{22} \quad + Q_{33} \cdot P_{32} \quad + Q_{34} \cdot P_{42}]$$

$$b'_{41} = [Q_{41} \cdot P_{11} \quad + Q_{42} \cdot P_{21} \quad + Q_{43} \cdot P_{31} \quad + Q_{44} \cdot P_{41}]$$

$$b'_{42} = [Q_{41} \cdot P_{12} \quad + Q_{42} \cdot P_{22} \quad + Q_{43} \cdot P_{32} \quad + Q_{44} \cdot P_{42}]$$

We have that

$$B_4 = 8s_1 s_2 s_3 (1-n_1)(1-n_2)(1-n_3) \cdot \frac{C_{22}}{C_{11} \cdot C_{22} - C_{12} \cdot C_{21}}$$

$$D_4 = -8s_1 s_2 s_3 (1-n_1)(1-n_2)(1-n_3) \cdot \frac{C_{21}}{C_{11} \cdot C_{22} - C_{12} \cdot C_{21}}$$

$$c_{22} = a_{21} \cdot b_{12} + a_{22} \cdot b_{22} + a_{23} \cdot b_{32} + a_{24} \cdot b_{42}$$

$$= e^x \cdot e^{s_1 x} \cdot e^{(z-y)} \cdot e^{s_3(z-y)} \left[ e^{-(y-x)} a'_{21} \cdot b'_{12} \right. \\ \left. + e^{(y-x)} \cdot a'_{22} \cdot b'_{22} + e^{-s_2(y-x)} a'_{23} \cdot b'_{32} + e^{s_2(y-x)} a'_{24} \cdot b'_{42} \right]$$

$$= e^x \cdot e^{s_1 x} \cdot e^{(y-x)} \cdot e^{s_2(y-x)} \cdot e^{(z-y)} \cdot e^{s_3(z-y)}$$

$$\left[ e^{-2(y-x)-s_2(y-x)} a'_{21} \cdot b'_{12} + e^{-s_2(y-x)} a'_{22} \cdot b'_{22} \right. \\ \left. + e^{-(y-x)-2s_2(y-x)} a'_{23} \cdot b'_{32} + e^{-(y-x)} a'_{24} \cdot b'_{42} \right]$$

$$c_{21} = a_{21} \cdot b_{11} + a_{22} \cdot b_{21} + a_{23} \cdot b_{31} + a_{24} \cdot b_{41}$$

$$= e^x \cdot e^{s_1 x} \cdot e^{(z-y)} \cdot e^{s_3(z-y)} \left[ e^{-(y-x)} a'_{21} \cdot b'_{11} \right. \\ \left. + e^{(y-x)} \cdot a'_{22} \cdot b'_{21} + e^{-s_2(y-x)} a'_{23} \cdot b'_{31} + e^{s_2(y-x)} a'_{24} \cdot b'_{41} \right]$$

$$= e^x \cdot e^{s_1 x} \cdot e^{(y-x)} \cdot e^{s_2(y-x)} \cdot e^{(z-y)} \cdot e^{s_3(z-y)}$$

$$\left[ e^{-2(y-x)-s_2(y-x)} a'_{21} \cdot b'_{11} + e^{-s_2(y-x)} a'_{22} \cdot b'_{21} \right. \\ \left. + e^{-(y-x)-2s_2(y-x)} a'_{23} \cdot b'_{31} + e^{-(y-x)} a'_{24} \cdot b'_{41} \right]$$

$$c_{22} = e^x \cdot e^{s_1 x} \cdot e^{(y-x)} \cdot e^{s_2(y-x)} \cdot e^{(z-y)} \cdot e^{s_3(z-y)} \cdot c'_{22}$$

$$c_{21} = e^x \cdot e^{s_1 x} \cdot e^{(y-x)} \cdot e^{s_2(y-x)} \cdot e^{(z-y)} \cdot e^{s_3(z-y)} \cdot c'_{21}$$

$$B_4 = 8s_1 s_2 s_3 (1-h_1)(1-h_2)(1-h_3) \cdot \frac{c'_{22}}{c_{12}}$$

$$D_4 = -8s_1 s_2 s_3 (1-h_1)(1-h_2)(1-h_3) \cdot \frac{c'_{21}}{c_{12}}$$

The numerators contain only negative exponents.

The denominator contains a constant and negative exponents.

#### 4. Values of the parameters $A_i, D_i$ .

##### 4.1. Values of the parameters $A_3, B_3, C_3, D_3$ .

The values of the parameters are obtained from the matrix equation in § 2.1.

$$\begin{pmatrix} A_3 \\ B_3 \\ C_3 \\ D_3 \end{pmatrix} = \frac{1}{2s_3(1-n_3)} \begin{pmatrix} P_{11} & P_{12} \\ P_{21}e^{(z-y)} & P_{22}e^{(z-y)} \\ P_{31} & P_{32} \\ P_{41}e^{(z-y) \cdot s_3} & P_{42}e^{s_3(z-y)} \end{pmatrix} \begin{pmatrix} B_4 \\ D_4 \end{pmatrix}$$

One obtains immediately the values of the parameters  $A_3$  and  $C_3$

$$A_3 = \frac{1}{2s_3(1-n_3)} [P_{11} \cdot B_4 + P_{12} \cdot D_4]$$

$$C_3 = \frac{1}{2s_3(1-n_3)} [P_{31} \cdot B_4 + P_{32} \cdot D_4]$$

The determination of the values of the parameters  $B_3$  and  $D_3$  need some more computation to insure convergency.

$$B_3 = \frac{e^{(z-y)}}{2s_3(1-n_3)} [P_{21} \cdot B_4 + P_{22} \cdot D_4]$$

$$D_3 = \frac{e^{s_3(z-y)}}{2s_3(1-n_3)} [P_{41} \cdot B_4 + P_{42} \cdot D_4]$$

Those relations contain positive exponents which much disappear to avoid overflow problems.

$$B_3 = \frac{e^{(z-y)} \cdot s_1 s_2 s_3 (1-n_1) (1-n_2) (1-n_3)}{2s_3(1-n_3) \cdot C_{12}} [P_{21} \cdot B'_4 - P_{22} \cdot D'_4]$$

$$D_3 = \frac{e^{s_3(z-y)} \cdot s_1 s_2 s_3 (1-n_1) (1-n_2) (1-n_3)}{2s_3(1-n_3) \cdot C_{12}} [P_{41} \cdot B'_4 - P_{42} \cdot D'_4]$$

$$P_{21} \cdot B'_4 - P_{22} \cdot D'_4 =$$

$$\begin{aligned} & P_{21} \left[ e^{-2(y-x)} e^{-s_1(y-x)} \alpha'_{21} \cdot b'_{12} + e^{-s_2(y-x)} \alpha'_{22} \cdot b'_{22} \right. \\ & \quad \left. + e^{-(y-x)} e^{-2s_2(y-x)} \alpha'_{23} \cdot b'_{32} + e^{-(y-x)} \alpha'_{24} \cdot b'_{42} \right] \\ & - P_{22} \left[ e^{-2(y-x)} e^{-s_1(y-x)} \alpha'_{21} \cdot b'_{11} + e^{-s_2(y-x)} \alpha'_{22} \cdot b'_{21} \right. \\ & \quad \left. + e^{-(y-x)} e^{-2s_2(y-x)} \alpha'_{23} \cdot b'_{31} + e^{-(y-x)} \alpha'_{24} \cdot b'_{41} \right] = \\ & = e^{-2(y-x)} e^{-s_1(y-x)} \alpha'_{21} \left[ P_{21} \cdot b'_{12} - P_{22} \cdot b'_{11} \right] \\ & \quad + e^{-s_2(y-x)} \alpha'_{22} \left[ P_{21} \cdot b'_{22} - P_{22} \cdot b'_{21} \right] \\ & \quad + e^{-(y-x)} e^{-2s_2(y-x)} \alpha'_{23} \left[ P_{21} \cdot b'_{32} - P_{22} \cdot b'_{31} \right] \\ & \quad + e^{-(y-x)} \alpha'_{24} \left[ P_{21} \cdot b'_{42} - P_{22} \cdot b'_{41} \right] \end{aligned}$$

$$P_{21} \cdot b'_{12} - P_{22} \cdot b'_{11} =$$

$$\begin{aligned} & P_{21} \left[ \Phi_{11} P_{12} e^{-2(z-y)} e^{-s_3(z-y)} + \Phi_{12} P_{22} e^{-s_3(z-y)} + \Phi_{13} P_{32} e^{-(z-y)} e^{-2s_3(z-y)} + \Phi_{14} P_{42} e^{-(z-y)} \right] \\ & - P_{22} \left[ \Phi_{11} P_{11} + \Phi_{12} P_{21} + \Phi_{13} P_{31} + \Phi_{14} P_{41} \right] = \\ & = e^{-(z-y)} P_{21} \left[ \Phi_{11} P_{12} e^{-(z-y)} e^{-s_3(z-y)} + \Phi_{13} P_{32} e^{-2s_3(z-y)} + \Phi_{14} P_{42} \right] \\ & - e^{-(z-y)} P_{22} \left[ \Phi_{11} P_{11} + \Phi_{13} P_{31} + \Phi_{14} P_{41} \right] \\ & = e^{-(z-y)} \cdot P B_{21} \end{aligned}$$

$$P_{21} \cdot b'_{22} - P_{22} \cdot b'_{21} =$$

$$\begin{aligned} & e^{-(z-y)} P_{21} \left[ \alpha'_{21} P_{12} e^{-(z-y)} e^{-s_3(z-y)} + \alpha'_{23} P_{32} e^{-2s_3(z-y)} + \alpha'_{24} P_{42} \right] \\ & - e^{-(z-y)} P_{22} \left[ \alpha'_{21} P_{11} + \alpha'_{23} P_{31} + \alpha'_{24} P_{41} \right] \\ & = e^{-(z-y)} \cdot P B_{22} \end{aligned}$$

$$\begin{aligned}
P_{21} \cdot b'_{32} - P_{22} \cdot b'_{31} &= \\
e^{-(z-y)} \cdot P_{21} [Q_{31} \cdot P_{12} e^{-(z-y)} e^{-s_3(z-y)} + Q_{33} \cdot P_{32} e^{-2s_3(z-y)} + Q_{34} \cdot P_{42}] \\
- e^{-(z-y)} \cdot P_{22} [Q_{31} \cdot P_{11} + Q_{33} \cdot P_{31} + Q_{34} \cdot P_{41}] \\
&= e^{-(z-y)} \cdot P B_{23}
\end{aligned}$$

$$\begin{aligned}
P_{21} \cdot b'_{42} - P_{22} \cdot b'_{41} &= \\
e^{-(z-y)} \cdot P_{21} [Q_{41} \cdot P_{12} e^{-(z-y)} e^{-s_3(z-y)} + Q_{43} \cdot P_{32} e^{-2s_3(z-y)} + Q_{44} \cdot P_{42}] \\
- e^{-(z-y)} \cdot P_{22} [Q_{41} \cdot P_{11} + Q_{43} \cdot P_{31} + Q_{44} \cdot P_{41}] \\
&= e^{-(z-y)} \cdot P B_{24}
\end{aligned}$$

The positive exponent  $e^{(z-y)}$  can now be eliminated

$$B_3 = 4s_1 s_2 (1-n_1) (1-n_2).$$

$$\begin{aligned}
& \left[ e^{-2(y-x)} \cdot e^{-s_2(y-x)} \cdot a'_{21} \cdot P B_{21} + e^{-s_2(y-x)} a'_{22} \cdot P B_{22} \right. \\
& \left. + e^{-(y-x)} \cdot e^{-2s_2(y-x)} \cdot a'_{23} \cdot P B_{23} + e^{-(y-x)} a'_{24} \cdot P B_{24} \right] \cdot \frac{1}{C_{12}}
\end{aligned}$$

and the numerator contains again only negative exponents.

$$\begin{aligned}
P_{41} \cdot B'_4 - P_{42} \cdot D'_4 &= \\
P_{41} [e^{-2(y-x)} \cdot e^{-s_2(y-x)} a'_{21} \cdot b'_{12} + e^{-s_2(y-x)} a'_{22} \cdot b'_{22} \\
+ e^{-(y-x)} e^{-2s_2(y-x)} a'_{23} \cdot b'_{32} + e^{-(y-x)} a'_{24} \cdot b'_{42}] \\
- P_{42} [e^{-2(y-x)} \cdot e^{-s_2(y-x)} a'_{21} \cdot b'_{11} + e^{-s_2(y-x)} a'_{22} \cdot b'_{21} \\
+ e^{-(y-x)} e^{-2s_2(y-x)} a'_{23} \cdot b'_{31} + e^{-(y-x)} a'_{24} \cdot b'_{41}] = \\
= e^{-2(y-x)} e^{-s_2(y-x)} a'_{21} [P_{41} \cdot b'_{12} - P_{42} \cdot b'_{11}] + e^{-s_2(y-x)} a'_{22} [P_{41} \cdot b'_{22} - P_{42} \cdot b'_{21}] \\
+ e^{-(y-x)} e^{-2s_2(y-x)} a'_{23} [P_{41} \cdot b'_{32} - P_{42} \cdot b'_{31}] + e^{-(y-x)} a'_{24} [P_{41} \cdot b'_{42} - P_{42} \cdot b'_{41}]
\end{aligned}$$

$$P_{41} \cdot b'_{12} - P_{42} \cdot b'_{11} =$$

$$P_{41} [Q_{11} \cdot P_{12} e^{-2(z-y)} e^{-s_3(z-y)} + Q_{12} P_{22} e^{-s_3(z-y)} + Q_{13} P_{32} e^{-(z-y)s_3} e^{-s_3(z-y)} + Q_{14} P_{42} e^{-(z-y)}] \\ - P_{42} [Q_{11} \cdot P_{11} + Q_{12} \cdot P_{21} + Q_{13} P_{31} + Q_{14} P_{41}] \\ = e^{-s_3(z-y)} \cdot P_{41} [Q_{11} \cdot P_{12} e^{-2(z-y)} + Q_{12} \cdot P_{22} + Q_{13} \cdot P_{32} e^{-(z-y)} e^{-s_3(z-y)}] \\ - e^{-s_3(z-y)} \cdot P_{42} [Q_{11} P_{11} + Q_{12} \cdot P_{21} + Q_{13} P_{31}] \\ = e^{-s_3(z-y)} P_{B41}$$

$$P_{41} \cdot b'_{22} - P_{42} \cdot b'_{21} =$$

$$= e^{-s_3(z-y)} \cdot P_{41} [Q_{21} \cdot P_{12} e^{-2(z-y)} + Q_{22} \cdot P_{22} + Q_{23} \cdot P_{32} e^{-(z-y)} e^{-s_3(z-y)}] \\ - e^{-s_3(z-y)} \cdot P_{42} [Q_{21} \cdot P_{11} + Q_{22} \cdot P_{21} + Q_{23} \cdot P_{31}] \\ = e^{-s_3(z-y)} P_{B42}$$

$$P_{41} \cdot b'_{32} - P_{42} \cdot b'_{31} =$$

$$= e^{-s_3(z-y)} \cdot P_{41} [Q_{31} \cdot P_{12} e^{-2(z-y)} + Q_{32} \cdot P_{22} + Q_{33} \cdot P_{32} e^{-(z-y)} e^{-s_3(z-y)}] \\ - e^{-s_3(z-y)} \cdot P_{42} [Q_{31} \cdot P_{11} + Q_{32} \cdot P_{21} + Q_{33} \cdot P_{31}] \\ = e^{-s_3(z-y)} P_{B43}$$

$$P_{41} \cdot b'_{42} - P_{42} \cdot b'_{41} =$$

$$= e^{-s_3(z-y)} P_{41} [Q_{41} \cdot P_{12} e^{-2(z-y)} + Q_{42} \cdot P_{22} + Q_{43} \cdot P_{32} e^{-(z-y)} e^{-s_3(z-y)}] \\ - e^{-s_3(z-y)} P_{42} [Q_{41} \cdot P_{11} + Q_{42} \cdot P_{21} + Q_{43} \cdot P_{31}] \\ = e^{-s_3(z-y)} P_{B44}$$

and the positive exponent  $e^{s_3(z-y)}$  can be eliminated

$$D_3 = 4s_1 s_2 (1-n_1)(1-n_2).$$

$$\left[ e^{-2(\gamma-x)} e^{-s_2(\gamma-x)} a'_{21} \cdot P_{B41} + e^{-s_2(\gamma-x)} a'_{22} \cdot P_{B42} \right. \\ \left. + e^{-(\gamma-x)} e^{-2s_2(\gamma-x)} a'_{23} \cdot P_{B43} + e^{-(\gamma-x)} a'_{24} \cdot P_{B44} \right] \cdot \frac{1}{C_{12}}$$

#### 4.2. Values of the parameters $A_2, B_2, C_2, D_2$ .

The values of the parameters  $A_2$  and  $C_2$  are immediately obtained from the matrix equation in § 2.2

$$A_2 = \frac{1}{2s_2(1-n_2)} \left[ \Phi_{11} \cdot A_3 e^{-(z-y)} + \Phi_{12} \cdot B_3 + \Phi_{13} \cdot C_3 e^{-s_3(z-y)} + \Phi_{14} \cdot D_3 \right]$$

$$C_2 = \frac{1}{2s_2(1-n_1)} \left[ \Phi_{31} \cdot A_3 e^{-(z-y)} + \Phi_{32} \cdot B_3 + \Phi_{33} \cdot C_3 e^{-s_3(z-y)} + \Phi_{34} \cdot D_3 \right]$$

The values of the parameters  $B_2$  and  $D_2$  are obtained from next matrix equation (§ 3):

$$\begin{pmatrix} A_2 \\ B_2 \\ C_2 \\ D_2 \end{pmatrix} = \frac{1}{4s_2s_3(1-n_2)(1-n_3)} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{pmatrix} \cdot \begin{pmatrix} B_4 \\ D_4 \end{pmatrix}$$

The necessity of convergency needs again some more computation.

$$B_2 = \frac{1}{4s_2s_3(1-n_2)(1-n_3)} \cdot [b_{21} \cdot B_4 + b_{22} \cdot D_4]$$

$$= \frac{2s_1(1-n_1)}{C_{12}} [c'_{22} \cdot b_{21} - c'_{21} \cdot b_{22}]$$

$$D_2 = \frac{1}{4s_2s_3(1-n_2)(1-n_3)} [b_{41} \cdot B_4 + b_{42} \cdot D_4]$$

$$= \frac{2s_1(1-n_1)}{C_{12}} [c'_{22} \cdot b_{41} - c'_{21} \cdot b_{42}]$$

$$c'_{22} \cdot b_{21} - c'_{21} \cdot b_{22} =$$

$$\begin{aligned} & \left[ e^{-2(y-x)} \cdot e^{-s_2(y-x)} a'_{21} \cdot b'_{12} + e^{-s_2(y-x)} a'_{22} \cdot b'_{22} \right. \\ & \quad \left. + e^{-(y-x)} \cdot e^{-2s_2(y-x)} a'_{23} \cdot b'_{32} + e^{-(y-x)} a'_{24} \cdot b'_{42} \right] b_{21} \\ & - \left[ e^{-2(y-x)} \cdot e^{-s_2(y-x)} a'_{21} \cdot b'_{11} + e^{-s_2(y-x)} a'_{22} \cdot b'_{21} \right. \\ & \quad \left. + e^{-(y-x)} \cdot e^{-2s_2(y-x)} a'_{23} \cdot b'_{31} + e^{-(y-x)} a'_{24} \cdot b'_{41} \right] b_{22} \end{aligned}$$

$$\begin{aligned} & = e^{-2(y-x)} \cdot e^{-s_2(y-x)} a'_{21} \left[ b'_{12} \cdot b_{21} - b'_{11} \cdot b_{22} \right] \\ & \quad + e^{-s_2(y-x)} a'_{22} \left[ b'_{22} \cdot b_{21} - b'_{21} \cdot b_{22} \right] \\ & \quad + e^{-(y-x)} \cdot e^{-2s_2(y-x)} a'_{23} \left[ b'_{32} \cdot b_{21} - b'_{31} \cdot b_{22} \right] \\ & \quad + e^{-(y-x)} a'_{24} \left[ b'_{42} \cdot b_{21} - b'_{41} \cdot b_{22} \right] \end{aligned}$$

$$\begin{aligned} & = -e^{-2(y-x)} \cdot e^{-s_2(y-x)} a'_{21} \cdot e^{(y-x)} \cdot B_{12} \\ & \quad + e^{-(y-x)} \cdot e^{-2s_2(y-x)} a'_{23} \cdot e^{(y-x)} \cdot B_{23} \\ & \quad + e^{-(y-x)} \cdot a'_{24} \cdot e^{(y-x)} \cdot B_{24} \end{aligned}$$

$$B_2 = \frac{2s_1(1-n_1)}{C_{12}} \left[ a'_{24} \cdot B_{24} + e^{-2s_2(y-x)} a'_{23} \cdot B_{23} - e^{-(y-x)} \cdot e^{-s_2(y-x)} a'_{21} \cdot B_{12} \right]$$

The positive exponent,  $e^{(y-x)}$  included in  $b_{21}$  and  $b_{22}$ , has disappeared.

$$c'_{22} \cdot b_{41} - c'_{21} \cdot b_{42} =$$

$$\begin{aligned} & e^{-2(y-x)} \cdot e^{-s_2(y-x)} a'_{21} \left[ b'_{12} \cdot b_{41} - b'_{11} \cdot b_{42} \right] \\ & \quad + e^{-s_2(y-x)} a'_{22} \left[ b'_{22} \cdot b_{41} - b'_{21} \cdot b_{42} \right] \\ & \quad + e^{-(y-x)} \cdot e^{-2s_2(y-x)} a'_{23} \left[ b'_{32} \cdot b_{41} - b'_{31} \cdot b_{42} \right] \\ & \quad + e^{-(y-x)} a'_{24} \left[ b'_{42} \cdot b_{41} - b'_{41} \cdot b_{42} \right] \end{aligned}$$

$$c'_{22} \cdot b_{41} - c'_{21} \cdot b_{42} =$$

$$- e^{-2(y-x)} \cdot e^{-s_2(y-x)} \cdot a'_{21} \cdot e^{s_2(y-x)} \cdot B_{14}$$

$$- e^{-s_2(y-x)} \cdot a'_{22} \cdot e^{s_2(y-x)} \cdot B_{24}$$

$$- e^{-(y-x)} \cdot e^{-2s_2(y-x)} \cdot a'_{23} \cdot e^{s_2(y-x)} \cdot B_{34}$$

$$D_2 = - \frac{2s_1(1-n_1)}{C_{12}} \left[ e^{-2(y-x)} a'_{21} B_{14} + a'_{22} B_{24} + a'_{23} e^{-(y-x)-s_2(y-x)} B_{34} \right]$$

The positive exponent,  $e^{s_2(y-x)}$  included in  $b_{41}$  and  $b_{42}$ , has disappeared.

The numerators of  $B_2$  and  $D_2$  contain only negative exponents because of the presence of the factors  $a'_{21}$ ,  $a'_{22}$ ,  $a'_{23}$  and  $a'_{24}$ .

#### 4.3. Values of the parameters $A_1$ , $B_1$ , $C_1$ , $D_1$ .

The values of the parameters  $A_1$  and  $C_1$  are immediately obtained from the matrix equation in § 2.3.

$$A_1 = \frac{1}{2s_1(1-n_1)} \left[ R_{11} e^{-(y-x)} A_2 + R_{12} B_2 + R_{13} e^{-s_2(y-x)} C_2 + R_{14} D_2 \right]$$

$$C_1 = \frac{1}{2s_1(1-n_1)} \left[ R_{31} e^{-(y-x)} A_2 + R_{32} B_2 + R_{33} e^{-s_2(y-x)} C_2 + R_{34} D_2 \right]$$

The values of the parameters  $B_1$  and  $D_1$  are obtained from the boundary conditions at the surface

$$B_1 = \frac{1}{s_1 - 1} \left[ s_1 - (1+s_1) A_1 e^{-x} - 2s_1 C_1 e^{s_1 x} \right]$$

$$D_1 = - \frac{1}{s_1 - 1} \left[ 1 - 2A_1 e^{-x} - (1+s_1) C_1 e^{s_1 x} \right]$$

## 5. Determination of the stresses and the displacements.

### 5.1. Mathematical procedure.

The stresses and the displacements are deduced from following relations wherein the notations of § 1. are utilized:

$$\sigma_z = pa \int_0^a J_0(mr) \cdot J_1(ma) \left[ A_i e^{-m(H_i-z)} + B_i e^{-m(z-H_{i-1})} + C_i e^{-ms_i(H_i-z)} + D_i e^{-ms_i(z-H_{i-1})} \right] dm$$

$$\frac{\sigma_r + \sigma_\theta}{2} = - \frac{pa}{2} \int_0^a J_0(mr) \cdot J_1(ma) \left[ A_i e^{-m(H_i-z)} + B_i e^{-m(z-H_{i-1})} + \frac{1-\mu_i}{n_i-\mu_i} C_i e^{-ms_i(H_i-z)} + \frac{1-\mu_i}{n_i-\mu_i} D_i e^{-ms_i(z-H_{i-1})} \right] dm$$

$$\frac{\sigma_r - \sigma_\theta}{2} = - \frac{pa}{2} \int_0^a J_1(ma) \left[ J_0(mr) - \frac{2J_1(mr)}{mr} \right] \cdot \left[ A_i e^{-m(H_i-z)} + B_i e^{-m(z-H_{i-1})} + \frac{1+\mu_i}{n_i+\mu_i} C_i e^{-ms_i(H_i-z)} + \frac{1+\mu_i}{n_i+\mu_i} D_i e^{-ms_i(z-H_{i-1})} \right] dm$$

$$\tau_{rz} = -pa \int_0^a J_1(mr) J_1(ma) \left[ A_i e^{-m(H_i-z)} - B_i e^{-m(z-H_{i-1})} + s_i C_i e^{-ms_i(H_i-z)} - s_i D_i e^{-ms_i(z-H_{i-1})} \right] dm$$

$$w = pa \frac{1+\mu_i}{E_i} \int_0^a \frac{J_0(mr) \cdot J_1(ma)}{m} \left[ A_i e^{-m(H_i-z)} - B_i e^{-m(z-H_{i-1})} + \frac{s_i(n_i+\mu_i)}{(1+\mu_i)} C_i e^{-ms_i(H_i-z)} - \frac{s_i(n_i+\mu_i)}{(1+\mu_i)} D_i e^{-ms_i(z-H_{i-1})} \right] dm$$

$$u = - \frac{(n_i+\mu_i)}{E_i} \cdot pa \int_0^a \frac{J_1(mr) \cdot J_1(ma)}{m} \left[ A_i e^{-m(H_i-z)} + B_i e^{-m(z-H_{i-1})} + \frac{1+\mu_i}{n_i+\mu_i} C_i e^{-ms_i(H_i-z)} + \frac{1+\mu_i}{n_i+\mu_i} D_i e^{-ms_i(z-H_{i-1})} \right] dm$$

The stresses in cartesian coordinates are calculated as explained in appendix 1.

5.2. Stresses and displacements at the surface ( $z = 0$ ).

$$\begin{aligned} \sigma_z &= p \quad (r < a) \\ &= p/2 \quad (r = a) \\ &= 0 \quad (r > a) \end{aligned}$$

$$\begin{aligned} \frac{\sigma_r + \sigma_\theta}{2} &= -\frac{pa}{2} \int_0^\infty J_0(mr) \cdot J_1(ma) \left[ A_1 e^{-s_1 x} + B_1 + \frac{1-\mu_1}{n_1-\mu_1} C_1 e^{-s_1 x} + \frac{1+\mu_1}{n_1+\mu_1} D_1 \right] dm \\ &= -\frac{pa}{2} \cdot \frac{[s_1(n_1-\mu_1) - (1+\mu_1)]}{(n_1-\mu_1)(s_1-1)} \int_0^\infty J_0(mr) \cdot J_1(ma) dm \\ &\quad - pa \cdot \frac{(1-n_1)}{(n_1-\mu_1)(s_1-1)} \int_0^\infty J_0(mr) \cdot J_1(ma) \left[ A_1 e^{-s_1 x} + s_1 C_1 e^{-s_1 x} \right] dm \end{aligned}$$

$$\begin{aligned} \frac{\sigma_r - \sigma_\theta}{2} &= -\frac{pa}{2} \int_0^\infty J_1(ma) \left[ J_0(mr) - \frac{2J_1(mr)}{mr} \right] \left[ A_1 e^{-s_1 x} + B_1 + \frac{1+\mu_1}{n_1+\mu_1} C_1 e^{-s_1 x} + \frac{1+\mu_1}{n_1+\mu_1} D_1 \right] dm \\ &= -\frac{pa}{2} \cdot \frac{[s_1(n_1+\mu_1) - (1+\mu_1)]}{(n_1+\mu_1)(s_1-1)} \int_0^\infty J_1(ma) \cdot \left[ J_0(mr) - \frac{2J_1(mr)}{mr} \right] dm \\ &\quad - pa \cdot \frac{(1-n_1)}{(n_1+\mu_1)(s_1-1)} \int_0^\infty J_1(ma) \left[ J_0(mr) - \frac{2J_1(mr)}{mr} \right] \left[ A_1 e^{-s_1 x} + s_1 C_1 e^{-s_1 x} \right] dm \end{aligned}$$

with

$$\frac{pa}{2} \int_0^\infty J_0(mr) \cdot J_1(ma) = \begin{cases} p/2 & (r < a) \\ p/4 & (r = a) \\ 0 & (r > a) \end{cases}$$

$$pa \int_0^\infty \frac{J_1(mr) \cdot J_1(ma)}{mr} = \begin{cases} p/2 & (r < a) \\ \frac{pa^2}{2r^2} & (r > a) \end{cases}$$

$$\cdot \text{Tr}2 = 0$$

$$\begin{aligned} \cdot w &= pa \frac{1+\mu_1}{E_1} \int_0^a \frac{J_0(mr) J_1(ma)}{m} \left[ A_1 e^{-s_1 x} - B_1 + \frac{s_1(1+\mu_1)}{1+\mu_1} C_1 e^{-s_1 x} - \frac{s_1(1+\mu_1)}{1+\mu_1} D_1 \right] dm \\ &= -pa \frac{s_1(1-\mu_1)}{E_1(s_1-1)} \int_0^a \frac{J_0(mr) \cdot J_1(ma)}{m} dm \\ &\quad + pa \frac{2s_1(1-\mu_1)}{E_1(s_1-1)} \int_0^a \frac{J_0(mr) \cdot J_1(ma)}{m} \left[ A_1 e^{-s_1 x} + C_1 e^{-s_1 x} \right] dm \end{aligned}$$

with

$$pa \int_0^a \frac{J_0(mr) \cdot J_1(ma)}{m} dm = \left. \begin{aligned} &1 \quad (r=0) \\ &F\left(\frac{1}{2}, -\frac{1}{2}; 1; \frac{r^2}{a^2}\right) \quad (r < a) \\ &2/\pi \quad (r=a) \\ &\frac{a}{2r} F\left(\frac{1}{2}, \frac{1}{2}; 2; \frac{a^2}{r^2}\right) \quad (r > a) \end{aligned} \right\}$$

$$\begin{aligned} \cdot u &= -pa \frac{1+\mu_1}{E_1} \int_0^a \frac{J_1(mr) \cdot J_1(ma)}{m} \left[ A_1 e^{-s_1 x} + B_1 + \frac{1+\mu_1}{1+\mu_1} C_1 e^{-s_1 x} + \frac{1+\mu_1}{1+\mu_1} D_1 \right] dm \\ &= -pa \frac{s_1(1+\mu_1) - (1+\mu_1)}{E_1(s_1-1)} \int_0^a \frac{J_1(mr) \cdot J_1(ma)}{m} dm \\ &\quad - 2pa \frac{(1-\mu_1)}{E_1(s_1-1)} \int_0^a \frac{J_1(mr) \cdot J_1(ma)}{m} \left[ A_1 e^{-s_1 x} + s_1 C_1 e^{-s_1 x} \right] dm \end{aligned}$$

with

$$pa \int_0^a \frac{J_1(mr) \cdot J_1(ma)}{m} dm = \left\{ \begin{aligned} &\frac{pr}{2} \quad (r < a) \\ &\frac{pa^2}{2r} \quad (r > a) \end{aligned} \right.$$

### 5.3. Stresses and displacements in the first layer ( $0 < h < H_1$ )

$$\begin{aligned} \sigma_z &= pa \int_0^a J_0(mr) \cdot J_1(ma) \left[ A_1 e^{-m(H_1-h)} + B_1 e^{-mh} + C_1 e^{-s_1 m(H_1-h)} + D_1 e^{-s_1 mh} \right] dm \\ &= \frac{pa}{(s_1-1)} \int_0^a J_0(mr) \cdot J_1(ma) \left[ s_1 e^{-mh} - e^{-s_1 mh} \right] dm \\ &\quad + pa \int_0^a J_0(mr) \cdot J_1(ma) \left[ A_1 e^{-m(H_1-h)} + C_1 e^{-s_1 m(H_1-h)} \right] dm \\ &\quad + \frac{pa}{(s_1-1)} \int_0^a J_0(mr) \cdot J_1(ma) \left[ 2A_1 e^{-m(H_1+h s_1)} - (1+s_1) A_1 e^{-m(H_1+h)} \right. \\ &\quad \left. + (1+s_1) C_1 e^{-s_1 m(H_1+h)} - 2s_1 C_1 e^{-m(s_1 H_1+h)} \right] dm \end{aligned}$$

$$\begin{aligned} \frac{\sigma_r + \sigma_\theta}{2} &= -\frac{pa}{2} \int_0^a J_0(mr) \cdot J_1(ma) \left[ A_1 e^{-m(H_1-h)} + B_1 e^{-mh} \right. \\ &\quad \left. + \frac{1-\mu_1}{n_1-\mu_1} C_1 e^{-s_1 m(H_1-h)} + \frac{1-\mu_1}{n_1-\mu_1} D_1 e^{-s_1 mh} \right] dm \\ &= -\frac{pa}{2(s_1-1)} \int_0^a J_0(mr) \cdot J_1(ma) \left[ s_1 e^{-mh} - \frac{1-\mu_1}{n_1-\mu_1} e^{-s_1 mh} \right] dm \\ &\quad - \frac{pa}{2} \int_0^a J_0(mr) \cdot J_1(ma) \left[ A_1 e^{-m(H_1-h)} + \frac{1-\mu_1}{n_1-\mu_1} C_1 e^{-s_1 m(H_1-h)} \right] dm \\ &\quad - \frac{pa}{2(s_1-1)} \int_0^a J_0(mr) \cdot J_1(ma) \left[ 2 \frac{1-\mu_1}{n_1-\mu_1} A_1 e^{-m(H_1+h s_1)} \right. \\ &\quad \left. - (1+s_1) A_1 e^{-m(H_1+h)} + \frac{1-\mu_1}{n_1-\mu_1} (1+s_1) C_1 e^{-s_1 m(H_1+h)} - 2s_1 C_1 e^{-m(s_1 H_1+h)} \right] dm \end{aligned}$$

with

$$pa \int_0^a J_0(mr) \cdot J_1(ma) s_1 e^{-mh} dm = \frac{ps_1 h}{\pi} \int_0^a \frac{(a^2-x^2)^{1/2}}{a(h^2+x^2+r^2-2xr)(h^2+a^2+r^2-2xr)^{1/2}} dx$$

$$pa \int_0^a J_0(mr) \cdot J_1(ma) e^{-s_1 mh} dm = \frac{ps_1 h}{\pi} \int_0^a \frac{(a^2-x^2)^{1/2}}{a(s_1^2 h^2+x^2+r^2-2xr)(s_1^2 h^2+a^2+r^2-2xr)^{1/2}} dx$$

$$\begin{aligned}
 \frac{\sigma_r - \sigma_\theta}{2} &= -\frac{\rho a}{2} \int_0^a J_1(ma) \left[ J_0(mr) - 2 \frac{J_1(mr)}{mr} \right] \left[ A_1 e^{-m(H,-h)} + B_1 e^{-mh} \right. \\
 &\quad \left. + \frac{1+\mu_1}{h+\mu_1} C_1 e^{-s_1 m(H,-h)} + \frac{1+\mu_1}{h+\mu_1} D_1 e^{-s_1 mh} \right] dm \\
 &= -\frac{\rho a}{2(s_1-1)} \int_0^a J_1(ma) \left[ J_0(mr) - 2 \frac{J_1(mr)}{mr} \right] \left[ s_1 e^{-mh} - \frac{1+\mu_1}{h+\mu_1} e^{-s_1 mh} \right] dm \\
 &\quad - \frac{\rho a}{2} \int_0^a J_1(ma) \left[ J_0(mr) - 2 \frac{J_1(mr)}{mr} \right] \left[ A_1 e^{-m(H,-h)} + \frac{1+\mu_1}{h+\mu_1} C_1 e^{-s_1 m(H,-h)} \right] dm \\
 &\quad - \frac{\rho a}{2(s_1-1)} \int_0^a J_1(ma) \left[ J_0(mr) - 2 \frac{J_1(mr)}{mr} \right] \left[ 2 \frac{1+\mu_1}{h+\mu_1} A_1 e^{-m(H,+hs_1)} \right. \\
 &\quad \left. - (1+s_1) A_1 e^{-m(H,+h)} + \frac{1+\mu_1}{h+\mu_1} (1+s_1) C_1 e^{-s_1 m(H,+h)} - 2s_1 C_1 e^{-m(s_1 H,+h)} \right] dm
 \end{aligned}$$

with

$$\rho a \int_0^a \frac{J_1(ma) J_1(mr)}{mr} s_1 e^{-mh} = \frac{\rho a^2 s_1}{\pi} \int_0^\pi \frac{1}{\omega^2} \left[ 1 - \frac{h}{(h^2 + \omega^2)^{1/2}} \right] \sin^2 \phi \, d\phi$$

$$\rho a \int_0^a \frac{J_1(ma) J_1(mr)}{mr} e^{-s_1 mh} = \frac{\rho a^2}{\pi} \int_0^\pi \frac{1}{\omega^2} \left[ 1 - \frac{s_1 h}{(s_1^2 h^2 + \omega^2)^{1/2}} \right] \sin^2 \phi \, d\phi$$

$$\omega^2 = a^2 + r^2 - 2ar \cos \phi$$

$$\begin{aligned}
 \tau_{rz} &= -\rho a \int_0^a J_1(mr) \cdot J_1(ma) \left[ A_1 e^{-m(H,-h)} - B_1 e^{-mh} + s_1 C_1 e^{-s_1 m(H,-h)} - s_1 D_1 e^{-s_1 mh} \right] dm \\
 &= -\frac{\rho a}{(s_1-1)} \int_0^a J_1(mr) \cdot J_1(ma) \left[ e^{-s_1 mh} - s_1 e^{-mh} \right] dm \\
 &\quad - \rho a \int_0^a J_1(mr) \cdot J_1(ma) \left[ A_1 e^{-m(H,-h)} + s_1 C_1 e^{-s_1 m(H,-h)} \right] dm \\
 &\quad - \frac{\rho a}{(s_1-1)} \int_0^a J_1(mr) \cdot J_1(ma) \left[ (1+s_1) A_1 e^{-m(H,+h)} - 2s_1 A_1 e^{-m(H,+hs_1)} \right. \\
 &\quad \left. + 2s_1 C_1 e^{-m(s_1 H,+h)} - s_1 (1+s_1) C_1 e^{-s_1 m(H,+h)} \right] dm
 \end{aligned}$$

with

$$\rho a \int_0^a s_1 J_1(mr) \cdot J_1(ma) e^{-mh} dm = \frac{\rho a^2 s_1}{\pi} \int_0^\pi \frac{\sin^2 \phi}{(h^2 + \omega^2)^{3/2}} d\phi$$

$$\rho a \int_0^a J_1(mr) \cdot J_1(ma) e^{-s_1 mh} dm = \frac{\rho a^2}{\pi} \int_0^\pi \frac{\sin^2 \phi}{(s_1^2 h^2 + \omega^2)^{3/2}} d\phi$$

$$\begin{aligned}
 W &= \rho a \frac{1+\mu_1}{E_1} \int_0^a J_0(mr) \frac{J_1(ma)}{m} \left[ A_1 e^{-m(H_1-h)} - B_1 e^{-mh} + \frac{s_1(h+\mu_1)}{(1+\mu_1)} \left( C_1 e^{-ms_1(H_1-h)} - D_1 e^{-ms_1 h} \right) \right] dm \\
 &= -\rho a \frac{s_1(1+\mu_1)}{E_1(s_1-1)} \int_0^a J_0(mr) \frac{J_1(ma)}{m} \left[ e^{-mh} - \frac{(h+\mu_1)}{(1+\mu_1)} e^{-ms_1 h} \right] dm \\
 &\quad + \rho a \frac{1+\mu_1}{E_1} \int_0^a J_0(mr) \frac{J_1(ma)}{m} \left[ A_1 e^{-m(H_1-h)} + \frac{s_1(h+\mu_1)}{(1+\mu_1)} C_1 e^{-ms_1(H_1-h)} \right] dm \\
 &\quad + \rho a \frac{1+\mu_1}{E_1(s_1-1)} \int_0^a J_0(mr) \frac{J_1(ma)}{m} \left[ (1+s_1) A_1 e^{-m(H_1+h)} + 2s_1 C_1 e^{-ms_1(H_1+h)} \right. \\
 &\quad \left. - 2 \frac{s_1(h+\mu_1)}{(1+\mu_1)} A_1 e^{-m(H_1+h s_1)} - (1+s_1) \frac{s_1(h+\mu_1)}{(1+\mu_1)} C_1 e^{-s_1 m(H_1+h)} \right] dm
 \end{aligned}$$

with

$$\rho a \int_0^a \frac{J_0(mr) \cdot J_1(ma)}{m} e^{-mh} = \frac{\rho}{2\pi} \int_{-a}^a \ln \frac{(r^2 - 2xr + a^2 + h^2)^{1/2} + (a^2 - x^2)^{1/2}}{(r^2 - 2xr + a^2 + h^2)^{1/2} - (a^2 - x^2)^{1/2}} dx$$

$$\rho a \int_0^a \frac{J_0(mr) \cdot J_1(ma)}{m} e^{-s_1 m h} = \frac{\rho}{2\pi} \int_{-a}^a \ln \frac{(r^2 - 2xr + a^2 + s_1^2 h^2)^{1/2} + (a^2 - x^2)^{1/2}}{(r^2 - 2xr + a^2 + s_1^2 h^2)^{1/2} - (a^2 - x^2)^{1/2}} dx$$

$$\begin{aligned}
 u &= -\frac{h+\mu_1}{E_1} \cdot \rho a \int_0^a \frac{J_1(mr) \cdot J_1(ma)}{m} \left[ A_1 e^{-m(H_1-h)} + B_1 e^{-mh} + \frac{1+\mu_1}{h+\mu_1} \left( C_1 e^{-ms_1(H_1-h)} + D_1 e^{-ms_1 h} \right) \right] dm \\
 &= -\frac{h+\mu_1}{E_1} \cdot \frac{\rho a}{(s_1-1)} \int_0^a \frac{J_1(mr) \cdot J_1(ma)}{m} \left[ s_1 e^{-mh} - \frac{1+\mu_1}{h+\mu_1} e^{-s_1 m h} \right] dm \\
 &\quad - \frac{h+\mu_1}{E_1} \cdot \rho a \int_0^a \frac{J_1(mr) \cdot J_1(ma)}{m} \left[ A_1 e^{-m(H_1-h)} + \frac{1+\mu_1}{h+\mu_1} C_1 e^{-s_1 m(H_1-h)} \right] dm \\
 &\quad - \frac{h+\mu_1}{E_1} \cdot \frac{\rho a}{(s_1-1)} \int_0^a \frac{J_1(mr) \cdot J_1(ma)}{m} \left[ 2 \frac{1+\mu_1}{h+\mu_1} A_1 e^{-m(H_1+h s_1)} \right. \\
 &\quad \left. - (1+s_1) A_1 e^{-m(H_1+h)} + \frac{1+\mu_1}{h+\mu_1} (1+s_1) C_1 e^{-s_1 m(H_1+h)} - 2s_1 C_1 e^{-ms_1(H_1+h)} \right] dm.
 \end{aligned}$$

with

$$p a \int_0^{\omega} \frac{J_1(mr) \cdot J_1(ma)}{m} s_i e^{-s_i h} = \frac{p a^2 r s_i}{\pi} \int_0^{\pi} \frac{1}{\omega^2} \left[ 1 - \frac{h^2}{(h^2 + \omega^2)^{1/2}} \right] \sin^2 \phi d\phi$$

$$p a \int_0^{\omega} \frac{J_1(mr) \cdot J_1(ma)}{m} e^{-s_i m h} = \frac{p a^2 r}{\pi} \int_0^{\pi} \frac{1}{\omega^2} \left[ 1 - \frac{s_i^2 h^2}{(s_i^2 h^2 + \omega^2)^{1/2}} \right] \sin^2 \phi d\phi$$

#### 5.4. Stresses and displacements in the other layers.

The stresses and displacements are immediately deduced from the relations in § 5.1.

The exponents associated to the parameters are:

- in the second layer ( $H_1 < h < H_1 + H_2$ )

$$A_2 e^{-m(H_1 + H_2 - h)}, B_2 e^{-m(h - H_1)}, C_2 e^{-s_2 m(H_1 + H_2 - h)}, D_2 e^{-s_2 m(h - H_1)}$$

- in the third layer ( $H_1 + H_2 < h < H_1 + H_2 + H_3$ )

$$A_3 e^{-m(H_1 + H_2 + H_3 - h)}, B_3 e^{-m(h - H_1 - H_2)}$$

$$C_3 e^{-s_3 m(H_1 + H_2 + H_3 - h)}, D_3 e^{-s_3 m(h - H_1 - H_2)}$$

- in the fourth layer ( $H_1 + H_2 + H_3 < h < H_1 + H_2 + H_3 + H_4$ )

$$D_4 e^{-m(h - H_1 - H_2 - H_3)}, D_4 e^{-s_4 m(h - H_1 - H_2 - H_3)}$$


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APPENDIX 4

Algebraical Analysis of a four-layer anisotropic System  
with fixed bottom, full slip condition at the first and  
second interface, full friction at the third interface.

## APPENDIX 4

Algebraical analysis of an anisotropic four layered structure with  
fixed bottom and full slip condition at the first and second interfaces.

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### 1. Boundary conditions.

To reduce the number of exponentials we write

$$A_i m^2 n_i (1 + \mu_i) e^{m(H_1 + H_2 + \dots + H_i)} = A_i$$

$$B_i m^2 n_i (1 + \mu_i) e^{-m(H_1 + H_2 + \dots + H_{i-1})} = B_i$$

$$C_i m^2 n_i s_i (n_i + \mu_i) e^{m s_i (H_1 + H_2 + \dots + H_i)} = C_i$$

$$D_i m^2 n_i s_i (n_i + \mu_i) e^{-m s_i (H_1 + H_2 + \dots + H_{i-1})} = D_i$$

Further we also write

$$F_1 = \frac{E_1}{E_2} \quad F = \frac{E_1}{E_2} \cdot \frac{s_2 (n_2 - 1)}{s_1 (n_1 - 1)}$$

$$K_1 = \frac{E_2}{E_3} \quad K = \frac{E_2}{E_3} \cdot \frac{s_3 (n_3 - 1)}{s_2 (n_2 - 1)}$$

$$L_1 = \frac{E_3}{E_4} \quad L_2 = \frac{E_3}{E_4}$$

$$s_i = \left( \frac{n_i - \mu_i^2}{n_i^2 - \mu_i^2} \right)^{\frac{1}{2}}$$

$$x = m H_1$$

$$y = m (H_1 + H_2)$$

$$z = m (H_1 + H_2 + H_3)$$

$$t = m (H_1 + H_2 + H_3 + H_4)$$

where  $H_1, H_2, H_3$  and  $H_4$  are the thicknesses of the four layers.

Boundary conditions at the surface ( $z = 0$ ):

$$v_2 = p: A_1 e^{-x} + B_1 + C_1 e^{-s_1 x} + D_1 = 1$$

$$\tau_{12} = 0: A_1 e^{-x} - B_1 + s_1 C_1 e^{-s_1 x} - s_1 D_1 = 0$$

Boundary conditions at the first interface ( $z = H_1$ ):

$$v_2: A_1 + B_1 e^{-x} + C_1 + D_1 e^{-s_1 x} = A_2 e^{-(y-x)} + B_2 + C_2 e^{-s_2(y-x)} + D_2$$

$$\tau_{12}: A_1 - B_1 e^{-x} + s_1 C_1 - s_1 D_1 e^{-s_1 x} = 0$$

$$A_2 e^{-(y-x)} - B_2 + s_2 C_2 e^{-s_2(y-x)} - s_2 D_2 = 0$$

$$w: (1+\mu_1) A_1 - (1+\mu_1) B_1 e^{-x} + s_1 (n_1+\mu_1) C_1 - s_1 (n_1+\mu_1) D_1 e^{-s_1 x} =$$

$$F_1 [(1+\mu_2) A_2 e^{-(y-x)} - (1+\mu_2) B_2 + s_2 (n_2+\mu_2) C_2 e^{-s_2(y-x)} - s_2 (n_2+\mu_2) D_2]$$

Boundary conditions at the second interface ( $z = H_1 + H_2$ ):

$$v_2: A_2 e + B_2 e^{-(y-x)} + C_2 + D_2 e^{-s_2(y-x)} = A_3 e^{-(z-y)} + B_3 + C_3 e^{-s_3(z-y)} + D_3$$

$$\tau_{22}: A_2 - B_2 e^{-(y-x)} + s_2 C_2 - s_2 D_2 e^{-s_2(y-x)} = 0$$

$$A_3 e^{-(z-y)} - B_3 + s_3 C_3 e^{-s_3(z-y)} - s_3 D_3 = 0$$

$$w: (1+\mu_2) A_2 - (1+\mu_2) B_2 e^{-(y-x)} + s_2 (n_2+\mu_2) C_2 - s_2 (n_2+\mu_2) D_2 e^{-s_2(y-x)} =$$

$$K_1 [(1+\mu_3) A_3 e^{-(z-y)} - (1+\mu_3) B_3 + s_3 (n_3+\mu_3) C_3 e^{-s_3(z-y)} - s_3 (n_3+\mu_3) D_3]$$

Boundary conditions at the third interface ( $z = H_1 + H_2 + H_3$ ):

$$v_2: A_3 + B_3 e^{-(z-y)} + C_3 + D_3 e^{-s_3(z-y)} = A_4 e^{-(t-z)} + B_4 + C_4 e^{-s_4(t-z)} + D_4$$

$$\tau_{32}: A_3 - B_3 e^{-(z-y)} + s_3 C_3 - s_3 D_3 e^{-s_3(z-y)} = A_4 e^{-(t-z)} - B_4 + s_4 C_4 e^{-s_4(t-z)} - s_4 D_4$$

$$w: (1+\mu_3) A_3 - (1+\mu_3) B_3 e^{-(z-y)} + s_3 (n_3+\mu_3) C_3 - s_3 (n_3+\mu_3) D_3 e^{-s_3(z-y)} =$$

$$L_1 [(1+\mu_4) A_4 e^{-(t-z)} - (1+\mu_4) B_4 + s_4 (n_4+\mu_4) C_4 e^{-s_4(t-z)} - s_4 (n_4+\mu_4) D_4]$$

$$u: (n_3+\mu_3) A_3 + (n_3+\mu_3) B_3 e^{-(z-y)} + (1+\mu_3) C_3 + (1+\mu_3) D_3 =$$

$$L_2 [(n_4+\mu_4) A_4 e^{-(t-z)} + (n_4+\mu_4) B_4 + (1+\mu_4) C_4 e^{-s_4(t-z)} + (1+\mu_4) D_4]$$

Boundary condition at the bottom ( $z = H_1 + H_2 + H_3 + H_4$ ):

$$w: (1+\mu_4) A_4 - (1+\mu_4) B_4 e^{-(t-z)} + s_4 (n_4+\mu_4) C_4 - s_4 (n_4+\mu_4) D_4 e^{-s_4(t-z)} = 0$$

If  $s_4 > 1$

$$C_4 = 0$$

$$A_4 e^{-(t-z)} = B_4 e^{-2(t-z)} + \frac{s_4 (n_4+\mu_4)}{(1+\mu_4)} D_4 e^{-(1+s_4)(t-z)}$$

If  $s_4 < 1$

$$A_4 = 0$$

$$C_4 e^{-s_4(t-z)} = \frac{(1+\mu_4)}{s_4 (n_4+\mu_4)} B_4 e^{-(1+s_4)(t-z)} + D_4 e^{-2s_4(t-z)}$$

## 2. Expression of the boundary conditions in matrixform.

### 2.1. At the third interface.

In the equations at the third interface,  $A_4$ , or  $C_4$ , is replaced by its value obtained from the fixed bottom condition.

We write the conditions at the third interface in matrixform

$$M_5 (A_3 B_3 C_3 D_3)^T = M_6 (B_4 D_4)^T$$

We invert  $M_5$

$$(A_3 B_3 C_3 D_3)^T = M_5^{-1} \cdot M_6 (B_4 D_4)^T$$

$$M_5^{-1} = \frac{1}{2s_3(1+n_3)} \begin{pmatrix} s_3(1+\mu_3) & -s_3(n_3+\mu_3) & s_3 & -s_3 \\ s_3(1+\mu_3)e^{(z-y)} & s_3(n_3+\mu_3)e^{(z-y)} & -s_3e^{(z-y)} & -s_3e^{(z-y)} \\ -s_3(n_3+\mu_3) & (1+\mu_3) & -1 & s_3 \\ -s_3(n_3+\mu_3)e^{s_3(z-y)} & -(1+\mu_3)e^{s_3(z-y)} & e^{s_3(z-y)} & s_3e^{s_3(z-y)} \end{pmatrix}$$

If  $s_4 > 1$

$$M_6 = \begin{pmatrix} 1 + e^{-2(t-z)} & 1 + \frac{s_4(n_4+\mu_4)}{(1+\mu_4)} e^{-(1+s_4)(t-z)} \\ -1 + e^{-2(t-z)} & -s_4 + \frac{s_4(n_4+\mu_4)}{(1+\mu_4)} e^{-(1+s_4)(t-z)} \\ L_1(1+\mu_4) [-1 + e^{-2(t-z)}] & L_1 s_4(n_4+\mu_4) [-1 + e^{-(1+s_4)(t-z)}] \\ L_2(n_4+\mu_4) [1 + e^{-2(t-z)}] & L_2(1+\mu_4) \left[ -1 + \frac{s_4(n_4+\mu_4)^2}{(1+\mu_4)^2} e^{-(1+s_4)(t-z)} \right] \end{pmatrix}$$

If  $s_4 < 1$

$$M_6 = \begin{pmatrix} 1 + \frac{1+\mu_4}{s_4(n_4+\mu_4)} e^{-(1+s_4)(t-z)} & 1 + e^{-2s_4(t-z)} \\ -1 + \frac{1+\mu_4}{(n_4+\mu_4)} e^{-(1+s_4)(t-z)} & -s_4 + e^{-2s_4(t-z)} s_4 \\ L_1(1+\mu_4) [-1 + e^{-(1+s_4)(t-z)}] & L_1 s_4(n_4+\mu_4) [-1 + e^{-2s_4(t-z)}] \\ L_2(n_4+\mu_4) \left[ 1 + \frac{(1+\mu_4)^2}{s_4(n_4+\mu_4)^2} e^{-(1+s_4)(t-z)} \right] & L_2(1+\mu_4) [1 + e^{-2s_4(t-z)}] \end{pmatrix}$$

We write  $A_3$ ,  $B_3$ ,  $C_3$  and  $D_3$  in function of  $B_4$  and  $D_4$ .

$$A_3 = \frac{\sum M_5(i1) \cdot M_6(i1) \cdot B_4 + \sum M_5(i1) \cdot M_6(i2) D_4}{2J_3(1-n_3)}$$

$$B_3 = \frac{\sum M_5(2i) \cdot M_6(i1) \cdot B_4 + \sum M_5(2i) \cdot M_6(i2) D_4}{2J_3(1-n_3)} \cdot e^{(2-y)}$$

$$C_3 = \frac{\sum M_5(3i) \cdot M_6(i1) \cdot B_4 + \sum M_5(3i) \cdot M_6(i2) D_4}{2J_3(1-n_3)}$$

$$D_3 = \frac{\sum M_5(4i) \cdot M_6(i1) \cdot B_4 + \sum M_5(4i) \cdot M_6(i2) D_4}{2J_3(1-n_3)} e^{j_3(2-y)}$$

where  $M_5(i,j)$  are the constants in  $M_5^{-1}$ .

We write

$$P_{j1} = \sum M_5(ji) M_6(i1)$$

$$P_{j2} = \sum M_5(ji) M_6(i2)$$

so that

$$\begin{pmatrix} A_3 \\ B_3 \\ C_3 \\ D_3 \end{pmatrix} = \frac{1}{2J_3(1-n_3)} \begin{pmatrix} P_{11} & P_{12} \\ P_{21}e^{(2-y)} & P_{22}e^{(2-y)} \\ P_{31} & P_{32} \\ P_{41}e^{j_3(2-y)} & P_{42}e^{j_3(2-y)} \end{pmatrix} \begin{vmatrix} B_4 \\ D_4 \end{vmatrix}$$

### 2.2. At the surface

Adding and subtracting the surface conditions we obtain

$$2A_1 e^{-x} = 1 - (1+s_1)C_1 e^{-s_1 x} - (1-s_1)D_1$$

$$2B_1 = 1 - (1-s_1)C_1 e^{-s_1 x} - (1+s_1)D_1$$

### 2.3. At the first interface.

We add and subtract the first two conditions

$$2A_1 + (1+s_1)C_1 + (1-s_1)D_1 e^{-s_1 x} = A_2 e^{-(y-x)} + B_2 + C_2 e^{-s_2(y-x)} + D_2$$

$$= [A_2]$$

$$2B_1 e^{-x} + (1-s_1)C_1 + (1+s_1)D_1 e^{-s_1 x} = [A_2]$$

We replace  $A_1$  and  $B_1$  by their values obtained from the surface conditions

$$(1+s_1)C_1 [1 - e^{-x} e^{-s_1 x}] + (1-s_1)D_1 [e^{-s_1 x} - e^{-x}] = [A_2] - e^{-x}$$

$$(1-s_1)C_1 [1 - e^{-x} e^{-s_1 x}] + (1+s_1)D_1 [e^{-s_1 x} - e^{-x}] = [A_2] - e^{-x}$$

We solve the system

$$C_1 = \left\{ [A_2] \cdot [2s_1 e^{-s_1 x} - (1+s_1)e^{-x} + (1-s_1)e^{-x}] \right. \\ \left. + [2s_1 - (1+s_1)e^{-x} e^{-s_1 x} + (1-s_1)e^{-x} e^{-s_1 x}] \right\} \cdot \frac{1}{e^{-x} \nabla_1}$$

$$= \left\{ [A_2] \cdot [2s_1 e^{-x} e^{-s_1 x} - (1+s_1)e^{-2x} + (1-s_1)] \right. \\ \left. + [2s_1 e^{-x} - (1+s_1)e^{-s_1 x} + (1-s_1)e^{-2x} e^{-s_1 x}] \right\} \cdot \frac{1}{\nabla_1}$$

The positive exponent  $e^x$  has disappeared.

$$D_1 = \left\{ [A_2] \cdot [2s_1 e^{-x} - (1+s_1)e^{-s_1 x} + (1-s_1)e^{-2x} e^{-s_1 x}] \right. \\ \left. + [2s_1 e^{-x} e^{-s_1 x} - (1+s_1)e^{-2x} + (1-s_1)] \right\} \cdot \frac{1}{\nabla_1}$$

$$\nabla_1 = 2s_1 e^{-x} e^{-s_1 x} - (1+s_1)^2 [e^{-2x} + e^{-2s_1 x}] + (1-s_1)^2 [1 + e^{-2x} e^{-2s_1 x}]$$

$$\text{For } m = \infty \quad \nabla_1 = (1-s_1)^2$$

We transform the w-condition utilizing the  $\tau_{r_2}$ -conditions

$$s_1(n_1-1)C_1 - s_1(n_1-1)D_1 e^{-s_1 x} = F_1 [s_2(n_2-1)C_2 e^{-s_2(y-x)} - s_2(n_2-1)D_2]$$

$$F = F_1 \frac{s_2(n_2-1)}{s_1(n_1-1)}$$

$$C_1 - D_1 e^{-s_1 x} = FC_2 e^{-s_2(y-x)} - FD_2$$

We replace  $C_1$  and  $D_1$  by their values

$$C_1 - D_1 e^{-s_1 x} =$$

$$\left\{ [A_2] \left[ (1-s_1) - (1+s_1)e^{-2x} + (1+s_1)e^{-2s_1 x} - (1-s_1)e^{-2x}e^{-2s_1 x} \right] \right. \\ \left. + [2s_1 e^x (1-e^{-2s_1 x}) + 2e^{-2x}e^{-s_1 x} - 2e^{-s_1 x}] \right\} \cdot \frac{1}{V_1}$$

$$= \left\{ [A_2] \cdot R_1 + R_2 \right\} \cdot \frac{1}{V_1}$$

$$\text{For } m=\infty \quad R_1 = (1-s_1), \quad R_2 = 0$$

$$\left\{ [A_2] \cdot R_1 + R_2 \right\} \cdot \frac{1}{V_1} = FC_2 e^{-s_2(y-x)} - FD_2$$

$$[A_2] - \frac{FV_1}{R_1} [C_2 e^{-s_2(y-x)} - D_2] = -\frac{R_2}{R_1}$$

Writing  $R_3 = \frac{FV_1}{R_1}$ , we obtain the system

$$A_2 e^{-(y-x)} + B_2 + C_2 e^{-s_2(y-x)} + D_2 - R_3 C_2 e^{-s_2(y-x)} + R_3 D_2 = -\frac{R_2}{R_1}$$

$$A_2 e^{-(y-x)} - B_2 + s_2 C_2 e^{-s_2(y-x)} - s_2 D_2 = 0$$

and by adding and subtracting

$$2A_2 e^{-(y-x)} = - (1+s_2-R_3) C_2 e^{-s_2(y-x)} - (1-s_2+R_3) D_2 - \frac{R_2}{R_1}$$

$$2B_2 = - (1-s_2-R_3) C_2 e^{-s_2(y-x)} - (1+s_2+R_3) D_2 - \frac{R_2}{R_1}$$

2.4. At the second interface.

We add and subtract the first two conditions

$$2A_2 + (1+r_2)C_2 + (1-s_2)D_2 e^{-s_2(y-x)} = A_3 e^{-(x-y)} + B_3 + C_3 e^{-s_3(x-y)} + D_3$$

$$= [A_3]$$

$$2B_2 e^{-(y-x)} + (1-s_2)C_2 + (1+r_2)D_2 e^{-s_2(y-x)} = [A_3]$$

We replace  $A_2$  and  $B_2$  by their values obtained from the first interface conditions

$$C_2 [(1+r_2) - (1+r_2-R_3) e^{(y-x)} e^{-s_2(y-x)}]$$

$$+ D_2 [(1-s_2) e^{-s_2(y-x)} - (1-s_2+R_3) e^{(y-x)}] = [A_3] + \frac{R_2}{R_1} e^{(y-x)}$$

$$C_2 [(1-s_2) - (1-s_2-R_3) e^{-(y-x)} e^{-s_2(y-x)}]$$

$$+ D_2 [(1+r_2) e^{-s_2(y-x)} - (1+r_2+R_3) e^{-(y-x)}] = [A_3] + \frac{R_2}{R_1} e^{-(y-x)}$$

We solve the system

$$C_2 = \left\{ [A_3] [2s_2 e^{-s_2(y-x)} - (1+r_2+R_3) e^{-(y-x)} + (1-s_2+R_3) e^{(y-x)}] \right.$$

$$\left. + \frac{R_2}{R_1} [(1+r_2) e^{(y-x)} e^{-s_2(y-x)} - 2s_2 - (1-s_2) e^{-(y-x)} e^{-s_2(y-x)}] \right\} \frac{1}{e^{(y-x)} \nabla_2}$$

$$= \left\{ [A_3] [2s_2 e^{-(y-x)} e^{-s_2(y-x)} - (1+r_2+R_3) e^{-2(y-x)} + (1-s_2+R_3)] \right.$$

$$\left. + \frac{R_2}{R_1} [(1+r_2) e^{-s_2(y-x)} - 2s_2 e^{-(y-x)} - (1-s_2) e^{-2(y-x)} e^{-s_2(y-x)}] \right\} \frac{1}{\nabla_2}$$

The positive exponent  $e^{(y-x)}$  has disappeared.

$$D_2 = \left\{ [A_3] [2s_2 e^{-(y-x)} - (1+r_2-R_3) e^{-s_2(y-x)} + (1-s_2-R_3) e^{-2(y-x)} e^{-s_2(y-x)}] \right.$$

$$\left. + \frac{R_2}{R_1} [(1+r_2) e^{-2(y-x)} - 2s_2 e^{-(y-x)} e^{-s_2(y-x)} - (1-s_2)] \right\} \frac{1}{\nabla_2}$$

$$\nabla_2 = 2s_2 e^{-(y-x)} e^{-s_2(y-x)} - (1+r_2) [(1+r_2+R_3) e^{-2(y-x)} + (1-s_2-R_3) e^{-2s_2(y-x)}]$$

$$+ (1-s_2) [(1-s_2+R_3) + (1-s_2-R_3) e^{-2(y-x)} e^{-2s_2(y-x)}]$$

$$\text{For } m=\infty, \quad \nabla_2 = (1-s_2)(1-s_2+R_3)$$

We transform the  $w$ -condition utilizing the  $\tau_{r2}$ -conditions

$$\begin{aligned} s_2(n_2-1)C_2 - s_2(n_2-1)D_2 e^{-s_2(y-x)} \\ = k_1 \left[ s_3(n_3-1)C_3 e^{-s_3(z-y)} - s_3(n_3-1)D_3 \right] \end{aligned}$$

$$k = k_1 \frac{s_3(n_3-1)}{s_2(n_2-1)}$$

$$C_2 - D_2 e^{-s_2(y-x)} = k C_3 e^{-s_3(z-y)} - k D_3$$

We replace  $C_2$  and  $D_2$  by their values

$$\begin{aligned} C_2 - D_2 e^{-s_2(y-x)} = \\ \left\{ [A_3] \left[ (1+s_2-R_2) e^{-2s_2(y-x)} + (1-s_2+R_2) \right. \right. \\ \left. \left. - (1+s_2+R_2) e^{-2(y-x)} - (1-s_2-R_2) e^{-2(y-x)-2s_2(y-x)} \right] \right. \\ \left. + \frac{R_2}{R_1} \left[ 2e^{-s_2(y-x)} - 2s_2 e^{-(y-x)} - 2e^{-2(y-x)} e^{-s_2(y-x)} + 2s_2 e^{-(y-x)-2s_2(y-x)} \right] \right\} \cdot \frac{1}{\nabla_2} \\ = \left\{ [A_3] Q_1 + Q_2 \right\} \frac{1}{\nabla_2} \end{aligned}$$

$$\text{For } m = \infty, \quad Q_1 = (1-s_2+R_2), \quad Q_2 = 0$$

Writing  $Q_3 = \frac{k \nabla_2}{Q_1}$ , we obtain the system

$$A_3 e^{-(z-y)} + B_3 + C_3 e^{-s_3(z-y)} + D_3 - Q_3 C_3 e^{-s_3(z-y)} + Q_3 D_3 = -\frac{Q_2}{Q_1}$$

$$A_3 e^{-(z-y)} - B_3 + s_3 C_3 e^{-s_3(z-y)} - s_3 D_3 = 0$$

and by adding and subtracting

$$2A_3 e^{-(z-y)} + (1+s_3-Q_3) C_3 e^{-s_3(z-y)} + (1-s_3+Q_3) D_3 = -\frac{Q_2}{Q_1}$$

$$2B_3 + (1-s_3-Q_3) C_3 e^{-s_3(z-y)} + (1+s_3+Q_3) D_3 = -\frac{Q_2}{Q_1}$$

### 3. Resolution of the system of boundary conditions.

We have from the boundary conditions at the third interface (§ 2.1)

$$A_3 = \frac{1}{2j_3(1-n_3)} [P_{11} \cdot B_4 + P_{12} \cdot D_4]$$

$$B_3 = \frac{1}{2j_3(1-n_3)} [P_{21} \cdot B_4 + P_{22} D_4] e^{(2-\gamma)}$$

$$C_3 = \frac{1}{2j_3(1-n_3)} [P_{31} \cdot B_4 + P_{32} D_4]$$

$$D_3 = \frac{1}{2j_3(1-n_3)} [P_{41} \cdot B_4 + P_{42} D_4] e^{j_3(z-\gamma)}$$

We replace  $A_3$ ,  $B_3$ ,  $C_3$  and  $D_3$  in the last equations of § 2.4

$$\begin{aligned} & [2P_{11} e^{-(z-\gamma)} + (1+j_3-\theta_3) P_{31} e^{-j_3(z-\gamma)} + (1-j_3+\theta_3) P_{41} e^{j_3(z-\gamma)}] B_4 \\ & + [2P_{12} e^{-(z-\gamma)} + (1+j_3-\theta_3) P_{32} e^{-j_3(z-\gamma)} + (1-j_3+\theta_3) P_{42} e^{j_3(z-\gamma)}] D_4 \\ & = -2j_3(1-n_3) \frac{\theta_2}{\theta_1} \end{aligned}$$

$$\begin{aligned} & [2P_{21} e^{(2-\gamma)} + (1-j_3-\theta_3) P_{31} e^{-j_3(z-\gamma)} + (1+j_3+\theta_3) P_{41} e^{j_3(z-\gamma)}] B_4 \\ & + [2P_{22} e^{(2-\gamma)} + (1-j_3-\theta_3) P_{32} e^{-j_3(z-\gamma)} + (1+j_3+\theta_3) P_{42} e^{j_3(z-\gamma)}] D_4 \\ & = -2j_3(1-n_3) \frac{\theta_2}{\theta_1} \end{aligned}$$

We solve the system in  $B_4$  and  $D_4$

$$B_4 = -2j_3(1-n_3) \frac{\theta_2}{\theta_1} \left[ \begin{aligned} & -2P_{12} e^{-(z-\gamma)} + 2P_{22} e^{(2-\gamma)} \\ & -2j_3 P_{32} e^{-j_3(z-\gamma)} + 2j_3 P_{42} e^{j_3(z-\gamma)} \end{aligned} \right] \frac{1}{2e^{(2-\gamma)} e^{j_3(z-\gamma)} \nabla_3}$$

$$B_4 = 2s_3(1-n_3) \frac{Q_2}{Q_1} \left[ P_{12} e^{-2(z-y)} e^{-s_3(z-y)} - P_{22} e^{-s_3(z-y)} \right. \\ \left. + s_3 \cdot P_{32} e^{-(z-y)} e^{-2s_3(z-y)} - s_3 P_{42} e^{-(z-y)} \right] \cdot \frac{1}{V_3}$$

The positive exponents  $e^{(z-y)}$ ,  $e^{s_3(z-y)}$  and  $e^{(z-y)} \cdot e^{s_3(z-y)}$  have disappeared.

$$D_4 = 2s_3(1-n_3) \frac{Q_2}{Q_1} \left[ P_{21} e^{-s_3(z-y)} - P_{11} e^{-2(z-y)} e^{-s_3(z-y)} \right. \\ \left. - s_3 P_{41} e^{-(z-y)} - s_3 P_{31} e^{-(z-y)} e^{-2s_3(z-y)} \right] \cdot \frac{1}{V_3}$$

$$V_3 = 2 \left[ (P_{11} \cdot P_{22} - P_{12} \cdot P_{21}) + s_3 (P_{31} \cdot P_{42} - P_{32} \cdot P_{41}) \right] e^{-(z-y)} e^{-s_3(z-y)} \\ + (1 + s_3 + Q_3) (P_{11} P_{42} - P_{12} P_{41}) e^{-2(z-y)} \\ + (1 + s_3 - Q_3) (P_{31} P_{22} - P_{32} P_{21}) e^{-2s_3(z-y)} \\ + (1 - s_3 - Q_3) (P_{11} P_{32} - P_{12} P_{31}) e^{-2(z-y)} e^{-2s_3(z-y)} \\ + (1 - s_3 + Q_3) (P_{22} P_{41} - P_{21} P_{42})$$

$$\text{For } m = \infty \quad V_3 = [(1 - s_3) + k(1 - s_2)] (P_{22} P_{41} - P_{21} P_{42})$$

The numerators of  $B_4$  and  $D_4$  tend both to zero and the denominator tends to a constant value.

#### 4. Values of the parameters $A_i, D_i$ .

##### 4.1. Values of the parameters $A_3, B_3, C_3, D_3$ .

The values of the parameters  $A_3, B_3, C_3$  and  $D_3$  are obtained from the boundary conditions at the third interface in which  $B_4$  and  $D_4$  are replaced by their values from § 3.

$$\begin{aligned}
 A_3 &= [P_{11} B_4 + P_{12} D_4] \cdot \frac{1}{2s_3(1-h_3)} \\
 &= \frac{Q_2}{Q_1} \left[ (P_{12} \cdot P_{31} - P_{11} \cdot P_{22}) \cdot e^{-s_3(z-y)} + s_3 (P_{11} P_{32} - P_{12} P_{31}) e^{-(z-y)} e^{-2s_3(z-y)} \right. \\
 &\quad \left. + s_3 (P_{12} \cdot P_{41} - P_{11} \cdot P_{42}) e^{-(z-y)} \right] \frac{1}{V_3} \\
 B_3 &= [P_{21} B_4 + P_{22} D_4] \cdot \frac{e^{(z-y)}}{2s_3(1-h_3)} \\
 &= \frac{Q_2}{Q_1} \left[ (P_{12} \cdot P_{21} - P_{11} \cdot P_{22}) e^{-2(z-y)} \cdot e^{-s_3(z-y)} \right. \\
 &\quad \left. + s_3 (P_{21} \cdot P_{32} - P_{22} \cdot P_{31}) e^{-(z-y)} e^{-2s_3(z-y)} + s_3 (P_{22} P_{41} - P_{21} P_{42}) e^{-(z-y)} \right] \frac{e^{(z-y)}}{V_3} \\
 &= \frac{Q_2}{Q_1} \left[ (P_{12} \cdot P_{21} - P_{11} \cdot P_{22}) e^{-(z-y)} e^{-s_3(z-y)} \right. \\
 &\quad \left. + s_3 (P_{21} \cdot P_{32} - P_{22} \cdot P_{31}) e^{-2s_3(z-y)} + s_3 (P_{22} P_{41} - P_{21} P_{42}) \right] \frac{1}{V_3}
 \end{aligned}$$

The positive exponent  $e^{(z-y)}$  has disappeared. Although the presence of the constant  $s_3(P_{22}P_{41} - P_{21}P_{42})$  the numerator converges to zero because of the factor  $Q_2$ .

$$\begin{aligned}
 C_3 &= [P_{31} B_4 + P_{32} D_4] \frac{1}{2s_3(1-h_3)} \\
 &= \frac{Q_2}{Q_1} \left[ (P_{31} \cdot P_{12} - P_{32} \cdot P_{11}) e^{-2(z-y)} \cdot e^{-s_3(z-y)} + (P_{32} \cdot P_{21} - P_{31} P_{22}) e^{-s_3(z-y)} \right. \\
 &\quad \left. + s_3 (P_{32} P_{41} - P_{31} P_{42}) e^{-(z-y)} \right] \frac{1}{V_3}
 \end{aligned}$$

$$D_3 = \left[ P_{41} \cdot B_4 + P_{42} \cdot D_4 \right] \frac{e^{s_3(z-y)}}{2s_3(1-h_3)}$$

$$= \frac{U_2}{Q_1} \left[ (P_{41} \cdot P_{12} - P_{11} \cdot P_{42}) e^{-2(z-y)} + (P_{42} \cdot P_{21} - P_{41} \cdot P_{22}) \right. \\ \left. + s_3 [P_{41} \cdot P_{32} - P_{31} \cdot P_{42}] e^{-(z-y)} e^{-s_3(z-y)} \right] \frac{1}{V_3}$$

#### 4.2. Values of the parameters $A_2, B_2, C_2, D_2$ .

The values of  $C_2$  and  $D_2$  are obtained from the relations established in § 2.4.

$$C_2 = \frac{1}{V_2} \left\{ \left[ 2s_2 e^{-(y-x)} \cdot e^{-s_2(y-x)} - (1+s_2+R_3) e^{-2(y-x)} + (1-s_2+R_3) \right] \cdot \right. \\ \left. [A_3 e^{-(z-y)} + B_3 + C_3 e^{-s_3(z-y)} + D_3] \right. \\ \left. + \frac{R_2}{R_1} \left[ (1+s_2) e^{-s_2(y-x)} - 2s_2 e^{-(y-x)} - (1-s_2) e^{-2(y-x)} e^{-s_2(y-x)} \right] \right\}$$

$$D_2 = \frac{1}{V_2} \left\{ \left[ 2s_2 e^{-(y-x)} - (1+s_2-R_3) e^{-s_2(y-x)} + (1-s_2-R_3) e^{-2(y-x)} \cdot e^{-s_2(y-x)} \right] \cdot \right. \\ \left. [A_3 e^{-(z-y)} + B_3 + C_3 e^{-s_3(z-y)} + D_3] \right. \\ \left. + \frac{R_2}{R_1} \left[ (1+s_2) e^{-2(y-x)} - 2s_2 e^{-(y-x)} e^{-s_2(y-x)} - (1-s_2) \right] \right\}$$

The numerator in  $D_2$  converges because of the presence of the factor  $R_2$ .

The value of  $A_2$  is obtained from the relation established in § 2.4.

$$A_2 = \frac{1}{2} \left[ A_3 e^{-(z-y)} + B_3 + C_3 e^{-s_3(z-y)} + D_3 - (1+s_2) C_2 - (1-s_2) D_2 e^{s_2(y-x)} \right]$$

The value of  $B_2$  is obtained from the last relation of § 2.3.

$$B_2 = -\frac{1}{2} \left[ (1-s_2-R_3) C_2 e^{-s_2(y-x)} + (1+s_2+R_3) D_2 + \frac{R_2}{R_1} \right]$$

#### 4.3. Values of the parameters $A_1, B_1, C_1, D_1$ .

The values of the parameters  $A_1$  and  $C_1$  are obtained from the relations established in § 2.3.

$$C_1 = \frac{1}{\nabla_1} \left\{ \begin{aligned} & [2s_1 e^{-x} e^{-s_1 x} - (1+s_1) e^{-2x} + (1-s_1)] \cdot \\ & [A_2 e^{-(y-x)} + B_2 + C_2 e^{-s_2(y-x)} + D_2] \\ & + [2s_1 e^{-x} - (1+s_1) e^{s_1 x} + (1-s_1) e^{-2x} e^{s_1 x}] \end{aligned} \right\}$$

For the determination of  $A_1$ , we need the value of  $D_1 e^{-s_1 x}$ .

$$D_1 e^{-s_1 x} = \frac{1}{\nabla_1} \left\{ \begin{aligned} & [2s_1 e^{-x} e^{-s_1 x} - (1+s_1) e^{-2s_1 x} + (1-s_1) e^{-2x} e^{-2s_1 x}] \\ & [A_2 e^{-(y-x)} + B_2 + C_2 e^{-s_2(y-x)} + D_2] \\ & + [2s_1 e^{-x} e^{-2s_1 x} - (1+s_1) e^{-2x} e^{-s_1 x} + (1-s_1) e^{s_1 x}] \end{aligned} \right\}$$

$$A_1 = \frac{1}{2} [A_2 e^{-(y-x)} + B_2 + C_2 e^{-s_2(y-x)} + D_2 - (1+s_1) C_1 - (1-s_1) D_1 e^{s_1 x}]$$

The values of  $B_1$  and  $D_1$  are obtained from the surface conditions.

$$B_1 = \frac{1}{s_1 - 1} [s_1 - (1+s_1) A_1 e^{-x} - 2s_1 C_1 e^{-s_1 x}]$$

$$D_1 = -\frac{1}{s_1 - 1} [1 - 2A_1 e^{-x} - (1+s_1) C_1 e^{s_1 x}]$$

#### 5. Relations for the stresses and the displacements.

The relations for the stresses and the displacements are completely the same as those developed in appendix 3, by replacing the parameters  $A_i, D_i$  by their adequate values.

The relation for the vertical displacement is again undetermined at the origin ( $m = 0$ ). The problem is solved in exactly the same way as developed in appendix 2 (§ 4.).

APPENDIX 5

EXPLANATORY NOTICE



\*\*\*\*\*

FLxy5.FOR	---> subroutines	BESJ1 BJOJ2 SURFA COUC1 COUC2 COUC3 COUC4
FLxy6.FOR	---> subroutines	TITRE MODIF SIMPS
FLxy7.FOR	---> subroutines	COMSU FONC CCOU1 FONC1 FONC2 FONC3 FONC4 FONC5 FONC6
FLxy8.FOR	---> subroutines	ECHEF IMDON IMRES
FLxy9.FOR	---> subroutines	MENU ECDON LEDON AFDMC AFDTR AFPOC

#### Configuration:

=====

IBM-PC (G,XT,AT) with at least 256KB, 1 ou 2 diskette drives,  
80 col. screen (monochrome or color), math. coprocessor,  
matrix printer.

#### Running of the program:

=====

In the next presentation <ENTER> means action of key <---> .  
Insert the EXECUTABLE disk into drive B.  
Load DOS, if necessary, then type DIR B: and PATH A:\  
FLB> FLxy <ENTER>

#### Printing of the results:

=====

on screen : FLB> type FLxy.LST <ENTER>  
on printer : FLB> type FLxy.LST >prn <ENTER>

#### Consultation of this notice:

=====

on screen : FLB> type FLxyNO.TXT <ENTER>  
on printer : FLB> type FLxyNO.TXT >prn <ENTER>

\*\*\*\*\*

## Disk preparation for normal running:

=====

Format a system disk with COMMAND.COM, and files needed for running AUTOEXEC.BAT.

Preparation of CONFIG.SYS with files=10, device=ansi, buffers=10

## Alterations in source files:

=====

By text editor EDLIN or any other text editor, program statements, may be altered.

## Commands for EDLIN are:

FLB> edlin xxxxn.for <ENTER>

where ---> file name to be modified (xxxxn.for)

nD --> erase line number n

n --> displays line n for alteration, type the correct statement.

n1,n2L --> displays lines between number n1 and number n2

E --> ends session and returns to DOS

## Compilation:

=====

After alteration, the new version of the module has to be compiled.

Insert Professional fortran compiler into drive A.

FLA> b: <ENTER>

FLB> path a:\ <ENTER>

FLB> profort xxxx/li <ENTER>

## Linkage:

=====

After correct compilation has taken place, a new executable program has to be created.

Insert diskette with FORTRAN libraries into drive A.

FLB> link FLxy1+FLxy2+...,FLxy,CON:; <ENTER>

## Note :

=====

To obtain introduction logo on printer

FLB> type FLxyND.LOG >PRN <ENTER>

\*\*\*\*\*

Presentation of different possible screens:

=====

FD-USAE-vers. 2.00 1986

MAIN MENU screen :

=====

Strains and stresses  
in  
a four-layered system.

USAE-2

\*\*\*\*\*

1. Data retrieval in a file
2. Data saving in a file
3. Screen displaying and/or alteration of system data
4. Screen displaying and/or alteration of traffic data
5. Screen displaying and/or alteration of computation coordinates
6. Input of intermediate depths
7. Program start

Your choice ----- 0 for stop -- :

-----

Screen of CHOICE 1:

=====

Strains and stresses  
in  
a four-layered system.

USAE-2

\*\*\*\*\*

Data retrieval in a file.

-----

A FLxy.DAT file contains base data.  
The user can define another file whose name has to be written in  
8 characters (format XXXX.DAT).

Name of chosen file or FLxy.DAT :

-----

\*\*\*\*\*

Screen of CHOICE 2:

=====

Strains and stresses

in

USAE-2

a four-layered system.

\*\*\*\*\*

Data saving in a file.

-----

A FLxy.DAT file contains base data.

The user can define another file whose name has to be written in 8 characters (format XXXX.DAT).

Name of chosen file or FLxy.DAT :

-----

Screen of CHOICE 3:

=====

Strains and stresses

in

USAE-2

a four-layered system.

\*\*\*\*\*

System data.

=====

Layer	Modulus	Poisson's r.	Thickness	Friction ra.
1.	400000.0	0.16	20.0	1.0000
2.	100000.0	0.25	20.0	1.0000
3.	10000.0	0.50	30.0	1.0000
4.	1000.0	0.50	9000.0	1.0000

For return =1 or alter. =2 ---- :

-----

\*\*\*\*\*

Screen of CHOICE 4:

=====

Strains and stresses

in

USAE-2

a four-layered system.

\*\*\*\*\*

Number of circular loads = 2

Radius	Pressure	Dist. x	Dist. y
-----	-----	-----	-----
11.450	7.900	0.000	0.000
11.450	7.900	34.350	0.000

For continuation = 1 or alter. = 2 ---- :

-----

Screen of CHOICE 5:

=====

Strains and stresses

in

USAE-2

a four-layered system.

\*\*\*\*\*

Number of computation coordinates = 2

x=	y=
0.000	0.000
17.175	0.000

For continuation= 1 or alter. = 2 ---- :

-----

Screen of CHOICE 6:

=====

Strains and stresses

in

USAE-2

a four-layered system.

\*\*\*\*\*

Positions of stress computations in depth out of interfaces.

(max 22)

number of positions :

-----

\*\*\*\*\*

Screen of CHOICE 7:

=====

Strains and stresses

in

USAE-2

a four-layered system.

\*\*\*\*\*

initial computation interval (0.1 is generally small enough) :

-----

Next screen:

=====

Strains and stresses

in

USAE-2

a four-layered system.

\*\*\*\*\*

choice of scale :

\*\*\*\*\*

allowed choices :

- 1 ---- thickness of 1st layer
- 2 ---- thickness of 2 first layers
- 3 ---- thickness of 3 first layers
- 4 ---- thickness of 4 layers
- 5 ---- load radius

suggested solution : 5

your choice :

-----

Screen for execution:

=====

Strains and stresses

in

USAE-2

a four-layered system.

\*\*\*\*\*

computation start..... be patient!!

m = 0.10

be even more patient...!!

-----

\*\*\*\*\*

Screen for results:

=====

Strains and stresses

in

USAE-2

a four-layered system.

\*\*\*\*\*

A file FLxy.LST contains base results.

The user can define another file whose name has to be written in  
8 characters (format XXXX.DAT).

Name of chosen file or FLxy.LST :

-----

sample of results (FLxy.LST)

four-layered system program isotropic - partial friction

USAE-2

departement des constructions

isic  
av.de l'hopital, 27 h  
7000 Mons belgique

mechanical data  
\*\*\*\*\*

Young's modulus	P.'s ratio	thickness	friction ratio
400000.0	0.16	20.000	1.00
100000.0	0.25	20.000	1.00
10000.0	0.50	30.000	1.00
1000.0	0.50	9000.000	1.00

traffic data  
\*\*\*\*\*

load	radius	pressure	x	y
1	11.450	7.900	0.000	0.000
2	11.450	7.900	34.350	0.000





Utilized symbols:

sx : normal stress in the x-direction  
sy : normal stress in the y-direction  
sz : normal stress in the z-direction  
tyz : shear stress in the yz plane, parallel to y or z  
txz : shear stress in the xz plane, parallel to x or z  
txy : shear stress in the xy plane, parallel to x or y  
s1 : maximum principal stress  
s2 : medium principal stress  
s3 : minimum principal stress  
eps1 : principal strain  
eps2 : principal strain  
eps3 : principal strain  
u(x) : displacement in the x-direction  
v(y) : displacement in the y-direction  
w(z) : displacement in the z-direction  
ex : strain in the x-direction  
ey : strain in the y-direction  
ez : strain in the z-direction

The normal stresses are taken positive when they produce compression and negative when they produce tension.

END  
FILMED

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DTIC