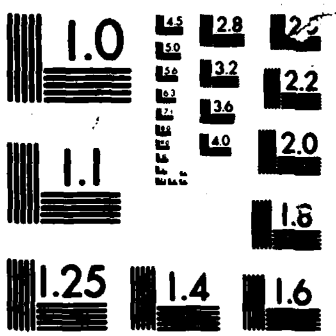
AD-A168 740 OPTIMAL SAMPLING RATES FOR AUTO REGRESSIVE MODELS(U) 1/1
AERONAUTICAL RESEARCH LABS MELBOURNE (AUSTRALIA)
T G RYALL APR 86 ARL-STRUC-TM-435

UNCLASSIFIED F/G 12/1 NL



MICROCOP' CHART

(12)

AD-A168 740

ARL-STRUC-TM-435 AR-004-458

DEPARTMENT OF DEFENCE

DEFENCE SCIENCE AND TECHNOLOGY ORGANISATION

AERONAUTICAL RESEARCH LABORATORIES

MELBOURNE, VICTORIA

Structures Technical Memorandum 435

OPTIMAL SAMPLING RATES FOR AUTO REGRESSIVE MODELS

by

T. G. RYALL

Approved for Public Release

APRIL 1986

AR-004-458

DEPARTMENT OF DEFENCE
DEFENCE SCIENCE AND TECHNOLOGY ORGANISATION
AERONAUTICAL RESEARCH LABORATORIES

Structures Technical Memorandum 435

OPTIMAL SAMPLING RATES FOR AUTO REGRESSIVE MODELS

by

T. G. RYALL

SUMMARY

A detailed examination is made of a simple auto-regressive model, so as to determine the optimal sampling rates for parameter identification. It is shown that sampling at approximately six times the frequency of interest has a number of optimal properties.

POSTAL ADDRESS: Director, Aeronautical Research Laboratories,
P.O. Box 4331, Melbourne, Victoria, 3001, Australia.

CONTENTS

1. INTRODUCTION

In time series analysis a particular model for a linear system is an auto-regression model. Auto-regression models are particularly well suited to structural linear systems which have been persistently excited by white noise. The white noise is assumed to be unmeasurable and it is assumed that there is no measurement noise.

The variability of the results obtained in practice tends to depend on the sampling rates used.

This memo examines in detail the asymptotic convariance matrix for estimates of the damping rate and frequency for a fixed length of record and variable sampling rates in a linear system which is completely described, in continuous time, by a single second order differential equation. It is shown that the selection of sampling rate plays an extremely important role in the quality of the estimates.

2. THE MODEL

Let x_t be the measured response of a linear system at time t and let ε_t be a stationary white-noise process. Then the auto-regression model to be studied is (Ref. 1).

$$x_t + a_1 x_{t-1} + a_2 x_{t-2} = \varepsilon_t \tag{2.1}$$

where $a_1 = -2\exp(-\alpha T)\cos\omega T$

$a_2 = \exp(-2\alpha T)$

α = decay rate of the corresponding continuous time process

ω = frequency of the corresponding continuous time process and

T = time between samples.

3. PARAMETER ESTIMATES

The least squares estimates of $a_1(\hat{a}_1)$ and $a_2(\hat{a}_2)$ minimise the expression $\sum_1^N (x_t + a_1 x_{t-1} + a_2 x_{t-2})^2$ where N is the number of independent estimates of the nosie.

The normal equations which result from this process are

$$\Sigma x_t x_{t-1} + \hat{a}_1 \Sigma x^2_{t-1} + \hat{a}_2 \Sigma x_{t-1} x_{t-2} = 0 \qquad (3.1)$$

$$\Sigma x_t x_{t-2} + \hat{a}_1 \Sigma x_{t-1} x_{t-2} + \hat{a}_2 \Sigma x^2_{t-2} = 0 \qquad (3.2)$$

Asymptotically equations (3.1) and (3.2) become

$$\hat{a}_1 \rho_0 + \hat{a}_2 \rho_1 + \rho_1 = 0 \qquad (3.3)$$

$$\hat{a}_1 \rho_1 + \hat{a}_2 \rho_0 + \rho_2 = 0 \qquad (3.4)$$

where ρ_k is the auto correlation of lag k.

Equations (3.3) and (3.4) are the Yule-Walker equations and may be solved using very efficient algorithms for higher order systems. In our example we have

$$\begin{bmatrix} \hat{a} \\ \hat{a}_2 \end{bmatrix} = - \begin{bmatrix} \rho_0 & \rho_1 \\ \rho_1 & \rho_0 \end{bmatrix}^{-1} \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} \qquad (3.5)$$

The convariance matrix of this estimator is

$$E\left[(\hat{\underline{a}} - \underline{a})(\hat{\underline{a}} - \underline{a})^T\right] = \frac{\sigma}{N}^2 \begin{bmatrix} \rho_0 & \rho_1 \\ \rho_1 & \rho_0 \end{bmatrix}^{-1} \qquad (3.6)$$

where $\underline{a} = [a_1, a_2]^T \qquad (3.7)$

and $\sigma^2 = E(\varepsilon_t{}^2)$

The parameters $\underset{\sim}{a}$ depend however on the sampling rate as well as the system parameters α, ω.

It is necessary to calculate the asymptotic convariance matrix of α and ω. Noting that

$$a_2 = \exp(-2\alpha T)$$

$$a_1 = -2 \exp(-\alpha T)\cos\omega T \qquad (3.9a)$$

and

$$\delta a_1 = 2T\cos(\omega T)\exp(-\alpha T)\delta\alpha$$

$$+ 2T\exp(-\alpha T)\sin\omega T \; \delta\omega \qquad (3.9b)$$

$$\delta a_2 = -2T \exp(-2\alpha T)\delta\alpha$$

we obtain

$$\begin{bmatrix} \delta a_1 \\ \delta a_2 \end{bmatrix} = -T \begin{bmatrix} a_1 & -\sqrt{4a_2 - a_1^2} \\ 2a_2 & 0 \end{bmatrix} \begin{bmatrix} \delta\alpha \\ \delta\omega \end{bmatrix} \qquad (3.10)$$

Using standard statistical theory it follows that the asymptotic convariance of α,ω is given by

$$\frac{\sigma^2}{NT^2} \begin{bmatrix} a_1 & -\sqrt{4a_2-a_1^2} \\ 2a_2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \rho_0 & \rho_1 \\ \rho_1 & \rho_0 \end{bmatrix}^{-1} \begin{bmatrix} a_1 & 2a_2 \\ -\sqrt{4a_2-a_1^2} & 0 \end{bmatrix}^{-1}$$

$$= \frac{1}{NT^2}\begin{bmatrix} a_1 & -\sqrt{4a_2-a_1^2} \\ 2a_2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1-a_2^2 & a_1(1-a_2) \\ a_1(1-a_2) & (1-a_2^2) \end{bmatrix} \begin{bmatrix} a_1 & 2a_2 \\ -\sqrt{4a_2-a_1^2} & 0 \end{bmatrix}^{-1} \tag{3.11}$$

and the asymptotic covariance matrix (Q) is

$$Q = \frac{(1-a_2)}{NT^2\ 4a_2^2(4a_2-a_1^2)} \begin{bmatrix} 0 & \sqrt{4a_2-a_1^2} \\ -2a_2 & a_1 \end{bmatrix} \begin{bmatrix} (1+a_2) & a_1 \\ a_1 & 1+a_2 \end{bmatrix} \cdot \begin{bmatrix} 0 & -2a_2 \\ \sqrt{4a_2-a_1^2} & a_1 \end{bmatrix} \tag{3.12}$$

As $NT = t_f$ = total time available for observations and defining

$\theta = \omega T$ and $G = \alpha/\omega$ it follows that

$$a_1 = -2\exp(-G\theta)\cos\theta \tag{3.13}$$

$$a_2 = \exp(-2G\theta) \tag{3.14}$$

The convariance matrix can now be expressed as a function of G and θ, except for a constant multiplier ω/t_f. The convariance matrix can be scaled up or down according to the value of ω/t_f.

It should be noted that as $t_f \to \infty$ the convariance matrix goes to zero for any fixed values of G and θ. This means that for a sufficiently long record time α and ω are essentially determined without error, provided that there is no aliasing in ω. However this is only true numerically if the computer were to work with infinite word lengths. For a finite record length it will be shown that the sampling rate is important.

4. MEASURES OF PERFORMANCE

Various measures of performance will now be studied. These measures reflect the fact that the convariance matrix must be small in some sense. The actual value of these measures at any particular sampling rate is unimportant due to the factor ω/t_f rather it is the minimum value of the measure as a function of θ that is sought. The matrix Q is given by (3.12).

These measures are

(i) the determinant of the covariance matrix $Q_{11}Q_{22}-Q_{12}^2$

(ii) the maximum eigenvalue of the covariance matrix $\lambda_{max}(Q)$

(iii) the variance of the damping estimate (Q_{11}).

(iv) the variance of the frequency estimate (Q_{22}).

The first measure is called the generalised variance by statisticians. The second measure finds the linear combination with the most uncertainty. The third and fourth measures are appropriate if the parameter of interest was purely damping or frequency respectively.

The various measures are plotted as a function of θ for $0 < \theta < \pi$ and G = .01, .10, .50, 1.0 and the optimal value of θ (ITHETA, in degrees) is printed for each measure above each graph.

In all measures that involve eigenvalues, plots of log (maximum eigenvalue) and log (minimum eigenvalue) versus θ are given. Here it should be noted that $\theta = 0$ corresponds to sampling infinitely fast and $\theta = 180^o$ corresponds to sampling at the NYQUIST rate.

All calculations were done extremely carefully as the condition number of the matrix varied from 1 to 50,000 and upwards. This meant that differences between numbers of a similar size were always avoided.

The vertical scale in all graphs is artibrary due to the presence of ω/t_f in the convariance matrix.

5. DISCUSSION

Figure 1 shows that the generalised variance or determinant is a monotonically increasing function of θ (This can be shown analytically for all G). Here it should be noted that the value of ITHETA should be zero (ie 0°) rather than 1° since 0° requires a special evaluation of the determinant which was not done.

The generalised variance is an inappropriate measure as the determinant is the product of two eigenvalues. The possibility exists that the determinant is small but that the matrix is very ill-conditioned. Indeed Figure 2 shows that this actually occurs. The top curve is the logarithm of the maximum eigenvalue and the bottom curve is the logarithm of the minimum eigenvalue.

The maximum eigenvalue is minimised for lightly damped modes at about $\theta = 60°$ which correspnds to sampling at six times the frequency of interest.

It should be noted that the condition number of the matrix is also at its optimum for this sampling rate.

For some value of G ($0.1<G<.5$) the optimal sampling rate occurs at $\theta = 0°$ with a corresponding increase in ill-conditioning. This implies that for modes which are heavily damped the sampling rate should be as high as possible consistent with the precision of the calculations.

Figure 3 illustrates the fact that the variance of the damping is a monotonic increasing function of θ.

This function is $K\left[\frac{\exp(4G\theta)-1}{\theta}\right]$

as substitution into (3.12) will show, where K is an arbitrary constant.

Figure 4 also resembles a monotonic function; however this is misleading due to the large values obtained near $\theta = 180°$.

6. CONCLUSION

The optimal sampling rate has been examined for a very simple model involving an autoregressive model. The optimal sampling rate for lightly damped modes is six times the frequency of interest. At this sampling rate the convariance matrix has a condition number approximately equal to one.

Throughout this work it has been assumed that the sampling rate has to be the same as the sampling rate used in the model. The model, however, could use a sampling rate much lower than the rate at which data arrive with no loss of information.

It is clear that in a more complicated system the optimal sampling rates will depend on the pole positions of the continuous time system in a complex manner and that different sampling rates need to be used for different modes.

Reference

1. Goodwin, G. C. and Sin, K.S., Adaptive Filtering Prediction and Control, Prentice Hall, 1983.

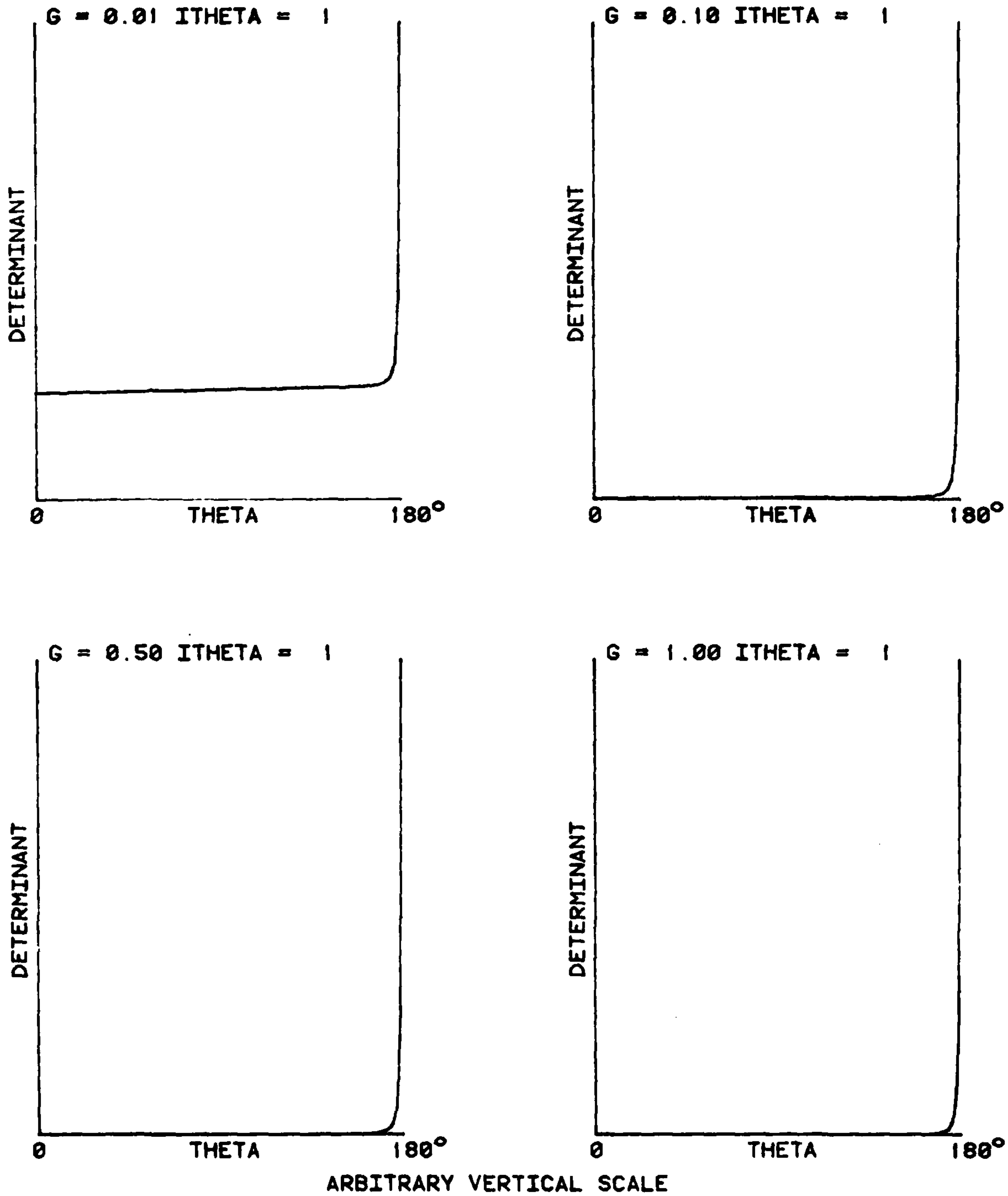


FIGURE 1 DETERMINANT OF COVARIANCE MATRIX VERSUS THETA

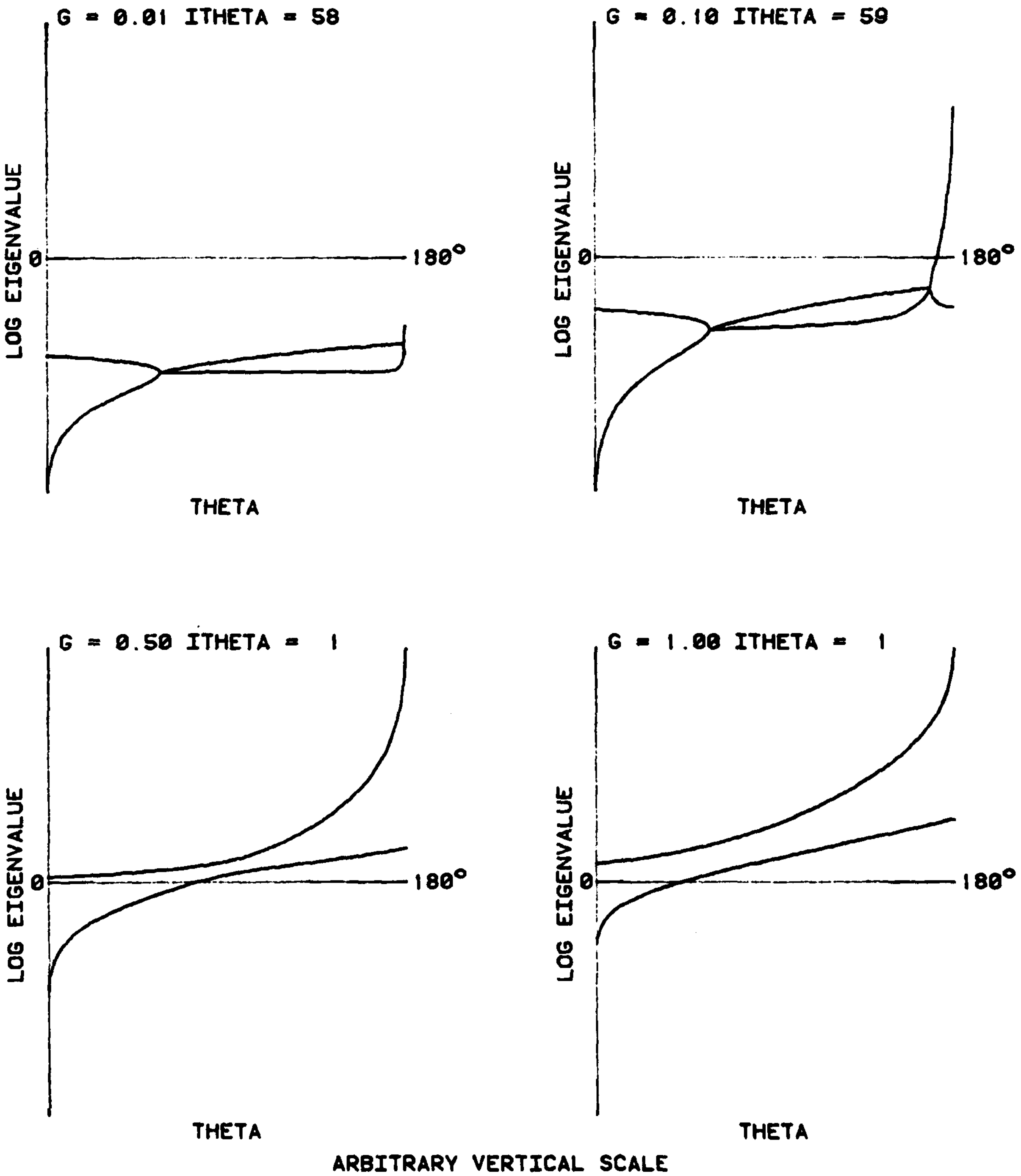


FIGURE 2 LOGARITHM OF MAXIMUM AND MINIMUM EIGENVALUE OF COVARANCE MATRIX VERSUS THETA

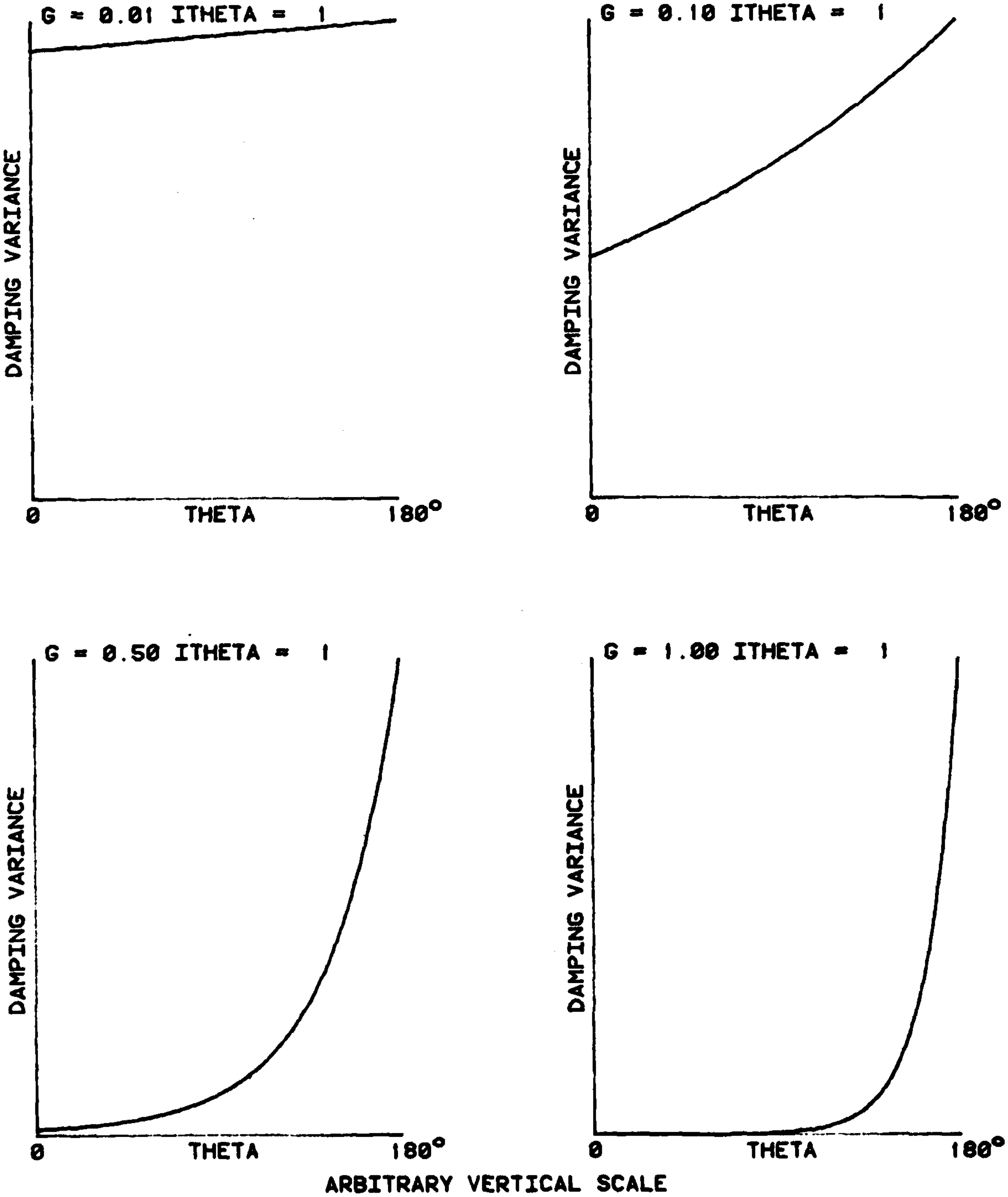


FIGURE 3 VARIANCE OF DAMPING ESTIMATE
VERSUS THETA

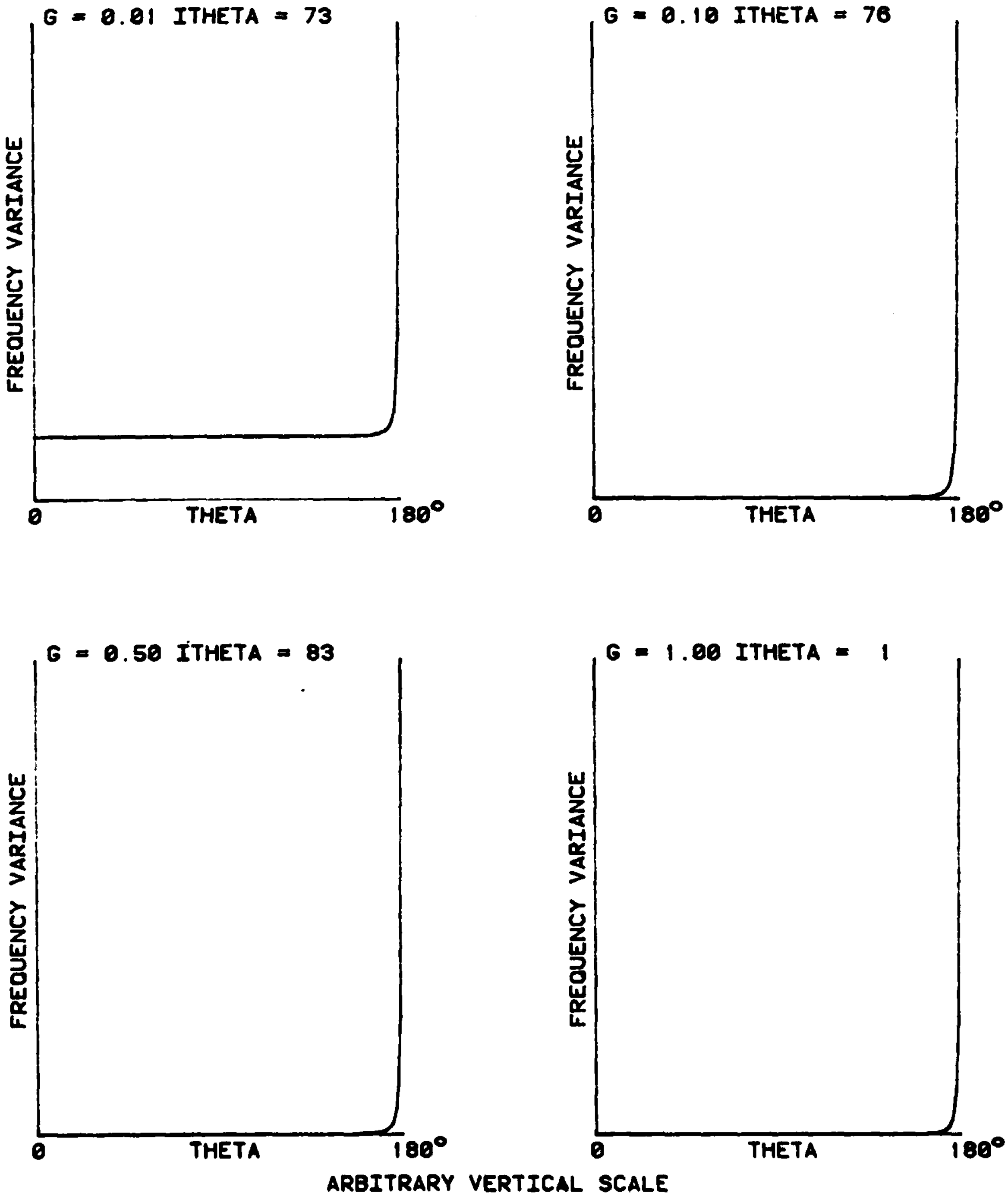


FIGURE 4 VARIANCE OF FREQUENCY ESTIMATE VERSUS THETA

DISTRIBUTION

AUSTRALIA

Department of Defence

Defence Central

Chief Defence Scientist)
Deputy Chief Defence Scientist)
Superintendent, Science and Program Administration) 1 Copy
Controller, External Relations, Projects and)
Analytical Studies)
Defence Science Adviser (UK) (Doc Data Sheet only)
Counsellor, Defence Science (USA) (Doc Data sheet only)
Defence Central Library
Document Exchange Centre, DISB (18 copies)
Joint Intelligence Organisation
Librarian H Block, Victoria Barracks, Melbourne

Aeronautical Research Laboratories

Director
Library
Superintendent Aero Propulsion
Superintendent Systems
Superintendent Aerodynamics
Divisional File - Structures
R A Feik
C Martin
C Guy
P Preston
B Emslie
P Farrell
R Jones
P Cox
Author: T G Ryall

Materials Research Laboratories

Director/Library

Defence Research Library

Library

RAN Research Laboratory

Library

Navy Office

Navy Scientific Adviser

Army Office

Scientific Adviser - Army
Engineering Development Establishment, Library
Royal Military College Library

Air Force Office

- Air Force Scientific Adviser
- Aircraft Research and Development Unit
  - Scientific Flight Group
  - Library
- Technical Division Library
- RAAF College, Point Cook

Government Aircraft Factories

- Manager
- Library

Department of Science

- Bureau of Meteorology, Library

Department of Aviation

- Library
- Flight Standards Division

Universities and Colleges

- Adelaide
  - Barr Smith Library
- Australian National
  - Prof. J. B. Moore, RS PhysS
- Flinders
  - Library
- La Trobe
  - Library
- Melbourne
  - Engineering Library
- Monash
  - Hargrave Library
- Newcastle
  - Library
  - Prof G. Goodwin, Elec. Engin. & Computing
- New England
  - Library
- Sydney
  - Engineering Library
  - Head, School of Civil Engineering
- NSW
  - Physical Sciences Library
  - Prof R A A Bryant, Mechanical Engineering
- Queensland
  - Library
- Tasmania
  - Engineering Library
- Western Australia
  - Library
  - Associate Prof J A Cole, Mechanical Eng
- RMIT
  - Library

SPARES (5 copies)

TOTAL (75 copies)

ADA 168740

Department of Defence

DOCUMENT CONTROL DATA

1. a. AR No	1. b. Establishment No	2. Document Date	3. Task No
AR-004-458	ARL-STRUC-TM-435	APRIL 1986	82/054

4. Title	5. Security a. document	6. No Pages
OPTIMAL SAMPLING RATES FOR AUTO REGRESSIVE MODELS	UNCLASSIFIED	9
	b. title: U — c. abstract: U	7. No Refs: 1

8. Author(s)	9. Downgrading Instructions
T. G. RYALL	

10. Corporate Author and Address	11. Authority (as appropriate) a.Sponsor b.Security c.Downgrading d.Approval
Aeronautical Research Laboratories PO Box 4331 MELBOURNE VIC 3001	

12. Secondary Distribution (of this document)

Approved for public release.

Overseas enquirers outside stated limitations should be referred through ASDIS, Defence Information Services Branch, Department of Defence, Campbell Park, CANBERRA ACT 2601

13. a. This document may be ANNOUNCED in catalogues and awareness services available to ...

No limitations

13. b. Citation for other purposes (ie casual announcement) may be (select) unrestricted (or) as for 13 a.

14. Descriptors	15. COSATI Group
Time series analysis Parameter identification Autoregression Least squares method Sampling rates	12010

16. Abstract

A detailed examination is made of a simple auto-regressive model, so as to determine the optimal sampling rates for parameter identification. It is shown that sampling at approximately six times the frequency of interest has a number of optimal properties.

This page is to be used to record information which is required by the Establishment for its own use but which will not be added to the DISTIS data base unless specifically requested.

16. Abstract *(Contd)*		
17. Imprint Aeronautical Research Laboratories, Melbourne		
18. Document Series and Number Structures Technical Memorandum 435	**19. Cost Code** 23 6925	**20. Type of Report and Period Covered**
21. Computer Programs Used		
22. Establishment File Ref(s)		

END
DTIC
7-86