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FIRST FAILURE TIME OF DEPENDENT PARALLEL SYSTEMS WITH
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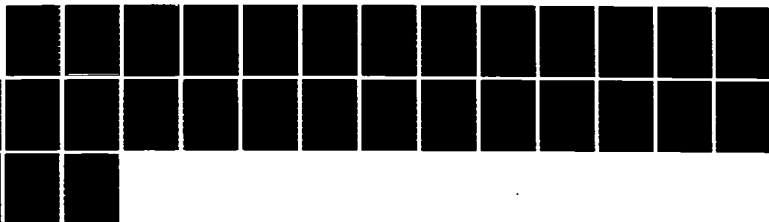
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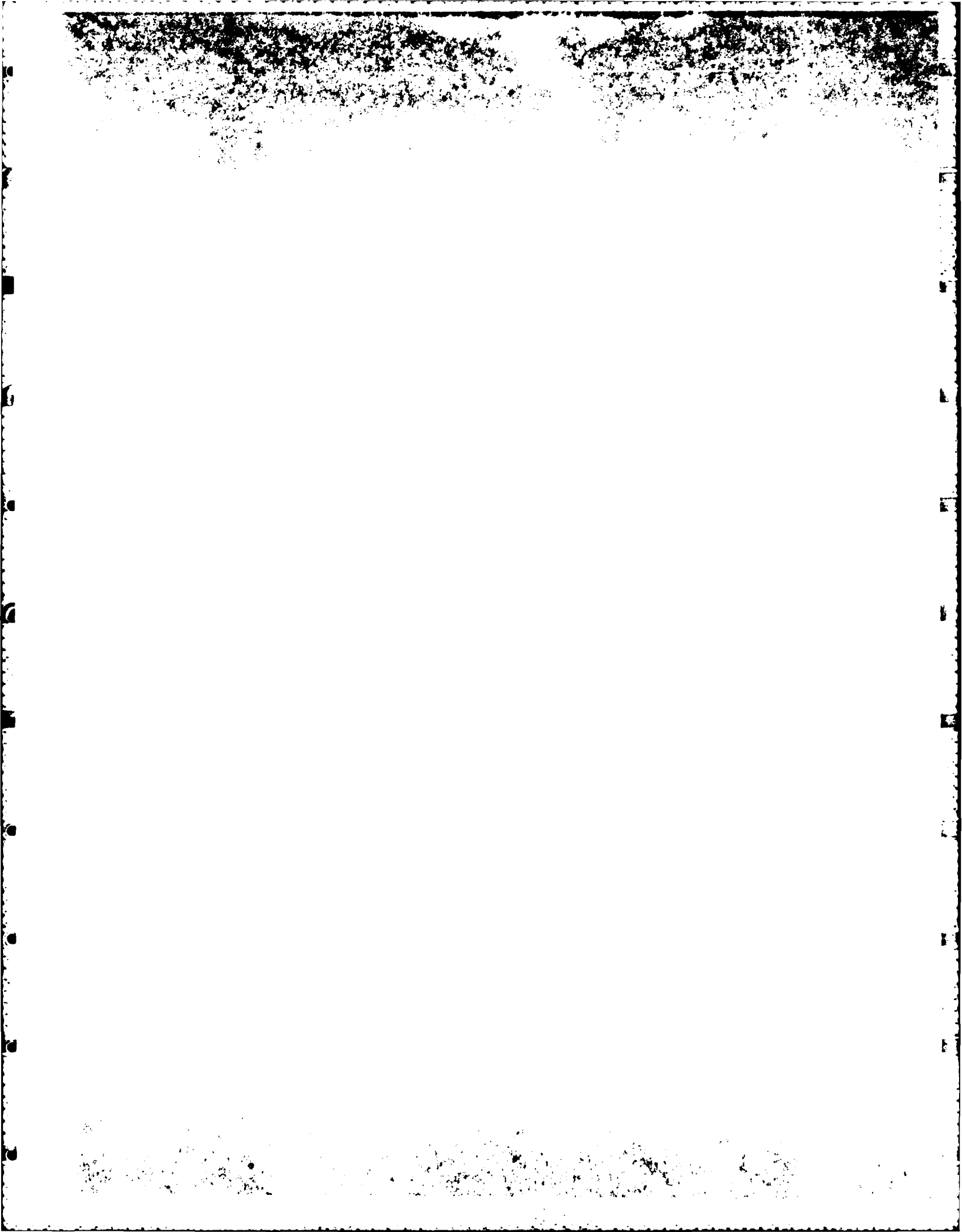
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Parallel Systems with Safety Periods

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FIRST FAILURE TIME OF DEPENDENT PARALLEL SYSTEMS WITH SAFETY PERIODS

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ABSTRACT

In this paper we consider the time to first failure of a parallel system in which the failure and repair rates of components depend on the state of the other components as well. A back-up unit with a random life time is employed whenever all the components of the system are down. The system fails when all the components of the system and the back-up unit are down. The first moment, the Laplace transform and the probability distribution of the time to first failure of this system are obtained. Sufficient conditions under which this distribution has the new better than used (NBU) and an exponential limit property are given. Special cases with phase type and deterministic back-up unit lifetimes are also considered. These results extend the results of Ross and Schechtman (1979).

Key words: Dependent parallel system, first failure time, back-up unit, safety periods, new better than used (NBU) distributions, exponential limit theorem, phase type distributions.

INTRODUCTION

In many parallel reliability systems with maintained components, a system failure may not occur immediately upon the failure of all of its components; rather a system failure may occur only after all of its components have been down continuously for a fixed or random time period. This time period is called the safety period. This safety period may be the lifetime of a back-up unit. For example consider a system serviced by two AC power sources. When both these power sources fail, a DC power source (battery) with a fixed or random lifetime is used to supply power to the system: the battery here serves as a cold-stand-by unit, providing a safety period for the AC power sources to be repaired. On the other hand, the safety period may itself represent the time required to cause actual damage to the system when all of its components are down. For example consider a vessel accommodating a constant heat source cooled by a system of two redundant blowers (Dunbar (1984)). In such a system it is usual to consider a system failure as structural damage of the vessel due to excessive temperature. Suppose the normal temperature of operation be much lower than the maximum that the vessel has been designed to withstand. In this case failure of both blowers is not sufficient to cause immediate failure of the system. The system will fail with a delay in time depending on the magnitude of the heat input, system heat capacity and the difference between the operating temperature and the temperature required to cause damage to the vessel. Furthermore, the system will fail only if neither of the blowers is repaired (or replaced) before damage occurs. Similar examples in the nuclear and perishable food industries are discussed in Ross and Schechtman (1979).

In this paper we consider such a parallel system in which the failure and repair rates of components depend on the state of the other components as well. A back-up unit with a random lifetime is employed whenever all the components of the system are down. The system fails when all the components of the system and the back-up unit are down. A detailed description of this model is given in Section 1. The first moment, the Laplace transform and the probability distribution of the time to first failure of this system are obtained in the same section. Section 1 also contains sufficient conditions under which the first failure time has the new better than used and an exponential limit property. Special cases with phase type and deterministic safety periods are considered in Sections 2 and 3 respectively.

1. DEPENDENT PARALLEL SYSTEMS WITH SAFETY PERIODS: THE MODEL

Consider a parallel system consisting of n components. Each component alternate between intervals in which they are up (i.e. working) and in which they are down (i.e. failed). Let $Z_i(t)$ be the state of component i at time t : i.e. $Z_i(t) = 0$ if component i is up at time t and $Z_i(t) = 1$ otherwise. Thus $\underline{Z}(t) = (Z_1(t), Z_2(t), \dots, Z_n(t))$ describes the performance of the components at time t . It is assumed that the vector performance process $\underline{Z} \equiv \{\underline{Z}(t), t \in R_+\}$ is Markov on the state space $S = \{0, 1\}^n$, and if $\{\hat{Z}_i, i \in N^+\}$ is the embedded process of $\underline{Z}$ embedded at its transition epochs, $\|\hat{Z}_i - \hat{Z}_{i-1}\| = 1$, a.s., where for $\underline{a}, \underline{b} \in S$, $\|\underline{a} - \underline{b}\| \equiv \sum_{i=1}^n (a_i - b_i)^2$. That is at any transition epoch of $\underline{Z}$ either one repair or one failure of a component takes place. We call such a process multivariate binary birth and death process. Note that if all n components are separately maintained and the failure (respectively repair)

rate of component i is λ_i (respectively μ_i), the vector performance process $\underline{Z}$ of such a system satisfies the above condition (e.g. Chiang and Niu (1980), Ross and Schechtman (1979)).

Our system is, however, dependent in the sense that the failure and repair rates of component i can be dependent on the state of the other components. Suppose the vector performance process $\underline{Z}$ is in state $\underline{a}$ (respectively $\underline{b}$) such that $a_i = 0$ (respectively $b_i = 1$). The failure (respectively repair) rate $\lambda_i(\underline{a})$ (respectively $\mu_i(\underline{b})$) of component i may depend on $\underline{a}$ (respectively $\underline{b}$) (e.g. Ross (1984), Schechner (1984)). Such a generalisation is needed when the working components share the overall workload or when the repair facility has limited capacity. Let Q be the infinitesimal generator of $\underline{Z}$. Two examples with $n = 3$ for (i) independently maintained and (ii) dependent system with equal load sharing of a constant load and proportional failure rate are shown in Figures 1 and 2.

When $\underline{Z}$ is in state $\underline{0} \equiv (0, 0, 0, \dots, 0)$ (i.e. all n components are working) we say that the system is in perfect condition. The system is said to be down whenever $\underline{Z}$ takes the value $\underline{1} \equiv (1, 1, \dots, 1)$ (i.e. all n components are failed) and it is said to be up otherwise (i.e. at least one component is working). Whenever the system reaches the down state an emergency back-up unit is brought in which provides a random safety period. If none of the n failed components are repaired during this safety period, the system fails at the end of this period. Otherwise at the time of the first repair after the system became down, the emergency back-up unit is removed and the system is placed in normal operation.

Let $0 < R_1 < R_2 < \dots$ (respectively $0 < S_1 < S_2 < \dots$) be the consecutive time epochs at which the process $\underline{Z}$ enters (respectively exits) state $\underline{1}$ starting from some initial state $\underline{Z}(0) \neq \underline{1}$. Then one sees that $0 < R_1 < S_1 < R_2 < S_2 < \dots$, a.s.

$$Q = \begin{matrix} & \begin{matrix} (0,0,0) & (1,0,0) & (0,1,0) & (0,0,1) & (1,1,0) & (1,0,1) & (0,1,1) & (1,1,1) \end{matrix} \\ \begin{matrix} (0,0,0) \\ (1,0,0) \\ (0,1,0) \\ (0,0,1) \\ (1,1,0) \\ (1,0,1) \\ (0,1,1) \\ (1,1,1) \end{matrix} & \begin{bmatrix} -(\lambda_1+\lambda_2+\lambda_3) & \lambda_1 & \lambda_2 & \lambda_3 & 0 & 0 & 0 & 0 \\ \mu_1 & -(\mu_1+\lambda_2+\lambda_3) & 0 & 0 & \lambda_2 & \lambda_3 & 0 & 0 \\ \mu_2 & 0 & -(\lambda_1+\mu_2+\lambda_3) & 0 & \lambda_1 & 0 & \lambda_3 & 0 \\ \mu_3 & 0 & 0 & -(\lambda_1+\lambda_2+\mu_3) & 0 & \lambda_1 & \lambda_2 & 0 \\ 0 & \mu_2 & \mu_1 & 0 & -(\mu_1+\mu_2+\lambda_3) & 0 & 0 & \lambda_3 \\ 0 & \mu_3 & 0 & \mu_1 & 0 & -(\mu_1+\lambda_2+\mu_3) & 0 & \lambda_2 \\ 0 & 0 & \mu_3 & \mu_2 & 0 & 0 & -(\lambda_1+\mu_2+\mu_3) & \lambda_1 \\ 0 & 0 & 0 & 0 & \mu_3 & \mu_2 & \mu_1 & -(\mu_1+\mu_2+\mu_3) \end{bmatrix} \end{matrix}$$

Figure 1: Infinitesimal Generator Q for a Separately Maintained System with 3 Components.

$$Q = \begin{matrix} & \begin{matrix} (0,0,0) & (1,0,0) & (0,1,0) & (0,0,1) & (1,1,0) & (1,0,1) & (0,1,1) & (1,1,1) \end{matrix} \\ \begin{matrix} (0,0,0) \\ (1,0,0) \\ (0,1,0) \\ (0,0,1) \\ (1,1,0) \\ (1,0,1) \\ (0,1,1) \\ (1,1,1) \end{matrix} & \begin{bmatrix} -\lambda & \lambda/3 & \lambda/3 & \lambda/3 & 0 & 0 & 0 & 0 \\ \mu_1 & -(\mu_1+\lambda) & 0 & 0 & \lambda/2 & \lambda/2 & 0 & 0 \\ \mu_2 & 0 & -(\mu_2+\lambda) & 0 & \lambda/2 & 0 & \lambda/2 & 0 \\ \mu_3 & 0 & 0 & -(\mu_3+\lambda) & 0 & \lambda/2 & \lambda/2 & 0 \\ 0 & \mu_2 & \mu_1 & 0 & -(\mu_1+\mu_2+\lambda) & 0 & 0 & \lambda \\ 0 & \mu_3 & 0 & \mu_1 & 0 & -(\mu_1+\lambda+\mu_3) & 0 & \lambda \\ 0 & 0 & \mu_3 & \mu_2 & 0 & 0 & -(\lambda+\mu_2+\mu_3) & \lambda \\ 0 & 0 & 0 & 0 & \mu_3 & \mu_2 & \mu_1 & -(\mu_1+\mu_2+\mu_3) \end{bmatrix} \end{matrix}$$

Figure 2: Infinitesimal Generator Q for a Dependent System with Equal Load Sharing, Proportional Failure Rate and Independent Repairs.

Let $X_0 = R_1$, $X_i = R_{i+1} - S_i$ and $Y_i = S_i - R_i$, $i = 1, 2, \dots$. Note that Y_i is the time required for the first repair after the i -th entrance of the process $\underline{Z}$ into the state $\underline{1}$. Thus $(Y_i)_{i=1}^{\infty}$ is a sequence of independent and identically distributed (i.i.d) exponential random variables. It can be easily verified that $(X_i)_{i=0}^{\infty}$ is a sequence of mutually independent phase type random variables (e.g. Neuts (1981)) with $(X_i)_{i=1}^{\infty}$ having identical distributions. Now let $(T_i)_{i=1}^{\infty}$ be the sequence of the length of the safety periods. That is, at the entrance of $\underline{Z}$ into state $\underline{1}$ for its i -th time, the emergency back-up unit will provide a safety period of length T_i . Then the time to first system failure V_T is given by:

$$V_T = \inf \{R_i + T_i : R_i + T_i < S_i, i=1, 2, \dots\} \quad (1)$$

In the remainder of this paper we will assume that $(T_i)_{i=1}^{\infty}$ is a sequence of independently distributed random variables that are independent of $\underline{Z}$. Note that V_T is nondecreasing in R_i and T_i . Then from (1) one easily sees

Lemma 1: For $X_i' \stackrel{st}{\leq} X_i''$, $i = 0, 1, 2, \dots$, $Y_i' \stackrel{st}{=} Y_i''$, and

$T_i' \stackrel{st}{\leq} T_i''$, $i = 1, 2, \dots$, one has

$$V_{T'} \stackrel{st}{\leq} V_{T''} ,$$

where for any two random variables W' and W'' , $W' \stackrel{st}{\leq} W''$ (respectively $\stackrel{st}{=}$) $W'' \Leftrightarrow P\{W' > t\} \leq$ (respectively $=$) $P\{W'' > t\}$ for all t .

It is of interest to obtain the mean and the probability distribution of V_T . In

Section 1.1 we will first obtain these results with no specific assumptions regarding the probability distributions of $(X_i)_{i=0}^{\infty}$, $(Y_i)_{i=1}^{\infty}$ and $(T_i)_{i=1}^{\infty}$.

1.1 General Results for Single Component Systems.

Consider a single component system where $(X_i)_{i=1}^{\infty}$, $(Y_i)_{i=1}^{\infty}$ and $(T_i)_{i=1}^{\infty}$ are the up, down and safety periods respectively. Then the time to the first system failure V_T as defined in (1) can be rewritten as:

$$V_T = \inf \left\{ \sum_{j=0}^{i-1} (X_j + Y_j) + T_i : T_i < Y_i, i=1,2,\dots \right\}, \quad (2)$$

where $Y_0 \equiv 0$ w.p.1. Let F_0 , F , G and H be the probability distribution functions of the mutually independent random variables X_0 , $(X_i)_{i=1}^{\infty}$, $(Y_i)_{i=1}^{\infty}$ and $(T_i)_{i=1}^{\infty}$ respectively. Define $K+1$ to be the smallest integer for which $T_i < Y_i$ is satisfied. That is

$$K + 1 = \min \{i : T_i < Y_i, i = 1,2,\dots\}. \quad (3)$$

Since $(T_i)_{i=1}^{\infty}$ and $(Y_i)_{i=1}^{\infty}$ are mutually independent sequences of i.i.d random variables, K has a geometric distribution on $\{0,1,2,\dots\}$ with paramete $P\{T_i < Y_i\}$. So

$$P\{K = i\} = p(1-p)^i, \quad i = 0,1,2,\dots$$

and

$$E\{K\} = (1 - p)/p,$$

where

$$p \equiv P\{T_1 < Y_1\} = \int_0^{\infty} (1 - G(x))dH(x),$$

and E is the expectation operator. Now combining (2) and (3) one obtains

$$V_T = X_0 + \sum_{j=1}^K (X_j + Y_j^*) + T_{K+1}^* \quad , \quad (4)$$

with the usual convention that $\sum_a^b x_i = 0$ for $b < a$ and where

$$Y_j^* = Y_j | Y_j < T_j \quad \text{and} \quad T_j^* = T_j | T_j < Y_j, \quad j = 1, 2, \dots$$

Note that for $j \leq K$, $Y_j^* = Y_j$ and $T_{K+1} = T_{K+1}^*$. The probability distribution functions G^* and H^* of $(Y_j^*)_{j=1}^{\infty}$ and $(T_j^*)_{j=1}^{\infty}$ respectively are given by

$$G^*(x) = \frac{1}{1-p} \int_0^x (1 - H(y))dG(y), \quad x > 0$$

and

$$H^*(x) = \frac{1}{p} \int_0^x (1 - G(y))dH(y), \quad x > 0.$$

Taking the expectations on both sides of (4) one obtains

$$E\{V_T\} = E\{X_0\} + \frac{1-p}{p} \{E\{X\} + E\{Y^*\}\} + E\{T^*\} \quad , \quad (5)$$

where $E\{X\} = E\{X_i\}$, $i = 1, 2, \dots$,

$$E\{Y^*\} = E\{Y_i^*\} = \frac{1}{1-p} \int_0^{\infty} y(1 - H(y))dG(y),$$

and

$$E\{T^*\} = E\{T_i^*\} = \frac{1}{p} \int_0^{\infty} y(1 - G(y))dH(y), \quad i=1, 2, \dots$$

Furthermore, taking the Laplace transform one has

$$L\{V_T\} = L\{X_0\} \left\{ \frac{p}{1 - (1-p)L\{X\}L\{Y^*\}} \right\} L\{T^*\}, \quad (6)$$

where for any nonnegative random variable W , $L\{W\} \equiv E\{\exp(-sW)\}$, $\text{Re}(s) > 0$.

Hence

$$P\{V_T \leq x\} = F_0 \circledast \left(p \sum_{j=0}^{\infty} (1-p)^j F^{(j)} \circledast G^{*(j)} \right) \circledast H^*(x), \quad x > 0, \quad (7)$$

where $F^{(0)}(x) = 1$, $x > 0$, $F^{(j)}$ is the j (≥ 1) fold convolution of F with itself and $\circledast$ stands for convolution. $G^{*(j)}$ is similarly defined. Equation (6) is a generalisation of Equation (2) of Ross and Schechtman (1979). When F_0 , F , G and H are all absolutely continuous with support $(0, \infty)$ one may use either the Laguerre transform (Keilson and Nunn (1979), and Sumita (1981)) or the generalised phase type (Shanthikumar (1985)) method to compute $P\{V_T \leq x\}$ given in (7). However, it seems appropriate that we also develop approximations and bounds for V_T that are easily computable. With this in

mind we will first provide sufficient conditions under which V_T has the new better than used (NBU) property. A nonnegative random variable W is

NBU iff $P\{W > t + s\} \leq P\{W > t\} P\{W > s\}$ or equivalently $W \stackrel{st}{\geq} W_{|W>t}^{-t}$, $t > 0$.

Theorem 2: Assume that X_0 and $(T_i)_{i=1}^{\infty}$ are NBU, $(Y_i)_{i=1}^{\infty}$ are exponentially

distributed and $X_0 \stackrel{st}{\geq} X_{0|X_0>t}^{-u}$, $u > 0$, $i = 1, 2, \dots$. Then V_T is NBU.

Proof: In order to establish that V_T is NBU we need to show that

$$V_T \stackrel{st}{\geq} V_{T|V_T>t}^{-t}, \quad t > 0. \quad (8)$$

Suppose $V_T > t$. Then there are three cases one can consider. They are:

Case I: $0 < t < R_1$.

In this case, from (4) one sees that

$$V_{T|\text{Case I}}^{-t} = X_{0|X_0>t}^{-t} + \sum_{j=1}^K (X_j + Y_j^*) + T_{K+1}^*. \quad (9)$$

Since $X_{0|X_0>t}^{-t} \stackrel{st}{\leq} X_0$, comparing (4) and (9) one sees that (8) is satisfied in this case.

Case II: $R_\ell < t < S_\ell$, $V_T > t$ and $T_\ell > t - R_\ell$ for some $\ell (\geq 1)$.

Let $u \equiv t - R_\ell$ and define

$$T'_1 \stackrel{st}{=} T_{\ell|T_\ell>u}^{-u}, \quad T'_j = T_{\ell+j-1}, \quad j = 2, 3, \dots,$$

$$Y_1^{\text{st}} = Y_{\ell | Y_{\ell} > u}^{-u}, \quad Y_j^{\text{st}} = Y_{\ell+j-1}, \quad j = 2, 3, \dots,$$

and $X_j^{\text{st}} = X_{\ell+j-1}$, $j = 1, 2, \dots$, and set $X_0^{\text{st}} = Y_0^{\text{st}} = 0$.

Then

$$V_{\text{TI Case II}}^{-t} = \inf \left\{ \sum_{j=0}^{i-1} (X_j^{\text{st}} + Y_j^{\text{st}}) + T_i^{\text{st}} : T_i^{\text{st}} < Y_i^{\text{st}}, \right. \\ \left. i = 1, 2, \dots \right\}. \quad (10)$$

Since $(T_j)_{j=1}^{\infty}$ is NBU and $(Y_j)_{j=1}^{\infty}$ is exponential one sees that $T_j^{\text{st}} \leq T_j$ and

$$Y_j^{\text{st}} = Y_j, \quad j = 1, 2, \dots. \quad \text{Furthermore, } X_j^{\text{st}} = X_j, \quad j = 0, 1, \dots. \quad \text{Then comparing (2)}$$

and (10), from Lemma 1 one sees that (8) is satisfied in this case too.

Case III: $S_{\ell} < t < R_{\ell+1}$, $V_T > t$ for some $(\ell \geq 1)$.

Let $u = t - S_{\ell}$ and define

$$X_0^{\text{st}} = X_{\ell | X_{\ell} > u}^{-u}, \quad X_j^{\text{st}} = X_{\ell+j}, \quad j = 1, 2, \dots,$$

$$Y_j^{\text{st}} = Y_{\ell+j-1}, \quad T_j^{\text{st}} = T_{\ell+j-1}, \quad j = 1, 2, \dots, \quad \text{and } Y_0^{\text{st}} = 0.$$

Then

$$V_{\text{TI Case III}}^{-t} = \inf \left\{ \sum_{j=0}^{i-1} (X_j^{\text{st}} + Y_j^{\text{st}}) + T_i^{\text{st}} : T_i^{\text{st}} < Y_i^{\text{st}}, \right. \\ \left. i = 1, 2, \dots \right\}. \quad (11)$$

Since $X_0^{\text{st}} \geq X_{\ell | X_{\ell} > u}^{-u}$ one sees that $X_j \geq X_j^{\text{st}}$, $j = 0, 1, 2, \dots$, $Y_j^{\text{st}} = Y_j$ and $T_j^{\text{st}} = T_j$, $j = 1, 2, \dots$. Then from Lemma 1 and Equations (2) and (11) one sees that (8) is satisfied in this case as well.

Now a routine unconditioning with respect to these three cases (e.g. Shanthikumar (1984)) completes the proof.



The above theorem appears to be weaker than Proposition 1 of Ross and Schechtman (1979). As we will see later in Section 1.2, application of Theorem 2 to the dependent parallel system results in a theorem that is stronger than Proposition 2 of Ross and Schechtman (1979). When T_i is degenerate (say equal to a constant A), one can use the proof of Proposition 1 of Ross and Schechtman (1979) and the above proof to establish

Theorem 3: When X_0 is NBU, $T_i = A$ is a constant and for all $u > 0$,

$$\begin{aligned} &\text{st} \\ &X_0 \geq X_{i|X_i > u}^{-u}, i = 1, 2, \dots, V_T \text{ is NBU.} \end{aligned}$$

Note that Theorem 3 is a slight generalisation of Proposition 1 of Ross and Schechtman (1979). Theorems 2 and 3 can be used to bound higher moments of V_T (e.g. Proposition 3 and Corollary 1 of Ross and Schechtman (1979)). Next we will provide an exponential limit theorem that can be used to approximate the distribution of V_T . Consider the sum

$$U_Y = \sum_{i=1}^{K_Y} (X_i + Y_i^Y)$$

where K_Y is a geometric random variable with parameter p_Y and support $\{0, 1, 2, \dots\}$,

$$p_Y = \int_0^{\infty} (1 - G(\gamma x)) dH(x)$$

and

$$Y_i^Y = Y_i | Y_i < \gamma T_i, \quad i = 1, 2, \dots,$$

for some $\gamma > 0$. Then

$$E\{U_Y\} = \frac{(1-p_Y)}{p_Y} \left\{ E\{X_i\} + E\{Y_i | Y_i < \gamma T_i\} \right\}.$$

Suppose $P\{T_i > 0\} = 1$ and Y_i has support $(0, \infty)$ and is absolutely continuous. Then as $\gamma \rightarrow \infty$ one has $p_Y \rightarrow 0$, $E\{U_Y\} \rightarrow \infty$ and $L\{Y_i^Y\} \rightarrow L\{Y_i\}$. Hence, using an analysis similar to that of Shanthikumar and Sumita (1983).

Theorem 1.A4, one obtains

$$P\{U_Y \leq t E\{U_Y\}\} \rightarrow 1 - e^{-t}, \quad t > 0 \text{ as } \gamma \rightarrow \infty.$$

Therefore

Theorem 4: When G is absolutely continuous with support $(0, \infty)$, $H(0) = 0$ and $0 < p = P\{Y_j > T_j\} < \varepsilon$, there exists an $M(\varepsilon)$ such that $M(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ and

$$|P\{V_T \leq x\} - F_0 \oplus B \oplus H^*(x)| < M(\varepsilon), \quad x > 0$$

where

$$B(x) = 1 - \exp \left\{ \frac{px}{(1-p)(E\{X\} + E\{Y^*\})} \right\}, \quad x > 0.$$

In the spirit of the above theorem one may therefore approximate $P\{V_T \leq x\}$

for small values of p by

$$P\{V_T \leq x\} \cong F_0 \otimes B \otimes H^*(x), \quad x > 0$$

Two simpler approximations of the above form (for small values of p) are:

$$P\{V_T \leq x\} = B_0 \otimes H^*(x), \quad x > 0, \quad (12)$$

and

$$P\{V_T \leq x\} = 1 - \exp \{x/E\{V_T\}\}, \quad x > 0, \quad (13)$$

where $E\{V_T\}$ is as given in (5) and

$$B_0(x) = 1 - \exp \{x/(E\{X_0\} + (\frac{1-p}{p})\{E\{X\} + E\{Y^*\}\})\}, \quad x > 0.$$

1.2 Results for Dependent Parallel Systems

In the remainder of this paper we will assume that the sequences $(X_i)_{i=0}^{\infty}$ and $(Y_i)_{i=1}^{\infty}$ are as defined for the dependent parallel system governed by the vector performance process $\underline{Z}$ with infinitesimal generator Q . We will first give the means and the probability distributions of $X_0, (X_i)_{i=1}^{\infty}$. Partition the Q matrix such that

$$Q = \begin{matrix} & S' & \underline{1} \\ \begin{matrix} S' \\ \underline{1} \end{matrix} & \begin{bmatrix} Q_0 & -Q_0 \underline{1}' \\ \underline{\mu}(\underline{1}) & -M \end{bmatrix} \end{matrix}$$

where Q_0 is the infinitesimal generator of the lossy process of $\underline{Z}$ defined over the state space $S' \equiv S - \underline{1}$ (Keilson (1979)), $\underline{\mu}(\underline{1})$ is the vector of rates at which the process $\underline{Z}$ leaves the state $\underline{1}$. The rate from state $\underline{1}$ to $(1, 1, \dots, 1, 0, 1, \dots, 1)$ when the 0 is in the i -th position is $\mu_i(\underline{1})$, $i = 1, 2, \dots, n$. All other rates are zero. So

$$M = \sum_{i=1}^n \mu_i(\underline{1}) = \underline{\mu}(\underline{1}) \cdot \underline{1}'.$$

For the example corresponding to Figure 2, $\underline{\mu}(\underline{1}) = (0, 0, 0, 0, \mu_3, \mu_2, \mu_1)$, $M = (\mu_1 + \mu_2 + \mu_3)$ and $Q_0 \underline{1}' = (0, 0, 0, 0, \lambda, \lambda, \lambda)'$. Then (e.g. Keilson (1979), Neuts (1981), Shanthikumar (1985)),

$$E\{X_0\} = \underline{a} Q_0^{-1} \underline{1}', \quad E\{X\} = \underline{b} Q_0^{-1} \underline{1}'$$

$$\bar{F}_0(x) = \underline{a} \left\{ \sum_{m=0}^{\infty} \frac{p^m e^{-\lambda x} (\lambda x)^m}{m!} \right\} \underline{1}', \quad x > 0$$

$$F(x) = \underline{b} \left\{ \sum_{m=0}^{\infty} \frac{p^m e^{-\lambda x} (\lambda x)^m}{m!} \right\} \underline{1}', \quad x > 0$$

where $\underline{a}$ is the initial probability distribution of $\underline{Z}(0)$ and $\underline{b}$ is the probability distribution of $\underline{Z}$ immediately after its exit from state $\underline{1}$. That is $\underline{b}(1, 1, \dots, 1, 0, 1, \dots, 1) = \mu_i(\underline{1})/M$, $i=1, 2, \dots, n$ and all other probabilities are zero: i.e. $\underline{b} = \underline{\mu}(\underline{1})/M$. Similarly if the system starts from perfection $P\{\underline{Z}(0) = \underline{0}\} = \underline{a}(0, 0, \dots, 0) = 1$. Furthermore,

$$\Lambda = \max \{-Q_0(i, i)\}$$

$$P = I + \frac{1}{\Lambda} Q_0$$

and I is an identity matrix of appropriate dimension. Also one can easily see that

$$E\{Y_j\} = 1/M$$

and

$$G(x) = 1 - \exp(-Mx), \quad x > 0.$$

We will next provide sufficient conditions under which Theorem 2 can be applied to this case of multivariate binary birth and death process $\underline{Z}$ defined on the state space $S = \{0, 1\}^n$. For each state $\underline{x} \in S$ define $W(\underline{x}) = \{i: X_i = 0, i = 1, 2, \dots, n\}$ and $\bar{W}(\underline{x}) = \{1, 2, \dots, n\} - W(\underline{x})$. That is $W(\underline{x})$ is the set of working components in state $\underline{x}$. Then Shaked and Shanthikumar (1985) has shown that if

$$(C1) \quad \left\{ \begin{array}{l} \underline{Z}(0) = \underline{0} \text{ and for all } \underline{x}, \underline{y} \in S \text{ such that} \\ W(\underline{x}) \subset W(\underline{y}), \lambda_i(\underline{x}) \leq \lambda_i(\underline{y}), i \in W(\underline{x}) \\ \text{and } \mu_i(\underline{x}) \geq \mu_i(\underline{y}), i \in \bar{W}(\underline{y}), \end{array} \right.$$

st

then X_0 is NBU and $X_0 \geq X_{i|X_i > u}^{-u}$, $u > 0$, $i = 1, 2, \dots$. Then from Theorem 2 one has

Corollary 5: Assume that $\underline{Z}$ satisfies Condition (C1). Then V_T is NBU, whenever T_i is NBU.

Note that when the components are independent (C1) is automatically satisfied. Since a degenerate random variable is NBU, one can easily see that Corollary 5 is a generalisation of Proposition 2 of Ross and Schechtman (1979). In fact the above result holds true for more general systems, more general than those considered in Chiang and Niu (1980). Such a generalisation is discussed in Shaked and Shanthikumar (1985).

2. DEPENDENT PARALLEL SYSTEMS WITH PHASE TYPE SAFETY PERIODS

In this section we assume that $(T_j)_{j=1}^{\infty}$ has a phase type distribution with representation $(\underline{c}, C)$ and m phases (e.g. Neuts (1981)). That is T_j has the same distribution as the time to exit from a lossy process (say) $N = \{N(t), t \in R_+\}$ with state space $\{1, 2, \dots, m\}$, infinitesimal generator C and initial probability vector $\underline{c}$. Then

$$E\{T_j\} = \underline{c} C^{-1} \underline{1}',$$

and

$$\bar{H}(x) = \underline{c} \left\{ \sum_{i=0}^{\infty} \left(I + \frac{1}{\eta} C \right)^i e^{-\eta x} \frac{(\eta x)^i}{i!} \underline{1}' \right\},$$

where

$$\eta = \max \{-C(i, i)\}.$$

The evolution of the parallel system with a back-up unit that has a phase type lifetime can be described as follows: the process $\underline{Z}$ evolves as before starting from some initial state $\underline{Z}(0)$ different from $\underline{1}$. Every time $\underline{Z}$ enters state $\underline{1}$ an auxillary process N is initiated into state i with probability c_i , $i = 1, 2, \dots, m$. The process N is lossy and is governed by the infinitesimal generator C . If the auxillary process N leaves the set of states $\{1, 2, \dots, m\}$ before $\underline{Z}$ exits state $\underline{1}$, the system fails. Otherwise, the auxillary process is terminated as soon as $\underline{Z}$ exits state $\underline{1}$. Let $\hat{\underline{Z}} = \{\hat{\underline{Z}}(t), t \in R_+\}$ be this combined process. The state space of $\hat{\underline{Z}}$ is then $S'U\{(\underline{1}, 1), (\underline{1}, 2), \dots, (\underline{1}, m)\}$. Note that a state $(\underline{1}, i)$ means that $\underline{Z}$ is in state $\underline{1}$, and N is in state i , $i = 1, 2, \dots, m$. The infinitesimal generator $\hat{Q}$ of this process $\hat{\underline{Z}}$ is given by

$$\hat{Q} = \begin{matrix} S' & & \\ & \begin{matrix} 1 \times \{1, 2, \dots, m\} \end{matrix} & \end{matrix} \begin{bmatrix} Q_0 & -Q_0 \underline{1}' \cdot \underline{c} \\ \underline{1}' \cdot \underline{\mu}(\underline{1}) & -M\underline{I} + C \end{bmatrix}.$$

An example with Q as given in Figure 2 and $m = 2$ is given in Figure 3.

Note that $\hat{Z}$ is a lossy process and the exit time is V_T . Therefore

$$E\{V_T\} = (\underline{a}, \underline{0}) \hat{Q}^{-1} \underline{1}', \quad (15)$$

and

$$P\{V_T > x\} = (\underline{a}, \underline{0}) \left\{ \sum_{i=0}^{\infty} \left(1 + \frac{1}{\beta} \hat{Q}\right)^i \frac{e^{-\beta x} (\beta x)^i}{i!} \right\} \underline{1}', \quad (16)$$

$$x > 0,$$

where

$$\beta = \max \{-\hat{Q}(i, i)\}.$$

Since the class of phase type distributions is weakly dense, one may use the above results to approximate the distribution of V_T when the safety period do not have a phase type distribution. We will next consider a very special case of this: the safety period is a fixed constant.

3. SYSTEM FAILURE TIME WITH FIXED SAFETY PERIOD

In this section we will assume that the lifetime of the back-up unit is a fixed constant A . That is $T_j = A$ w.p.1, $j = 1, 2, \dots$. Then from (5) and (14) one obtains

$$Q = \begin{matrix} & (0,0,0) & (1,0,0) & (0,1,0) & (0,0,1) & (1,1,0) & (1,0,1) & (0,1,1) & (1,1,1,1) & (1,1,1,2) \\ \begin{matrix} (0,0,0) \\ (1,0,0) \\ (0,1,0) \\ (0,0,1) \\ (1,1,0) \\ (1,0,1) \\ (0,1,1) \\ (1,1,1,1) \\ (1,1,1,2) \end{matrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{matrix}$$

Figure 3: Infinitesimal Generator $\hat{Q}$ with Phase Type Safety Period with Representation $(\underline{c}, C)$ where

$$\underline{c} = (c_1, c_2) \text{ and } C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$E\{V_A\} = (\underline{a} + e^{MA} \underline{b} - \underline{b}) Q_0^{-1} \underline{1} + e^{MA} \int_0^A Mx e^{-Mx} dx + A.$$

The probability distribution $P\{V_A \leq x\}$ can be obtained from (7) with appropriate substitution. Specifically,

$$P\{V_A \leq x\} = 0, \quad x \leq A$$

and

$$P\{V_A \leq x\} = \{e^{-MA} \sum_{j=0}^{\infty} (1-e^{-MA})^j F^{(j)} \otimes G^{*(j)}\} \otimes F_0(x-A), \quad x \geq A,$$

where F and F_0 are given by (14) and

$$G^*(x) = (1-e^{-Mx})/(1-e^{-MA}), \quad 0 \leq x \leq A.$$

In the spirit of Theorem 4 and Equation (12) one may also use the following exponential approximations for $P\{V_A \leq x\}$ when e^{-MA} is very small.

$$P\{V_A \leq x\} \cong F_0 \otimes B(x-A), \quad x \geq A \quad (17)$$

and

$$P\{V_A \leq x\} \cong 1 - \exp\{-(x-A)/(E\{V_A\}-A)\}, \quad x \geq A,$$

where

$$B(x) = 1 - \exp\{-xe^{-MA}/\{(E\{X\} + E\{Y^*\})(1-e^{-MA})\}\}, \quad (18)$$

and

$$E\{Y^*\} = (1-e^{-MA})^{-1} \int_0^A Mx e^{-Mx} dx.$$

In (17) we approximate the time to first failure distribution by the

convolution of a phase type distribution F_0 given by (14) and an exponential distribution given by (18). Next we will look at an alternative approximation for $P\{V_A \leq x\}$.

It is well known that a degenerate distribution can be arbitrarily closely approximated by an Erlang- m distribution with sufficiently large m . Let $(T_j^m)_{j=1}^\infty$ be an i.i.d. sequence of safety periods with Erlang- m distribution: equivalently phase type distribution with representation $(\underline{c}_m, C_m)$ where $\underline{c}_m = (1, 0, 0, \dots, 0)$, $C_m(i, i+1) = m/A = -C_m(i, i)$, $i = 1, 2, \dots, m-1$; $C_m(m, m) = -m/A$ for $m = 1, 2, \dots$. Note that $E\{T_j^m\} = A$ for all $m = 1, 2, \dots$ and $\text{Var}\{T_j^m\} = A^2/m$, $m = 1, 2, \dots$. Now let V_{T^m} be the system lifetime V_T obtained from (4) with $(T_j)_{j=1}^\infty$ replaced by $(T_j^m)_{j=1}^\infty$. Now combining the above observations with the results in Section 2 (Equation (15) and (16)) one obtains

$$E\{V_{T^m}\} = (\underline{a}, \underline{0}) \hat{Q}_m^{-1} \underline{1}',$$

and

$$P\{V_{T^m} \leq x\} = (\underline{a}, \underline{0}) \left\{ \sum_{i=0}^{\infty} \left(1 + \frac{1}{\beta_m} \hat{Q}_m\right) \frac{e^{-\beta_m x} (\beta_m x)^i}{i!} \right\} \underline{1}', \quad x > 0, \\ m = 1, 2, \dots,$$

where

$$\hat{Q}_m = \begin{bmatrix} Q_0 & -Q_0 \underline{1}' \cdot \underline{c}_m \\ \underline{1}' \cdot \underline{\mu}(\underline{1}) & -M \underline{I} + C_m \end{bmatrix}$$

and $\beta_m = \max \{-\hat{Q}_m(i, i)\}$. Thus, for large m one has

$$E\{V_A\} \cong E\{V_{T^m}\}$$

and

$$P\{V_A \leq x\} \cong P\{V_{T^m} \leq x\}.$$

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