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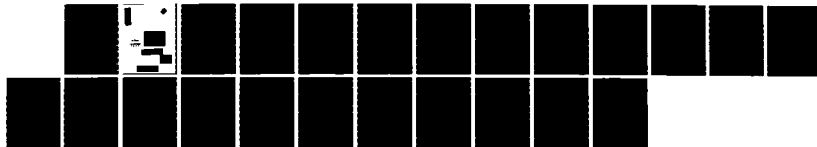
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MATHEMATICS J G SHANTHIKUMAR JAN 86 AFOSR-TR-86-0275

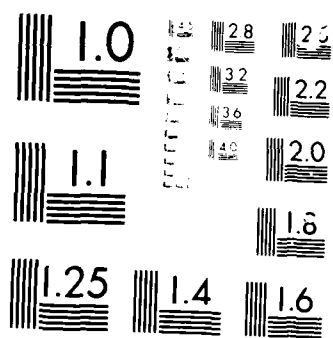
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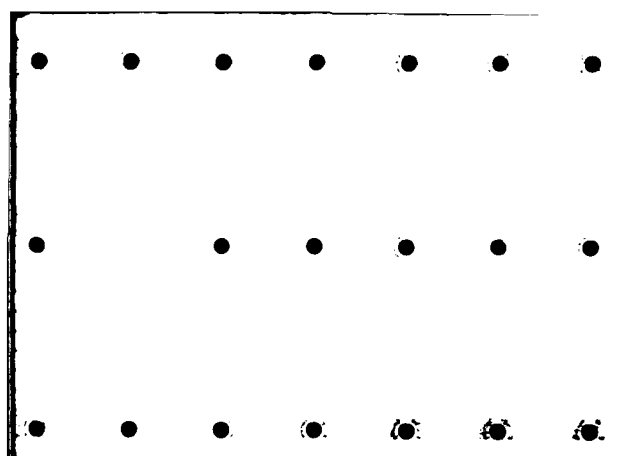
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**AFOSR-TR- 86 - 0275**  
Management Science  
Working Paper No. MS-33

DFR Property of First Passage Times  
and Its Preservation  
under Geometric Compounding

J. George Shanthikumar

January 1986

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## REPORT DOCUMENTATION PAGE

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| 1a. REPORT SECURITY CLASSIFICATION<br>UNCLASSIFIED                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |       |                                            | 1b. RESTRICTIVE MARKINGS                                                                       |                                                                                                                            |                          |
| 2a. SECURITY CLASSIFICATION AUTHORITY                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 |       |                                            | 3. DISTRIBUTION/AVAILABILITY OF REPORT<br>Approved for public release; distribution unlimited. |                                                                                                                            |                          |
| 2b. DECLASSIFICATION/DOWNGRADING SCHEDULE                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             |       |                                            |                                                                                                |                                                                                                                            |                          |
| 4. PERFORMING ORGANIZATION REPORT NUMBER(S)                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           |       |                                            | 5. MONITORING ORGANIZATION REPORT NUMBER(S)<br>AFOSR-TR- 86-0275                               |                                                                                                                            |                          |
| 6a. NAME OF PERFORMING ORGANIZATION<br>The University of Arizona                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      |       | 6b. OFFICE SYMBOL<br>(If applicable)       |                                                                                                | 7a. NAME OF MONITORING ORGANIZATION<br>Air Force Office of Scientific Research                                             |                          |
| 6c. ADDRESS (City, State and ZIP Code)<br>Department of Mathematics<br>Building #89<br>Tucson, Arizona 85721                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          |       |                                            |                                                                                                | 7b. ADDRESS (City, State and ZIP Code)<br>Directorate of Mathematical & Information<br>Sciences, Bolling AFB DC 20332-6448 |                          |
| 8a. NAME OF FUNDING/SPONSORING ORGANIZATION<br>AFOSR                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  |       | 8b. OFFICE SYMBOL<br>(If applicable)<br>NM |                                                                                                | 9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER<br>AFOSR-84-0205                                                           |                          |
| 8c. ADDRESS (City, State and ZIP Code)<br>Bolling AFB DC 20332-6448                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   |       | 10. SOURCE OF FUNDING NOS.                 |                                                                                                |                                                                                                                            |                          |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       |       | PROGRAM<br>ELEMENT NO.<br>61102F           |                                                                                                | PROJECT<br>NO.<br>2304                                                                                                     | TASK<br>NO.<br>A5        |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       |       |                                            |                                                                                                | WORK UNIT<br>NO.                                                                                                           |                          |
| 11. TITLE (Include Security Classification)<br>DFR PROPERTY OF FIRST PASSAGE TIMES AND ITS PRESERVATION UNDER<br>GEOMETRIC COMPOUNDING                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                |       |                                            |                                                                                                |                                                                                                                            |                          |
| 12. PERSONAL AUTHOR(S)<br>J. George Shanthikumar                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      |       |                                            |                                                                                                |                                                                                                                            |                          |
| 13a. TYPE OF REPORT<br>Technical                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      |       | 13b. TIME COVERED<br>FROM _____ TO _____   |                                                                                                | 14. DATE OF REPORT (Yr., Mo., Day)<br>Jan. 86                                                                              |                          |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       |       |                                            |                                                                                                | 15. PAGE COUNT<br>17                                                                                                       |                          |
| 16. SUPPLEMENTARY NOTATION                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                            |       |                                            |                                                                                                |                                                                                                                            |                          |
| 17. COSATI CODES                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      |       |                                            | 18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)              |                                                                                                                            |                          |
| FIELD                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 | GROUP | SUB GR                                     |                                                                                                |                                                                                                                            |                          |
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| 19. ABSTRACT (Continue on reverse if necessary and identify by block number)                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          |       |                                            |                                                                                                |                                                                                                                            |                          |
| <h2 style="text-align: center;">Abstract</h2> <p>It is shown that if the transition kernel of a discrete time Markov chain with state space <math>\{0, 1, \dots\}</math> is totally positive of order two (<math>TP_2</math>), the first passage time from state 1 to state 0 has decreasing failure rate (DFR). This result is used to show that (i) the sum of a geometric number (i.e., geometric compound) of i.i.d. DFR random variables is DFR and (ii) the number of customers served during a busy period in an M/G/1 queue with increasing failure rate service times is DFR. Recent results of Szekli (1986) and the closure property of i.i.d. DFR random variables under geometric compounding are combined to show that the stationary waiting time in a GI/G/1 (M/G/1) queue with DFR (increasing mean residual) service times is DFR. We also provide sufficient conditions on the inter-renewal times under which the renewal function is concave. These results shed some light on a conjecture of Brown (1981).</p> |       |                                            |                                                                                                |                                                                                                                            |                          |
| 20. DISTRIBUTION/AVAILABILITY OF ABSTRACT<br>UNCLASSIFIED/UNLIMITED <input checked="" type="checkbox"/> SAME AS RPT. <input type="checkbox"/> DTIC USERS <input type="checkbox"/>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     |       |                                            | 21. ABSTRACT SECURITY CLASSIFICATION<br>UNCLASSIFIED                                           |                                                                                                                            |                          |
| 22a. NAME OF RESPONSIBLE INDIVIDUAL<br>Major Brian W. Woodruff                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        |       |                                            | 22b. TELEPHONE NUMBER<br>(Include Area Code)<br>(202) 767- 5027                                |                                                                                                                            | 22c. OFFICE SYMBOL<br>NM |

# **DFR Property of First Passage Times and Its Preservation under Geometric Compounding**

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**January 1986**

**Supported by the Air Force Office of Scientific Research,  
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## Abstract

It is shown that if the transition kernel of a discrete time Markov chain with state space  $\{0, 1, \dots\}$  is totally positive of order two ( $TP_2$ ), the first passage time from state 1 to state 0 has decreasing failure rate (DFR). This result is used to show that (i) the sum of a geometric number (i.e., geometric compound) of i.i.d. DFR random variables is DFR and (ii) the number of customers served during a busy period in an M/G/1 queue with increasing failure rate service times is DFR. Recent results of Szekli (1986) and the closure property of i.i.d. DFR random variables under geometric compounding are combined to show that the stationary waiting time in a GI/G/1 (M/G/1) queue with DFR (increasing mean residual) service times is DFR. We also provide sufficient conditions on the inter-renewal times under which the renewal function is concave. These results shed some light on a conjecture of Brown (1981).

*AMS 1970 subject classification:* Primary 60J27, secondary 60K10.

*Key words and phrases:* DFR distributions, geometric compounding, renewal function, GI/G/1 queues.

## 1. INTRODUCTION

Brown (1980) proved that the renewal function for a renewal process with decreasing failure rate (DFR) inter-renewal times is concave. In a subsequent paper (Brown 1981) he conjectured that DFR is also a necessary condition for concavity. A consequence of this conjecture, as pointed out by Brown, is that DFR distributions are closed under geometric compounding (i.e., the sum of a geometric number of i.i.d. DFR random variables is DFR). Thus a counterexample to this (possible) closure property of DFR random variables under geometric compounding would provide a counterexample to the concavity conjecture.

In an attempt to verify Brown's conjecture we obtained sufficient conditions for (i) the first passage time from state 1 to state 0 of a Markov chain with state space  $\{0, 1, \dots\}$  to be DFR and (ii) the renewal function to be concave. These results led to progress in two areas which shed light on Brown's conjecture:

- (i) It is shown that DFR distributions are closed under geometric compounding,
- (ii) An example is given which shows that the conjecture does not hold in the discrete time case.

The discrete time example does not generalize to the continuous case, and the truth of the conjecture is still unresolved. In the continuous case, however, it is shown that a consequence of Brown's conjecture, if true, is that the first passage time from state 1 to state 0 in a stochastically monotone Markov process with state space  $\{0, 1, \dots\}$  is DFR. This property is known to hold for birth and death processes (Keilson 1979) and in general for any Markov process that can be uniformized such that the embedded Markov chain has a totally positive of order two ( $TP_2$ ) transition matrix (Assaf, Shaked and Shanthikumar 1985). But it is not

known whether it holds for this larger class. For this class of Markov processes (i.e., stochastically monotone) it is easily verified that the first passage time from state 1 to state 0 has the new worse than used (NWU) property. The NWU property is weaker than the DFR property.

The results presented in this paper, apart from providing some useful insights into Brown's conjecture, have other applications. The DFR property of the first passage times of Markov chains is used to show that the number of customers served during a busy period in an M/G/1 queue with increasing failure rate service times is DFR. Geometric compounding of i.i.d. random variable arises naturally in many applied probability models. A recent paper of Gertsbakh (1984) discusses a wide range of applications in reliability theory. In the queueing theory context it is well known that the stationary waiting time in a GI/G/1 queue can be represented as a geometric compound of i.i.d. random variables (e.g., Feller 1971). Related results are that the class of completely monotone (CM) distributions is closed under geometric compounding and the stationary waiting time in an M/G/I queue with CM service times is CM (Keilson 1978) and that the distribution function of a geometric convolution of DFR distributions is concave and the stationary waiting time in a GI/G/1 (M/G/1) queue with DFR (increasing mean residual) service times has a concave distribution function (Szekli 1986). We will strengthen the results of Szekli.

One aspect of our methodology which appears to be new is the consideration of a monotonicity property for a Markov chain which is stronger than stochastic monotonicity and weaker than  $TP_2$ . Stochastic monotonicity is based on the partial ordering of stochastic ordering,  $TP_2$  on ordering by monotone likelihood ratio. An intermediate ordering we use is the hazard rate ordering (e.g., Ross 1983). This ordering is utilized by Brown (1980, 1983, 1984) to study properties of IMRL and DFR distributions.

## 2. PRELIMINARIES

A random variable  $X$  or its distribution  $F$  on  $[0, \infty)$  is said to be DFR (IFR) if its failure rate  $r_X(t) \equiv f(t) / \bar{F}(t)$  is decreasing (increasing) in  $t \in [0, \infty)$ :  $f$  and  $\bar{F}$  are the density and survival functions of  $X$  ("increasing" and "decreasing" are not used in the strict sense).  $X$  is said to be IMRL if  $E(X - t | X > t)$  is increasing in  $t \in [0, \infty)$ .

Two non-negative random variables  $X_1$  and  $X_2$  or their distributions  $F_1$  and  $F_2$  are ordered in the sense of usual stochastic (hazard rate) ordering if  $\bar{F}_1(t) \geq \bar{F}_2(t), t \geq 0$  ( $\bar{F}_1(t) / \bar{F}_2(t)$  is increasing in  $t \in [0, \infty)$ ). Denote  $X_1 \geq_{st} X_2$  ( $X_1 \geq_h X_2$ ). Note that (e.g. Ross 1983)  $X_1 \geq_h X_2$  implies  $\{X_1 | X_1 > t\} \geq_{st} \{X_2 | X_2 > t\}, t \geq 0$ .

A function  $\underline{a} = [a(i, j)]$  of two real variables ranging over linearly ordered sets  $X$  and  $Y$ , respectively is  $TP_2$  if for any  $n_1 < n_2$  and  $m_1 < m_2$  ( $n_i \in X, m_i \in Y$ ),  $a(n_1, m_1)a(n_2, m_2) \geq a(n_1, m_2)a(n_2, m_1)$ . Equivalently using the convention  $0/0 = 0$  one has  $\frac{a(n_2, m_2)}{a(n_2, m_1)} - \frac{a(n_1, m_2)}{a(n_1, m_1)} \geq 0$  (when defined; otherwise set the difference equal to zero).

Let  $\underline{t} = [t(i, j)]_{i, j \in N_+}$ ;  $t(i, j) = 1, i \geq j$  and  $t(i, j) = 0$  otherwise.  $\underline{a} = [a(i, j)]_{i, j \in N_+}$  and  $\underline{b} = [b(i, j)]_{i, j \in N_+}$  are non-negative matrices (here  $N_+ = \{1, 2, \dots\}$ ).

*Lemma 2.1:*  $\underline{a} \underline{t} \in TP_2, \underline{r} \underline{t} \in TP_2$  and  $\underline{t}^{-1} \underline{r} \underline{t} \geq \underline{0}$ . Then  $\underline{a} \underline{r} \underline{t} \in TP_2$ .

*Proof:* Let  $\underline{A} = \underline{a} \underline{t}, \underline{R} = \underline{r} \underline{t}, \underline{B} = \underline{a} \underline{r} \underline{t}$  and for  $1 \leq n_1 < n_2, 1 \leq m_1 < m_2$  and  $B(n_1, m_1) > 0$  consider

$$(2.1) \quad \frac{B(n_1, m_2)}{B(n_1, m_1)} = \frac{\sum_{k=1}^{\infty} a(n_1, k) R(k, m_2)}{\sum_{\ell=1}^{\infty} a(n_1, \ell) R(\ell, m_1)}$$

$$\sum_{k=1}^{\infty} \frac{R(k, m_2)}{R(k, m_1)} \left[ \frac{a(n_1, k) R(k, m_1)}{\sum_{\ell=1}^{\infty} a(n_1, \ell) R(\ell, m_1)} \right]$$

$$= \sum_{k=1}^{\infty} \left[ \frac{R(k, m_2)}{R(k, m_1)} - \frac{R(k-1, m_2)}{R(k-1, m_1)} \right] \frac{\sum_{j=k}^{\infty} a(n_1, j) R(j, m_1)}{\sum_{\ell=1}^{\infty} a(n_1, \ell) R(\ell, m_1)}$$

where  $\frac{R(0, m_2)}{R(0, m_1)} \equiv 0$ . Since  $\underline{R} \in TP_2$ , the expression in the square bracket in the right hand side of (2.1) is non-negative. Consider

$$(2.2) \quad \frac{\sum_{j=k}^{\infty} a(n_1, j) R(j, m_1)}{\sum_{\ell=1}^{\infty} a(n_1, \ell) R(\ell, m_1)} = \frac{1}{1 + \frac{\sum_{\ell=1}^{k-1} a(n_1, \ell) R(\ell, m_1)}{\sum_{j=k}^{\infty} a(n_1, j) R(j, m_1)}}$$

Now for  $A(n_1, k) > 0$ , set  $R(0, m_1) = 0$  and consider

$$\frac{\sum_{\ell=1}^{k-1} a(n_1, \ell) R(\ell, m_1)}{\sum_{j=k}^{\infty} a(n_1, j) R(j, m_1)}$$

$$= \frac{\sum_{\ell=1}^k A(n_1, \ell) [R(\ell, m_1) - R(\ell - 1, m_1)] - A(n_1, k) R(k, m_1)}{\sum_{j=k}^{\infty} A(n_1, j) [R(j, m_1) - R(j - 1, m_1)] + A(n_1, k) R(k - 1, m_1)}$$

$$= \frac{\sum_{\ell=1}^k \frac{A(n_1, \ell)}{A(n_1, k)} [R(\ell, m_1) - R(\ell - 1, m_1)] - R(k, m_1)}{\sum_{j=k}^{\infty} \frac{A(n_1, j)}{A(n_1, k)} [R(j, m_1) - R(j - 1, m_1)] + R(k - 1, m_1)}$$

$$\geq \frac{\sum_{\ell=1}^k \frac{A(n_2, \ell)}{A(n_2, k)} [R(\ell, m_1) - R(\ell - 1, m_1)] - R(k, m_1)}{\sum_{j=k}^{\infty} \frac{A(n_2, j)}{A(n_2, k)} [R(j, m_1) - R(j - 1, m_1)] + R(k - 1, m_1)}$$

$$= \frac{\sum_{\ell=1}^{k-1} a(n_2, \ell) R(\ell, m_1)}{\sum_{j=k}^{\infty} a(n_2, j) R(j, m_1)},$$

since  $\underline{t}^{-1} \underline{r} \underline{t} \geq \underline{0}$  implies  $R(i, m_1) - R(i - 1, m_1) \geq 0, i = 1, 2, \dots$  and  $\underline{A} = \underline{a} \underline{t} \in TP_2$  implies  $A(n_1, \ell) / A(n_1, k) \geq (\leq) A(n_2, \ell) / A(n_2, k)$  for  $\ell \leq (\geq) k$  and  $n_1 < n_2$ . Then from (2.2) one gets

$$(2.3) \quad \frac{\sum_{j=k}^{\infty} a(n_1, j) R(j, m_1)}{\sum_{\ell=1}^{\infty} a(n_1, \ell) R(\ell, m_1)} \leq \frac{\sum_{j=k}^{\infty} a(n_2, j) R(j, m_1)}{\sum_{\ell=1}^{\infty} a(n_2, \ell) R(\ell, m_1)}.$$

When  $A(n_1, k) = 0$ , the left hand side of (2.3) is zero and the above inequality is trivially satisfied. Combining (2.1) and (2.3) one has

$$(2.4) \quad \frac{B(n_1, m_2)}{B(n_1, m_1)} \leq \frac{B(n_2, m_2)}{B(n_2, m_1)}, \quad 1 \leq n_1 < n_2; \quad 1 \leq m_1 < m_2.$$

Since  $\underline{a} \underline{r} \geq 0$ ,  $B(i, j)$  is increasing in  $j$  and therefore  $B(n_1, m_1) = 0$  implies  $B(n_1, m_2) = 0$ . Hence (2.4) in this case is trivially satisfied.  $\square$

*Remark 2.2:* Keilson and Kester (1978) show that if  $\underline{a} \underline{t} \in TP_2$  and  $\underline{t}^{-1} \underline{r} \underline{t} \in TP_2$  then  $\underline{a} \underline{r} \underline{t} \in TP_2$ . This follows from the observations that  $\underline{a} \underline{r} \underline{t} = \underline{a} \underline{t} \underline{t}^{-1} \underline{r} \underline{t}$  and that the class of  $TP_2$  matrices is closed under multiplication. Since  $\underline{t}^{-1} \underline{r} \underline{t} \in TP_2$  implies  $\underline{r} \underline{t} \in TP_2$ , our condition  $\underline{r} \underline{t} \in TP_2$  and  $\underline{t}^{-1} \underline{r} \underline{t} \geq \underline{0}$  is weaker than the  $TP_2$  condition of  $\underline{t}^{-1} \underline{r} \underline{t}$ .

### 3. DFR FIRST PASSAGE TIMES

Let  $\underline{X} = \{X_n, n = 0, 1, \dots\}$  be a temporally homogeneous discrete time Markov chain with state space  $\underline{N} = \{0, 1, \dots\}$  and transition probability matrix  $\underline{P} = [P(i, j)]_{i, j \in \underline{N}}$  ( $P(i, j) = P\{X_n = j \mid X_{n-1} = i\}, i, j \in \underline{N}$ ). Define the first passage time

$$(3.1) \quad T = \{\min\{n: X_n = 0, n = 1, 2, \dots\} \mid X_0 = 1\}.$$

Let  $\underline{Q} = \underline{P} \underline{t}$  be the transition kernel of  $\underline{X}$  (i.e.,  $Q(i, j) = \sum_{k=j}^{\infty} P(i, k)$ ,  $i, j \in \mathbb{N}$ ). One has

*Theorem 3.1:*  $\underline{Q} \in TP_2$  implies  $T \in \text{DFR}$

*Proof:* The failure rate  $r_T$  of  $T$  is given by  $r_T(n) \equiv P\{T = n | T \geq n\} = P\{X_n = 0 | X_{n-1} \geq 1\} = \sum_{i=1}^{\infty} P\{X_n = 0 | X_{n-1} = i, X_{n-1} \geq 1\} P\{X_{n-1} = i | X_{n-1} \geq 1\}$ . That is

$$(3.2) \quad r_T(n) = 1 - E[Q(\tilde{X}_{n-1}, 1)],$$

where  $\tilde{X}_{n-1} \stackrel{d}{=} \{X_{n-1} | X_{n-1} \geq 1\}$  ( $\stackrel{d}{=}$  stands for equality in law.) Let  $\underline{P}_i = [P(i, j)]_{i, j \in \mathbb{N}_+}$  be the transition probability matrix of the lossy process  $\underline{X}^\ell = \{X_n^\ell, n = 0, 1, \dots\}$  of  $\underline{X}$  on the state space  $\mathbb{N}_+$  (e.g. Keilson 1979). Let  $\underline{v}_n$  and  $\underline{v}_n^\ell$  be the probability vector of  $\tilde{X}_n$  and  $X_n^\ell$  respectively (i.e.,  $v_n(k) = P\{\tilde{X}_n = k\}$  and  $v_n^\ell(k) = P\{X_n^\ell = k\}$ ). Since state 0 is absorbing it is easily verified that

$$(3.3) \quad \underline{v}_n = \underline{v}_n^\ell / \underline{v}_n^\ell \underline{e} = \underline{v}_0^\ell \underline{P}_i^n / \underline{v}_0^\ell \underline{P}_i^n \underline{e},$$

where  $\underline{e} = (1, 1, \dots)'$  and  $\underline{v}_0 = \underline{v}_0^\ell = (1, 0, 0, \dots)$ . Now consider  $(\underline{v}_n^\ell, n = 0, 1, \dots)$ . Clearly

$\begin{pmatrix} \underline{v}_0^\ell \\ \underline{v}_1^\ell \end{pmatrix} \in TP_2$  and therefore from the closure property of  $TP_2$  matrices under multiplication

$$\begin{pmatrix} \underline{v}_1^\ell \\ \underline{v}_2^\ell \end{pmatrix} \stackrel{t}{=} \begin{pmatrix} \underline{v}_0^\ell \\ \underline{v}_1^\ell \end{pmatrix} \underline{P}_i \stackrel{t}{=} \begin{pmatrix} \underline{v}_0^\ell \\ \underline{v}_1^\ell \end{pmatrix} \underline{Q}_i$$

is  $TP_2$  since  $\underline{Q}_\ell = [Q(i, j)]_{i, j \in \mathbb{N}_+}$  is  $TP_2$  ( $\underline{Q} \in TP_2$  implies  $\underline{Q}_\ell \in TP_2$ ). Now as an induction

hypothesis assume that  $\begin{pmatrix} v_{n-1}^\ell \\ v_n^\ell \end{pmatrix} \underline{t} \in TP_2$ . We have shown that this is true for  $n = 1, 2$ .

Now observing that  $\underline{t}^{-1} \underline{P}_\ell \underline{t} \geq 0$ , since  $\underline{P}_\ell \underline{t}$  is  $TP_2$  and  $\underline{P}$  is a transition probability matrix one has from Lemma 2.1

$$\begin{pmatrix} v_n^\ell \\ v_{n+1}^\ell \end{pmatrix} \underline{t} = \begin{pmatrix} v_{n-1}^\ell \\ v_n^\ell \end{pmatrix} \underline{P}_\ell \underline{t}$$

is  $TP_2$ . Observe that this  $TP_2$  property implies

$$(3.4) \quad \frac{P\{X_{n+1}^\ell \geq k+1\}}{P\{X_n^\ell \geq k+1\}} \geq \frac{P\{X_{n+1}^\ell \geq k\}}{P\{X_n^\ell \geq k\}}, \quad k = 1, 2, \dots$$

Combining (3.3) and (3.4) one sees that

$$(3.5) \quad \tilde{X}_{n+1} \geq_h \tilde{X}_n, \quad n = 0, 1, \dots$$

Since  $\geq_h$  implies stochastic ordering and  $Q(i, 1)$  is increasing in  $i$  one has from (3.2) and (3.5),  $r_T(n)$  is decreasing in  $n \in \mathbb{N}_+$ .  $\square$

*Remark 3.2:* It is known that if  $\underline{P}$  is  $TP_2$  then  $T$  has log-convex probability mass function which implies  $T \in \text{DFR}$  (Assaf, Shaked and Shanthikumar 1985). However, our condition  $\underline{Q}$  is  $TP_2$  is weaker since  $\underline{P} \in TP_2$  implies  $\underline{Q}$  is  $TP_2$  and not necessarily in the reverse direction.

*Remark 3.3:* Let  $T_k^- = \min[n: X_n \leq k-1, n = 1, 2, \dots] | X_0 = k, (k \geq 1)$ . Considering a modification of  $\underline{X}$  such that its states  $\{0, 1, \dots, k-1\}$  are lumped into one absorbing state, from Theorem 3.1 one sees that  $\underline{Q} \in TP_2$  implies  $T_k^- \in \text{DFR}$ .

Next consider an absorbing, right-continuous continuous time Markov chain  $\underline{Y} = \{Y(t), t \geq 0\}$  with state space  $N$ , where 0 is the absorbing state. Let  $\underline{\mu} = [\mu(i, j)]_{i, j \in N}$  be the transition rate matrix of  $\underline{Y}$  ( $\mu(i, j)$  is the transition rate from state  $i$  to state  $j$ ).

$$\mu_i = \sum_{j=0}^{\infty} \mu_{ij}; \quad \underline{D}_{\mu} = \text{diag} \{\mu_0, \mu_1, \dots\}.$$

$$(3.6) \quad T^* = \left\{ \inf [t: Y(t) = 0, t \geq 0] \mid Y(0) = 1 \right\}$$

Define  $\mu^* = \sup\{\mu_i, i = 0, 1, \dots\}$ . Then using uniformization (e.g., Keilson 1979, Chapter 2 or Assaf, Shaked and Shanthikumar 1985, Section 3) one has from Theorem 3.1

*Corollary 3.4:* If there exists a  $\lambda$  ( $\mu^* \leq \lambda < \infty$ ) such that  $\left[ \underline{I} + \frac{1}{\lambda}(\underline{\mu} - \underline{D}_{\mu}) \right] \underline{I}$  is  $TP_2$  then  $T^* \in \text{DFR}$ .

We now present some applications of the above results. Let  $\{Z_n, n = 1, 2, \dots\}$  be a sequence of i.i.d. random variables with support  $N^+$  and  $K$  be a geometric random variable with  $P\{K = k\} = (1-p)^{k-1}p, k \in N^+$ .  $\{Z_n\}$  and  $K$  are mutually independent.

*Theorem 3.5:*  $Z_n \in \text{DFR}$  implies  $Z^* = \sum_{n=1}^K Z_n \in \text{DFR}$ .

*Proof:* Let  $\underline{X}$  be a temporally homogeneous Markov chain with state space  $N$  and transition probability matrix  $\underline{P}$  where  $P(0, 0) = 1, P(i, i+1) = 1 - r_Z(i), P(i, 1) = (1-p)r_Z(i), P(i, 0) = pr_Z(i), i \in N^+$  and all other entries are zero.

$r_Z(i) = P\{Z_n = i \mid Z_n \geq i\}$  is the failure rate of  $Z_n$ . Let  $T$  be as defined in (3.1) and define

$$(3.7) \quad T_1 = \{\min [n: X_n \leq 1, n = 1, 2, \dots] \mid X_0 = 1\}.$$

Simple calculation shows that

$$(3.8) \quad P\{T_1 = k\} = \prod_{\ell=1}^{k-1} (1 - r_Z(\ell)) r_Z(k)$$

$$= P\{Z_n = k\}, k = 1, 2, \dots$$

It can also be verified that  $P\{X_{T_1} = 1\} = (1 - p)$  and  $P\{X_{T_1} = 0\} = p$ . Since state 0 is absorbing, employing the Markov property of  $\underline{X}$ , (3.8) and the fact that  $T_1$  is a stopping time of  $\underline{X}$ , it is not hard to see that

$$(3.9) \quad T \stackrel{d}{=} Z^*.$$

Computing  $\underline{Q}$  one sees that  $Q(i, 0) = 1, i \in \mathbb{N}, Q(i, 1) = 1 - pr_Z(i), Q(i, j) = 1 - r_Z(i), 2 \leq j \leq i; i \in \mathbb{N} +$  and all other entries are zero. In this case  $\underline{Q} \in TP_2$  as long as  $r_Z(i)$  is decreasing in  $i \in \mathbb{N} +$ . The required result now follows from (3.9) and Theorem 3.1.

□

Consider a GI/G/1 queue at which customers arrive according to a renewal process with rate  $\lambda$ . The service times form a sequence of i.i.d. random variables with a common distribution function  $F$ . The arrival process and service times are mutually independent. We

will first consider an M/G/1 queue which is a special case of GI/G/1 queue with Poisson arrival process.

*Theorem 3.6:* The number of customers served during a busy period of an M/G/1 queue with IFR service times is DFR.

*Proof:* Let  $\tilde{X}_n$  be the number of customers in the M/G/1 queueing system just after the  $n$ -th customer departure. Then  $\tilde{X} = \{\tilde{X}_n, n = 0, 1, \dots\}$  is a temporally homogeneous Markov chain with transition probabilities  $\tilde{P}(i, j) = g(j + 1 - i)$ ,  $i \in \mathbb{N} +$ ;  $\tilde{P}(0, j) = g(j)$ ,  $j \in \mathbb{N}$ , where

$$g(k) = \begin{cases} 0 & k = -1, -2, \dots \\ \int_0^\infty \frac{e^{-\lambda t} (\lambda t)^k}{k!} dF(t), & k = 0, 1, \dots \end{cases}$$

(e.g., Ross 1983). Consider a modification  $\underline{X}$  of  $\tilde{X}$  such that state 0 is absorbing in  $\underline{X}$  (i.e.,  $P(i, j) = \tilde{P}(i, j)$ ,  $i \in \mathbb{N} +$ ,  $j \in \mathbb{N}$  and  $P(0, 0) = 1$ ). Then if  $T$  is as defined in (3.1), it is the number of customers served during the first busy period. Consider  $Q(i, j) = \bar{G}(j + 1 - i)$ ,  $i, j \in \mathbb{N} +$  ( $Q(0, 0) = 1$ ,  $Q(0, j) = 0$ ,  $j \in \mathbb{N} +$ ),

$$\bar{G}(k) = \begin{cases} 1 & k = -1, -2, \dots \\ \sum_{\ell=k}^{\infty} g(\ell), & k = 0, 1, \dots \end{cases}$$

It is known that if  $\bar{F}(t+x)/\bar{F}(t)$  is decreasing in  $t$  (i.e.,  $F$  is IFR) then  $\bar{G}(n+k)/\bar{G}(k)$  is decreasing in  $k$  (Block and Savits 1980). This observation leads to a straightforward verification that  $\underline{Q} \in TP_2$ . The DFR property of  $T$  now follows from Theorem 3.1.  $\square$

#### 4. CLOSURE OF DFR UNDER GEOMETRIC COMPOUNDING

$\{W_n, n = 1, 2, \dots\}$  is a sequence i.i.d. random variables with support  $[0, \infty)$  and survival function  $\bar{F}$  and  $K$  is a geometric random variable with  $P\{K = k\} = (1 - p)^{k-1}p, k = 1, 2, \dots$ .  $\{W_n\}$  and  $K$  are mutually independent. Then one has

*Theorem 4.1:*  $W_n \in \text{DFR}$  implies  $W^* = \sum_{n=1}^K W_n \in \text{DFR}$ .

*Proof:* For some  $\lambda > 0$  let  $\bar{G}(k) = \bar{F}(\frac{k}{\lambda}), k = 0, 1, \dots$ . Then if  $\{Z_n^\lambda, n = 1, 2, \dots\}$  is a sequence of i.i.d. random variables with a common survival function  $\bar{G}$ ,  $r_{Z^\lambda}(k) = [\bar{G}(k-1) - \bar{G}(k)] / \bar{G}(k-1)$  is decreasing in  $k = 1, 2, \dots$  (i.e.,  $Z_n^\lambda \in \text{DFR}$ ). Let  $\{E_n^\lambda, n = 1, 2, \dots\}$  be a sequence of i.i.d. exponential random variables with mean  $\frac{1}{\lambda}$ . Now define a sequence of i.i.d. random variables  $\{W_n^\lambda, n = 1, 2, \dots\}$  such that  $W_n^\lambda \stackrel{d}{=} \sum_{k=1}^{Z_n^\lambda} E_k^\lambda$ . That is, if  $\bar{F}^\lambda$  is the survival function of  $W_n^\lambda$ ,

$$(4.1) \quad \bar{F}^\lambda(t) = \sum_{k=0}^{\infty} \bar{F}(\frac{k}{\lambda}) \frac{e^{-\lambda t} (\lambda t)^k}{k!}.$$

Then it is easily verified that

$$(4.2) \quad \text{as } \lambda \rightarrow \infty, \bar{F}^\lambda(t) \rightarrow \bar{F}(t) \text{ at every continuity point of } \bar{F}$$

(Feller 1971). Then consider

$$\sum_{n=1}^K W_n^\lambda \stackrel{d}{=} \sum_{n=1}^{\sum_{k=1}^K Z_k^\lambda} E_n^\lambda.$$

Since  $Z_k^\lambda \in \text{DFR}$  implies  $\sum_{k=1}^K Z_k^\lambda \in \text{DFR}$  (Theorem 3.4) and a sum of DFR number of i.i.d. exponential random variables is DFR (Esary, Marshall and Proschan (1973)),  $\sum_{n=1}^K W_n^\lambda \in \text{DFR}$ . Since DFR property is preserved under limits as  $\lambda \rightarrow \infty$  one has  $\sum_{n=1}^K W_n \in \text{DFR}$ .  $\square$

*Remark 4.2:* Szekli (1986) shows that  $W^*$  has convex survival function (say  $\bar{F}^*$ ). However,  $W^* \in \text{DFR}$  is equivalent to that  $\bar{F}^*$  is log-convex: a stronger property than convexity.

The above result combined with the results of Szekli (1986) lead to

*Corollary 4.3:* The stationary waiting time in a GI/G/1 (M/G/1) queue with DFR (IMRL) service times is DFR.

## 5. CONCAVE RENEWAL FUNCTIONS

Consider a discrete time renewal process with the inter-renewal time having the first passage time distribution of  $X$  from state 1 to state 0. Let  $\gamma(n)$  be the probability that a renewal occurs at time  $n = 1, 2, \dots$  ( $\gamma$  is the renewal density). Then the expected number of renewals  $M(n)$  during  $\{1, 2, \dots, n\}$  is equal to  $\sum_{k=1}^n \gamma(k)$  ( $M$  is the renewal function.)

*Theorem 5.1:*  $\underline{\underline{t}}^{-1}Q \geq \underline{\underline{0}}$  implies  $\gamma(n)$  is decreasing in  $n \in \mathbb{N}^+$  (i.e.,  $M$  is concave on  $\mathbb{N}^+$ ).

*Proof:* Let  $X^* = \{X_n^*, n = 0, 1, \dots\}$  be a modification of  $X$  such that as soon as  $X$  reaches 0 it is placed back to state 1 (representing a renewal). Then  $Q^*(i, j) = Q(i, j)$ ,  $i \in \mathbb{N}^+$ ,  $j = 2, 3, \dots$ , and  $Q^*(i, 1) = 1$ ,  $i \in \mathbb{N}^+$ . So  $\underline{\underline{t}}^{-1}Q \geq \underline{\underline{0}}$  implies  $\underline{\underline{t}}^{-1}Q^* \geq \underline{\underline{0}}$  and therefore  $X^*$  is stochastically monotone. Specifically  $\{X_n^* | X_0^* = 1\} \geq_{st} \{X_{n-1}^* | X_0^* = 1\}$ ,  $n = 1, 2, \dots$  (note that the state space of  $X^*$  is  $\mathbb{N}^+$ ). Then the observation that

$$\gamma(n) = 1 - E\{Q(X_{n-1}^*, 1)\}$$

and that  $Q(i, 1)$  is increasing in  $i$  leads to the desired conclusion.  $\square$

We will next see that the inter-renewal time need not be DFR for the renewal function to be concave. This will be achieved by showing that  $\underline{I}^{-1}\underline{Q} \geq \underline{0}$  is not sufficient for the DFR property of  $T$ .

*Counterexample 5.2:* Consider the first passage time  $T$  (defined in 3.1) of a Markov chain with transition probability matrix

$$\underline{P} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix}.$$

Then

$$\underline{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{4} & \frac{1}{4} \\ 1 & 1 & \frac{1}{4} \end{bmatrix}$$

and  $\underline{I}^{-1}\underline{Q} \geq \underline{0}$ . So the renewal density is decreasing (Theorem 5.1). Computing one gets  $r_T(1) = \frac{3}{4}$ ;  $r_T(2) = 0$ ;  $r_T(3) = \frac{9}{16}$ . So  $T$  is not DFR. Then one sees that in the discrete time case the *DFR property of inter-renewal times is sufficient but not necessary* for a concave renewal function.

We next turn our attention to the continuous time case. The inter-renewal times have the first passage time distribution of  $\underline{Y}$  from state 1 to state 0.  $\gamma$  and  $M$  are the renewal density and renewal function, respectively, of the renewal process. Similar to the discrete time case one has

*Theorem 5.3:*  $\underline{Y}$  is stochastically monotone implies  $\gamma(t)$  is decreasing in  $t \in [0, \infty)$  (i.e.,  $M$  is concave on  $[0, \infty)$ ).

*Remark 5.4:* An interesting consequence of Theorem 5.3 and the conjecture of Brown (1981), if true, is that the first passage time  $T^*$  of a stochastically monotone Markov process  $\underline{Y}$  with state space  $N$  (i.e., for any  $k \in N$ ,  $\sum_{j \geq k} \mu(i, j)$  is increasing in  $i < k$  and  $\sum_{j \leq k} \mu(i, j)$  is decreasing in  $i > k$ ) from state 1 to state 0 is DFR.

## ACKNOWLEDGEMENT

The author is very grateful for the helpful suggestions of Mark Brown and Moshe Shaked.

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