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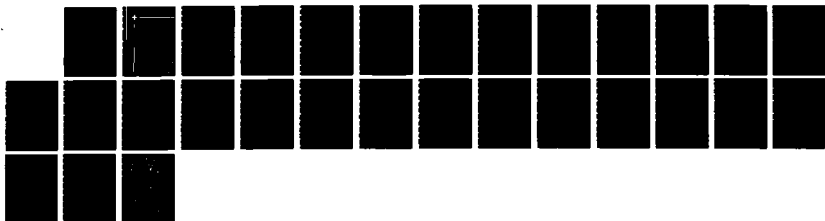
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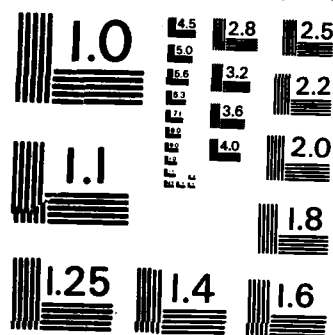
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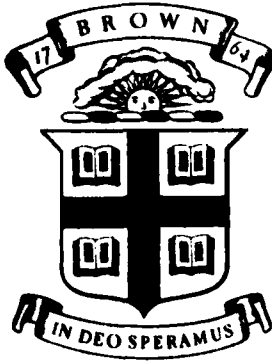
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THE STRESS INTENSITY FACTOR HISTORY
DUE TO THREE DIMENSIONAL TRANSIENT LOADING
OF THE FACES OF A CRACK

by

L.B. Freund
Division of Engineering
Brown University
Providence, RI 02912 USA

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NATIONAL SCIENCE FOUNDATION
Grant MSM-85 13096

OFFICE OF NAVAL RESEARCH
Contract N00014-85-K-0597

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L. B. Freund
Division of Engineering
Brown University
Providence, RI 02912 USA

ABSTRACT

A procedure is described for determining dynamic stress intensity factor histories for a half plane crack in an otherwise unbounded elastic body, with the crack faces subjected to tractions that result in variation of the stress intensity factor along the crack edge. The procedure is based on integral transform methods and the properties of analytic functions of a complex variable. The procedure is illustrated for the case of a pair of opposed line loads suddenly applied on the crack faces along a line perpendicular to the crack edge. An exact expression is obtained for the resulting mode I stress intensity factor as a function of time for any point along the crack edge. Some features of the solution, as well as possible extensions of the procedure, are discussed.

Key words: boundary value problems; three dimensional



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1. INTRODUCTION

In this article, a procedure is described for determining stress intensity factor histories for a class of three dimensional elastodynamic crack problems. The geometrical configuration studied is a half plane crack in an otherwise unbounded solid, with the crack faces subjected to tractions that result in variation of the transient stress intensity factors along the crack edge. Three dimensional phenomena have received relatively little attention in dynamic fracture mechanics. The reflection of Rayleigh waves travelling on the face of a crack and obliquely incident on the crack edge has been studied (Freund, 1971). Because the fields are steady as seen by an observer moving along the crack edge at an appropriate speed, problems of this type may be reduced to two dimensional problems. This general approach was also followed by Achenbach and Gautesen (1977) in their study of oblique reflection of plane harmonic waves from the edge of a crack. The latter authors went on to study harmonic three dimensional loading of the crack faces, obtaining some asymptotic results for very large or very small frequencies. In some respects, the present analysis is similar to the work of Achenbach and Gautesen (1977). The main differences are that the present analysis allows for transient loading and the results are exact.

In section 2, the boundary value problem is stated for general three dimensional crack face tractions that result in the opening mode of crack edge deformation, or mode I. The general approach to be followed in extracting three dimensional stress intensity factors is outlined in section 3. In section 4, attention is restricted to the case of a pair of line loads that are suddenly applied to the crack faces along a line *perpendicular* to the crack edge, and the results for this case are presented in section 5. Some possible extensions of the procedure are mentioned in section 6.

2. FORMULATION

Consider the elastic body containing a half plane crack depicted in Fig. 1. The body is initially stress free and at rest. The material is characterized by the shear modulus μ , the Poisson ratio ν , and the mass density ρ . The speeds of dilatational, shear and Rayleigh waves are c_d , c_s and c_r , respectively. At a certain instant, tractions begin to act on the crack faces, resulting in a three dimensional stress wave field in the material. The purpose is to determine exact expressions for stress intensity factors as functions of time and position along the crack edge. To illustrate the approach, the same transient normal pressure distribution is applied to each crack face, and the tangential or shear traction components are zero. For this type of loading, the mode of deformation is mode I for each point along the crack edge. Next, a mathematical formulation of a problem corresponding to the foregoing qualitative description is given.

A right handed rectangular coordinate system is introduced in the body, oriented so that the z axis coincides with the crack edge, and the half plane crack occupies $y = 0$ for $x < 0$. The normal traction on the crack faces begins to act at time $t = 0$. For the time being, consider some general traction variation, say $T_y(x, z, t) = \mp \sigma_-(x, z, t)$ on $y = \pm 0$, where $\sigma_-(x, z, t)$ is a given function of position on the faces and of time, and $\sigma_- > 0$ corresponds to tensile traction. The subscript minus sign is intended to imply that the function is defined for $x < 0$. The other components of traction are zero, that is, $T_x(x, z, t) = T_z(x, z, t) = 0$.

In view of the symmetry of the configuration and of the applied traction, the wave fields are expected to have reflective symmetry with respect to the plane $y = 0$, and attention may be limited to the region $y \geq 0$. The displacement fields satisfy $u_x(x, -y, z, t) = u_x(x, y, z, t)$, $u_y(x, -y, z, t) = -u_y(x, y, z, t)$, and $u_z(x, -y, z, t) = u_z(x, y, z, t)$. These properties, in turn, imply the symmetry conditions $\sigma_{xy}(x, 0, z, t) =$

0, $\sigma_{yz}(x, 0, z, t) = 0$ and $u_y(x, 0, z, t) = 0$ for $x > 0$ and for all z, t . Thus, the complete set of boundary conditions to be satisfied by the stress wave fields is

$$\begin{aligned}\sigma_{yy}(x, 0, z, t) &= \sigma_-(x, z, t) + \sigma_+(x, z, t) \\ \sigma_{xy}(x, 0, z, t) &= 0 \\ \sigma_{yz}(x, 0, z, t) &= 0 \\ u_y(x, 0, z, t) &= u_-(x, z, t)\end{aligned}\tag{2.1}$$

for $-\infty < x, z < \infty$ and $0 \leq t$. The definition of the function σ_- , which describes the imposed traction, is extended so that $\sigma_- \equiv 0$ for all $x > 0$. The function σ_+ represents the unknown normal component of stress σ_{yy} on $x > 0$ and $\sigma_+ \equiv 0$ for $x < 0$. Likewise, u_- is the unknown y component of displacement of material particles on the crack faces for $x < 0$ and $u_- \equiv 0$ for all $x > 0$.

For this system, the normal stress on the plane $y = 0$ ahead of the crack edge is expected to have the form

$$\sigma_+(x, z, t) = \frac{k_I(t, z)}{\sqrt{2\pi x}} + O(1) \quad \text{as } x \rightarrow 0^+ \tag{2.2}$$

where $k_I(t, z)$ is the stress intensity factor history at any point z along the crack edge. The main purpose here is to determine this history.

The Helmholtz representation of the displacement vector $\vec{u} = \nabla\phi + \nabla \times \vec{\psi}$ is introduced, where ∇ is the three dimensional gradient operator, ϕ is the scalar dilatational wave potential, and $\vec{\psi}$ is the vector shear wave potential. The wave potential functions are governed by the partial differential equations

$$c_d^2 \nabla^2 \phi - \phi_{,tt} = 0 \quad c_s^2 \nabla^2 \vec{\psi} - \vec{\psi}_{,tt} = 0 \quad \nabla \cdot \vec{\psi} = 0 \tag{2.3}$$

in $y > 0$. The conditions that the material is stress free and at rest everywhere for $t \leq 0$ are expressed in terms of the potential functions by

$$\phi(x, y, z, 0) = \frac{\partial \phi(x, y, z, 0)}{\partial t} = \vec{\psi}(x, y, z, 0) = \frac{\partial \vec{\psi}(x, y, z, 0)}{\partial t} = 0 \quad (2.4)$$

in $y > 0$. Finally, it is noted that the boundary conditions (2.1) are interpreted as though the stress and displacement components were replaced by their expressions in terms of the potentials ϕ and $\vec{\psi}$.

3. SOLUTION PROCEDURE

The mathematical problem described by the differential equations (2.3), the initial conditions (2.4), and the boundary conditions (2.1) is linear, and integral transforms, along with certain powerful theorems from the theory of analytic functions of a complex variable, are used to extract stress intensity factor histories. In some respects, the procedure is similar to that introduced by deHoop (1958) in his original analysis of two dimensional systems. Certain interpretations of the transformed fields at key points in the analysis, however, make it possible to extract exact results for this class of three dimensional problems. The first step is to apply the one sided Laplace transform on time to the differential equations (2.3) and the boundary conditions (2.1), taking into account the initial conditions (2.4). The transform parameter is s and the transform of any function, say $\phi(x, y, z, t)$, is denoted by a superposed hat, that is,

$$\hat{\phi}(x, y, z, s) = \int_0^{\infty} \phi(x, y, z, t) e^{-st} dt \quad (3.1)$$

The parameter s is considered to be a positive real parameter for the time being. Next, a two sided Laplace transform (which is equivalent to the more common Fourier transform but which results in slightly less cumbersome algebraic expressions) is applied to suppress the dependence on z . The complex transform parameter is $s\zeta$, and the transform of any function, say $\hat{\phi}(x, y, z, s)$, is denoted by the corresponding upper case symbol with a superposed hat,

$$\hat{\Phi}(x, y, \zeta, s) = \int_{-\infty}^{\infty} \hat{\phi}(x, y, z, s) e^{-s\zeta z} dz \quad (3.2)$$

Suppose that σ_- vanishes for $|z| > z_0$. Then it is expected that

$$\hat{\phi}(x, y, z, s) = o\left(e^{-s(\pm z - z_0 - \epsilon)/c_d}\right) \quad (3.3)$$

as $z \rightarrow \pm\infty$ for any $\epsilon > 0$. This condition, in turn, implies that the transform integral in (3.2) converges for $-a < \text{Re}(\zeta) < a$ where $a = 1/c_d$. Consequently, the integral defines an analytic function in this strip of the complex ζ plane. From the property of uniqueness of an analytic function in its domain of analyticity (Hille, 1959, p. 199), the analytic function in the strip is completely specified by the function defined only on the portion of the real axis in the strip $-a < \text{Re}(\zeta) < a$. Furthermore, it turns out to be of great advantage to restrict the range of ζ to this real interval in proceeding with the analysis. At an appropriate later stage, the definition of the functions of ζ may be continued into the entire complex ζ plane by invoking the property of uniqueness of an analytic function.

Finally, the two sided Laplace transform suppressing dependence on x is applied. The complex transform parameter is $s\xi$, and the transform of any function, say $\hat{\Phi}(x, y, \zeta, s)$, is denoted by the same upper case symbol but without the superposed hat, $\Phi(\xi, y, \zeta, s)$. To be able to continue with the solution procedure, some statement on the domain of convergence of the transform integrals over x is required. If the applied loading is nonzero for indefinitely large distance from the crack edge along the crack faces, then it may be concluded that the integral over negative values of x in the definition of $\Phi(\xi, y, \zeta, s)$ will converge for $\text{Re}(\xi) < 0$. For $x > 0$, the wave fields are zero beyond a cylindrical wavefront expanding from the y axis at speed c_d with instantaneous radius $z_0 + \sqrt{x^2 + z^2}$. To be more specific, consider the elementary wave field $\phi(x, y, z, t) = H(c_d t - z_0 - \sqrt{x^2 + z^2})$, where H denotes the unit step function. If the Laplace transform integrals over t , then z , and then x are formed, it is easily demonstrated that the triple integral converges if $\{\text{Re}(\xi)\}^2 + \zeta^2 < a^2$. This condition, when coupled with the convergence condition $\text{Re}(\xi) < 0$ suggested above, provides the basis for expecting that the transforms will be analytic in the strip $-\sqrt{a^2 - \zeta^2} < \text{Re}(\xi) < 0$ in the complex ξ plane, with ζ real and in the interval $-a < \zeta < a$.

It is not possible to consider arbitrary distributions of traction $\sigma_-(x, z, t)$ applied to the crack faces. From this point onward, attention is limited to those distributions for which the triple transform has the form

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\sigma}_-(x, z, s) e^{-s\zeta z - s\xi x} dz dx = \frac{1}{s^m} \Sigma_-(\xi, \zeta) \quad (3.4)$$

where m is a real number and $\Sigma_-(\xi, \zeta)$ does not depend on s . The condition (3.4) essentially restricts $\sigma_-(x, z, t)$ to be within a class of certain separable or homogeneous functions of x, z, t .

The transforms are applied to the partial differential equations (2.3) to reduce them to ordinary differential equations in y . The transformed boundary conditions (2.1) are imposed on the solutions of the ordinary differential equations to determine the unknown parameters of integration. As usual in a problem of this general type, there are more unknown functions than there are algebraic equations to determine them. However, certain of the unknown functions are sectionally analytic functions (of ξ in this case), and the Wiener-Hopf factorization (Noble, 1958, p. 11) makes it possible to complete the solution by determining two unknown functions from a single equation. Only a few of the intermediate steps are included here.

The solutions of the ordinary differential equations resulting from Laplace transformation of the partial differential equations in (2.3) that are bounded as $y \rightarrow \infty$ are

$$\Phi = \frac{1}{s^{m+2}} P(\xi, \zeta) e^{-s\alpha y}, \quad \bar{\Psi} = \frac{1}{s^{m+2}} \bar{Q}(\xi, \zeta) e^{-s\beta y} \quad (3.5)$$

where the parameters of integration P and $\bar{Q}$ are unknown functions of their arguments, the number m follows from (3.4), and

$$\alpha = \alpha(\xi, \zeta) = (a^2 - \zeta^2 - \xi^2)^{1/2}, \quad \beta = \beta(\xi, \zeta) = (b^2 - \zeta^2 - \xi^2)^{1/2} \quad (3.6)$$

where $a = 1/c_d$ and $b = 1/c_s$. The complex ξ plane is cut along $\sqrt{a^2 - \zeta^2} < |\operatorname{Re}(\xi)| < \infty$, $\operatorname{Im}(\xi) = 0$ so that $\operatorname{Re}(\alpha) \geq 0$ in the entire cut ξ plane for each value of ζ , and likewise for β . The transformation of the condition $\nabla \cdot \vec{\psi} = 0$ yields

$$\xi Q_x - \beta Q_y + \zeta Q_z = 0 \quad (3.7)$$

where Q_x, Q_y, Q_z are the rectangular components of $\vec{Q}$. The relation (3.7) is obviously a linear algebraic equation for the unknown functions. The Laplace transformation of the boundary conditions provides four additional linear algebraic equations for the six unknown functions P, Q_x, Q_y, Q_z, U_- and Σ_+ , where

$$\begin{aligned} U_-(\xi, \zeta) &= s^{m+1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{u}_-(x, z, s) e^{-s(\zeta z + \xi x)} dz dx \\ \Sigma_+(\xi, \zeta) &= s^m \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\sigma}_+(x, z, s) e^{-s(\zeta z + \xi x)} dz dx. \end{aligned} \quad (3.8)$$

These four equations are

$$\begin{aligned} (b^2 - 2\xi^2 - 2\zeta^2)P - 2\zeta\beta Q_x + 2\xi\beta Q_z &= \mu^{-1}(\Sigma_- + \Sigma_+) \\ -2\xi\alpha P + \zeta\xi Q_x + \zeta\beta Q_y + (b^2 - \zeta^2 - 2\xi^2)Q_z &= 0 \\ 2\zeta\alpha P + (b^2 - 2\zeta^2 - \xi^2)Q_x + \xi\beta Q_y + \zeta\xi Q_z &= 0 \\ -\alpha P + \zeta Q_x - \xi Q_z &= U_-. \end{aligned} \quad (3.9)$$

If P, Q_x, Q_y, Q_z are eliminated from the five equations (3.7) and (3.9), the result is a single equation involving the two remaining unknown functions U_- and Σ_+ , namely,

$$-\frac{\mu}{b^2} \frac{R(\xi, \zeta)}{\alpha(\xi, \zeta)} U_-(\xi, \zeta) = \Sigma_-(\xi, \zeta) + \Sigma_+(\xi, \zeta) \quad (3.10)$$

where $\Sigma_-(\xi, \zeta)$ is given, and

$$R(\xi, \zeta) = 4(\xi^2 + \zeta^2)\alpha(\xi, \zeta)\beta(\xi, \zeta) + (b^2 - 2\zeta^2 - 2\xi^2)^2 \quad (3.11)$$

The function defined in (3.11) is recognized as a modified form of the Rayleigh wave function. In particular, it has zeros in the complex ξ plane at $\xi = \pm\sqrt{c^2 - \zeta^2}$, where $c = 1/c_r$ is defined by $R(\pm c, 0) = 0$, and it has no other zeros in the cut plane. Note that (3.11) is identical to the standard Rayleigh wave function when $\zeta = 0$. Furthermore, Σ_+ is analytic in $\text{Re}(\xi) > -\sqrt{a^2 - \zeta^2}$ and U_- is analytic in $\text{Re}(\xi) < 0$. The equation (3.10) holds in the strip $-\sqrt{a^2 - \zeta^2} < \text{Re}(\xi) < 0$. Thus, for any fixed value of ζ , (3.10) may be solved by factorization in much the same way that an equation of the standard Wiener-Hopf type is solved. While the dependence on ζ must be taken into account in the solution procedure, it requires no special consideration, and it is simply understood to be a real parameter in the interval $-a < \zeta < a$ for the time being.

4. A PARTICULAR TRACTION DISTRIBUTION

To make further progress, it is necessary to choose a particular applied traction distribution. A case suitable for demonstrating the general approach is that of a uniform line load which suddenly begins to act along a line perpendicular to the crack edge. Thus, it is assumed that

$$\sigma_-(x, z, t) = -p^* \delta(z) H(-x) H(t) \quad (4.1)$$

where $H(\cdot)$ is the unit step function and $\delta(\cdot)$ is the Dirac delta function. The amplitude p^* has physical dimensions of (force)/(length), and $p^* > 0$ corresponds to traction that tends to separate the crack faces. Based on dimensional considerations alone, it is expected that this loading will produce a stress intensity factor history that depends on z and t according to

$$k_I(t, z) = p^* z^{-1/2} g(t/z) \quad (4.2)$$

Thus, the problem is to determine the dimensionless function $g(\cdot)$.

The ensemble of wavefronts that results from application of the traction specified by (4.1) includes a variety of space surfaces. Among the wavefronts are cylindrical dilatational and shear wavefronts centered on the load lines in $x > 0$, plus planar head wavefronts that intersect the dilatational wavefronts on the surfaces $y = \pm 0$ and terminate at the line of tangency with the cylindrical shear wavefronts. There are also spherical dilatational and shear wavefronts centered at the origin of coordinates, plus conical headwaves, each with its apex at a point where the spherical dilatational wave meets the z axis and its terminus along a circle of tangency with the spherical shear wavefront. There are also conical headwaves that intersect the spherical wavefront on the surfaces $y = \pm 0$ for $x < 0$ and extend to circles of tangency with the spherical shear wave.

If the relevant integral transforms indicated in (3.4) are applied to (4.1), it is found that $m = 2$ and that

$$\Sigma_-(\xi, \varsigma) = \frac{p^*}{\xi}. \quad (4.3)$$

The relationship between the sectionally analytic functions (3.10) becomes

$$-\frac{\mu\kappa}{b^2} \frac{(c^2 - \varsigma^2 - \xi^2)S(\xi, \varsigma)}{\alpha(\xi, \varsigma)} U_-(\xi, \varsigma) = \frac{p^*}{\xi} + \Sigma_+(\xi, \varsigma) \quad (4.4)$$

where the modified Rayleigh wave function $R(\xi, \varsigma)$ has been replaced by the function

$$S(\xi, \varsigma) = \frac{R(\xi, \varsigma)}{\kappa(c^2 - \varsigma^2 - \xi^2)}, \quad \kappa = 2(b^2 - a^2) \quad (4.5)$$

The function $S(\xi, \varsigma) \rightarrow 1$ as $|\xi| \rightarrow \infty$ and it has neither zeros nor poles in the finite ξ plane. The only singularities of $S(\xi, \varsigma)$ are the branch points at $\xi = \pm\sqrt{a^2 - \varsigma^2}$ and at $\xi = \pm\sqrt{b^2 - \varsigma^2}$ which are shared with $R(\xi, \varsigma)$, and it is single valued in the ξ plane cut along $\sqrt{a^2 - \varsigma^2} \leq |\operatorname{Re}(\xi)| \leq \sqrt{b^2 - \varsigma^2}$, $\operatorname{Im}(\xi) = 0$. The property that $S(\bar{\xi}, \varsigma) = \overline{S(\xi, \varsigma)}$ for the restricted range of ς may be exploited to show that

$$S_{\pm}(\xi, \varsigma) = \exp \left\{ -\frac{1}{\pi} \int_a^b \tan^{-1} \left(\frac{4\eta^2 \sqrt{(b^2 - \eta^2)(\eta^2 - a^2)}}{(b^2 - 2\eta^2)^2} \right) \frac{\eta d\eta}{\sqrt{\eta^2 - \varsigma^2} (\sqrt{\eta^2 - \varsigma^2} \pm \xi)} \right\} \quad (4.6)$$

where the overbar denotes complex conjugate. The functions $S_+(\xi, \varsigma)$ and $S_-(\xi, \varsigma)$ are analytic and nonzero in the half planes $\operatorname{Re}(\xi) > -\sqrt{a^2 - \varsigma^2}$ and $\operatorname{Re}(\xi) < \sqrt{a^2 - \varsigma^2}$, respectively.

With the factorization of $S(\xi, \varsigma)$ complete, the relationship (4.4) may be rewritten in the form

$$\frac{p^*}{\xi} \left(F_+(\xi, \varsigma) - F_+(0, \varsigma) \right) + F_+(\xi, \varsigma) \Sigma_+(\xi, \varsigma) = -\frac{p^* F_+(0, \varsigma)}{\xi} - \frac{\mu\kappa}{b^2} \frac{U_-(\xi, \varsigma)}{F_-(\xi, \varsigma)} \quad (4.7)$$

where

$$F_{\pm}(\xi, \varsigma) = \frac{(\sqrt{a^2 - \varsigma^2} \pm \xi)^{1/2}}{(\sqrt{c^2 - \varsigma^2} \pm \xi) S_{\pm}(\xi, \varsigma)} \quad (4.8)$$

The relationship (4.7) is valid in the strip $-\sqrt{a^2 - \varsigma^2} < \text{Re}(\xi) < 0$. Furthermore, each side of the equation is analytic in one of the overlapping half planes. Consequently, each side of (4.7) is the unique analytic continuation of the other into a complementary half plane, and together the two sides represent a single entire function $E(\xi)$, parametric in ς (Noble, 1958, p. 36). It remains to determine the entire function.

As $|\xi| \rightarrow \infty$ in the respective half planes, $|F_{\pm}(\xi, \varsigma)| = O(|\xi|^{-1/2})$. Furthermore, in view of (2.2), $\hat{\sigma}_+(x, z, s)$ is expected to be square root singular as $x \rightarrow 0^+$ for any z , and $\hat{u}_-(x, z, s)$ is expected to vanish as $x \rightarrow 0^-$ for any z to ensure continuity of displacement. Consequently, from the Abel theorem concerning asymptotic properties of transforms (Noble, 1958, p. 36)

$$\lim_{\xi \rightarrow +\infty} \sqrt{2s\xi} \Sigma_+(\xi, \varsigma) = \lim_{x \rightarrow 0^+} s^2 \sqrt{2\pi x} \int_{-\infty}^{\infty} \hat{\sigma}_+(x, \varsigma, s) e^{-s\varsigma z} dz = \hat{K}_I(s, \varsigma) \quad (4.8)$$

where

$$\hat{K}_I(s, \varsigma) = s^2 \int_{-\infty}^{\infty} \hat{k}_I(s, z) e^{-s\varsigma z} dz, \quad \hat{k}(s, z) = \lim_{x \rightarrow 0^+} \sqrt{2\pi x} \sigma_+(x, z, s) \quad (4.9)$$

Therefore, the left side of (4.7) vanishes as $|\xi| \rightarrow \infty$ in the left half plane, and the same conclusion may be drawn concerning the right side. According to Liouville's theorem, a bounded entire function is a constant (Hille, 1959, p. 204). In this case, $E(\xi)$ is bounded in the finite plane and $E(\xi) \rightarrow 0$ as $|\xi| \rightarrow \infty$ so that the constant must have the value zero; thus, $E(\xi) \equiv 0$. This completes the solution of the Wiener-Hopf equation (3.10) for the case at hand, and the two unknown functions in that equation

have been determined to be

$$\begin{aligned}\Sigma_+(\xi, \varsigma) &= \frac{p^*}{\xi} \left(\frac{F_+(0, \varsigma)}{F_+(\xi, \varsigma)} - 1 \right) \\ U_-(\xi, \varsigma) &= - \frac{b^2}{\xi \mu \kappa} p^* F_+(0, \varsigma) F_-(\xi, \varsigma)\end{aligned}\tag{4.10}$$

5. THE STRESS INTENSITY FACTOR HISTORY ALONG THE EDGE

It is concluded from (4.9) and (4.11) that the double transform of $k_I(t, z)$, the variation of stress intensity factor history along the edge of the crack, is

$$\hat{K}_I(s, \zeta) = p^* \sqrt{\frac{2\kappa s(a^2 - \zeta^2)^{1/2}}{R(0, \zeta)}} \quad (5.1)$$

where $\kappa = 2(b^2 - a^2)$, as before.

Up to this point, the variable ζ has been understood to be a real variable in the interval between $-a$ and a . Because an *explicit* expression for $\hat{K}_I(s, \zeta)$ is now in hand, however, the definition can be extended into the whole complex ζ plane in a straightforward way. The identity theorem for analytic functions (Hille, 1959, p. 199) assures that the extension or continuation is unique. When viewed as a function of the complex variable ζ , $\hat{K}_I$ is obviously analytic in the strip $-a < \text{Re}(\zeta) < a$. Consequently, it may be viewed as a convergent two sided Laplace transform over the spatial variable z . Furthermore, extension of the definition of $\hat{K}_I(s, \zeta)$ into the complex ζ plane reveals that it is analytic everywhere except at branch points $\zeta = \pm a, \pm b, \pm c$. The function is made single valued by introducing the branch cuts extending from each branch point in a direction away from the origin to infinity along the real axis. The branches of $(a^2 - \zeta^2)^{1/2}$, $(b^2 - \zeta^2)^{1/2}$ and $(c^2 - \zeta^2)^{1/2}$ with positive real part are selected.

The inverse two sided Laplace transform of (5.2) is

$$\hat{k}_I(s, z) = \frac{1}{2\pi i} \int_{\zeta_0 - i\infty}^{\zeta_0 + i\infty} s^{-1} \hat{K}_I(s, \zeta) e^{s\zeta z} d\zeta \quad (5.2)$$

where ζ_0 is any real number between $-a$ and a .

The central idea in the next step is to convert the integral in (5.1) to a form which will allow inversion of the one-sided Laplace transform on time *by observation*.

Suppose that $z > 0$. Then the path of integration is augmented by two arcs, quarter circles of indefinitely large radius, in the left half plane, plus a path running from the remote point on the negative real axis inward along the upper side of the branch cut to $\zeta = -a$, around the branch point, and then outward along the lower side of the cut to the remote point; see Fig. 2. These paths, together with the inversion path itself, form a closed contour, and the integrand of (5.2) is analytic inside and on this contour. According to Cauchy's integral theorem (Hille, 1959, p. 163), the integral of the function along the closed contour is zero. An immediate inference is that the integral in (5.2) is equal to the integral of the same function taken along the alternate path consisting of the two arcs of large radius and the branch line integral. According to Jordan's lemma, an exponential integral of this type along a remote circular arc vanishes as the radius of the arc becomes indefinitely large if the function multiplying the exponential in the integrand vanishes uniformly on the arc as the radius becomes indefinitely large. In the present case, $|\hat{K}_I(s, \zeta)| \rightarrow 0$ uniformly as $|\zeta| \rightarrow \infty$ in the left half plane and, consequently, the value of the integral along the remote arcs is zero. Furthermore, $\hat{K}_I(s, \bar{\zeta}) = \overline{\hat{K}_I(s, \zeta)}$ so that the branch line integral may be written as a real integral,

$$\hat{k}_I(s, z) = \frac{1}{\pi s} \int_{-a}^{-\infty} \text{Im} \left\{ \hat{K}_I(s, \zeta + i0) \right\} e^{s\zeta z} d\zeta \quad (5.3)$$

where the fact that $\hat{K}_I(s, \zeta)$ is to be evaluated on the *upper* side of the branch cut in the left half plane has been made explicit. In view of the fact that $\hat{K}_I$ is an even function of ζ , the integral takes the form

$$\hat{k}_I(s, z) = -\frac{1}{\pi s z} \int_a^{\infty} \text{Im} \left\{ \hat{K}_I(s, \zeta + i0) \right\} e^{-s\zeta z} d\zeta \quad (5.4)$$

and, indeed, the inversion of the one-sided transform becomes obvious. That is, (5.4) is a product of two transforms, so that $k_I(t, z)$ is a convolution of the inverses of the

two transforms,

$$k_I(t, z) = \frac{p^*}{\pi} \sqrt{\frac{2\kappa}{\pi}} \int_a^{t/z} \operatorname{Im} \left\{ \sqrt{\frac{\alpha(0, \zeta + i0)}{R(0, \zeta + i0)}} \right\} \frac{d\zeta}{\sqrt{t - \zeta z}} H(t - az) \quad (5.5)$$

for $z > 0$, and $K_I(t, -z) = K_I(t, z)$. This completes the process of inversion of the double transform for the stress intensity factor history at each point along the crack edge. Note that the result has the form anticipated in (4.2). The result is in the form of a real integral, and some of its features are now examined.

The integral in (5.5) cannot be evaluated in terms of elementary functions. Its general features are nonetheless evident. For example, the imaginary part of the quantity in brackets in the integrand is negative for $a < \zeta < c$, the it is positive for $\zeta > c$, and the integral is singular at $t/z = c$. The integral has been evaluated numerically, and the result of numerical evaluation for a Poisson ratio of $\nu = 0.3$ is shown in Fig. 3. Knowing the asymptotic behavior of the generalized Rayleigh wave function, namely, that $R(0, \zeta) \rightarrow -\kappa\zeta^2$ as $\zeta \rightarrow +\infty$, the integral (5.5) implies that

$$\lim_{t \rightarrow \infty} k_I(t, z) = k_I(\infty, z) = \frac{p^*}{\sqrt{\pi z}} \quad (5.6)$$

for any fixed $z > 0$. The result of the limiting process is consistent with the independently calculated equilibrium stress intensity factor distribution along the crack edge for this configuration (Tada et al, 1973, p. 23.1). The stress intensity factor history in Fig. 3 for any fixed value of z is normalized by its long time limiting value $k_I(\infty, z)$.

The general features of $k_I(t, z)$ derive from the nature of the wave fields. For any value of $z > 0$, the stress intensity factor is zero up until the arrival of the first dilatational wave at time $t = z/c_d$. Upon sudden application of the compressive line loads on the faces of the crack, the initial response is dilatational and the surfaces at points adjacent to the line loads tend to bulge outward. The first wave arriving at the

observation point along the crack edge carries this outward bulge. The crack faces tend to move toward each other, and this tendency is reflected in the tendency for the stress intensity factor to become *negative* following arrival of the leading dilatational wave. This effect, which is similar to that discussed by Freund (1974) in a two dimensional situation, persists until the arrival of the Rayleigh wave at time $t = z/c_r$ ($c_d/c_r = 2.02$ for $\nu = 0.3$). The stress intensity factor history is logarithmically singular at this instant, and it tends to increase thereafter. The transient stress intensity factor history $k_I(t, z)$ approaches its long time limit $k_I(\infty, z)$ quite slowly, with the ratio of the former to the latter being about 0.81 at $c_d t/z = 20$ and about 0.92 at $c_d t/z = 100$.

6. CONCLUDING REMARKS

This completes the analysis of the three dimensional stress intensity factor history for the particular applied traction distribution (4.1). Results could be derived for a number of other traction distributions in a similar way. For example, suppose that crack face traction in the form of a pair of opposed concentrated forces of intensity p^* , one acting on each face of the crack, begins at the crack edge and then moves at uniform speed v in the negative x direction. In this case, (4.1) is replaced by

$$\sigma_-(x, z, t) = -p^* \delta(x + vt) \delta(z) H(t) \quad (6.1)$$

Similarly, if a pair of line loads of intensity p^* acts over $z = 0$, $-vt < x < 0$ so that the step moves at uniform speed v in the negative x direction,

$$\sigma_-(x, z, t) = -p^* H(x + vt) \delta(z) H(t) \quad (6.2)$$

Yet another, slightly more complex, illustration is obtained if a pair of line loads of intensity p^* begins to act along lines inclined at an angle ω to the x direction. In this case,

$$\sigma_-(x, z, t) = -p^* \delta(x \sin \omega + z \cos \omega) H(-x) H(t) \quad (6.3)$$

and so on. The present analysis opens the way for examination of these and other boundary value problems connected with three dimensional phenomena.

A far more challenging situation is presented if an opposed pair of concentrated forces of intensity p^* suddenly begins to act on the crack faces at a point at some fixed distance $\ell > 0$ from the crack edge, that is,

$$\sigma_-(x, z, t) = -p^* \delta(x + \ell) \delta(z) H(t) \quad (6.4)$$

The means of obtaining an exact stress intensity factor solution in this case has been elusive. The equivalent two dimensional problem was solved by means of the dynamic weight function method (Freund and Rice, 1974). In principle, the weight function method is also applicable for this three dimensional situation. However, the procedure requires that the dependence of $\hat{u}(x, z, s)$ on x and z be known quite explicitly, and the way to extract this dependence from the double inversion integral (4.11)₂ has not yet been found.

ACKNOWLEDGEMENTS

Some helpful comments on an earlier version of this paper were provided by J. C. Ramirez, a research assistant at Brown University. The research support of the National Science Foundation, Solid Mechanics Program Grant No. MSM-8513096, and of the Office of Naval Research, Mechanics Division Contract No. N00014-85-K-0597, is gratefully acknowledged.

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FIGURE CAPTIONS

Fig. 1 Geometrical configuration of the elastic solid.

Fig. 2 The complex ς plane showing the singularities of $\hat{K}_I(s, \varsigma)$ and the integration path for evaluation of (5.2).

Fig. 3 Normalized stress intensity factor $k_I(t, z)\sqrt{\pi z}/p^*$ versus $c_d t/z$.

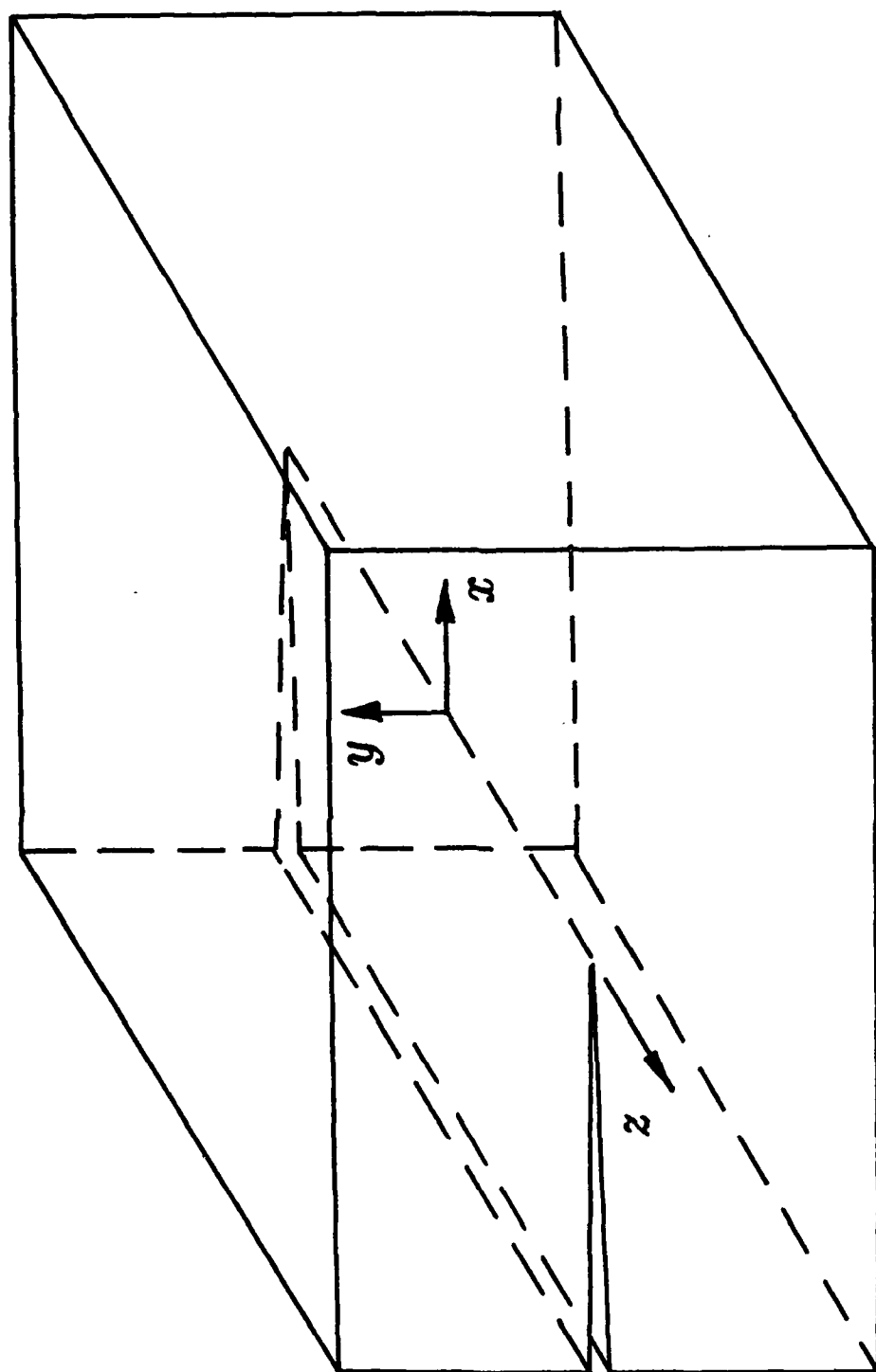


Figure 1

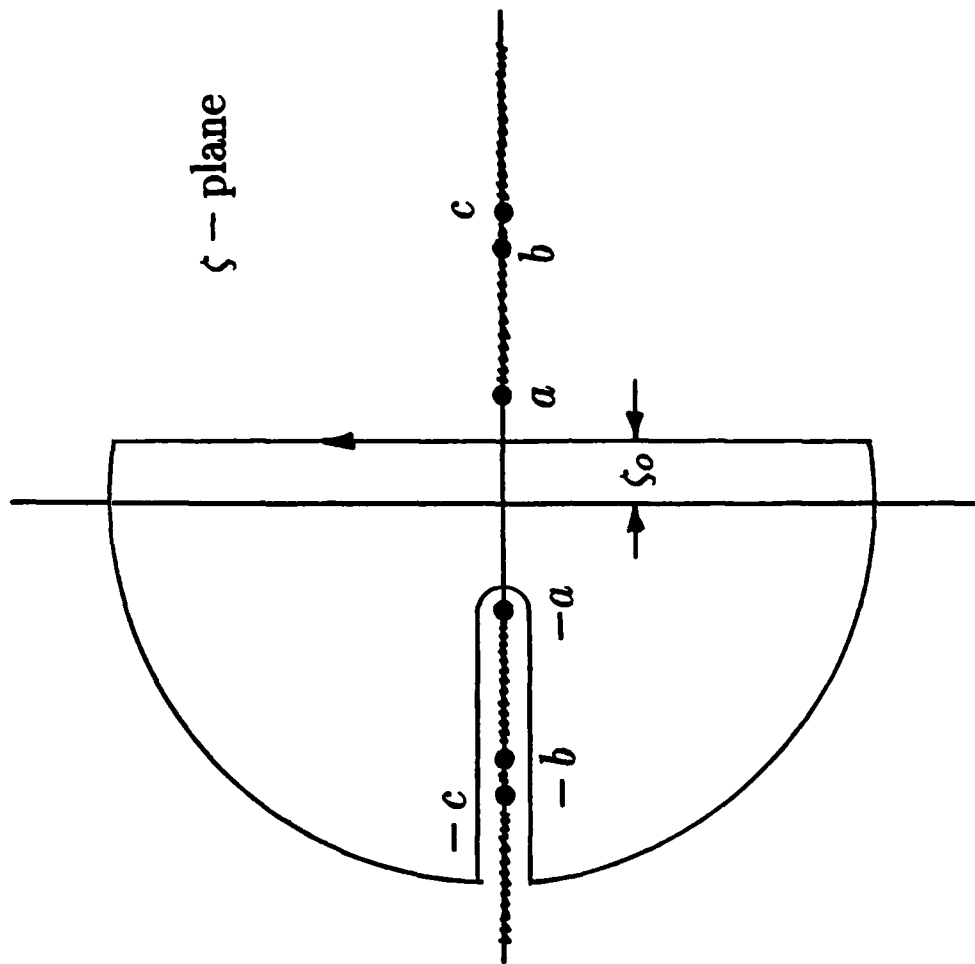
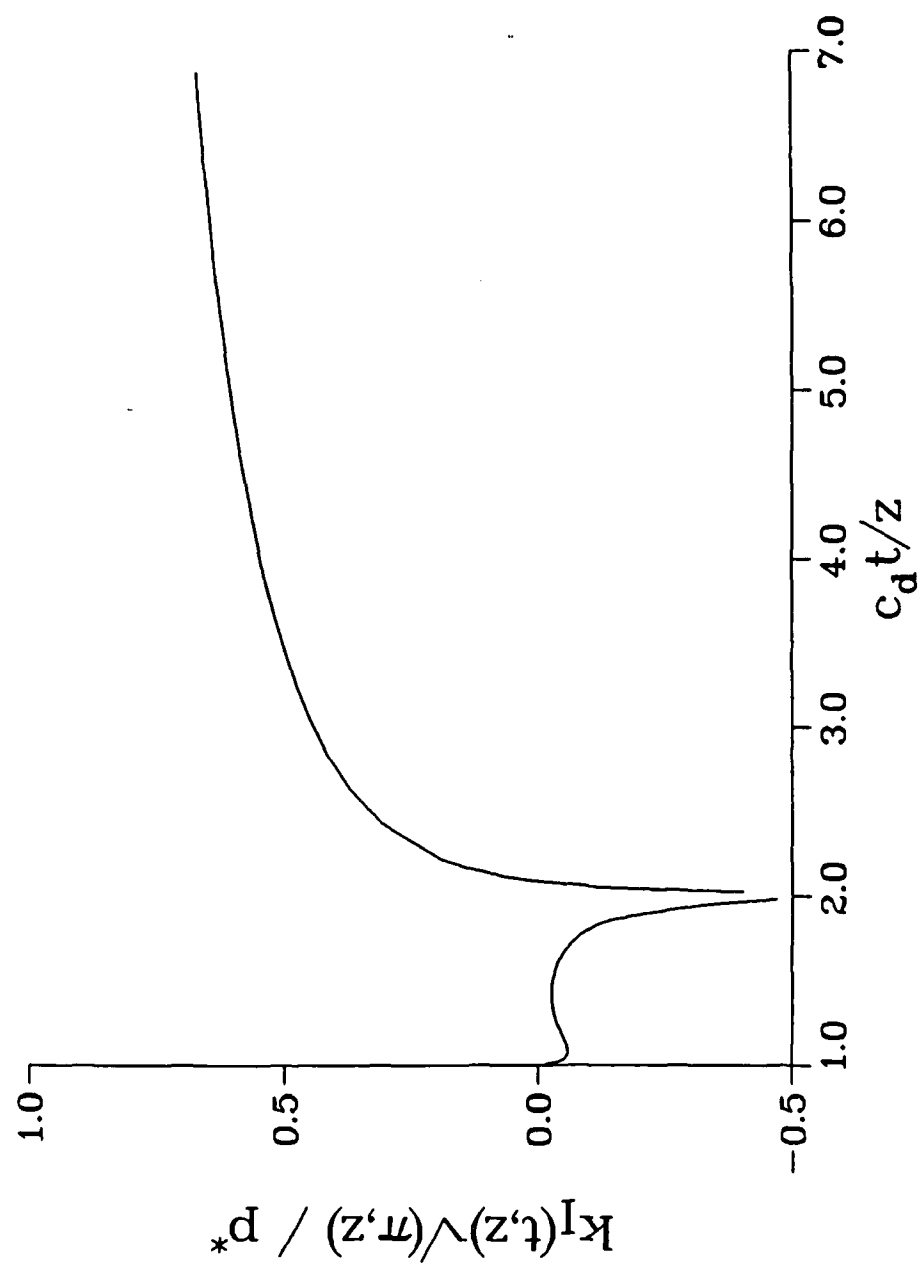


Figure 2



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