

AD-A168 823

EXACT TESTS FOR THE MAIN EFFECTS' VARIANCE COMPONENTS
IN AN UNBALANCED RA. (U) FLORIDA UNIV GAINESVILLE DEPT
OF STATISTICS A I KHURI ET AL. JUN 86 TR-268

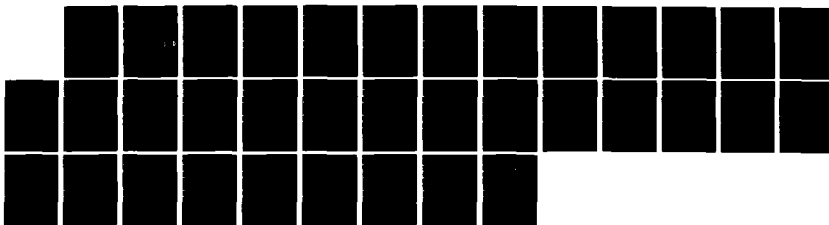
1/1

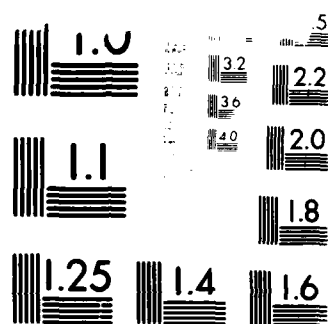
UNCLASSIFIED

N00014-86-K-0059

F/G 12/1

NL





1. *Chlorophyll a* (Chl *a*)

AD-A168 823

EXACT TESTS FOR THE MAIN EFFECTS' VARIANCE
COMPONENTS IN AN UNBALANCED RANDOM TWO-WAY MODEL

By

A. I. Khuri and R. C. Littell

Department of Statistics, University of Florida
Gainesville, FL. 32611

DTIC
SELECTE
JUN 20 1986
S D

Technical Report Number 268

June, 1986

ATTC FILE COPY

PREPARED UNDER GRANT NO.N00014-86-K-0059 FROM
THE OFFICE OF NAVAL RESEARCH. ANDRE' I. KHURI
AND RAMON C. LITTELL, PRINCIPAL INVESTIGATORS.

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 268	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Exact Tests for the Main Effects' Variance Components in an Unbalanced Random Two-Way Model		5. TYPE OF REPORT & PERIOD COVERED
7. AUTHOR(s) Andre' I. Khuri and Ramon C. Littell		6. PERFORMING ORG. REPORT NUMBER
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Statistics Nuclear Sciences Center, University of Florida Gainesville, FL 32611		8. CONTRACT OR GRANT NUMBER(s) N00014-86-K-0059 R&T 4114552---01 Acct. No. 121623459
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Mathematical Sciences Division (Code 411) Arlington, VA 22217-5000		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		12. REPORT DATE June 1986
		13. NUMBER OF PAGES 31
		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for Public Release: Distribution Unlimited		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Variance Components testing, Unbalanced random two-way classification model, Exact tests, Approximate tests, Power of exact tests.		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) Exact tests are presented for testing the significance of the random effects' variance components in an unbalanced random two-way classification model. A study is made concerning the power of the proposed tests, including a comparison with other approximate tests.		

EXACT TESTS FOR THE MAIN EFFECTS'
VARIANCE COMPONENTS IN AN UNBALANCED RANDOM TWO-WAY MODEL

By

A. I. Khuri and R. C. Littell

University of Florida

TECHNICAL REPORT NUMBER 268

Department of Statistics
University of Florida
Gainesville, Florida 32611

June 1986

EXACT TESTS FOR THE MAIN EFFECTS'
VARIANCE COMPONENTS IN AN UNBALANCED RANDOM TWO-WAY MODEL

A. I. KHURI and R. C. LITTELL

Department of Statistics, University of Florida
Gainesville, Florida 32611, U.S.A.

SUMMARY

Exact tests are derived for testing hypotheses concerning the variance components of the main effects in an unbalanced random two-way crossed classification with interaction model. The tests are based on four sums of squares which are distributed independently as scalar multiples of chi-squared variates. These sums of squares can also be used to find an exact test concerning the interaction variance component, and to obtain simultaneous confidence intervals on all continuous functions of the model's variance components. A study is made concerning the power of the proposed tests, including a comparison with other approximate tests.

Key words: Variance components; Unbalanced random model; Two-way crossed classification with interaction model; Hypothesis testing; Power of exact tests.

1. Introduction

There are very few exact tests available concerning the variance components in an unbalanced random or mixed model. This is mainly attributed to the fact that in an unbalanced data situation, the traditional partitioning of the total sum of squares does not in general yield independent and chi-squared type sums of squares. Furthermore, such partitioning is not unique as is the case with balanced models.

Wald (1940, 1941) was the first author to introduce exact testing procedures for the unbalanced one-way and two-way crossed classification without interaction models. A generalization of Wald's tests was recently discussed by Seely and El-Bassiouni (1983). See also Harville and Fenech (1985). Using two different approaches, Spjøtvoll (1968) and Thomsen (1975) developed exact tests concerning the variance components in an unbalanced random two-way crossed classification with interaction model. However, unless the interaction variance component is zero, neither Spjøtvoll's tests nor those of Thomsen can be used to make inferences about the main effects' variance components. In this paper we managed to overcome this difficulty by producing exact tests concerning the latter variance components which apply even when the interaction variance component is different from zero. A comparison of the power of the exact tests with simulated powers of approximate ANOVA-based F tests shows that in most cases the



2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
13	
14	
15	
16	
17	
18	
19	
20	
21	
22	
23	
24	
25	
26	
27	
28	
29	
30	
31	
32	
33	
34	
35	
36	
37	
38	
39	
40	
41	
42	
43	
44	
45	
46	
47	
48	
49	
50	
51	
52	
53	
54	
55	
56	
57	
58	
59	
60	
61	
62	
63	
64	
65	
66	
67	
68	
69	
70	
71	
72	
73	
74	
75	
76	
77	
78	
79	
80	
81	
82	
83	
84	
85	
86	
87	
88	
89	
90	
91	
92	
93	
94	
95	
96	
97	
98	
99	
100	

odes

Dist	Avail and/or Spec
A-1	

exact tests are at least as powerful as the approximate tests. One main disadvantage of the latter tests is that their true critical values are unknown since they depend on variance components other than those under consideration. Thus, in practice the critical values of the approximate tests must be estimated using some procedure such as Satterthwaite's approximation. The simulation study shows that in some cases this approximation is highly unreliable for producing the critical values. Only in such cases were the approximate tests observed to be more powerful than the exact tests.

2. The Development of the Exact Tests

We shall adopt the same notation as in Thomsen (1975). Consider the unbalanced random two-way crossed classification model

$$y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + e_{ijk}, \quad (2.1)$$

$i = 1, 2, \dots, r$; $j = 1, 2, \dots, s$; $k = 1, 2, \dots, n_{ij}$, where μ is an unknown constant parameter; α_i , β_j , $(\alpha\beta)_{ij}$, and e_{ijk} are independent normally distributed random variables with zero means and variances σ_α^2 , σ_β^2 , $\sigma_{\alpha\beta}^2$, and σ_e^2 , respectively. Alternatively, (2.1) can be written in matrix form as

$$\underline{y} = \mu \underline{1}_{n_{..}} + \underline{X}_1 \underline{\alpha} + \underline{X}_2 \underline{\beta} + \underline{X}_3 (\underline{\alpha\beta}) + \underline{e}, \quad (2.2)$$

where $\underline{y}$ is the vector of observations of dimension $n_{..} = \sum_{i,j} n_{ij}$, $\underline{1}_{n_{..}}$ is a vector of ones of dimension $n_{..}$, $\underline{X}_1, \underline{X}_2$, and $\underline{X}_3$ are

matrices of zeros and ones of orders $n_{..} \times r$, $n_{..} \times s$, and $n_{..} \times rs$, respectively. The variance-covariance matrix of $\bar{y}$, denoted by Σ , is

$$\Sigma = \tilde{X}_1 \tilde{X}_1' \sigma_\alpha^2 + \tilde{X}_2 \tilde{X}_2' \sigma_\beta^2 + \tilde{X}_3 \tilde{X}_3' \sigma_{\alpha\beta}^2 + \tilde{I}_{n_{..}} \sigma_e^2, \quad (2.3)$$

where $\tilde{I}_{n_{..}}$ is the identity matrix of order $n_{..} \times n_{..}$.

Let $\bar{y}_{ij.}$ be the $(i,j)^{th}$ sample cell mean ($i = 1, 2, \dots, r$; $j = 1, 2, \dots, s$). From (2.1) we have

$$\bar{y}_{ij.} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \bar{e}_{ij.}, \quad (2.4)$$

$i = 1, 2, \dots, r$; $j = 1, 2, \dots, s$, where $\bar{e}_{ij.} = \sum_{k=1}^{n_{ij}} e_{ijk} / n_{ij}$. In matrix form, (2.4) may be written as

$$\bar{\bar{y}} = \mu \tilde{1}_{rs} + \tilde{B}_1 \alpha + \tilde{B}_2 \beta + \tilde{I}_{rs} (\alpha\beta) + \bar{\bar{e}}, \quad (2.5)$$

where $\tilde{B}_1 = \tilde{I}_r \otimes \tilde{1}_s$, $\tilde{B}_2 = \tilde{1}_r \otimes \tilde{I}_s$, and $\otimes$ is the direct product symbol. The variance-covariance matrix of $\bar{\bar{y}}$ is

$$\text{Var } \bar{\bar{y}} = \tilde{A}_1 \sigma_\alpha^2 + \tilde{A}_2 \sigma_\beta^2 + \tilde{I}_{rs} \sigma_{\alpha\beta}^2 + \tilde{K} \sigma_e^2,$$

where $\tilde{A}_1 = \tilde{B}_1 \tilde{B}_1'$, $\tilde{A}_2 = \tilde{B}_2 \tilde{B}_2'$, and $\tilde{K} = \text{diag}(n_{11}^{-1}, n_{12}^{-1}, \dots, n_{rs}^{-1})$.

Let $\bar{z} = \tilde{P} \bar{\bar{y}}$, where $\tilde{P}$ is an orthogonal matrix of order $rs \times rs$ whose first row is $(rs)^{-1} \tilde{1}_{rs}'$ and simultaneously diagonalizes $\tilde{A}_1$ and $\tilde{A}_2$. The vector $\bar{z}$ can be partitioned as $(z_1, z_\alpha', z_\beta', z_{\alpha\beta}')'$, where z_1 is the first element of $\bar{z}$; z_α , z_β , and $z_{\alpha\beta}$ are vectors of dimensions $r-1$, $s-1$, and $(r-1)(s-1)$, respectively. The latter

three vectors are normally distributed with zero means and have the following variance-covariance matrices (see Thomsen 1975, p. 259):

$$\text{Var } \underline{z}_\alpha = (s\sigma_\alpha^2 + \sigma_{\alpha\beta}^2) \underline{I}_{r-1} + \underline{K}_1 \sigma_e^2$$

$$\text{Var } \underline{z}_\beta = (r\sigma_\beta^2 + \sigma_{\alpha\beta}^2) \underline{I}_{s-1} + \underline{K}_2 \sigma_e^2$$

$$\text{Var } \underline{z}_{\alpha\beta} = \sigma_{\alpha\beta}^2 \underline{I}_{(r-1)(s-1)} + \underline{K}_3 \sigma_e^2,$$

where $\underline{K}_1$, $\underline{K}_2$, and $\underline{K}_3$ are the submatrices of $\underline{PKP}'$ which correspond to $\underline{z}_\alpha$, $\underline{z}_\beta$, and $\underline{z}_{\alpha\beta}$, respectively. Let $\underline{u}$ be the vector

$$\underline{u} = (\underline{z}'_\alpha, \underline{z}'_\beta, \underline{z}'_{\alpha\beta})'. \quad (2.6)$$

Then,

$$\text{Var } \underline{u} = \text{diag}(\delta_1 \underline{I}_{r-1}, \delta_2 \underline{I}_{s-1}, \delta_3 \underline{I}_{(r-1)(s-1)}) + \underline{L} \sigma_e^2, \quad (2.7)$$

where δ_1 , δ_2 , and δ_3 are given by

$$\begin{aligned} \delta_1 &= s\sigma_\alpha^2 + \sigma_{\alpha\beta}^2 \\ \delta_2 &= r\sigma_\beta^2 + \sigma_{\alpha\beta}^2 \\ \delta_3 &= \sigma_{\alpha\beta}^2, \end{aligned} \quad (2.8)$$

and $\underline{L}$ is the $(rs-1) \times (rs-1)$ submatrix of $\underline{PKP}'$ which corresponds to $\underline{u}$ and is expressible in the form

$$\underline{L} = \begin{bmatrix} \underline{K}_1 & \underline{K}_{12} & \underline{K}_{13} \\ \underline{K}'_{12} & \underline{K}_2 & \underline{K}_{23} \\ \underline{K}'_{13} & \underline{K}'_{23} & \underline{K}_3 \end{bmatrix}, \quad (2.9)$$

where $K_{12}\sigma_e^2 = E(z_\alpha z_\beta')$, $K_{13}\sigma_e^2 = E(z_\alpha z_{\alpha\beta}')$, and $K_{23}\sigma_e^2 = E(z_\beta z_{\alpha\beta}')$.

The matrix L is of rank $rs-1$ and is, therefore, nonsingular.

The random vectors z_α , z_β , and $z_{\alpha\beta}$ are not independent.

However, they are independent of the error sum of squares,

$$Q = \sum_{i,j,k} (y_{ijk} - \bar{y}_{ij})^2, \quad (2.10)$$

which can also be written as

$$Q = y' R y, \quad (2.11)$$

where y is the vector of observations and R is the $n_{..} \times n_{..}$ matrix

$$R = I_{n_{..}} - \sum_{i,j} (J_{n_{ij}} / n_{ij}). \quad (2.12)$$

In (2.12) $J_{n_{ij}}$ is the matrix of ones of order $n_{ij} \times n_{ij}$ ($i=1,2,\dots,r$; $j=1,2,\dots,s$) and the second term is the direct sum of the $J_{n_{ij}}/n_{ij}$'s. It is easy to verify that R is idempotent of rank $n_{..} - rs$ and that

$$R X_1 = R X_2 = R X_3 = Q, \quad (2.13)$$

where X_1 , X_2 , and X_3 are the matrices of zeros and ones in (2.2). Hence, Q/σ_e^2 has a chi-squared distribution with $n_{..} - rs$ degrees of freedom (see also Thomsen 1975, p. 260).

Since R is symmetric it can be written as

$$R = C \Lambda C', \quad (2.14)$$

where C is an orthogonal matrix and Λ is a diagonal matrix of

eigenvalues of R , both of order $n_{..} \times n_{..}$. We shall assume that

$$n_{..} > 2rs - 1. \quad (2.15)$$

This is not an unreasonable assumption and can, for example, be satisfied if each cell contains at least two observations. In this case and because R is idempotent of rank $n_{..} - rs > rs - 1$, Λ and C can be partitioned as

$$\Lambda = \text{diag}(\underline{I}_{v_1}, \underline{I}_{v_2}, \underline{0}) \quad (2.16)$$

$$C = [\underline{C}_1: \underline{C}_2: \underline{C}_3],$$

where

$$v_1 = rs - 1 \quad (2.17)$$

$$v_2 = n_{..} - 2rs + 1,$$

$\underline{Q}$ is a zero matrix of order $rs \times rs$, and $\underline{C}_1, \underline{C}_2, \underline{C}_3$ are of orders $n_{..} \times v_1, n_{..} \times v_2$, and $n_{..} \times rs$, respectively. Note that

$$\underline{C}_i' \underline{C}_i = \underline{I}, \quad i = 1, 2, 3, \quad (2.18)$$

$$\underline{C}_i' \underline{C}_j = \underline{0}, \quad i \neq j.$$

Formula (2.14) can then be rewritten as

$$R = \underline{C}_1 \underline{C}_1' + \underline{C}_2 \underline{C}_2'. \quad (2.19)$$

From (2.11) and (2.19), the error sum of squares Q can be partitioned as

$$Q = Q_1 + Q_2, \quad (2.20)$$

where

$$Q_1 = Y' \underline{C}_1 \underline{C}_1' Y \quad (2.21)$$

$$Q_2 = Y' C_2 C_2' Y. \quad (2.22)$$

The sums of squares Q_1 and Q_2 , and the random vector u in (2.6) are independent. Furthermore, Q_1/σ_e^2 and Q_2/σ_e^2 have the chi-squared distribution with v_1 and v_2 degrees of freedom, respectively.

We now define the random vector w as

$$w = u + (\lambda_{\max} I_{v_1} - L)^{\frac{1}{2}} C_1' Y, \quad (2.23)$$

where L is the matrix in (2.9) and $\lambda_{\max}$ is its largest eigenvalue. The matrix $\lambda_{\max} I_{v_1} - L$ is positive semidefinite, hence the matrix $(\lambda_{\max} I_{v_1} - L)^{\frac{1}{2}}$ is well defined with eigenvalues equal to the square roots of the eigenvalues of $\lambda_{\max} I_{v_1} - L$. Let w be partitioned just like u in (2.6) as

$$w = (w_\alpha', w_\beta', w_{\alpha\beta}')', \quad (2.24)$$

where the vectors w_α , w_β , and $w_{\alpha\beta}$ are of dimensions $r-1$, $s-1$, and $(r-1)(s-1)$, respectively.

Lemma 1

- (i) $Ew_\alpha = Ew_\beta = Ew_{\alpha\beta} = 0$.
- (ii) w_α , w_β , and $w_{\alpha\beta}$ are independent normally distributed random vectors and have the following variance-covariance matrices:

$$\begin{aligned} \text{Var } w_\alpha &= (s\sigma_\alpha^2 + \sigma_{\alpha\beta}^2 + \lambda_{\max} \sigma_e^2) I_{r-1}, \\ \text{Var } w_\beta &= (r\sigma_\beta^2 + \sigma_{\alpha\beta}^2 + \lambda_{\max} \sigma_e^2) I_{s-1}, \\ \text{Var } w_{\alpha\beta} &= (\sigma_{\alpha\beta}^2 + \lambda_{\max} \sigma_e^2) I_{(r-1)(s-1)}. \end{aligned} \quad (2.25)$$

(iii) ω_α , ω_β , and $\omega_{\alpha\beta}$ are independent of Q_2 , where Q_2 is the sum of squares in (2.22).

Proof. (i) From (2.12) it can be seen that $R \frac{1}{n_{..}} = 0$. Thus, by (2.19) we can write

$$(C_1 C_1' + C_2 C_2') \frac{1}{n_{..}} = 0. \quad (2.26)$$

Using (2.18) in (2.26) we get $C_1' \frac{1}{n_{..}} = 0$. It follows that $E(C_1' \chi) = C_1' \frac{1}{n_{..}} \mu = 0$. Since y in (2.6) has also a zero mean, we conclude that the mean of w in (2.23) is zero.

(ii) It is obvious that w in (2.23) is normally distributed. We now claim that y is independent of $C_1' \chi$. To show this we note that the vector $\bar{\chi}$ in (2.5); which can be written as $\bar{\chi} = D y$, where $D = \sum_{i,j} \frac{1}{n_{ij}} / n_{ij}$; is independent of Q and hence of Q_1 (see 2.20 and 2.21). Consequently,

$$D \Sigma C_1 C_1' = 0, \quad (2.27)$$

where Σ is the variance-covariance matrix of χ given in (2.3) (see Searle 1971, p. 59). From (2.18) and (2.27) we conclude that

$$D \Sigma C_1 = 0. \quad (2.28)$$

Hence, $\text{Cov}(\bar{\chi}, \chi' C_1) = D \Sigma C_1 = 0$. Since y is a subvector of $z = P \bar{\chi}$, then y is independent of $C_1' \chi$ as was claimed.

The variance-covariance matrix of w in (2.23) can then be written as

$$\text{Var } w = \text{Var } y + (\lambda_{\max \sim v_1} - L)^{\frac{1}{2}} C_1' E C_1 (\lambda_{\max \sim v_1} - L)^{\frac{1}{2}}. \quad (2.29)$$

But from (2.13) and (2.19) we have

$$(\tilde{C}_1 \tilde{C}_1' + \tilde{C}_2 \tilde{C}_2') \tilde{X}_i = \tilde{0}, \quad i = 1, 2, 3. \quad (2.30)$$

Using (2.18), equalities (2.30) yield

$$\tilde{C}_1' \tilde{X}_i = \tilde{0}, \quad i = 1, 2, 3. \quad (2.31)$$

It follows that

$$\tilde{C}_1' \tilde{\Sigma} \tilde{C}_1 = \tilde{C}_1' \tilde{C}_1 \sigma_e^2 = \tilde{I}_{v_1} \sigma_e^2. \quad (2.32)$$

From (2.7), (2.29), and (2.32) we then obtain

$$\text{Var } \tilde{\omega} = \text{diag}(\delta_1 \tilde{I}_{r-1}, \delta_2 \tilde{I}_{s-1}, \delta_3 \tilde{I}_{(r-1)(s-1)}) + \tilde{L} \sigma_e^2 + (\lambda_{\max \tilde{I}_{v_1}} - \tilde{L}) \sigma_e^2,$$

that is,

$$\begin{aligned} \text{Var } \tilde{\omega} = \text{diag}((\delta_1 + \lambda_{\max \sigma_e^2}) \tilde{I}_{r-1}, (\delta_2 + \lambda_{\max \sigma_e^2}) \tilde{I}_{s-1}, \\ (\delta_3 + \lambda_{\max \sigma_e^2}) \tilde{I}_{(r-1)(s-1)}). \end{aligned} \quad (2.33)$$

From (2.33) we conclude that $\tilde{\omega}_\alpha$, $\tilde{\omega}_\beta$, and $\tilde{\omega}_{\alpha\beta}$ are independent and have the variance-covariance matrices described in (2.25).

(iii) Q_2 is independent of $\tilde{y}$ (since Q is) and is also independent of $\tilde{C}_1' \tilde{y}$ since $\tilde{C}_1' \tilde{\Sigma} \tilde{C}_2 = \tilde{0}$, which follows from (2.18) and (2.31) after noting the formula for $\tilde{\Sigma}$ in (2.3). Thus, Q_2 is independent of $\tilde{\omega}$.

From lemma 1 we conclude that the sums of squares,

$S_\alpha = \tilde{\omega}_\alpha' \tilde{\omega}_\alpha$, $S_\beta = \tilde{\omega}_\beta' \tilde{\omega}_\beta$, $S_{\alpha\beta} = \tilde{\omega}_{\alpha\beta}' \tilde{\omega}_{\alpha\beta}$, and Q_2 are distributed independently, and

$$S_\alpha / (s\sigma_\alpha^2 + \sigma_{\alpha\beta}^2 + \lambda_{\max \sigma_e^2}) \sim \chi_{r-1}^2$$

$$S_\beta / (r\sigma_\beta^2 + \sigma_{\alpha\beta}^2 + \lambda_{\max \sigma_e^2}) \sim \chi_{s-1}^2$$

$$S_{\alpha\beta}/(\sigma_{\alpha\beta}^2 + \lambda_{\max} \sigma_e^2) \sim \chi_{(r-1)(s-1)}^2$$

$$Q_2/\sigma_e^2 \sim \chi_{v_2}^2,$$

where v_2 is given in (2.17). A test statistic for testing the hypothesis $H_0: \sigma_{\alpha}^2 = 0$ vs. $H_a: \sigma_{\alpha}^2 \neq 0$ is, therefore, $F = MS_{\alpha}/MS_{\alpha\beta}$, where $MS_{\alpha} = S_{\alpha}/(r-1)$ and $MS_{\alpha\beta} = S_{\alpha\beta}/(r-1)(s-1)$. Under H_0 this statistic has the F distribution with $r-1$ and $(r-1)(s-1)$ degrees of freedom. The hypothesis H_0 can be rejected at the α -level of significance if $F \geq F_{\alpha, r-1, (r-1)(s-1)}$, the upper $\alpha 100\%$ point of the corresponding F distribution. Similarly, to test the hypothesis $H_0: \sigma_{\beta}^2 = 0$ vs. $H_a: \sigma_{\beta}^2 \neq 0$, we use the statistic $F = MS_{\beta}/MS_{\alpha\beta}$, where $MS_{\beta} = S_{\beta}/(s-1)$. Furthermore, the statistic $F = (v_2/\lambda_{\max})(MS_{\alpha\beta}/Q_2)$ can be used to test the hypothesis $H_0: \sigma_{\alpha\beta}^2 = 0$ vs. $H_a: \sigma_{\alpha\beta}^2 \neq 0$. We do not, however, recommend using this test since it has fewer denominator degrees of freedom than the exact test for $\sigma_{\alpha\beta}^2$ given by Thomsen (1975).

We note that if the data set is balanced, then $\mathbf{K} = \text{diag}(n_{11}^{-1}, \dots, n_{rs}^{-1}) = \mathbf{I}_{rs}/n$, where n is the number of observations per cell. Hence, $\mathbf{PKP}' = \mathbf{I}_{rs}/n$ and $\mathbf{L} = \mathbf{I}_{v_1}/n$, where v_1 is given in (2.17). Consequently, $\lambda_{\max} = 1/n$ and the vectors $\mathbf{u}$ and $\mathbf{y}$ in (2.23) become identical. Furthermore, the sums of squares nS_{α} , nS_{β} , and $nS_{\alpha\beta}$ reduce to the balanced ANOVA sums of squares associated with the main and interaction effects, respectively.

The following lemma is useful for the power study in Section 5 and is proved in the Appendix:

Lemma2

The largest eigenvalue, $\lambda_{\max}$, of the matrix L in (2.9) satisfies the double inequality

$$\frac{1}{rs} \sum_{i,j} \frac{1}{n_{ij}} < \lambda_{\max} < \frac{1}{n^{(1)}}, \quad (2.34)$$

where $n^{(1)}$ is the smallest cell frequency.

We note that the lower bound in (2.34) is the reciprocal of the harmonic mean of the cell frequencies.

3. A Numerical Example

Layton (1985) studied variation in fusiform rust in Southern pine tree plantations. Trees with female parents from different families were evaluated in several test locations. We extract data from five families and four test locations, and disregard the male parents (which were from a different set of families) for purpose of illustration. The number of plots in each family $\times$ test combination ranged from one to four. Proportions of symptomatic trees in each plot are recorded in the following table:

Test number	Family number				
	288	352	19	141	60
34	.804	.734	.967	.917	.850
	.967	.817	.930		
	.970	.833	.889		
		.304			
35	.867	.407	.896	.952	.486
	.667	.511	.717		.467
	.793	.274			
	.458	.428			
40	.409	.411	.919	.408	.275
	.569	.646	.669	.435	.256
	.715	.310	.669	.500	
	.487		.450		
41	.587	.304	.928	.367	.525
	.538	.428	.855		
	.961		.655		
	.300		.800		

We analyze variation due to family and test according to the model in (2.1), where y_{ijk} = arcsin-square root of the k^{th} observed proportion in family i and test j . The exact test will be used to test, for example, the null hypothesis $H_0: \sigma_\alpha^2 = 0$ regarding the family variance component.

The first step is to obtain the matrix $\underline{P}$. This can be done using the algorithm given by Graybill (1983, p. 406). Alternatively, $\underline{P}$ may be constructed with rows 2 through 20 given

by sets of normalized orthogonal contrasts for α effect, β effect and $\alpha\beta$ effect for balanced data. Computation of other components of ω in (2.23) are straightforward. For the present example, we obtain $MS_{\alpha}/MS_{\alpha\beta} = 0.1543/0.0415 = 3.718$, which has an observed significance level of 0.034.

4. Simultaneous Confidence Intervals on the Variance Components

One of the interesting consequences of Lemmal is that simultaneous confidence intervals on all continuous functions of σ_{α}^2 , σ_{β}^2 , $\sigma_{\alpha\beta}^2$, and σ_e^2 can be as easily obtained as in a balanced data situation. To see this, let us denote the expected values of MS_{α} , MS_{β} , $MS_{\alpha\beta}$, and Q_2/v_2 by τ_{α} , τ_{β} , $\tau_{\alpha\beta}$, and τ_e , respectively.

Then

$$\tau_{\alpha} = s\sigma_{\alpha}^2 + \sigma_{\alpha\beta}^2 + \lambda_{\max} \sigma_e^2, \quad (4.1)$$

$$\tau_{\beta} = r\sigma_{\beta}^2 + \sigma_{\alpha\beta}^2 + \lambda_{\max} \sigma_e^2, \quad (4.2)$$

$$\tau_{\alpha\beta} = \sigma_{\alpha\beta}^2 + \lambda_{\max} \sigma_e^2, \quad (4.3)$$

$$\tau_e = \sigma_e^2. \quad (4.4)$$

Individual $(1-\alpha)100\%$ confidence intervals on τ_{α} , τ_{β} , $\tau_{\alpha\beta}$, and τ_e are, respectively

$$C_{\alpha} = \{\tau_{\alpha} : S_{\alpha}/\chi_{\alpha/2, r-1}^2 \leq \tau_{\alpha} \leq S_{\alpha}/\chi_{1-\alpha/2, r-1}^2\}$$

$$C_{\beta} = \{\tau_{\beta} : S_{\beta}/\chi_{\alpha/2, s-1}^2 \leq \tau_{\beta} \leq S_{\beta}/\chi_{1-\alpha/2, s-1}^2\}$$

$$C_{\alpha\beta} = \{\tau_{\alpha\beta} : S_{\alpha\beta}/\chi_{\alpha/2, (r-1)(s-1)}^2 \leq \tau_{\alpha\beta} \leq S_{\alpha\beta}/\chi_{1-\alpha/2, (r-1)(s-1)}^2\}$$

$$C_e = \{\tau_e : Q_2/\chi_{\alpha/2, v_2}^2 \leq \tau_e \leq Q_2/\chi_{1-\alpha/2, v_2}^2\}, \text{ where } \chi_{\alpha, m}^2 \text{ denotes the upper } \alpha 100\% \text{ point of the chi-squared distribution with } m$$

degrees of freedom. Since S_α , S_β , $S_{\alpha\beta}$, and Q_2 are independently distributed, the Cartesian product, $C = C_\alpha \times C_\beta \times C_{\alpha\beta} \times C_e$, represents an exact rectangular confidence region on $\underline{\tau} = (\tau_\alpha, \tau_\beta, \tau_{\alpha\beta}, \tau_e)'$ with a confidence coefficient equal to $1 - \alpha^* = (1 - \alpha)^4$.

Let us now suppose that $\gamma = f(\sigma_\alpha^2, \sigma_\beta^2, \sigma_{\alpha\beta}^2, \sigma_e^2)$ is a continuous function of the variance components. This function can be expressed as $\gamma = g(\tau_\alpha, \tau_\beta, \tau_{\alpha\beta}, \tau_e)$, where g is obtained from f by substituting the variance components by $\tau_\alpha, \tau_\beta, \tau_{\alpha\beta}, \tau_e$ using equations (4.1) - (4.4). By the method described in Khuri (1981) for balanced data, the interval

$$B_g = \{\gamma : \min_{\underline{\tau} \in C} g(\underline{\tau}) < \gamma < \max_{\underline{\tau} \in C} g(\underline{\tau})\}$$

is a confidence interval on γ with a confidence coefficient greater than or equal to $1 - \alpha^*$. Furthermore, if g belongs to a family G of continuous functions of $\tau_\alpha, \tau_\beta, \tau_{\alpha\beta}, \tau_e$, then

$$P[\chi \in B_g, \forall g \in G] > 1 - \alpha^*.$$

Thus, for $g \in G$, the intervals B_g are conservative simultaneous confidence intervals on the values of all continuous functions of the variance components for model (2.1).

5. The Power of the Exact Tests

Power values for each of the exact tests described in Section 2 can be easily computed using the F distribution. We shall only consider the power of the test concerning σ_α^2 . A similar power study can be made regarding the test for σ_β^2 .

Let Ψ denote the power of the test for σ_α^2 . Then

$$\Psi = P\left[\frac{MS_{\alpha}}{MS_{\alpha\beta}} > F_{\alpha, r-1, (r-1)(s-1)} \mid H_a\right], \quad (5.1)$$

where $H_a : \sigma_{\alpha}^2 \neq 0$. Under H_a ,

$$\frac{MS_{\alpha}}{s\sigma_{\alpha}^2 + \sigma_{\alpha\beta}^2 + \lambda_{\max} \sigma_e^2} \frac{\sigma_{\alpha\beta}^2 + \lambda_{\max} \sigma_e^2}{MS_{\alpha\beta}} \sim F_{r-1, (r-1)(s-1)}.$$

Hence, (5.1) can be rewritten as

$$\Psi = P\left[F_{r-1, (r-1)(s-1)} > \frac{1}{1 + s\theta} F_{\alpha, r-1, (r-1)(s-1)}\right], \quad (5.2)$$

where

$$\theta = \sigma_{\alpha}^2 / (\sigma_{\alpha\beta}^2 + \lambda_{\max} \sigma_e^2).$$

From (5.2) it can be seen that Ψ is a function of the level of significance, α , $\lambda_{\max}$, which depends on the design used, and the variance ratios $\sigma_{\alpha}^2/\sigma_e^2$, $\sigma_{\alpha\beta}^2/\sigma_e^2$ through θ . The latter variance ratio is considered a nuisance parameter. Since Ψ is a monotone increasing function of θ , it follows that Ψ is

- i) a monotone increasing function of $\sigma_{\alpha}^2/\sigma_e^2$ for a fixed value of $\sigma_{\alpha\beta}^2/\sigma_e^2$ and a fixed design.
- ii) a monotone decreasing function of the nuisance parameter $\sigma_{\alpha\beta}^2/\sigma_e^2$ for a fixed value of $\sigma_{\alpha}^2/\sigma_e^2$ and a fixed design.
- iii) a monotone decreasing function of $\lambda_{\max}$ for fixed ratios of the variance components. Consequently, if n., the total of the cell frequencies, is fixed, then by Lemma2 higher power values are expected for smaller values of d , where d is

$$d = \frac{1}{n^{(1)}} - \frac{1}{rs} \sum_{i,j} \frac{1}{n_{ij}} .$$

For a balanced data set, $d = 0$ and maximum power is achieved. We can, therefore, regard the quantity $n^{(1)}d$, which belongs to the interval $[0,1)$, as a measure of imbalance. Small values of this measure are associated with designs that are nearly balanced. For a more general discussion concerning measures of imbalance for unbalanced models, the interested reader is referred to Khuri (1986).

6. A Power Comparison With Other Approximate Tests

In this section, we compare the power of the exact test statistic, $MS_{\alpha}/MS_{\alpha\beta}$, given by formula (5.1) with powers of tests that are most commonly used in practice, namely, the ANOVA-based approximate F tests.

There are several analyses of variance, each using a different method of computing sums of squares. Two of these methods, expressed in "reduction in SS" notation, are:

Source of Variation	Degrees of Freedom	Type I SS	Type II SS
A	$r-1$	$R(\alpha \mu)$	$R(\alpha \mu, \beta)$
B	$s-1$	$R(\beta \mu, \alpha)$	$R(\beta \mu, \alpha)$
A*B	$(r-1)(s-1)$	$R(\alpha\beta \mu, \alpha, \beta)$	$R(\alpha\beta \mu, \alpha, \beta)$
Residual	$n.. - rs$	Q	Q

See Searle (1971, Section 6.3) for a description of the "reduction" notation. The terminology, "Type I" and "Type II", is

consistent with that of the SAS (1982) System of statistical software. An approximate F test statistic for $H_0: \sigma_\alpha^2 = 0$ based on the Type I SS ($i = I, i = II$) has the form $F(i) = MS_\alpha(i)/MS_{\alpha\beta}^*(i)$. The numerator, $MS_\alpha(i)$, is the Type I mean square for α . The denominator, $MS_{\alpha\beta}^*(i)$, is

$$MS_{\alpha\beta}^*(i) = \hat{\sigma}_e^2 + k_1(i) \hat{\sigma}_{\alpha\beta}^2 + k_2(i) \hat{\sigma}_\beta^2,$$

where $k_1(i)$ and $k_2(i)$ are the coefficients in the expected mean square

$$E[MS_\alpha(i)] = \sigma_e^2 + k_1(i) \sigma_{\alpha\beta}^2 + k_2(i) \sigma_\beta^2 + k_3(i) \sigma_\alpha^2$$

and $\hat{\sigma}_e^2$, $\hat{\sigma}_{\alpha\beta}^2$, and $\hat{\sigma}_\beta^2$ are, respectively, the analysis of variance estimators of σ_e^2 , $\sigma_{\alpha\beta}^2$, and σ_β^2 , based upon Q , $R(\alpha\beta|\mu, \alpha, \beta)$, and $R(\beta|\mu, \alpha)$. (Note: $k_2(II) = 0$).

Powers of the approximate test statistics $F(I)$ and $F(II)$ were estimated via computer simulation. The simulation study required two steps; the first to estimate critical values of $F(I)$ and $F(II)$ under $H_0: \sigma_\alpha^2 = 0$; the second to estimate the power for $\sigma_\alpha^2 > 0$. All simulations were conducted using PROC MATRIX of the SAS (1982) System. The SAS functions RANNOR and RANGAM were used to generate pseudo-random normal and chi-squared variates, respectively. Powers were estimated for 25 combinations of values of the variance components and six n_{1j} patterns, making 150 cases in all. Without loss of generality, $\sigma_e = 1.0$ was used in all combinations. Values of $\sigma_{\alpha\beta}$ and σ_β constituted a "response surface design" containing a 2×2 factorial and an interior point,

namely, (0.2,0.2), (0.2,5.0), (5.0,0.2), (5.0,5.0) and (1.0,1.0). For each of these five combinations, five values of $\sigma_\alpha = 0.2, 0.5, 1.0, 2.0$ and 5.0 were considered to produce the 25 combinations of $\sigma_{\alpha\beta}$, σ_β , and σ_α .

The six n_{ij} patterns contained three "near balance" patterns (NB) and three "highly unbalanced" patterns (HU), each containing 5×5 , 5×10 and 10×5 arrays. The six patterns, with rows representing levels of factor A and columns representing levels of factor B, are:

	NB (near balance)					HU (highly unbalanced)				
r=5	5	5	5	5	6	9	2	9	1	2
	4	4	6	4	5	10	1	2	9	10
	6	4	4	4	4	1	8	1	2	2
s=5	4	6	5	5	6	9	10	1	9	3
	6	5	4	5	6	8	3	2	10	1
r=10	5	5	5	5	6	9	2	9	1	2
	4	4	6	4	5	10	1	2	9	10
	6	4	4	4	4	1	8	1	2	2
s=5	4	6	5	5	6	9	10	1	9	3
	6	5	4	5	6	8	3	2	10	1
	5	5	5	5	6	9	2	9	1	2
r=5	4	4	6	4	5	10	1	2	9	10
	6	4	4	4	4	1	8	1	2	2
	4	6	5	5	6	9	10	1	9	3
s=10	6	5	4	5	6	8	3	2	10	1
	5	5	5	5	6	9	2	9	1	2
	4	4	6	4	5	10	1	2	9	10
r=5	6	4	4	6	4	1	8	1	2	2
	4	6	5	5	6	9	10	1	9	3
	6	5	4	5	6	8	3	2	10	1

The distributions of $F(I)$ and $F(II)$ depend on the true values of $\sigma_{\alpha\beta}$ and σ_{β} , even under the null hypothesis $H_0: \sigma_{\alpha}^2 = 0$. Therefore, it was necessary to estimate the critical values of $F(I)$ and $F(II)$ for all values of $\sigma_{\alpha\beta}$ and σ_{β} , and all n_{ij} patterns involved in the power study. This was done as follows: For each of the five combinations of $\sigma_{\alpha\beta}$ and σ_{β} and each of the six n_{ij} patterns (thirty cases in all), 1000 sets of cell means $\bar{y}_{ij}$ and Q values were generated according to the model $\bar{y}_{ij} = \mu + \beta_j + (\alpha\beta)_{ij} + \bar{e}_{ij}$, where β_j , $(\alpha\beta)_{ij}$, and $\bar{e}_{ij}$ are independently distributed as normal variates with zero means and variances σ_{β}^2 , $\sigma_{\alpha\beta}^2$, and σ_e^2/n_{ij} , respectively, Q/σ_e^2 has the chi-squared distribution with $n-1$ degrees of freedom, and, without loss of generality, $\mu = 0$. (Note the absence of α_i in the model, corresponding to $\sigma_{\alpha}^2 = 0$). For each set of $\bar{y}_{ij}$ and Q values, $F(I)$ and $F(II)$ were calculated, and the 95% sample quantiles of $F(I)$ and $F(II)$ were recorded from the 1000 sets. This process was repeated ten times, and the mean and standard deviation of the ten $F(I)$ and $F(II)$ quantiles were computed to estimate the upper 5% critical values for $F(I)$ and $F(II)$. These are reported in Tables 6.1 and 6.2.

The estimated critical values in Tables 6.1 and 6.2 demonstrate the degree of dependence of the null distributions of $F(I)$ and $F(II)$ upon the nuisance parameters, $\sigma_{\alpha\beta}$ and σ_{β} . The most

serious disturbance of the distributions is for small values of $\sigma_{\alpha\beta}$ (≈ 0.2), especially in "highly unbalanced" cases (Table 6.2).

In practice, the true critical values of $F(I)$ and $F(II)$ would not be available because $\sigma_{\alpha\beta}$ and σ_{β} are not known. Instead, the calculated values of $F(I)$ and $F(II)$ would typically be referred to an F distribution with denominator degrees of freedom given by a Satterthwaite-type approximation such as illustrated by Milliken and Johnson (1984, Section 20.1.2). Actual Rejection probabilities (type 1 error rates) corresponding to a nominal $\alpha = .05$ for $F(I)$ and $F(II)$ using these approximate degrees of freedom were estimated in the simulation study. These are also reported in Tables 6.1 and 6.2. The results show that the Satterthwaite-type approximate procedures produce true type 1 error rates that are far less than the nominal .05 for some cases, particularly those with small values of $\sigma_{\alpha\beta}$ in the highly unbalanced situation.

Estimation of power for the statistics $F(I)$ and $F(II)$ followed a process similar to that used to estimate the critical values. For each of the 25 selected combinations of $\sigma_{\alpha\beta}$, σ_{β} and σ_{α} and each of the six n_{ij} patterns, 2000 sets of cell means $\bar{y}_{ij}$ and Q values were generated according to model (2.4) with μ taken equal to zero. The statistics $F(I)$ and $F(II)$ were calculated for each set of $\bar{y}_{ij}$ and Q values. The proportion of times, out of the 2000, that $F(I)$ and $F(II)$ exceeded the estimated critical

values in Tables 6.1 and 6.2 was computed. These proportions are estimates of the powers of F(I) and F(II) for testing $H_0: \sigma_\alpha^2 = 0$, and are recorded in Tables 6.3 and 6.4. Powers of the exact test statistic $MS_\alpha/MS_{\alpha\beta}$ are also reported in the latter tables. These results show that, with a few exceptions, the power of the exact test is better or essentially as good as the power of either of the approximate procedures. The exceptions are for small values of σ_α (0.2 and 0.5) and small values of $\sigma_{\alpha\beta}$ (0.2).

It must be remembered that the approximate tests whose powers are shown in Tables 6.3 and 6.4 could not be computed in practice because their critical values depend on the unknown $\sigma_{\alpha\beta}$ and σ_β . The dependence is most severe for small values of $\sigma_{\alpha\beta}$. These are the same values of $\sigma_{\alpha\beta}$ for which the power of the exact test was inferior to the approximate tests. Therefore, the power of the exact test appears to be generally as good or better than powers of the approximate tests except in cases for which valid critical values of the approximate tests are most unreliable.

ACKNOWLEDGEMENTS

This research was supported in part by the Office of Naval Research under Grant No. N00014-86-K-0059.

REFERENCES

- Graybill, F. A. (1983). Matrices With Applications in Statistics, Second Edition. Belmont, California: Wadsworth International Group
- Harville, D. A. and Fenech, A. P. (1985). Confidence intervals for a variance ratio, or for heritability, in an unbalanced mixed linear model. Biometrics, 41, 137-152.
- Khuri, A. I. (1981). Simultaneous confidence intervals for functions of variance components in random models. J. Amer. Statist. Assoc., 76, 878-885.
- Khuri, A. I. (1986). Measures of imbalance for unbalanced models. Technical Report No. 255, Department of Statistics, University of Florida, Gainesville, Florida 32611.
- Layton, P. A. (1985). Genetic Variation in Symptomology of Slash Pine in Response to Fusiform Rust. Unpublished Ph.D. Dissertation, Department of Forestry, University of Florida, Gainesville, Florida 32611.
- Milliken, G. A. and Johnson, D. E. (1984). Analysis of Messy Data. Belmont, California: Lifetime Learning Publications.
- Searle, S. R. (1971). Linear Models. New York: Wiley.
- Seely, J. F. and El-Bassiouni, Y. (1983). Applying Wald's variance component test. Ann. Statist., 11, 197-201.

- Spjøtvoll, E. (1968). Confidence intervals and tests for variance ratios in unbalanced variance components models. Rev. Internat. Statist. Inst., 36, 37-42.
- Thomsen, I. (1975). Testing hypotheses in unbalanced variance components models for two-way layouts. Ann. Statist., 3, 257-265.
- Wald, A. (1940). A note on the analysis of variance with unequal class frequencies. Ann. Math. Statist., 11, 96-100.
- Wald, A. (1941). On the analysis of variance in case of multiple classifications with unequal class frequencies. Ann. Math. Statist., 12, 346-350.

APPENDIX

Proof of Lemma2

Consider the orthogonal matrix P whose first row is $(rs)^{-1/2} \mathbf{1}_{rs}'$ and simultaneously diagonalizes $A_1 = B_1 B_1'$, $A_2 = B_2 B_2'$, where B_1 and B_2 are the matrices in (2.5). Let P_1 be the submatrix of P obtained by deleting the first row. It is easy to verify that

$$P_1 P_1' = I_{rs-1}, \quad (A.1)$$

$$P_1' P_1 + \frac{1}{rs} J_{rs} = I_{rs}. \quad (A.2)$$

Now, $\lambda_{\max}$ is the largest eigenvalue of $L = P_1 K P_1'$, that is, $\lambda_{\max} = e_{\max}(P_1 K P_1')$. If $n^{(1)}$ is the smallest cell frequency, then the matrix $(1/n^{(1)}) P_1 P_1' - P_1 K P_1'$ is positive semidefinite. Using (A.1) we get

$$e_{\max}(\underline{P}_1 \underline{K} \underline{P}_1') \leq \frac{1}{n(1)} e_{\max}(\underline{P}_1 \underline{P}_1') = \frac{1}{n(1)}. \quad (\text{A.3})$$

It is also true that $\lambda_{\max}$ is greater than or equal to the sum of the eigenvalues of $\underline{P}_1 \underline{K} \underline{P}_1'$ divided by $rs-1$ (this is the average eigenvalue of $\underline{P}_1 \underline{K} \underline{P}_1'$). Thus,

$$e_{\max}(\underline{P}_1 \underline{K} \underline{P}_1') > \frac{1}{(rs-1)} \text{tr}(\underline{P}_1 \underline{K} \underline{P}_1'). \quad (\text{A.4})$$

But,

$$\begin{aligned} \text{tr}(\underline{P}_1 \underline{K} \underline{P}_1') &= \text{tr}(\underline{P}_1' \underline{P}_1 \underline{K}) \\ &= \text{tr}\left[\left(\underline{I}_{rs} - \frac{1}{rs} \underline{J}_{rs}\right) \underline{K}\right] \text{ (using A.2)} \\ &= \text{tr}(\underline{K}) - \frac{1}{rs} \text{tr}(\underline{1}_{rs} \underline{1}_{rs}' \underline{K}) \\ &= \text{tr}(\underline{K}) - \frac{1}{rs} \underline{1}_{rs}' \underline{K} \underline{1}_{rs} \\ &= \sum_{i,j} \frac{1}{n_{ij}} - \frac{1}{rs} \sum_{i,j} \frac{1}{n_{ij}}. \end{aligned} \quad (\text{A.5})$$

From (A.4) and (A.5) we conclude that

$$e_{\max}(\underline{P}_1 \underline{K} \underline{P}_1') > \frac{1}{rs} \sum_{i,j} \frac{1}{n_{ij}}. \quad (\text{A.6})$$

The proof of Lemma2 follows from inequalities (A.3) and (A.6).

Table 6.1

Estimated 95% quantiles of F(I) and F(II), and estimated type I error rates for F(I) and F(II) using Satterthwaite's approximate degrees of freedom for a nominal $\alpha = .05$ for "near balance" cases

Design	$\sigma_{\alpha\beta}$	σ_{β}	95% Quantiles		Type I Error Rates	
			F(I)	F(II)	F(I)	F(II)
5x5	0.2	0.2	3.01	3.01	0.048	0.047
	0.2	5.0	2.32	3.07	0.005	0.051
	1.0	1.0	2.90	2.92	0.046	0.045
	5.0	0.2	2.98	2.97	0.048	0.048
	5.0	5.0	2.94	2.94	0.049	0.047
10x5	0.2	0.2	2.18	2.18	0.053	0.052
	0.2	5.0	2.04	2.16	0.004	0.089
	1.0	1.0	2.17	2.20	0.054	0.055
	5.0	0.2	2.16	2.15	0.051	0.052
	5.0	5.0	2.15	2.17	0.051	0.051
5x10	0.2	0.2	2.67	2.68	0.052	0.053
	0.2	5.0	2.52	2.66	0.029	0.051
	1.0	1.0	2.59	2.61	0.048	0.049
	5.0	0.2	2.63	2.63	0.050	0.050
	5.0	5.0	2.58	2.63	0.049	0.049

Table 6.2
 Estimated 95% quantiles of F(I) and F(II), and estimated type 1
 error rates for F(I) and F(II) using Satterthwaite's approximate
 degrees of freedom for a nominal $\alpha = .05$, for "highly unbalanced"
 cases

Design	$\sigma_{\alpha\beta}$	σ_{β}	95% Quantiles		Type 1 Error Rates	
			<u>F(I)</u>	<u>F(II)</u>	<u>F(I)</u>	<u>F(II)</u>
5x5	0.2	0.2	4.45	4.76	0.003	0.002
	0.2	5.0	2.11	4.64	0.000	0.002
	1.0	1.0	2.81	3.26	0.037	0.049
	5.0	0.2	3.00	3.00	0.054	0.049
	5.0	5.0	2.76	3.08	0.037	0.053
10x5	0.2	0.2	2.61	2.67	0.019	0.018
	0.2	5.0	2.03	2.70	0.000	0.017
	1.0	1.0	2.12	2.23	0.035	0.050
	5.0	0.2	2.20	2.21	0.057	0.056
	5.0	5.0	2.10	2.16	0.036	0.052
5x10	0.2	0.2	2.97	3.04	0.035	0.036
	0.2	5.0	2.58	3.15	0.009	0.041
	1.0	1.0	2.57	2.70	0.046	0.051
	5.0	0.2	2.67	2.70	0.055	0.055
	5.0	5.0	2.54	2.65	0.046	0.052

Table 6.3
Estimated powers of F(I) and F(II) and
exact powers (E) of $MS_{\alpha}/MS_{\alpha\beta}$ for "near balance" cases.

Design	$\sigma_{\alpha\beta}$	σ_{β}		σ_{α}				
				0.2	0.5	1.0	2.0	5.0
5x5	0.2	0.2	I	0.200	0.740	0.965	0.997	1.000
			II	0.200	0.739	0.964	0.997	1.000
			E	0.196	0.720	0.960	0.996	0.999
5x5	0.2	5.0	I	0.124	0.479	0.841	0.980	0.999
			II	0.209	0.737	0.959	0.997	0.999
			E	0.196	0.720	0.960	0.996	0.999
5x5	1.0	1.0	I	0.082	0.266	0.688	0.952	0.998
			II	0.090	0.269	0.696	0.953	0.997
			E	0.076	0.252	0.673	0.948	0.998
5x5	5.0	0.2	I	0.053	0.060	0.084	0.211	0.733
			II	0.053	0.062	0.085	0.215	0.736
			E	0.051	0.057	0.082	0.203	0.732
5x5	5.0	5.0	I	0.063	0.059	0.085	0.196	0.727
			II	0.066	0.063	0.097	0.207	0.732
			E	0.051	0.057	0.082	0.203	0.732
10x5	0.2	0.2	I	0.329	0.962	1.000	1.000	1.000
			II	0.331	0.962	1.000	1.000	1.000
			E	0.317	0.943	0.999	1.000	1.000
10x5	0.2	5.0	I	0.154	0.649	0.999	0.999	1.000
			II	0.351	0.952	0.998	1.000	1.000
			E	0.317	0.943	0.999	1.000	1.000
10x5	1.0	1.0	I	0.088	0.415	0.909	0.997	1.000
			II	0.080	0.405	0.910	0.997	1.000
			E	0.091	0.410	0.914	0.998	1.000
10x5	5.0	0.2	I	0.049	0.053	0.110	0.312	0.946
			II	0.050	0.058	0.110	0.317	0.947
			E	0.051	0.060	0.101	0.322	0.945
10x5	5.0	5.0	I	0.050	0.058	0.104	0.317	0.924
			II	0.051	0.062	0.106	0.323	0.927
			E	0.051	0.060	0.101	0.322	0.945

Table 6.3 (continued)

Design	$\sigma_{a\beta}$	σ_{β}		σ_a				
				0.2	0.5	1.0	2.0	5.0
5x10	0.2	0.2	I	0.416	0.920	0.991	0.999	1.000
			II	0.410	0.920	0.991	0.999	1.000
			E	0.398	0.907	0.991	0.999	1.000
5x10	0.2	5.0	I	0.182	0.630	0.930	0.995	0.999
			II	0.428	0.923	0.992	1.000	1.000
			E	0.398	0.907	0.991	0.999	1.000
5x10	1.0	1.0	I	0.119	0.485	0.887	0.988	0.998
			II	0.117	0.492	0.895	0.988	0.998
			E	0.117	0.493	0.884	0.988	0.999
5x10	5.0	0.2	I	0.057	0.062	0.125	0.392	0.912
			II	0.055	0.061	0.128	0.392	0.912
			E	0.052	0.067	0.133	0.409	0.912
5x10	5.0	5.0	I	0.057	0.074	0.136	0.415	0.919
			II	0.055	0.075	0.136	0.418	0.924
			E	0.052	0.067	0.133	0.409	0.912

Table 6.4
Estimated powers of F(I) and F(II) and
exact powers (E) of $MS_{\alpha}/MS_{\alpha\beta}$ for "highly unbalanced" cases.

Design	$\sigma_{\alpha\beta}$	σ_{β}		σ_{α}				
				0.2	0.5	1.0	2.0	5.0
5x5	0.2	0.2	I	0.142	0.529	0.895	0.980	0.990
			II	0.126	0.498	0.873	0.976	0.987
			E	0.101	0.413	0.837	0.981	0.999
5x5	0.2	5.0	I	0.063	0.127	0.296	0.602	0.927
			II	0.132	0.518	0.881	0.975	0.987
			E	0.101	0.413	0.837	0.981	0.999
5x5	1.0	1.0	I	0.067	0.179	0.511	0.685	0.995
			II	0.059	0.178	0.520	0.901	0.998
			E	0.069	0.198	0.578	0.919	0.996
5x5	5.0	0.2	I	0.040	0.054	0.070	0.154	0.606
			II	0.036	0.062	0.077	0.154	0.609
			E	0.051	0.057	0.082	0.201	0.727
5x5	5.0	5.0	I	0.059	0.055	0.065	0.136	0.551
			II	0.051	0.044	0.065	0.150	0.609
			E	0.051	0.057	0.082	0.201	0.727
10x5	0.2	0.2	I	0.218	0.853	0.997	1.000	1.000
			II	0.231	0.854	0.996	1.000	1.000
			E	0.136	0.667	0.983	0.999	1.000
10x5	0.2	5.0	I	0.063	0.144	0.363	0.713	0.989
			II	0.227	0.861	0.998	1.000	1.000
			E	0.136	0.667	0.983	0.999	1.000
10x5	1.0	1.0	I	0.078	0.261	0.736	0.991	1.000
			II	0.081	0.294	0.794	0.996	1.000
			E	0.079	0.314	0.844	0.996	1.000
10x5	5.0	0.2	I	0.052	0.056	0.088	0.216	0.854
			II	0.051	0.054	0.085	0.214	0.852
			E	0.051	0.060	0.100	0.317	0.943
10x5	5.0	5.0	I	0.046	0.081	0.080	0.182	0.792
			II	0.052	0.070	0.091	0.220	0.852
			E	0.051	0.060	0.100	0.317	0.943

Table 6.4 (continued)

Design	$\sigma_{\alpha\beta}$	σ_{β}		σ_{α}				
				0.2	0.5	1.0	2.0	5.0
5x10	0.2	0.2	I	0.330	0.865	0.983	0.999	1.000
			II	0.311	0.860	0.985	0.999	1.000
			E	0.190	0.711	0.958	0.996	0.999
5x10	0.2	5.0	I	0.061	0.107	0.292	0.668	0.966
			II	0.289	0.856	0.984	0.999	1.000
			E	0.190	0.711	0.958	0.996	0.999
5x10	1.0	1.0	I	0.087	0.311	0.754	0.967	0.999
			II	0.095	0.373	0.801	0.975	0.999
			E	0.099	0.406	0.833	0.981	0.999
5x10	5.0	0.2	I	0.050	0.053	0.095	0.303	0.814
			II	0.057	0.059	0.099	0.309	0.815
			E	0.052	0.067	0.131	0.405	0.910
5x10	5.0	5.0	I	0.046	0.056	0.089	0.227	0.773
			II	0.051	0.060	0.107	0.275	0.834
			E	0.052	0.067	0.131	0.405	0.910

END

Dtic

7-86