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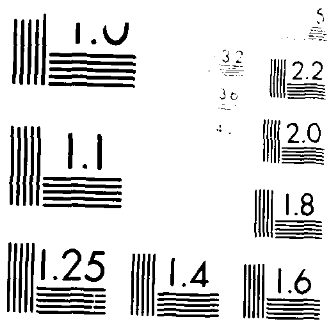
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Let F denote the life distribution of a coherent structure of independent components. Suppose that we have a sample of independent systems, each having the structure φ. Each system is continuously observed until it fails. For every component in each system, either a failure time or a censoring time is recorded. A failure time is recorded if the component fails before or at the time of system failure; otherwise a censoring time is recorded. We introduce a method for finding estimates for F, quantiles, and other functionals of F, based on the censorship of the component lives by system failure. We present limit theorems that enable the construction of confidence intervals for large samples.

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ESTIMATING JOINTLY SYSTEM AND COMPONENT RELIABILITIES USING A MUTUAL CENSORSHIP APPROACH

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ABSTRACT

Let F denote the life distribution of a coherent structure of independent components. Suppose that we have a sample of independent systems, each having the structure ϕ. Each system is continuously observed until it fails. For every component in each system, either a failure time or a censoring time is recorded. A failure time is recorded if the component fails before or at the time of system failure; otherwise a censoring time is recorded. We introduce a method for finding estimates for F(t), quantiles, and other functionals of F, based on the censorship of the component lives by system failure. We present limit theorems that enable the construction of confidence intervals for large samples.

1. INTRODUCTION AND SUMMARY.

Consider a system of independent components labeled 1 through m. We assume that the system forms a coherent structure, which we denote by ϕ. Specifically, the system and each component are in either a functioning state or a failed state, and the state of the system depends only on the states of the components; see Barlow and Proschan (1981, Chapters 1 and 2) for definitions and basic facts relating to coherent systems. Let F_j denote the life distribution of component j, $j = 1, 2, \ldots, m$, and F_ϕ, or simply F, denote the life distribution of the system.

Suppose that we have a sample of n independent systems, each with the same structure ϕ. Each system is continuously observed until it fails. For every component in each system, either a failure time or a censoring time is recorded. A failure time is recorded if the component fails before or at the time of system failure. A censoring time is recorded if the component is still functioning at the time of system failure. From these failure times and censoring times we wish to estimate F.

In order to distinguish between components and systems, we index systems with the letter i and components with the letter j; i ranges over 1, ..., n, and j over 1, ..., m. All random variables are non-negative. We define the following random variables:

T_i is the lifelength of system i,

X_{ij} is the lifelength of component j in system i,

for each j, $X_{1j}, X_{2j}, \ldots, X_{nj}$ are iid $\sim F_j$,

$Z_{ij} = \min(X_{ij}, T_i)$,

and

$\delta_{ij} = I(X_{ij} \le T_i)$, where $I(A)$ is the indicator function of the set A.

Z_{ij} records the time on test of component j of system i, and δ_{ij} indicates whether component j in system i is uncensored ($\delta_{ij} = 1$) or censored ($\delta_{ij} = 0$). For each j, $Z_{1j}, \ldots, Z_{nj}$ are iid with distribution H_j. The sequence $\{(Z_{ij}, \delta_{ij});$ $1 \le j \le m,\ 1 \le i \le n\}$ contains all the information used in estimating F, and thus is called *the sample information.*

The system life distribution F can be estimated by the empirical estimator $\hat{F}^{emp}$ defined for $t \ge 0$ by

$$\hat{F}^{emp}(t) = \frac{1}{n} \sum_{i=1}^{n} I(T_i \le t). \tag{1.1}$$

$\hat{F}^{emp}$ does not fully utilize the sample information. Specifically, it does not use the following information: the identity of the components still functioning at system failure time, and the failure times of the components failing before system failure time.

We propose an estimator $\hat{F}$ of F that uses the information described above. The estimator, described next, is based upon estimators of the component life distributions $F_1, \ldots, F_m$. For each j, let $Z_{(1)j} \le Z_{(2)j} \le \cdots \le Z_{(n)j}$ be the ordered values of $Z_{1j}, Z_{2j}, \ldots, Z_{nj}$. Define

$$\delta_{(i)j} = \begin{cases} 1 & \text{if } Z_{(i)j} \text{ corresponds to an uncensored lifetime} \\ 0 & \text{if } Z_{(i)j} \text{ corresponds [illegible]} \end{cases} \tag{1.2}$$

(When an uncensored and a censored observation [illegible] observation is considered to have occurred [illegible] Meier estimator of F:

$$\hat{F}_j(t) = 1 - \prod_{i: Z_{(i)j} \le t} \text{[illegible]} \tag{1.3}$$

The definition above differs from the usual definition [illegible] Meier estimator in that $\hat{F}_j(t)$ is not arbitrarily defined to be [illegible]

For each coherent structure ϕ of independent components, there corresponds a function h_ϕ, called the *reliability function*, such that

$$\bar{F}_\phi(t) = h_\phi(\bar{F}_1(t), \ldots, \bar{F}_m(t)). \tag{1.4}$$

Here, $\bar{F}_\phi(t) = 1 - F_\phi(t)$ and $\bar{F}_j(t) = 1 - F_j(t)$. A more detailed description of reliability functions is provided in Chapter 2 of Barlow and Proschan (1981). The estimator $\hat{F}$ is defined by

$$\hat{F}(t) = \begin{cases} 1 - h_\phi(\hat{\bar{F}}_1(t), \ldots, \hat{\bar{F}}_m(t)) & \text{if } t < T_{(n)} \\ 1 & \text{if } t \geq T_{(n)}. \end{cases} \tag{1.5}$$

Here, $T_{(n)} = \max(T_1, T_2, \ldots, T_n)$. The estimator $\hat{F}$ has obvious intuitive appeal.

The properties of the Kaplan-Meier estimator have been studied extensively by various authors. Under the assumption that the censoring variables and the lifelengths are independent, the Kaplan-Meier estimator is the maximum likelihood estimate (Kaplan and Meier, 1958; Johansen, 1978). Regarded as a stochastic process, it is strongly uniformly consistent (Földes, Rejtö, and Winter, 1980) and converges weakly to a Gaussian process (Breslow and Crowley, 1974, Aalen, 1976, and Gill, 1983).

The main results of this paper can now be stated. Let $D[0,T]$ be the space of all real valued functions defined on $[0,T]$ that are right continuous and have left limits, with the Skorohod metric topology. $D^m[0,T]$ denotes the product metric space.

<u>THEOREM 1</u>. Suppose F_1, F_2, ..., F_m are continuous, and let T be such that $F_j(T) < 1$ for $j = 1, 2, \ldots, m$. Then as $n \to \infty$

$$n^{\frac{1}{2}}(\hat{F}_1 - F_1, \hat{F}_2 - F_2, \ldots, \hat{F}_m - F_m) \to (W_1, W_2, \ldots, W_m)$$

weakly in $D^m[0,T]$, where $W_1, \ldots, W_m$ are *independent* mean 0 Gaussian processes. The covariance structure of W_j is given by

$$\mathrm{Cov}(W_j(t_1), W_j(t_2)) = \bar{F}_j(t_1)\bar{F}_j(t_2) \int_0^{t_1} \frac{dF_j(u)}{\bar{H}_j(u)\bar{F}_j(u)} \quad \text{for } 0 \leq t_1 \leq t_2 \leq T. \tag{1.6}$$

Since in general the dependence among the $\hat{F}_j$'s may be complex, Theorem 1 is not a trivial extension of the corresponding result for the individual Kaplan-Meier estimators $\hat{F}_j$.

The next theorem gives a central result regarding the estimator $\hat{F}$.

<u>THEOREM 2</u>. Suppose $F_1, F_2, \ldots, F_m$ are continuous, and suppose T is such that $F_j(T) < 1$ for $j = 1, 2, \ldots, m$. Then as $n \to \infty$

$$n^{\frac{1}{2}}(\hat{F} - F) \to W \quad \text{weakly in } D[0,T],$$

where W is a mean 0 Gaussian process with covariance structure given by

$$\begin{aligned} \mathrm{Cov}(W(t_1), W(t_2)) = \sum_{j=1}^{m} & \left(\frac{\partial h_\phi}{\partial u_j}(u_1,\ldots,u_m) \Bigg|_{\substack{(u_1,\ldots,u_m) = \\ (\bar{F}_1(t_1),\ldots,\bar{F}_m(t_1))}} \right) \\ & \left(\frac{\partial h_\phi}{\partial u_j}(u_1,\ldots,u_m) \Bigg|_{\substack{(u_1,\ldots,u_m) = \\ (\bar{F}_1(t_2),\ldots,\bar{F}_m(t_2))}} \right) \\ & \bar{F}_j(t_1)\bar{F}_j(t_2) \int_0^{t_1} \frac{dF_j(u)}{\bar{H}_j(u)\bar{F}_j(u)} \quad \text{for } 0 \le t_1 \le t_2 \le T. \end{aligned} \tag{1.7}$$

The commonly used estimate of the variance of the Kaplan-Meier estimate is given by Greenwood's formula (see Chapter 3 of Miller, 1981). Since this estimate is known to be consistent (see Hall and Wellner, 1980), it follows that for fixed t, the variance of $\hat{F}(t)$ given in Theorem 2 can be consistently estimated. This enables the construction of confidence intervals for $F(t)$ in large samples.

The method of estimating F proposed here has an additional advantage: the estimates $\hat{F}_1, \hat{F}_2, \ldots, \hat{F}_m$ can be used to estimate the life distribution of any structure ψ whose components form a subset of the components of ϕ. Specifically, if h_ψ is the reliability function of ψ, then the estimate $\hat{F}_\psi$ defined by

$$\hat{F}_\psi(t) = 1 - h_\psi(\hat{\bar{F}}_1(t),\ldots,\hat{\bar{F}}_m(t)), \tag{1.8}$$

when suitably normalized, converges to a Gaussian process; this fact will be clear from the proof of Theorem 2.

In the large literature on point and interval estimation of system reliability it is always assumed that the components are tested separately; for a survey and references see Mann, Schafer, and Singpurwalla (1974). The idea of basing the estimate of the system life distribution on estimates of the component life distributions with the resulting censoring considerations is new in reliability theory. This approach extends and gives a novel application of censoring methodology.

The competing risks model corresponds to a series system. Aalen (1976) showed that for this model, the vector of Kaplan-Meier estimates $(\hat{F}_1, \ldots, \hat{F}_m)$, when normalized, converges to a multidimensional Gaussian process, whose components are independent. This result corresponds to our Theorem 1 for the case of a series structure.

This paper is organized as follows. Section 2 gives some definitions, preliminary results including the strong consistency of $\hat{F}$, and results without proof concerning the Kaplan-Meier estimator to be used subsequently. Section 3 is technical, and applies martingale theory to obtain the results of Theorems 1 and 2. It contains all the terminology and facts concerning martingales that are needed to prove Theorems 1 and 2. Section 4 gives an application of the results of Section 3 to system design methods. The Appendix gives a proof of a result used in Sections 2 and 3.

2. PRELIMINARIES AND DEFINITIONS.

Corresp nding to a generic system, we define generic random variables X_j, Z_j, δ_j, and T, such that the random vector (X_j, Z_j, δ_j, T) has the same distribution as $(X_{ij}, Z_{ij}, \delta_{ij}, T_i)$ for $i = 1, 2, \ldots, n$, and $j = 1, 2, \ldots, m$.

In Section 1 it is noted that the strong consistency and weak convergence results for the Kaplan-Meier estimator are valid under the assumption that the lifelengths and the censoring random variables are independent. In our model, for each j, X_j is censored by T, and for a coherent structure these two random variables are dependent. However, it is possible to redefine the censoring variables to circumvent this difficulty. This is best explained in terms of a simple example. Consider the structure shown diagrammatically in Figure 2.1.

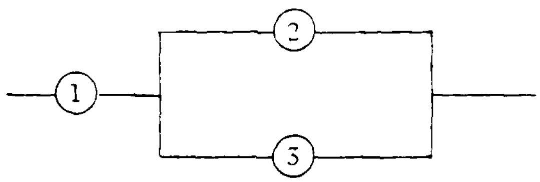

Figure 2.1.

In the example $T = X_1 \wedge (X_2 \vee X_3)$, where $x \wedge y = \min(x,y)$ and $x \vee y = \max(x,y)$. Consider now component 1. Clearly X_1 is censored by $Y_1 = X_2 \vee X_3$, which *is* independent of X_1. Similarly, X_2 is censored by $Y_2 = X_1$, and X_3 by $Y_3 = X_1$.

In the Appendix it is shown that in general, for each $j = 1, \ldots, m$ there is a nonnegative random variable Y_j such that

$$(Z_j, \delta_j) = (X_j \wedge Y_j,\ I(X_j \le Y_j)), \tag{2.1}$$

and

$$X_j \text{ and } Y_j \text{ are independent.} \tag{2.2}$$

We refer to Y_j as the *censoring variable of X_j*. Statements (2.1) and (2.2) imply that the censoring of a component lifelength is described by the random censorship model (Gilbert, 1962). Roughly speaking, Y_j is the lifelength of the system if X_j is replaced by ∞.

In order to describe the distribution of Y_j we introduce some notation. For $y = (y_1, \ldots, y_m) \in [0,1]^m$, $\alpha \in [0,1]$, and $j = 1, \ldots, m$, let

$$(\alpha_j, y) = (y_1, \ldots, y_{j-1}, \alpha, y_{j+1}, \ldots, y_m). \tag{2.3}$$

Let $\underline{\bar{F}}(t) = (\bar{F}_1(t),\ldots,\bar{F}_m(t))$ and recall that H_j is the distribution of Z_j. In the Appendix it is shown that

$$P(Y_j > t) = h_\phi(1_j, \underline{\bar{F}}(t)), \tag{2.4}$$

where h_ϕ is the reliability function (see (1.4)). Thus,

$$\bar{H}_j(t) = \bar{F}_j(t) h_\phi(1_j, \underline{\bar{F}}(t)). \tag{2.5}$$

We now review some terminology from reliability theory (see e.g. Barlow and Proschan, 1981) to be used in the proof of consistency of $\hat{F}$ and in later sections. For a coherent system of m components, the states of the components correspond to a vector $U = (U_1,\ldots,U_m)$, where $U_j = I$(component j is in a functioning state). The *structure function* is defined by $\phi(U) = I$(System functions when U describes the states of the components) for $U \in A_m$, where $A_m = \{0,1\}^m$. It is well-known (and easy to see) that for $p = (p_1,\ldots,p_m) \in [0,1]^m$,

$$h_\phi(p) = \sum_{U \in A_m} \phi(U) \prod_{j=1}^{m} p_j^{U_j} (1-p_j)^{1-U_j}, \tag{2.6}$$

where $0^0 = 1$ by definition.

The Kaplan-Meier estimates $\hat{F}_j$ given by (1.3) will be denoted $\hat{F}_j^n$ when we want to emphasize the dependence on n; similarly for the estimate $\hat{F}$ of system life distribution. Also, $\underline{\hat{F}}^n$ will denote the vector $(\hat{F}_1^n,\ldots,\hat{F}_m^n)$.

The following propositions specialize the properties of the Kaplan-Meier estimator investigated in the literature to the estimators $\hat{F}_j^n$.

PROPOSITION 2.1. (Földes, Rejtö, and Winter, 1980). If $T > 0$ is such that $H_j(T) < 1$, then

$$P\left\{\sup_{0 \le t \le T} \left|\hat{F}_j^n(t) - F_j(t)\right| = o\left(\frac{(\ln n)^{\frac{1}{2}}}{n^{\frac{1}{4}}}\right)\right\} = 1.$$

Thus, $\hat{F}_j^n$ is a strongly uniformly consistent estimator of F_j on the interval $[0,T]$.

Under the assumption of continuity of the distributions of the component lifelengths and the censoring random variables, the rate of convergence is improved.

PROPOSITION 2.2. (Földes and Rejtö, 1981). If F_j and $h(1_j, \underline{\bar{F}})$ are continuous and $T > 0$ is such that $H_j(T) < 1$, then

$$P\left\{ \sup_{0 \le t \le T} \left| \hat{F}_j^n(t) - F_j(t) \right| = O\left(\sqrt{\frac{\ln \ln n}{n}} \right) \right\} = 1.$$

We note that if $F_1, \ldots, F_m$ are continuous, then $h(1_j, \underline{\bar{F}})$ is continuous; see Lemma 2.1 below.

PROPOSITION 2.3. (Breslow and Crowley, 1974). If F_j and $h(1_j, \underline{\bar{F}})$ are continuous and $T > 0$ is such that $H_j(T) < 1$, then $n^{\frac{1}{2}}(F_j^n - F_j)$ converges weakly to a zero mean Gaussian process W on $D[0,T]$ whose covariance function is given by

$$\mathrm{Cov}(W(t_1), W(t_2)) = \bar{F}_j(t_1)\bar{F}_j(t_2) \int_0^{t_1 \wedge t_2} \frac{dF_j(u)}{\bar{H}_j(u)\bar{F}_j(u)} .$$

The following lemma is needed in the proofs of strong uniform consistency and weak convergence of $\hat{F}$.

LEMMA 2.1. For any structure of m independent components, the corresponding reliability function h_ϕ is twice continuously differentiable over $[0,1]^m$, and the first and second partial derivatives are bounded in absolute value by 1 uniformly over $[0,1]^m$.

Proof: For $p = (p_1, \ldots, p_m) \in [0,1]^m$ and $k = 1, 2, \ldots, m$, we have by (2.6),

$$\frac{\partial h_\phi}{\partial p_k}(p) = h_\phi(1_k, p) - h_\phi(0_k, p). \tag{2.7}$$

From (2.7) we have

$$\frac{\partial^2 h_\phi}{\partial p_k^2}(p) = 0, \tag{2.8}$$

and for $\ell \neq k$,

$$\left.\frac{\partial^2 h_\delta}{\partial p_k \partial p_\ell}\right|_p = \{h_\delta(1_k,1_\ell,p) - h_\delta(1_k,0_\ell,p)\} - \{h_\delta(0_k,1_\ell,p) - h_\delta(0_k,0_\ell,p)\}, \qquad (2.9)$$

in an obvious extension of the notation (2.3). By (2.6) h_δ is continuous over $[0,1]^m$. This fact together with (2.7), (2.8), and (2.9) imply that the first and second partial derivatives are continuous on $[0,1]^m$; hence, by Theorem 6.18 of Apostol (1964), h_δ is twice continuously differentiable on $[0,1]^m$. Equation (2.7) implies that the first partials are bounded in absolute value by 1. Since each of the two quantities inside the braces on the right side of (2.9) is between 0 and 1, it follows that the second partials are also bounded in absolute value by 1. The lemma follows since p is arbitrary. $\square$

We now establish the strong uniform consistency of $\hat{F}$ and give the rate of convergence.

PROPOSITION 2.4. Let $T > 0$ be such that for $j = 1, \ldots, m$, $\min(F_j(T), \bar{F}_j(T)) > 0$. Then

(a) $$P\left\{\sup_{0\le t\le T}\left|\hat{F}(t) - F(t)\right| = o\left(\frac{(\ell n\ n)^{\frac{1}{2}}}{n^{\frac{1}{2}}}\right)\right\} = 1.$$

(b) If $F_1, \ldots, F_m$ are continuous, the rate

$$o\left(\frac{(\ell n\ n)^{\frac{1}{2}}}{n^{\frac{1}{2}}}\right) \text{ may be replaced by } O\left(\sqrt{\frac{\ell n\ \ell n\ n}{n}}\right).$$

Proof: We fix $t \in [0,T]$ and consider $h_\delta(\underline{\bar{F}}^n(t)) - h_\delta(\underline{\bar{F}}(t))$. Since h_δ is continuously differentiable over $[0,1]^m$ by Lemma 2.1, we can apply the Mean Value Theorem (see for example Apostol, 1964, Theorem 6.17): there exists a point X_t^* lying on the line segment joining $\underline{\bar{F}}^n(t)$ and $\underline{\bar{F}}(t)$ such that

$$h_\delta(\underline{\bar{F}}^n(t)) - h_\delta(\underline{\bar{F}}(t)) = \left\{\left.\nabla h_\delta(p)\right|_{p=X_t^*}\right\}\cdot\left\{\underline{\bar{F}}^n(t) - \underline{\bar{F}}(t)\right\}, \qquad (2.10)$$

where ∇h_ϕ is the gradient of h_ϕ. In view of Lemma 2.1, Proposition 2.1 proves Part (a), and Proposition 2.2 proves Part (b). $\square$

3. WEAK CONVERGENCE RESULTS.

In Section 3.1 we outline the proofs of Theorems 1 and 2 and indicate where the theory of stochastic integration and counting processes is needed. Section 3.2 reviews the elements of this theory that are needed in this paper. Section 3.3 uses the results stated in Section 3.2 to give rigorous proofs of Theorems 1 and 2.

Throughout Section 3 we adopt the convention that $\frac{0}{0} = 0$. The index n used in defining a process is suppressed whenever possible.

3.1. Sketch of the Proofs of Theorems 1 and 2.

To prove Theorem 1, we show that for any $T > 0$ satisfying $\max_{1 \le j \le m} F_j(T) < 1$, that

$$n^{\frac{1}{2}}\left(\frac{\hat{F}_1 - F_1}{\bar{F}_1}, \ldots, \frac{\hat{F}_m - F_m}{\bar{F}_m}\right) \xrightarrow{d} (W_1^*, \ldots, W_m^*), \tag{3.1}$$

where $W_1^*, \ldots, W_m^*$ are independent mean zero Gaussian processes with covariance given by

$$\operatorname{Cov}(W_j^*(t_1), W_j^*(t_2)) = \int_0^{t_1} \frac{dF_j(u)}{\bar{H}_j(u)\bar{F}_j(u)} \quad \text{for } 0 \le t_1 \le t_2 \le T. \tag{3.2}$$

(Now and henceforth, the symbol d signifies weak convergences in $D^m[0,T]$.) Theorem 1 is an easy consequence of (3.1) and Theorem 5.1 of Billingsley (1968).

We prove (3.1) by a general method introduced by Aalen (1978) and later refined by Gill (1980). We define the stopped process F_j^{*n} on $[0,\infty)$, $j = 1, 2, \ldots, m$, by

$$F_j^*(t) = F_j(t \wedge Z_{(n)j}), \text{ and } \bar{F}_j^*(t) = 1 - F_j^*(t), \tag{3.3}$$

and use the following decomposition:

$$n^{\frac{1}{2}}\left(\frac{\hat{F}_1 - F_1}{\bar{F}_1}, \ldots, \frac{\hat{F}_m - F_m}{\bar{F}_m}\right)$$
$$= n^{\frac{1}{2}}\left(\frac{\hat{F}_1 - F_1^*}{\bar{F}_1^*}, \ldots, \frac{\hat{F}_m - F_m^*}{\bar{F}_m^*}\right) + n^{\frac{1}{2}}\left(\frac{\hat{F}_1 - F_1}{\bar{F}_1} - \frac{\hat{F}_1 - F_1^*}{\bar{F}_1^*}, \ldots, \frac{\hat{F}_m - F_m}{\bar{F}_m} - \frac{\hat{F}_m - F_m^*}{\bar{F}_m^*}\right). \tag{3.4}$$

It is easy to see that

$$n^{\frac{1}{2}}\left(\frac{\hat{F}_1 - F_1}{\bar{F}_1} - \frac{\hat{F}_1 - F_1^*}{\bar{F}_1^*}, \ldots, \frac{\hat{F}_m - F_m}{\bar{F}_m} - \frac{\hat{F}_m - F_m^*}{\bar{F}_m^*}\right) \overset{p}{\to} 0. \tag{3.5}$$

in $D^m[0,T]$. (Now and henceforth, the symbol p signifies convergence in probability.) Thus, the proof consists in showing that

$$n^{\frac{1}{2}}\left(\frac{\hat{F}_1 - F_1^*}{\bar{F}_1^*}, \ldots, \frac{\hat{F}_m - F_m^*}{\bar{F}_m^*}\right) \overset{d}{\to} (W_1^*, \ldots, W_m^*). \tag{3.6}$$

To show (3.6) we first establish that for each j and for all n, $\left\{n^{\frac{1}{2}}\left(\frac{\hat{F}_j(t) - F_j^*(t)}{\bar{F}_j^*(t)}\right); t \in [0,T]\right\}$ is a martingale with respect to $\{\mathcal{F}_t;\ t \in [0,T]\}$, where $\mathcal{F}_t$ is the σ-field generated by the observed component failure times up to time t. Formally, $\mathcal{F}_t$ is the completion of

$$\sigma\{I(Z_{ij} \le s,\ \delta_{ij} = 1);\ 1 \le i \le n,\ 1 \le j \le m,\ s \le t\}. \tag{3.7}$$

We complete the proof of (3.6) by applying a multivariate version of a martingale central limit theorem due to Rebolledo (1980).

To show that $\left\{\left(n^{\frac{1}{2}}\left(\frac{\hat{F}_j(t) - F_j^*(t)}{\bar{F}_j^*(t)}\right), \mathcal{F}_t\right); t \in [0,T]\right\}$ is a martingale, we define the following processes on $[0,\infty)$.

$$J_j(t) = I(Z_{(n)j} \ge t). \tag{3.8}$$

$$N_{ij}(t) = I(Z_{ij} \le t,\ \delta_{ij} = 1). \tag{3.9}$$

$$N_j^n(t) = \sum_{i=1}^{n} N_{ij}(t). \tag{3.10}$$

$$V_{ij}(t) = I(Z_{ij} \ge t). \tag{3.11}$$

$$V_j^n(t) = \sum_{i=1}^{n} V_{ij}(t). \tag{3.12}$$

$$A_{ij}(t) = \int_0^t \left[\frac{V_{ij}(s)}{\bar{F}_j(s)}\right] dF_j(s). \tag{3.13}$$

$$A_j^n(t) = \int_0^t \left[\frac{V_j^n(s)}{\bar{F}_j(s)}\right] dF_j(s) \qquad \left(= \sum_{i=1}^{n} A_{ij}(t)\right). \tag{3.14}$$

$$M_{ij}(t) = N_{ij}(t) - A_{ij}(t). \tag{3.15}$$

$$M_j^n(t) = N_j^n(t) - A_j^n(t) \qquad \left(= \sum_{i=1}^{n} N_{ij}(t)\right). \tag{3.16}$$

The process $N_j(t)$ records the number of uncensored failures of component j up to time t. The process $V_j(t)$ records the number of systems in which component j is at risk at time t^-. We note that for each t, F_t is the completion of

$$\sigma\{N_{ij}(s);\ 1 \le i \le n,\ 1 \le j \le m,\ s \le t\}. \tag{3.17}$$

The following proposition is fundamental in establishing that

$$\left\{ \left(n^{\frac{1}{2}} \left(\frac{\hat{F}_j(t) - F_j^*(t)}{\bar{F}_j^*(t)} \right),\ F_t \right);\ t \in [0,T] \right\} \text{ is a martingale.}$$

PROPOSITION 3.1.1. (Gill, 1980). Suppose $F_1, \ldots, F_n$ are continuous and $t \ge 0$ is such that $\max_{1\le j\le m} F_j(t) < 1$. Then for each j and for all n,

$$n^{\frac{1}{2}} \left(\frac{\hat{F}_j(t) - F_j^*(t)}{\bar{F}_j^*(t)} \right) = n^{\frac{1}{2}} \int_0^t \left[\frac{J_j(s)}{V_j(s)} \, \frac{\hat{\bar{F}}_j(s-)}{\bar{F}_j(s)} \right] dM_j(s), \tag{3.18}$$

where $\hat{\hat{F}}_j(s-)$ is the left limit of $\hat{\hat{F}}_j(s)$.

The theory of counting processes is needed to show that $\{(M_j(t), F_t);\ t \in [0,T]\}$ is a martingale. We apply the results in Section 3.2 to show that

$$\left\{\left(n^{\frac{1}{2}}\int_0^t \left[\frac{J_j(s)}{Y_j(s)}\ \frac{\hat{\hat{F}}_j(s-)}{\bar{F}_j(s)}\right] dM_j(s),\ F_t\right);\ t \in [0,T]\right\}$$ is also a martingale, and to verify

the conditions of the martingale central limit theorem.

For fixed t, Theorem 1 and a standard application of the delta method (see for example Section 6.a.2 of Rao, 1973) yields the result that $n^{\frac{1}{2}}(\hat{F}(t) - F(t))$ converges weakly to a normal distribution. We generalize this argument for the process to prove Theorem 2.

3.2. Review of the Theory of Counting Processes and Martingales.

References for the material below are Chapter 2 of Gill (1980) and Chapters 1 - 4 of Chung and Williams (1983). A very accessible review is Andersen and Borgan (1984).

For a complete probability space (Ω', G, P'), a family of sub-sigma-fields $\{G_t;\ t \geq 0\}$ of G is called a standard filtration if for each t,

(i) (Ω', G_t, P') is a complete probability space,

(ii) $G_s \subset G_t$ for $s \leq t$,

(iii) $G_t = \bigcap_{u>t} G_u$.

A process $Y = \{Y(t);\ t \geq 0\}$ is said to be adapted to the standard filtration $\{G_t;\ t \geq 0\}$ if for each t, $Y(t)$ is G_t-measurable. The word adapted will refer to the filtration $\{G_t;\ t \geq 0\}$.

A special class of adapted processes is the class of predictable processes. Rougly speaking, Y is predictable if its value at t is determined by its values at times up to but not including t. A formal definition is given on page 28 of Chung and Williams (1983). We will use the fact that an adapted process that has a.s. left continuous paths is predictable.

A martingale Y is defined to be square integrable if $\sup_t E\, Y^2(t) < \infty$. Henceforth, we assume that all martingales in this section are with respect to $\{G_t;\ t \geq 0\}$ and are square integrable.

It is well known that for a martingale Y, there exists a unique predictable process with nondecreasing paths, called the quadratic variation process of Y and denoted $\langle Y \rangle$, such that $Y^2 - \langle Y \rangle$ is a local martingale (see page 19 of Chung and Williams, 1983, for a definition of a local martingale). It follows from Proposition 1.8 of Chung and Williams (1983) that uniformly bounded local martingales are martingales. Since all local martingales encountered in Section 3.3 are uniformly bounded, the reader may substitute the word martingale for local martingale throughout this section without affecting the material in Section 3.3.

If Y_1 and Y_2 are martingales then their covariation process $\langle Y_1, Y_2 \rangle$ is defined by $\langle Y_1, Y_2 \rangle = \frac{1}{4}(\langle Y_1 + Y_2 \rangle - \langle Y_1 - Y_2 \rangle)$. Kunita and Watanabe (1967) showed that $\langle Y_1, Y_2 \rangle$ is the unique predictable process with paths of bounded variation, such that the process $Y_1 Y_2 - \langle Y_1, Y_2 \rangle$ is a local martingale. It is easily seen that for every martingale Y, we have $\langle Y \rangle = \langle Y, Y \rangle$.

For a process of bounded variation Y, define $\|Y\|$ to be the process such that $\|Y\|(t)$ is the variation of the paths of Y on $[0,t]$. The following proposition is used to show that the integral in (3.18) is a martingale.

<u>PROPOSITION 3.2.1</u>. (Doléans-Dadé and Meyer, 1970). Let G_1' and G_2' be predictable processes and Y_1 and Y_2 be square integrable martingales of bounded variation such that
$$E \int_0^t \left| G_k'(s) \right| d\|Y_k\|(s) < \infty, \quad k = 1,\ 2.$$
Then the processes X_1 and X_2 defined on $[0,t]$ whose paths are defined by the Lebesgue-Stieltjes integral
$$X_k(s) = \int_0^s G_k'(u)\, dY_k(u) \quad s \leq t,$$
is a local martingale on $[0,t]$. Furthermore,

$$\langle X_1, X_2\rangle(s) = \int_0^s G_1^*(u) G_2^*(u)\, d\langle Y_1, Y_2\rangle(u),$$

for $s \le t$.

A vector of adapted processes $(Y_1, \ldots, Y_m)$ is called a counting process if the following hold a.s.

(i) $Y_j(0) = 0$, $j = 1, 2, \ldots, m$. (3.19)

(ii) The paths of each process Y_j are nondecreasing, right continuous and have jumps of size +1 only. (3.20)

(iii) No two processes jump at the same time. (3.21)

Theorem I.9 of Meyer (1976) implies that for each process Y_j, there is a unique predictable process B_j with right continuous and nondecreasing paths originating at 0, such that $Y_j - B_j$ is a local martingale. The process B_j is called the compensator of Y_j.

The following proposition is adapted from a theorem of Murali-Rao (1969) and is useful for identifying the compensator.

<u>PROPOSITION 3.2.2</u>. (Gill, 1980). Let Y be a univariate counting process and let $t \in (0,\infty)$ satisfy $E\, Y(t) < \infty$. Define

$$t_{k,\ell} = \ell 2^{-k} t,\ k = 1, 2, \ldots,\ \ell = 0, 1, \ldots, 2^k$$

and

$$U_k = \sum_{\ell=0}^{2^k - 1} E(Y(t_{k,\ell+1}) - Y(t_{k,\ell}) \mid G_{t_{k,\ell}}),\quad k = 1, 2, \ldots .$$

Then there exists a subsequence of integers $\{r_k\}$, $r_k \to \infty$ as $k \to \infty$, and a unique random variable U, such that for all bounded random variables X,

$$E(XU_{r_k}) \to E(XU)$$

as $k \to \infty$. The compensator B of Y satisfies

$$B(t) = U \quad \text{a.s.}.$$

We note that this result holds for each fixed value of t, and that special care needs to be taken when dealing with a continuum of t's.

The following proposition is a special case of Theorem 2.3.1 of Gill (1980).

PROPOSITION 3.2.3. (Gill, 1980). Suppose $(Y_1,\ldots,Y_m)$ is a counting process with compensators $(B_1,\ldots,B_m)$. Define the process Z_j by $Z_j = Y_j - B_j$, $j = 1, 2, \ldots, m$. If the processes $(B_1,\ldots,B_m)$ have a.s. continuous paths then the following hold.

(i) Z_j is a local square integrable martingale, $j = 1, 2, \ldots, m$.

(ii) $$\langle Z_{j_1}, Z_{j_2}\rangle = \begin{cases} B_{j_1} & \text{if } j_1 = j_2 \\ 0 & \text{if } j_1 \neq j_2 \end{cases}.$$

We use a multivariate extension of a martingale central limit theorem due to Rebolledo (1980) in Section 3.3. The conditions of the proposition below are stronger than those used by Rebolledo. We use the stronger conditions because they are easier to understand and do not take much effort to verify in Section 3.3. Let $f_1, \ldots, f_m$ *be positive functions on* $[0,\infty)$.

PROPOSITION 3.2.4. (Gill, 1980). Suppose that the sequence of vector processes $(Z_1^n,\ldots,Z_m^n)$ $n = 1, 2, \ldots,$ satisfy the following conditions. For every $\varepsilon > 0$, $1 \le j_1, j_2 \le m$, $s \in [0,t]$, and every n,

(i) Z_j^n is a square integrable martingale, (3.22)

(ii) $$\langle Z_{j_1}^n, Z_{j_2}^n\rangle(s) \xrightarrow{p} \begin{cases} f_j(s) & \text{if } j_1 = j_2 \\ 0 & \text{if } j_1 \neq j_2 \end{cases}, \tag{3.23}$$

there exists processes $\bar{Z}_j^{n\varepsilon}$, $\underline{Z}_j^{n\varepsilon}$ such that

(iii) $\bar{Z}_j^{n\varepsilon}$ and $\underline{Z}_j^{n\varepsilon}$ are square integrable martingales, (3.24)

(iv) $Z_j^n = \bar{Z}_j^{n\varepsilon} + \underline{Z}_j^{n\varepsilon}$, (3.25)

(v) $\underline{Z}_j^{n\varepsilon}$ has no jumps larger than ε, (3.26)

(vi) $\bar{Z}_j^{n\varepsilon}$ has a.s. paths that are of bounded variation, (3.27)

(vii) the processes $\bar{\bar{Z}}_j^{n\varepsilon}$ and $\underline{\underline{Z}}_j^{n\varepsilon}$ do not jump at the same time, (3.28)

(viii) $\langle \underline{\underline{Z}}_j^{n\varepsilon}, \bar{\bar{Z}}_j^{n\varepsilon} \rangle (s) \overset{p}{\to} 0$, as $n \to \infty$. (3.29)

Then,

$$(Z_1^n, \ldots, Z_m^n) \overset{d}{\to} (Z_1^\infty, \ldots, Z_m^\infty)$$

where $Z_1^\infty, \ldots, Z_m^\infty$ are independent Gaussian processes with mean zero, and covariance structure given by

$$\mathrm{Cov}(Z_j^\infty(s_1), Z_j^\infty(s_2)) = f_j(s_1) \quad 0 \le s_1 \le s_2 \le t.$$

3.3. Weak Convergence Results.

All families of σ-fields defined in this section are standard filtrations. This fact is a consequence of Theorem A.2.1 of Gill (1980).

LEMMA 3.3.1. Suppose that $F_1, \ldots, F_m$ are continuous and suppose T is such that $F_j(T) < 1$ for $j = 1, 2, \ldots, m$. Then for each $n = 1, 2, \ldots$, we have

(i) The variation of the paths of M_j^n is bounded by $m\left(\frac{1+\bar{F}_j(T)}{\bar{F}_j(T)}\right)$ on $[0,T]$. (3.30)

(ii) $\{(M_j^n(t), F_t); t \in [0,T]\}$ is a martingale for each j, where F_t is defined by (3.17). (3.31)

(iii) $\langle M_{j_1}^n(t), M_{j_2}^n(t) \rangle = \begin{cases} A_{j_1}^n(t) & \text{if } j_1 = j_2 \\ 0 & \text{if } j_1 \ne j_2 \end{cases}$. (3.32)

Proof: The proof of (i) is immediate.

To prove (ii), we show that for $i = 1, 2, \ldots, n$, M_{ij} is a martingale on $[0,T]$. Part (ii) then follows since a sum of martingales is a martingale. If we assume that F_j is absolutely continuous with a continuous derivative and that the compensator of N_{ij} is absolutely continuous with a left continuous derivative whose right limit exists, a simple proof that M_{ij} is a martingale on $[0,T]$ can be given as follows.

It is not difficult to verify for every $t \in [0,T]$ that

$$\lim_{h\to 0}\frac{1}{h}P(N_{ij}(t+h) - N_{ij}(t) \geq 1 | F_t) = V_{ij}(t)\ \frac{f_j(t)}{\bar{F}_j(t)},$$

where f_j denotes the density of F_j. This is enough to show that A_{ij} is the compensator of N_{ij}. (See Lemma 3.3 of Aalen, 1978.) It now follows that M_{ij} is a martingale.

We now consider the general case. For measurability reasons that are indicated later on in the proof, we first show that M_{ij} is a martingale with respect to the filtration $\{G_t;\ t \in [0,T]\}$, where for each $t \in [0,T]$, G_t is the completion of $\sigma\{I(X_{ij} \leq s);\ 1 \leq i \leq n,\ i \leq j \leq m,\ s \leq t\}$. We then use this fact in a simple argument to show that M_{ij} is a martingale with respect to $\{F_t;\ t \in [0,T]\}$.

To prove that M_{ij} is a martingale with respect to $\{G_t;\ t \in [0,T]\}$, we show that A_{ij} is the compensator of N_{ij} with respect to $\{G_t;\ t \in [0,T]\}$. Let U_{ij} denote the compensator of N_{ij} with respect to $\{G_t;\ t \in [0,T]\}$. The key step in proving that $A_{ij} = U_{ij}$ a.s. involves the use of Proposition 3.2.2 to show for any fixed $t \in [0,T]$ that

$$P(A_{ij}(t) = U_{ij}(t)) = 1. \tag{3.33}$$

Since A_{ij} has a.s. continuous paths, it follows that $A_{ij} = U_{ij}$ a.s. on $[0,T]$. To prove (3.33) we use the following definitions. For $t \in [0,T]$ define

$$t_{k,\ell} = \ell 2^{-k} t,\ k = 1,\ 2,\ \ldots,\ \ell = 1,\ 2,\ \ldots,\ 2^k,$$

and

$$U_k = \sum_{\ell=0}^{2^k-1} E(N_{ij}(t_{k,\ell+1}) - N_{ij}(t_{k,\ell}) | G_{t_{k,\ell}})\quad k = 1,\ 2,\ \ldots\ .$$

A key step in proving (3.33) is to show that

$$U_k \xrightarrow{p} A_{ij}(t),\ \text{as } k \to \infty. \tag{3.34}$$

Assume that this has been done. Proposition 3.2.2 then implies that there exists a subsequence of integers $\{r_k\}$, $r_k \to \infty$ as $k \to \infty$ and a random variable V satisfying

$$E(XU_{r_k}) \to E(XV),\ \text{as } k \to \infty \tag{3.35}$$

for all bounded random variables X, and

$$P(V = U_{ij}(t)) = 1. \tag{3.36}$$

It follows from (3.34) and (3.35) by standard probability arguments given below that $A_{ij}(t) = V$ a.s.. Statement (3.33) now follows from (3.36).

For the sake of completeness we show that (3.34) and (3.35) imply that $A_{ij}(t) = V$ a.s.. It follows from (3.34) and (3.35) that there exists a subsequence of integers $\{\lambda_\ell\}$, $\lambda_\ell \to \infty$ as $\ell \to \infty$ such that

$$U_{\lambda_\ell} \to A_{ij}(t) \text{ a.s.,} \tag{3.37}$$

and

$$E\ XU_{\lambda_\ell} \to E\ XV, \tag{3.38}$$

as $\ell \to \infty$ for all bounded random variables X. Let $\varepsilon > 0$ be arbitrary. Define the sets $C_1 \subset C_2 \subset \ldots$, as follows:

$$C_\alpha = \bigcap_{\ell=\alpha}^{\infty} \{|U_{\lambda_\ell} - A_{ij}(t)| \le \varepsilon\}, \quad \alpha = 1, 2, \ldots \ . \tag{3.39}$$

Statement (3.37) implies that

$$\lim_{\alpha\to\infty} P(C_\alpha) = 1. \tag{3.40}$$

Let D^+ and D^- denote the sets $\{V > A_{ij}(t) + \varepsilon\}$ and $\{V < A_{ij}(t) - \varepsilon\}$, respectively. For each α, (3.38) implies that

$$\lim_{\ell\to\infty} E(I(D^+ \cap C_\alpha)U_{\lambda_\ell}) = E\ I(D^+ \cap C_\alpha)V. \tag{3.41}$$

Since the definition of C_α implies that

$$\lim_{\ell\to\infty} E(I(D^+ \cap C_\alpha)U_{\lambda_\ell}) \le E(I(D^+ \cap C_\alpha)(A_{ij}(t) + \varepsilon)),$$

it follows from (3.41) that

$$E(I(D^+ \cap C_\alpha)V) \le E(I(D^+ \cap C_\alpha)(A_{ij}(t) + \varepsilon)). \tag{3.42}$$

We conclude from the definition of D^+ and (3.42) that $P(D^+ \cap C_\alpha) = 0$ for each α. Statement (3.40) implies that $P(D^+) = 0$. Similar arguments show that $P(D^-) = 0$. Since ε was arbitrary, we conclude that $V = A_{ij}(t)$ a.s..

We now prove (3.34). For each $k = 1, 2, \ldots, \ell = 1, 2, \ldots$, define the set

$$B_k^{\ell+1} = \{X_{ij} \in (t_{k,\ell}, t_{k,\ell+1}], Z_{ij} > t_{k,\ell}\},$$

and the random variable

$$\xi_{k,\ell+1} = I(B_k^{\ell+1}) - (N_{ij}(t_{k,\ell+1}) - N_{ij}(t_{k,\ell})).$$

We note that

$$\xi_{k,\ell+1} = \begin{cases} 1 & \text{if } t_{k,\ell} < Y_{ij} < X_{ij} \le t_{k,\ell+1} \\ 0 & \text{otherwise} \end{cases}, \tag{3.43}$$

(where Y_{ij} is defined by (2.1) and (2.2)) and that

$$\xi_{k,\ell} \text{ is } G_{t_{k,\ell}}\text{-measurable for all } k \text{ and } \ell. \tag{3.44}$$

It is easy to see that $\xi_{k,\ell}$ is not $F_{t_{k,\ell}}$-measurable and this is the reason we use the filtration $\{G_t;\ t \in [0,T]\}$. To prove (3.34) we first show that

$$\sum_{\ell=0}^{2^k-1} \xi_{k,\ell+1} \xrightarrow{P} 0, \text{ as } k \to \infty. \tag{3.45}$$

Assuming the validity of (3.45), Lemma 2.5 of Rootzén (1983) together with (3.43) and (3.44) immediately imply that

$$\left|\sum_{j=1}^{2^k-1} E(I(B_k^{\ell+1})|G_{t_{k,\ell}}) - \sum_{j=1}^{2^k-1} E(N_{ij}(t_{k,\ell+1}) - N_{ij}(t_{k,\ell})|G_{t_{k,\ell}})\right| \xrightarrow{P} 0, \text{ as } k \to \infty. \tag{3.46}$$

We then show that

$$\left|A_{ij}(t) - \sum_{\ell=0}^{2^k-1} E(I(B_k^{\ell+1})|G_{t_{k,\ell}})\right| \to 0 \text{ a.s.}. \tag{3.47}$$

Statements (3.46) and (3.47) now imply (3.34).

We now prove (3.45). For each k,

$$P\left(\sum_{\ell=0}^{2^k-1} \xi_{k,\ell+1} \neq 0\right) \le \sum_{\ell=0}^{2^k-1} P(\xi_{k,\ell+1} \neq 0) = \sum_{\ell=0}^{2^k-1} P(t_{k,\ell} < Y_{ij} < X_{ij} \le t_{k,\ell+1})$$

$$= -\sum_{\ell=0}^{2^k-1} \int_{t_{k,\ell}}^{t_{k,\ell+1}} (F_j(t_{k,\ell+1}) - F_j(u))dh_\phi(1_j, \underline{\bar{F}}(u))$$

$$\le \sum_{\ell=0}^{2^k-1} [F_j(t_{k,\ell+1}) - F_j(t_{k,\ell})][h_\phi(1_j, \underline{\bar{F}}(t_{k,\ell})) - h_\phi(1_j, \underline{\bar{F}}(t_{k,\ell+1}))]$$

$$\le \sup_{s \in [0,t-\frac{1}{2^k}]} (F_j(s + \frac{1}{2^k}) - F_j(s)) \to 0 \quad \text{as } k \to \infty,$$

since F_j is uniformly continuous on $[0,t]$.

We now prove (3.47). Since for each k and ℓ $I(Z_{ij} > t_{k,\ell})$ is $G_{t_{k,\ell}}$-measurable, we have

$$E(I(B_k^{\ell+1}) \Big| G_{t_{k,\ell}}) = I(Z_{ij} > t_{k,\ell})P(t_{k,\ell} < X_{ij} \le t_{k,\ell+1} \Big| G_{t_{k,\ell}}).$$

It is easy to verify that

$$P(t_{k,\ell} < X_{ij} \le t_{k,\ell+1} | G_{t_{k,\ell}}) = I(X_{ij} > t_{k,\ell}) \int_{t_{k,\ell}}^{t_{k,\ell+1}} \frac{dF_j(u)}{\bar{F}_j(t_{k,\ell})} .$$

Thus,

$$E(I(B_k^{\ell+1}) | G_{t_{k,\ell}}) = I(Z_{ij} > t_{k,\ell}) \int_{t_{k,\ell}}^{t_{k,\ell+1}} \frac{dF_j(u)}{\bar{F}_j(t_{k,\ell})} .$$

Define the random variables Y_k by

$$Y_k = \sum_{\ell=0}^{2^k-1} I(Z_{ij} > t_{k,\ell}) \int_{t_{k,\ell}}^{t_{k,\ell+1}} \frac{dF_j(u)}{\bar{F}_j(u)} .$$

We prove (3.47) by showing that

$$\sum_{\ell=0}^{2^k-1} E(I(B_k^{\ell+1}) | G_{t_{k,\ell}}) - Y_k \to 0 \text{ a.s. as } k \to \infty,$$

and

$$Y_k \to A_{ij}(t) \text{ a.s. as } k \to \infty.$$

We proceed to prove these assertions.

$$\left|\sum_{i=0}^{2^k-1} E(I(B_k^{i+1})\,|\,G_{t_{k,i}}) - Y_k\right| \le \sum_{i=0}^{2^k-1} \int_{t_{k,i}}^{t_{k,i+1}} \left[\frac{1}{\bar{F}_j(u)} - \frac{1}{\bar{F}_j(t_{k,i})}\right] dF_j(u)$$

$$\le \sup_{0\le s\le t-\frac{1}{2^k}} \left[\frac{1}{\bar{F}_j(s+\frac{1}{2^k})} - \frac{1}{\bar{F}_j(s)}\right] \to 0 \text{ as } k\to\infty,$$

by the uniform continuity of $\bar{F}_j$ on $[0,t]$. Next,

$$|A_{ij}(t) - Y_k| \le \sup_{0\le s\le t-\frac{1}{2^k}} \left|\int_s^{s+\frac{1}{2^k}} \frac{dF_j(u)}{\bar{F}_j(u)}\right|$$

$$= \frac{1}{\bar{F}_j(t)} \sup_{0\le s\le t-\frac{1}{2^k}} (F_j(s+\tfrac{1}{2^k}) - F_j(s)) \to 0 \text{ as } k\to\infty.$$

This proves (3.47), consequently (3.34), and hence (3.33).

We now prove that M_{ij} is a martingale with respect to the filtration $\{F_t;\ t\in[0,T]\}$. It is easy to verify that

$$F_t \subset G_t \quad \text{for each } t\in[0,T], \tag{3.48}$$

$$N_{ij} \text{ is adapted to } \{F_t;\ t\in[0,T]\}, \tag{3.49}$$

and that

$$V_{ij} \text{ is adapted to } \{F_t;\ t\in[0,T]\}. \tag{3.50}$$

It follows from (3.49) and (3.50) that

$$M_{ij} \text{ is adapted to } \{F_t;\ t\in[0,T]\}. \tag{3.51}$$

It now follows from (3.48) and (3.51) that for each j and all $s\le t\le T$ that

$$E(M_{ij}(t)\,|\,F_s) = E(E(M_{ij}(t)\,|\,G_s)\,|\,F_s) = M_{ij}(s).$$

Thus, M_{ij} is a martingale with respect to $\{F_t;\ t\in[0,T]\}$, and Part (iii) has been proved.

A consequence of (ii) is that A_j^n is the compensator of N_j^n for each j. Part (iii) now follows from Proposition 3.2.3.

An alternative way to prove that M_{ij} is a martingale is to use Lemma 3.1.1 of Gill (1980) to show that A_{ij} is the compensator of N_{ij}. While shorter, this approach is not as straightforward as the method used above.

We use the filtration $\{F_t;\ t \in [0,T]\}$ because it has a natural interpretation: F_t represents all the information available at time t. It is also the filtration most widely used in the literature. However it is clear that Lemma 3.3.1 is valid for any filtration satisfying (3.48), (3.49), and (3.50), and furthermore the processes defined by (3.8)-(3.16) are adapted to this filtration. For the remainder of the proofs of this section, the filtration enters only via Lemma 3.3.1 and the measurability of the processes defined by (3.8)-(3.16). Thus, the results of this section can be proved using any filtration satisfying (3.48), (3.49), and (3.50). However, for the reasons stated above we shall continue to use $\{F_t;\ t \in [0,T]\}$.

<u>Proof of Theorem 1</u>: We proceed to prove (3.6). We first show that $n^{\frac{1}{2}} \int_0^t \left[\frac{J_j(s)\hat{\bar{F}}_j(s-)}{V_j(s)\bar{F}_j(s)}\right] dM_j(s)$ is a square integrable martingale on $[0,T]$. It is easy to see that for each j and n, the process $n^{\frac{1}{2}} \frac{J_j(t)\hat{\bar{F}}_j(t-)}{V_j(t)\bar{F}_j(t)}$ has left continuous paths a.s. and is uniformly bounded by $\frac{n^{\frac{1}{2}}}{\bar{F}_j(T)}$ on $[0,T]$. A consequence of (3.50) is that J_j is adapted to F_t. We note that an equivalent formula to (1.3) for $\hat{\bar{F}}_j$ can be given by

$$\hat{F}_j(t) = 1 - \prod_{s \le t} \left(1 - \frac{N_j(s) - N_j(s-)}{V_j(s)} \right), \tag{3.52}$$

where in the product only a finite number of the terms are not equal to 1. It is easy to see from (3.52) that $\hat{F}_j$ is adapted to F_t. It follows from Theorem 3.1 of Chung and Williams (1983) that the above processes are predictable. For each n it follows from (3.30) that

$$n^{\frac{1}{2}} \int_0^t \left[\frac{J_j(s)\hat{\bar{F}}_j(s-)}{V_j(s)\bar{F}_j(s)}\right] d\|M_j(s)\| \le \frac{n^{\frac{3}{2}}(1 + \bar{F}_j(T))}{(\bar{F}_j(T))^2} < \infty. \tag{3.53}$$

Proposition 3.2.1 together with (3.53) and (3.18) imply that for each n and j, $n^{\frac{1}{2}}\left(\frac{\hat{F}_j - F_j^*}{\bar{F}_j^*}\right)$ is a local martingale on [0,T]. Since these local martingales are uniformly bounded for each n, they are square integrable martingales on [0,T]. The proof of (3.6) follows from Proposition 3.2.4, whose conditions we now verify. We have shown that (3.22) holds for $n^{\frac{1}{2}}\left(\frac{\hat{F}_j - F_j^*}{\bar{F}_j^*}\right)$, and we now check (3.23). Proposition 3.2.1 implies that for $1 \le j_1, j_2 \le m$,

$$\left\langle n^{\frac{1}{2}}\left(\frac{\hat{F}_{j_1} - F_{j_1}^*}{\bar{F}_{j_1}^*}\right), n^{\frac{1}{2}}\left(\frac{\hat{F}_{j_2} - F_{j_2}^*}{\bar{F}_{j_2}^*}\right)\right\rangle$$
$$= n\int_0^t \left[\frac{J_{j_1}(s)\hat{\bar{F}}_{j_1}(s-)}{V_{j_1}(s)\bar{F}_{j_1}(s)}\right]\left[\frac{J_{j_2}(s)\hat{\bar{F}}_{j_2}(s-)}{V_{j_2}(s)\bar{F}_{j_2}(s)}\right] d\langle M_{j_1}, M_{j_2}\rangle(s). \tag{3.54}$$

Thus, it follows from (3.32) that for $j_1 \ne j_2$

$$\left\langle n^{\frac{1}{2}}\left(\frac{\hat{F}_{j_1} - F_{j_1}^*}{\bar{F}_{j_1}^*}\right), n^{\frac{1}{2}}\left(\frac{\hat{F}_{j_2} - F_{j_2}^*}{\bar{F}_{j_2}^*}\right)\right\rangle(t) = 0. \tag{3.55}$$

Statements (3.54) and (3.32) yield for each $t \in [0,T]$ that

$$\left\langle n^{\frac{1}{2}}\left(\frac{\hat{F}_j - F_j^*}{\bar{F}_j^*}\right), n^{\frac{1}{2}}\left(\frac{\hat{F}_j - F_j^*}{\bar{F}_j^*}\right)\right\rangle(t) = n\int_0^t \left[\frac{J_j(s)\hat{\bar{F}}_j(s-)}{V_j(S)\bar{F}_j(s)}\right]^2 dA_j(s)$$
$$= n\int_0^t \left[\frac{J_j(s)(\hat{\bar{F}}_j(s-))^2}{V_j(s)(\bar{F}_j(s))^3}\right] dF_j(s). \tag{3.56}$$

The uniform consistency of the Kaplan-Meier estimator (Proposition 2.2) and an application of the Glivenko-Cantelli Theorem (modified in a minor way) together with the fact that $\bar{H}_j(t) > 0$ now give that

$$n\int_0^t \left[\frac{J_j(s)\hat{\bar{F}}_j^2(s-)}{V_j(s)\bar{F}_j^3(s)}\right] dF_j(s) \to \int_0^t \frac{dF_j(s)}{\bar{H}_j(s)\bar{F}_j(s)} \text{ a.s. as } n\to\infty. \tag{3.57}$$

Thus, condition (3.23) follows from (3.55) and (3.57).

For each $\varepsilon > 0$ define the following processes.

$$n^{\frac{1}{2}}\bar{Z}^{n\varepsilon}(t) = n^{\frac{1}{2}}\int_0^t \left[\frac{J_j(s)\hat{\bar{F}}_j(s-)}{V_j(s)\bar{F}_j(s)}\right] I\left(\frac{n^{\frac{1}{2}}J_j(s)\hat{\bar{F}}_j(s-)}{V_j(s)\bar{F}_j(s)} \geq \varepsilon\right) dM_j(s).$$

$$n^{\frac{1}{2}}\underline{Z}^{n\varepsilon}(t) = n^{\frac{1}{2}}\int_0^t \left[\frac{J_j(s)\hat{\bar{F}}_j(s-)}{V_j(s)\bar{F}_j(s)}\right] I\left(\frac{n^{\frac{1}{2}}J_j(s)\hat{\bar{F}}_j(s-)}{V_j(s)\bar{F}_j(s)} < \varepsilon\right) dM_j(s).$$

It is clear that $\bar{Z}_j^{n\varepsilon}$ and $\underline{Z}_j^{n\varepsilon}$ are square integrable martingales, and that their sum is equal to the right side of (3.18). Conditions (3.26), (3.27), and (3.28) are trivially satisfied.

We now check (3.29). Propositions 3.2.1 and (3.32) imply that for each $t \in [0,T]$,

$$\langle \bar{Z}_j^{n\varepsilon}, \bar{Z}_j^{n\varepsilon}\rangle(t) = n\int_0^t \left[\frac{J_j(s)}{V_j(s)}\,\frac{\hat{\bar{F}}^2(s-)}{\bar{F}^3(s)}\right] I\left(n^{\frac{1}{2}}\frac{J_j(s)}{V_j(s)}\,\frac{\hat{\bar{F}}_j(s-)}{\bar{F}_j(s)} \geq \varepsilon\right) dF_j(s). \tag{3.58}$$

Almost surely, the indicator inside the integral is 0 for all large n, by the Glivenko-Cantelli Theorem. This proves (3.29) and concludes the proof of Theorem 1. □

Originally, we proved the asymptotic normality of the vector $(\hat{F}_1 - F_1, \ldots, \hat{F}_m - F_m)$ using the method of Breslow and Crowley (1974). The proof was conceptually simpler, not requiring the introduction of various families of σ-fields and the heavy machinery of stochastic integration and martingale central limit theorems. However, we were unable to obtain the covariance terms in the asymptotic covariance matrix.

Suppose the life distributions of the components all have infinite supports. It is straightforward to show that Theorem 1 is equivalent to

$$n^{\frac{1}{2}}(\hat{F}_1 - F_1, \ldots, \hat{F}_m - F_m) \to (W_1, \ldots, W_m) \text{ weakly on } D^m[0,\infty), \tag{3.59}$$

where $D[0,\infty)$ has been equipped with the standard metric for convergence on compacta (see Definition 1, page 123 of Pollard, 1984). It is not hard to prove that a functional f defined on $D^m[0,\infty)$ is continuous with respect to the above metric if and only if f is continuous with respect to $D^m[0,T]$, for each $T > 0$. Thus, (3.59) offers no advantage over Theorem 1 in obtaining via Theorem 5.1 of Billingsley (1968), the asymptotic distributions of functionals of $\hat{F}_1, \ldots, \hat{F}_m$.

Proof of Theorem 2: The uniform bound for the first two partial derivatives of h_ϕ given by Lemma 2.1 together with Taylor's Theorem imply that for each $t \in [0,T]$,

$$n^{\frac{1}{2}} \left| \hat{F}(t) - F(t) - \sum_{j=1}^{m} \left(\frac{\partial h_\phi}{\partial u_j} \Bigg|_{\substack{(u_1,\ldots,u_m) = \\ (\bar{F}_1(t),\ldots,\bar{F}_m(t))}} \right) (\hat{F}_j(t) - F_j(t)) \right|$$

$$\le \frac{n^{\frac{1}{2}}}{2} \sum_{j_1=1}^{m} \sum_{j_2=1}^{m} \left(\sup_{0\le t\le T} \left| \hat{F}_{j_1}(t) - F_{j_1}(t) \right| \right) \cdot \left(\sup_{0\le t\le T} \left| \hat{F}_{j_2}(t) - F_{j_2}(t) \right| \right). \tag{3.60}$$

It follows from Proposition 2.2 that the right side of (3.60) converges to 0 a.s.. We use the fact that convergence in sup norm implies convergence in the Skorohod topology (see page 111 of Billingsley, 1968) to conclude that the process

$$n^{\frac{1}{2}} \left| \hat{F}(t) - F(t) - \sum_{j=1}^{m} \left(\frac{\partial h_\phi}{\partial u_j} \Bigg|_{\substack{(u_1,\ldots,u_m) = \\ (\bar{F}_1(t),\ldots,\bar{F}_m(t))}} \right) (\hat{F}_j(t) - F_j(t)) \right| \to 0 \text{ a.s. in } D[0,T].$$

Thus, the proof follows by showing that

$$n^{\frac{1}{2}} \sum_{j=1}^{m} \left(\frac{\partial h_\phi}{\partial u_j} \Bigg|_{\substack{(u_1,\ldots,u_m) = \\ (\bar{F}_1(t),\ldots,\bar{F}_m(t))}} \right) (\hat{F}_j(t) - F_j(t)) \xrightarrow{d} W,$$

which is the consequence of Theorem 5.1 of Billingsley (1968) and Theorem 1. □

To construct confidence intervals for $F(t)$, we define the following functions and processes on $[0,\infty)$.

$$G_j(t) = (\bar{F}_j(t))^2 \int_0^t \frac{dF_j(s)}{\bar{F}_j(s)\bar{H}_j(s)} ;$$

$$\hat{G}_j^n(t) = \frac{(\hat{\bar{F}}_j(t))^2}{n} \int_0^t \frac{dN_j(s)}{\hat{\bar{H}}_j(s-)\hat{\bar{H}}_j(s)} \left(= n\,(\hat{\bar{F}}_j(t))^2 \sum_{i:Z_{(i)j} \le t} \frac{\delta_{(i)j}}{(n-i+1)(n-i)} \right),$$

where

$$\hat{\bar{H}}_j(t) = \frac{1}{n} \sum_{i=1}^n I(Z_{ij} > t);$$

$$h_j(t) = \left(\left. \frac{\partial h_\phi}{\partial u_j} \right|_{\substack{(u_1,\ldots,u_m) = \\ (\bar{F}_1(t),\ldots,\bar{F}_m(t))}} \right);$$

$$\hat{h}_j^n(t) = \left(\left. \frac{\partial h_\phi}{\partial u_j} \right|_{\substack{(u_1,\ldots,u_m) = \\ (\hat{\bar{F}}_1(t),\ldots,\hat{\bar{F}}_m(t))}} \right).$$

The quantity $\frac{\hat{G}_j(t)}{n}$ is called Greenwood's estimator of the variance of $\hat{F}_j(t)$.

<u>LEMMA 3.3.2</u>. Suppose $F_1, \ldots, F_m$ are continuous and $T > 0$ is such that $\max_{1 \le j \le m} F_j(T) < 1$. Then $\sum_{j=1}^m \hat{h}_j^2 \hat{G}_j$ is a strongly consistent estimator of $\sum_{j=1}^m h_j^2 G_j$.

We note that in view of (1.7), Theorem 2 and Lemma 3.3.2 allow the formation of asymptotic confidence intervals for $F(t)$, $t \in [0,T]$.

<u>Proof</u>: Part (a) of the proposition in Section 2 of Hall and Wellner (1980) together with Proposition 2.2 imply that $\hat{G}_j$ is a strongly consistent estimator of G_j. Lemma 2.1 implies that the partial derivatives of h_ϕ are continuous. Thus, it follows from Proposition 2.2 that $\hat{h}_j$ is a strongly consistent estimator of h_j. The proof follows. □

4. ESTIMATION OF THE RELIABILITY IMPORTANCE OF COMPONENTS.

The reliability importance $I_j(t)$ of component j at time t is defined by

$$I_j(t) = \frac{\partial}{\partial u_j} h(u_1,\dots,u_m) \Bigg|_{\substack{(u_1,\dots,u_m) = \\ (\bar{F}_1(t),\dots,\bar{F}_m(t))}} . \qquad (4.1)$$

Let $\varepsilon_1, \dots, \varepsilon_m$ be small numbers. Note that

$$h(\bar{F}_1(t)+\varepsilon_1,\dots,\bar{F}_m(t)+\varepsilon_m) - h(\bar{F}_1(t),\dots,\bar{F}_m(t)) \doteq \sum_{j=1}^{m} \varepsilon_j I_j(t).$$

Thus, the reliability importance of components may be used to evaluate the effect of an improvement in component reliability on system reliability, and can therefore by very useful in system analysis in determining those components on which additional research can be most profitably expended. For details, see pages 26-28 of Barlow and Proschan (1981), and the review by Natvig (1984).

We estimate $I_j(t)$ by replacing $(\bar{F}_1(t),\dots,\bar{F}_m(t))$ with $(\hat{\bar{F}}_1(t),\dots,\hat{\bar{F}}_m(t))$ in (4.1). *Formally, define* $\hat{I}_j$ *by*

$$\hat{I}_j(t) = \frac{\partial}{\partial u_j} h(u_1,\dots,u_m) \Bigg|_{\substack{(u_1,\dots,u_m) = \\ (\hat{\bar{F}}_1(t),\dots,\hat{\bar{F}}_m(t))}} . \qquad (4.2)$$

PROPOSITION 4.1. Suppose $F_1, \dots, F_m$ are continuous and $T > 0$ is such that $F_j(T) < 1$, $j = 1, 2, \dots, m$. Then

$$\sqrt{n}(\hat{I}_1 - I_1,\dots,\hat{I}_m - I_m) \xrightarrow{d} (Y_1,\dots,Y_m),$$

where $(Y_1,\dots,Y_m)$ is a vector of mean zero Gaussian processes whose covariance structure is given by

$$\mathrm{Cov}(Y_{j_1}(t_1),\ Y_{j_2}(t_2))$$

$$= \sum_{k=1}^{m} \left\{ \frac{\partial^2 h}{\partial u_{j_1} \partial u_k} \Bigg|_{\substack{(u_1,\dots,u_m) = \\ (\bar{F}_1(t_1),\dots,\bar{F}_m(t_1))}} \right\} \left\{ \frac{\partial^2 h}{\partial u_{j_2} \partial u_k} \Bigg|_{\substack{(u_1,\dots,u_m) = \\ (\bar{F}_1(t_2),\dots,\bar{F}_m(t_2))}} \right\} \qquad (4.3)$$

$$\bar{F}_k(t_1)\bar{F}_k(t_2) \int_0^{t_1} \frac{dF_k(u)}{\bar{H}_k(u)\bar{F}_k(u)}, \quad \text{for } 0 \le t_1 \le t_2 \le T \text{ and } j_1,\ j_2 = 1, \dots, m.$$

As before, the covariance terms in (4.3) can be estimated consistently, enabling the construction of confidence intervals for $I_j(t)$.

Proof: For each j define the function g_j by

$$g_j(p_1,\ldots,p_m) = \frac{\partial h}{\partial u_j}(u_1,\ldots,u_m)\Bigg|_{\substack{(u_1,\ldots,u_m)\\=(p_1,\ldots,p_m)}} \quad \text{for} \quad 0 \le p_k \le 1,\ k = 1, 2, \ldots, m. \tag{4.4}$$

Assume that we can show that

> g_j is twice continuously differentiable on $[0,1]^m$ with first and second partials bounded in absolute value by 1 uniformly over $[0,1]^m$, (4.5)

(cf. Lemma 2.1). The proposition then follows by a straightforward multivariate extension of the proof of Theorem 2 with h replaced by g_j, $j = 1, 2, \ldots, m$.

We now prove (4.5). It follows from (2.7) that

$$\frac{\partial g_j}{\partial p_k}\Bigg|_p = \frac{\partial^2 h_\phi}{\partial p_j \partial p_k}\Bigg|_p, \tag{4.6}$$

and that

$$\frac{\partial^2 g_j}{\partial p_k \partial p_\ell}\Bigg|_p = \frac{\partial^3 h_\phi}{\partial p_j \partial p_k \partial p_\ell}\Bigg|_p, \tag{4.7}$$

for each $p \in [0,1]^m$. It follows from Lemma 2.1 that g_j is continuously differentiable and its first partial derivatives are uniformly bounded in absolute value by 1 on $[0,1]^m$. Statement (2.9) implies that for distinct indices j, k, ℓ,

$$\begin{aligned}\frac{\partial^3 h_\phi}{\partial p_j \partial p_k \partial p_\ell}\Bigg|_p &= \{h_\phi(1_j,1_k,1_\ell,p) - h_\phi(1_j,1_k,0_\ell,p)\} - \{h_\phi(1_j,0_k,1_\ell,p) - h_\phi(1_j,0_k,0_\ell,p)\} \\ &\quad - \{h_\phi(0_j,1_k,1_\ell,p) - h_\phi(0_j,1_k,0_\ell,p)\} + \{h_\phi(0_j,0_k,1_\ell,p) - h_\phi(0_j,0_k,0_\ell,p)\},\end{aligned} \tag{4.8}$$

in an obvious extension of the notation (2.3). Statement (2.8) implies that if at least two of the indices j, k, ℓ are equal then

$$\left.\frac{\partial^3 h_\phi}{\partial p_j p_k p_\ell}\right|_p = 0. \tag{4.9}$$

It follows in the same manner as in the proof of Lemma 2.1 with (4.8) and (4.9) replacing (2.7) and (2.8), that h_ϕ has a continuous third derivative on $[0,1]^m$ and the third partial derivatives are bounded in absolute value by 1 uniformly over $[0,1]^m$. Thus, for each j, g_j has a continuous second derivative and the second partial derivatives are bounded in absolute value by 1 uniformly over $[0,1]^m$. □

APPENDIX: RANDOM CENSORSHIP.

In Section 2 we assumed the existence of a censoring random variable Y_j that satisfies (2.1), (2.2), and (2.4). Here we define Y_j and formally prove that it satisfies (2.1), (2.2), and (2.4). Define the binary function ϕ_j by

$$\phi_j(u_1,\ldots,u_m) = \phi(1_j,u_1,\ldots,u_m),\ u_k = 0,\ 1,\ k = 1,\ 2,\ \ldots,\ m, \tag{A1}$$

where ϕ is the structure function. (See the paragraph preceding equation (2.6).) The censoring random variable Y_{ij} is defined as follows:

$$Y_{ij} = \sup\{t\colon\ \phi_j(I(X_{i1} > t),\ \ldots,\ I(X_{im} > t)) = 1\}. \tag{A2}$$

<u>PROPOSITION A.1</u>. For each j, Y_{1j}, Y_{2j}, ..., are i.i.d. random variables satisfying (2.1), (2.2), and (2.4).

<u>Proof</u>: It follows from (A2) that Y_{ij} is a function of the vector $(1_j,\ I(X_{i1} > t),\ \ldots,\ I(X_{im} > t))$. Thus it follows that Y_{1j}, Y_{2j}, ..., are i.i.d. and that Y_{ij} satisfies (2.2).

We proceed to prove (2.4). The structure function ϕ is increasing in its arguments (see Definition 2.1, page 6 of Barlow and Proschan, 1981) and hence a fortiori ϕ_j is increasing in its arguments. Thus

$$P(Y_{ij} > t) = P(\phi_j(I(X_{i1} > t),\ \ldots,\ I(X_{im} > t)) = 1). \tag{A3}$$

It is easy to see that the right side of (A3) is equal to $h(1_j,\ \underline{\bar{F}}(t))$ and so Y_{ij}

satisfies (2.4). To prove that Y_{ij} satisfies (2.1), we consider two cases: $\delta_{ij} = 1$ and $\delta_{ij} = 0$. We first prove (2.1) for the case $\delta_{ij} = 1$. Since ϕ is increasing in its arguments,

$$\begin{aligned} &\sup\{t:\ \phi(I(X_{i1} > t), \ldots, I(X_{im} > t)) = 1\} \\ &\le \sup\{t:\ \phi_j(I(X_{i1} > t), \ldots, I(X_{im} > t)) = 1\}. \end{aligned} \tag{A4}$$

It is clear that the left side of (A4) equals T_i and the right side of (A4) equals Y_{ij}. Hence

$$T_i \le Y_{ij}. \tag{A5}$$

Since $\delta_{ij} = 1$,

$$X_{ij} \le T_i. \tag{A6}$$

It is immediate from (A5) and (A6) that $X_{ij} \le Y_{ij}$, which implies that (2.1) holds for this case. We now prove that (2.1) is satisfied if $\delta_{ij} = 0$. Since $\delta_{ij} = 0$, it follows that $X_{ij} > T_i = Z_{ij}$. Hence $0 = \phi_j(I(X_{i1} > Z_{ij}), \ldots, I(X_{im} > Z_{ij}))$. Thus it follows from (A2) that

$$Y_{ij} \le Z_{ij}. \tag{A7}$$

It is easy to see that (A5) holds for this case. Thus $Y_{ij} = Z_{ij}$, which implies that (2.1) is satisfied for this case. $\square$

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