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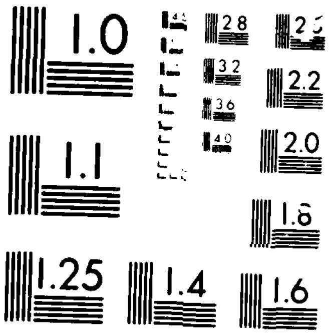
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ABSTRACT

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ABSTRACT

Let $(\theta_1, x_1), \ldots, (\theta_n, x_n)$ be independent and identically distributed random vectors with $E(x|\theta) = \theta$ and $\mathrm{Var}(x|\theta) = a + b\theta + c\theta^2$. Let t_i be the linear Bayes estimator of θ_i and $\tilde{\theta}_i$ be the linear empirical Bayes estimator of θ_i as proposed in Robbins (1983), when Ex and $\mathrm{Var}\, x$ are unknown to the statistician. The regret of using $\tilde{\theta}_i$ instead of t_i because of ignorance of the mean and the variance is $r_i = E(\tilde{\theta}_i - \theta_i)^2 - E(t_i - \theta_i)^2$. Under appropriate conditions cumulative regret $R_n = r_1 + \ldots + r_n$ is shown to have a finite limit even when n tends to infinity. The limit can be explicitly computed in terms of a, b, c and the first four moments of x.

1. INTRODUCTION AND SUMMARY

In the first Jerzy Neyman Memorial Lecture, Robbins (1983) has outlined a wide class of problems concerning the general empirical Bayes approach and the linear empirical Bayes approach to estimation. In this paper we shall study a special case which includes several important standard distributions. Specifically let (θ,x) be a random vector such that θ has a distribution function G, and the conditional expectation of x given θ satisfies

$$E(x/\theta) = \theta. \tag{1.1}$$

Suppose it is desired to use a linear function $A+Bx$ of the observed x to estimate the unknown parameter θ. If the loss function is squared error, the best linear estimate is

$$t(x) = E\theta + \frac{\mathrm{Cov}(\theta,x)}{\mathrm{Var}\ x}(x-Ex) . \tag{1.2}$$

and the mean squared error is

$$E(t-\theta)^2 = \mathrm{Var}\ \theta - \frac{\mathrm{Cov}^2(\theta,x)}{\mathrm{Var}\ x} . \tag{1.3}$$

Assume, in addition to (1.1), that

$$\mathrm{Var}(x|\theta) = a + b\theta + c\theta^2 . \tag{1.4}$$

for some known constants a, b, and c. Then (1.2) can be written as

$$t(x) = Ex + \left(1 - \frac{c\ \mathrm{Var}\ x + a + b\ Ex + cE^2x}{(c+1)\ \mathrm{Var}\ x}\right)(x-Ex) \tag{1.5}$$

which is computable if Ex and $\mathrm{Var}\ x$ are known.

We shall be dealing with the case when Ex and $\mathrm{Var}\ x$ are unknown. However we are faced with a large number n of independent versions of the component problem: $(\theta_1,x_1),\ldots,(\theta_n,x_n)$ are independent random vectors having the same distributions as (θ,x). Robbins (1983) has proposed to estimate Ex and $\mathrm{Var}\ x$ respectively by

$$\bar{x} = \frac{1}{n}\sum_1^n x_i, \quad \text{and} \quad s^2 = \frac{1}{n-1}\sum_1^n (x_i-\bar{x})^2, \tag{1.6}$$

and use the following statistic (with $h = cs^2 + a + b\bar{x} + c\bar{x}^2$),

$$\tilde{\theta}_i = \bar{x} + \left(1 - \frac{h}{(c+1)s^2}\right)^+ (x_i - \bar{x}) \tag{1.7}$$

to estimate θ_i, for each $i=1,\ldots,n$. He has also hoped that under some mild restrictions on the nature of G, with some reasonable rapidity as n tends to infinity

$$E(\tilde{\theta}_i - \theta_i)^2 \to E(t-\theta)^2 . \tag{1.8}$$

We shall assume that the best linear estimate (1.2) is also the best general estimate $E(\theta|x)$. This assumption will reduce the class of possible distributions for θ. For instance, if x has a distribution from an exponential family with parameter θ, then the above assumption will limit the class of the prior distributions to the conjugate family. See Diaconis and Ylvisaker (1979). However, even this special case will be wide enough to include many standard distributions used in practice. In this case we shall verify (1.8). Indeed we shall consider the cumulative regret

$$R_n = \sum_1^n (E(\tilde{\theta}_i - \theta_i)^2 - E(t(x_i) - \theta_i)^2) \tag{1.9}$$

of using $\tilde{\theta}_i$ instead of $t(x_i)$ because of the statistician's ignorance of Ex and $\text{Var}\, x$. It can be shown that even as n goes to infinity R_n remains bounded so that (1.8) will hold. We summarize the main results in the following theorem and leave the proof to the next section.

Theorem 1. Let (θ,x), (θ_1,x_1), ... be independent and identically distributed nondegenerate random vectors such that

(i) $E(x|\theta) = \theta$, $\text{Var}(x|\theta) = a+b\theta+c\theta^2$.

(ii) $E(\theta|x) = E\theta + (x-Ex)\,\text{Cov}(\theta,x)/\text{Var}\,x$.

(iii) $Ex^6 < \infty$.

For each $n=2,\ldots,$ and for each $i=1,2,\ldots,n,$ define

$$\bar{x} = \bar{x}_n = \frac{1}{n}\sum_1^n x_i \quad \text{and} \quad s^2 = s_n^2 = \frac{1}{n-1}\sum_1^n (x_i-\bar{x})^2,$$

$$t_i = Ex + \left(1 - \frac{c\ \mathrm{Var}\ x+a+b\ Ex+cE^2x}{(c+1)\ \mathrm{Var}\ x}\right)(x_i-Ex), \tag{1.10}$$

$$\tilde{\theta}_i = \bar{x} + \left(1 - \frac{cs^2+a+b\bar{x}+c\bar{x}^2}{(c+1)s^2}\right)^+ (x_i-\bar{x}), \quad \text{and}$$

$$R_n = \sum_1^n \left(E(\tilde{\theta}_i - \theta_i)^2 - E(t_i - \theta_i)^2\right).$$

Then

$$\lim_{n\to\infty} R_n = \frac{H^2}{(c+1)^2\gamma^2} + \frac{\sigma^2}{(c+1)^2\gamma^6} \tag{1.11}$$

where $\mu = Ex$, $\gamma^2 = \mathrm{Var}\ x$, $\mu_3 = E(x-\mu)^3$, $\mu_4 = E(x-\mu)^4$,

$$H = c\gamma^2 + a + b\mu + c\mu^2, \quad \text{and} \tag{1.12}$$

$$\sigma^2 = (\mu_4 - \gamma^4)(a+b\mu+c\mu^2)^2 + \gamma^6(b+2c\mu)^2$$
$$- 2\gamma^2\mu_3(a+b\mu+c\mu^2)(b+2c\mu)$$

For the special case when $b = c = 0$, a slightly more general result can be established under weaker conditions. The result is in Theorem 2.

Theorem 2. Let (θ_1,x_1), (θ_2,x_2), ... be independent random vectors satisfying the following conditions:

(A) For all i,

(i) $Ex_i = \mu$ and $\mathrm{Var}\ x_i = \gamma^2 > 0$.

(ii) $E(x_i|\theta_i) = \theta_i$ and $\mathrm{Var}(x_i|\theta_i) = a < \gamma^2$.

(iii) $E(\theta_i|x_i) = E\theta_i + (x_i-\mu)\ \mathrm{Cov}(\theta_i,x_i)/\gamma^2$.

(B) (iv) $\{(x_i-\mu)^2,\ i \geq 1\}$ satisfies the Lindeberg condition.

(v) $\frac{1}{n}\sum_{1}^{n}(x_i-\mu)^2$ converges in probability to γ^2,

and $\lim \frac{1}{n}\sum_{1}^{n}\mathrm{Var}(x_i-\mu)^2$ is finite.

For each $n = 2,\ldots,$ and each $i=1,2,\ldots,n$, put

$$\bar{x} = \bar{x}_n = \frac{1}{n}\sum_{1}^{n} x_i, \quad s^2 = s_n^2 = \frac{1}{n-1}\sum_{1}^{n}(x_i-\bar{x})^2, \tag{1.13}$$

$$t_i = Ex_i + \left(1-\frac{a}{\mathrm{Var}\ x_i}\right)(x_i - Ex_i),$$

$$\tilde{\theta}_i = \bar{x} + \left(1-\frac{a}{s^2}\right)^+(x_i-\bar{x}), \text{ and}$$

$$R_n = \sum_{1}^{n}\left(E(\tilde{\theta}_i-\theta_i)^2 - E(t_i-\theta_i)^2\right).$$

Then

$$\lim_{n\to\infty} R_n = \frac{a^2}{\gamma^2} + \frac{a^2}{\gamma^6}K, \tag{1.14}$$

where $K = \lim_{n\to\infty}\frac{1}{n}V_n$, and $V_n = \sum_{1}^{n}\mathrm{Var}(x_i-\mu)^2$.

2. PROOF OF THE MAIN RESULTS

We need some preliminary results for Theorem 1.

Lemma 1. Let $x,x_1,x_2,\ldots,$ be independent and identically distributed random variables with $Ex^4 < \infty$. Let the following notation be used: $\mu = Ex$, $\gamma^2 = \mathrm{Var}\ x$, $\mu_3 = E(x-\mu)^3$ and $\mu_4 = E(x-\mu)^4$. For each $n \ge 1$, put

$$W_{1n} = \frac{1}{\sqrt{n}}\left(\sum_{1}^{n}(x_i-\mu)^2 - \gamma^2\right), \tag{2.1}$$

$$W_{2n} = \frac{1}{\sqrt{n}}\left(\sum_{1}^{n}\mu(x_i-\mu)^2 - \gamma^2 x_i\right), \text{ and}$$

$$W_{3n} = \sqrt{n}\,(\bar{x}^2-\mu^2).$$

Then as n tends to infinity, (W_{1n},W_{2n},W_{3n}) converges in distribution to a multivariate normal distribution with mean $\underset{\sim}{0}$ and covariance matrix $\Sigma = (\sigma_{ij})$, where

$$\sigma_{11} = \mu_4 - \gamma^4, \quad \sigma_{22} = \mu^2(\mu_4 - \gamma^4) + \gamma^6 - 2\gamma^2\mu\mu_3, \tag{2.2}$$

$$\sigma_{33} = 4\mu^2\gamma^2, \quad \sigma_{12} = \mu(\mu_4 - \gamma^4) - \gamma^2\mu_3,$$

$$\sigma_{13} = 2\mu\mu_3, \text{ and } \sigma_{23} = 2\mu^2\mu_3 - 2\mu\gamma^4.$$

The proof of the Lemma is straightforward and is omitted.

Corollary 1. Under the same conditions as Lemma 1, as $n \to \infty$, $(a+c\mu^2)W_{1n} + bW_{2n} - c\gamma^2 W_{3n}$ has an asymptotic normal distribution with mean 0 and variance

$$\sigma^2 = (\mu_4-\gamma^4)(a+b\mu+c\mu^2)^2 + \gamma^6(b+2c\mu)^2 \tag{2.3}$$

$$- 2\gamma^2\mu_3(a+b\mu+c\mu^2)(b+2c\mu).$$

Lemma 2. Let $x, x_1, \ldots,$ be independent and identically distributed random variables with mean μ and variance

$$\gamma^2 = a + bEx + cE^2x + d, \tag{2.4}$$

where a,b,c and d are constants and $d > 0$. Assume $Ex^4 < \infty$. For each $n \geq 2$, put

$$\bar{x} = \frac{1}{n}\sum_1^n x_i \text{ and } s^2 = \frac{1}{n-1}\sum_1^n (x_i-\bar{x})^2. \tag{2.5}$$

Then as n tends to infinity

$$nP[s^2 \leq a+b\bar{x}+c\bar{x}^2] \to 0. \tag{2.6}$$

Proof: Choose $\delta > 0$, such that $\varepsilon = d-|c+1|\delta-|b+2\mu c|\sqrt{\delta} > 0$. Then

$$P[s^2 \leq a+b\bar{x}+c\bar{x}^2] \tag{2.7}$$

$$\leq P[\Sigma((x_i-\mu)^2 - \gamma^2) \leq na + nb\bar{x} + nc\bar{x}^2 - n\gamma^2 + n(\bar{x}-\mu)^2]$$

$$\leq P[\Sigma((x_i-\mu)^2 - \gamma^2) \leq -\varepsilon n, \ (\bar{x}-\mu)^2 < \delta]$$

$$+ P[(\bar{x}-\mu)^2 \geq \delta].$$

Let $B = \{\Sigma((x_i-\mu)^2 - \gamma^2) \le -\varepsilon n\}$. Then the first term in (2.7) is less than or equal to

$$P[|\Sigma((x_i-\mu)^2 - \gamma^2)| \ge \varepsilon n] \tag{2.8}$$
$$\le \frac{1}{\varepsilon^2 n^2} \int_B \left((x_i-\mu)^2 - \gamma^2\right)^2 dP$$

which is $o(1/n)$ as n tends to infinity by the uniform integrability of $\{|(\Sigma((x_i-\mu)^2 - \gamma^2))^2/n, n \ge 1\}$ implied by $Ex^4 < \infty$. For the second term in (2.7)

$$P[(\bar{x}-\mu)^2 \ge \delta] \tag{2.9}$$
$$\le \frac{1}{\delta^2 n^4} (n(E(x-\mu)^4 - \gamma^4) + (3n^2 - 2n)\gamma^4)$$

which is $o(1/n)$ as n tends to infinity. This concludes the proof of Lemma 2.

Lemma 3. Let $x, x_1, \ldots$ be independent and identically distributed random variables with mean μ and variance γ^2. Assume that $Ex^6 < \infty$. For each $n \ge 2$, put

$$\bar{x} = \frac{1}{n}\sum_1^n x_i \quad \text{and} \quad s^2 = \frac{1}{n-1}\sum_1^n (x_i-\bar{x})^2. \tag{2.10}$$

Then the following families of random variables are uniformly integrable:

$$\begin{aligned} &\text{(i)} && \{ns^2(s^2-\gamma^2)^2, n \ge 2\}, \\ &\text{(ii)} && \{ns^2(\bar{x}-\mu)^2, n \ge 2\}, \\ &\text{(iii)} && \{ns^2(\bar{x}^2-\mu^2)^2, n \ge 2\}. \end{aligned} \tag{2.11}$$

Proof. We shall verify (i) and (iii). The verification for (ii) is entirely analogous and hence omitted. For (i),

$$ns^2(s^2-\gamma^2)^2 \tag{2.12}$$
$$= n(s^2-\gamma^2)^3 + n\gamma^2(s^2-\gamma^2)^2$$
$$\le \frac{K}{n^2} |\Sigma((x_i-\mu)^2 - \gamma^2)|^3 + \frac{K}{n^2}|\gamma^2 - n(\bar{x}-\mu)^2|^3$$
$$+ \frac{K\gamma^2}{n} |\Sigma((x_i-\mu)^2 - \gamma^2)|^2 + \frac{K\gamma^2}{n} |\gamma^2 - n(\bar{x}-\mu)^2|^2,$$

for some constant K.

Since $Ex^6 < \infty$, it is clear the four terms on the bottom line of (2.12) are uniformly integrable. For (iii), for any event D with P[D] small,

$$Ens^2(\bar{x}^2-\mu^2)^2 I_D \tag{2.13}$$

$$\le K(E(\frac{\Sigma(x_i-\mu)^2}{n})^3 I_D)^{1/3} (E(n(\bar{x}^2-\mu^2)^2)^{3/2} I_D)^{2/3} ,$$

which can be made small uniformly in n.

Now we are ready to give the proofs. That the convergence in distribution of random variables together with uniform integrability implies moment convergence is used. For a reference, see (Chow and Teicher (1978), Section 8.1).

Proof of Theorem 1. Let

$$h = cs^2 + a+b\bar{x}+c\bar{x}^2. \tag{2.14}$$

From the identity

$$(\tilde{\theta}_i-\theta_i)^2 - (t_i-\theta_i)^2 \tag{2.15}$$

$$= (\tilde{\theta}_i-t_i)^2 + 2(\tilde{\theta}_i-t_i)(t_i-\theta_i) ;$$

taking expectation and summation and by assumptions (i) and (ii) we have the cumulative regret

$$R_n = \frac{H^2}{(c+1)^2\gamma^2} + E(\frac{H}{(c+1)\gamma^2} - \frac{h}{\max(h,(c+1)s^2)})^2 \Sigma(x_i-\bar{x})^2 \tag{2.16}$$

As n tends to infinity s^2 and h will go to γ^2 and H respectively with probability one. Since, (see Robbins (1983)),

$$\gamma^2 = \frac{H}{c+1} + \text{Var } \theta, \tag{2.17}$$

and θ is nondegenerate, asymptotically the term inside the expectation sign in (2.16) is equivalent to

$$\frac{n}{(c+1)^2\gamma^4}\frac{(s^2H-\gamma^2h)^2}{s^2} \tag{2.18}$$

$$= \frac{1}{(c+1)^2\gamma^4 s^2}((a+c\mu^2)\sqrt{n}(s^2-\gamma^2)+b\sqrt{n}(\mu s^2-\gamma^2\bar{x}) - c\gamma^2\sqrt{n}(\bar{x}^2-\mu^2))^2$$

$$= \frac{1}{(c+1)^2\gamma^6}((a+c\mu^2)\frac{\Sigma((x_i-\mu)^2-\gamma^2)}{\sqrt{n}} + b\,\frac{\Sigma\mu(x_i-\mu)^2-\gamma^2 x_i}{\sqrt{n}}$$

$$- c\gamma^2\sqrt{n}(\bar{x}^2-\mu^2) + o_p(1))^2 .$$

By Corollary 1 and (2.18), (χ_1^2 denoting chi-squared with 1 d.f.),

$$T = (\frac{H}{(c+1)\gamma^2} - \frac{h}{\max(h,(c+1)s^2)})^2\,\Sigma(x_i-\bar{x})^2 \tag{2.19}$$

converges in distribution to $\frac{\sigma^2}{(c+1)^2\gamma^6}\chi_1^2$, where σ^2 is defined in (2.3) Next we shall show that $\{T, n \geq 2\}$ is uniformly integrable so that as n tends to infinity

$$ET \to \frac{\sigma^2}{(c+1)^2\gamma^6} . \tag{2.20}$$

Let A be the event $\{(c+1)s^2 \leq h\}$, then for some positive K

$$ETI_A \leq K \int_A (na+nb\bar{x}+nc\bar{x}^2)\,dP \tag{2.21}$$

$$\leq K(a+b\mu+c\mu^2)nP[A] + Kbn\int_A|\bar{x}-\mu|\,dP + Kcn\int_A|\bar{x}^2-\mu^2|\,dP$$

$$\leq K(a+b\mu+c\mu^2)nP[A] + Kb\sqrt{n}\,((En(\bar{x}-\mu)^2)P[A])^{\frac{1}{2}}$$

$$+ Kc\sqrt{n}(En(\bar{x}^2-\mu^2)^2P[A])^{\frac{1}{2}},$$

which is $o(1)$ by Lemma 2. On the complement of A,

$$\frac{(c+1)^2\gamma^4}{2}T \tag{2.22}$$

$$= \frac{(c+1)^2\gamma^4}{2}(\frac{H}{(c+1)\gamma^2} - \frac{h}{(c+1)\gamma^2} + \frac{h}{(c+1)\gamma^2} - \frac{h}{(c+1)s^2})^2\Sigma(x_i-\bar{x})^2$$

$$\leq (H-h)^2 \Sigma(x_i-\bar{x})^2 + \frac{nh^2}{s^2}(s^2-\gamma^2)^2$$

$$\leq 4c^2(s^2-\gamma^2)^2\Sigma(x_i-\bar{x})^2 + 4b^2(\bar{x}-\mu)^2\Sigma(x_i-\bar{x})^2$$

$$+ 4c^2(\bar{x}^2-\mu^2)^2\Sigma(x_i-\bar{x})^2 + (c+1)^2 ns^2(s^2-\gamma^2)^2.$$

By Lemma 3, the four terms on the bottom line of (2.22) are uniformly integrable. Therefore we have the regret

$$\lim_{n\to\infty} R_n = \frac{H^2}{(c+1)^2\gamma^2} + \frac{\sigma^2}{(c+1)^2\gamma^6}. \tag{2.23}$$

Remark. The expression for σ^2 contains terms up to the fourth moment. Although it has terms of the eighth order (e.g. $c^2\mu^4\mu_4$), the sixth moment assumption is to ensure the uniform integrability of $\{(\frac{1}{\sqrt{n}}\Sigma((x_i-\mu)^2-\gamma^2))^3,\ n \geq 1\}$ in Lemma 3. Nevertheless it is reasonable to conjecture that condition (iii) in Theorem 1 can be replaced by $Ex^4 < \infty$.

Proof of Theorem 2. From the identity

$$(\tilde{\theta}_i-\theta_i)^2 - (t_i-\theta_i)^2 = (\tilde{\theta}_i-t_i)^2 \tag{2.24}$$

$$+ 2(\tilde{\theta}_i-t_i)(t_i-\theta_i);$$

taking summation and expectation and by assumptions (i), (ii), and (iii), and definition (1.13) we have the regret equal to

$$R_n = \frac{a^2}{\gamma^2} + a^2\, E\left(\frac{1}{\gamma^2} - \frac{1}{\max(s^2,a)}\right)^2\Sigma(x_i-\bar{x})^2. \tag{2.25}$$

Let $v_i = \mathrm{Var}(x_i-\mu)^2$ and $V_n = v_1+v_2+\ldots+v_n$. Consider the event $A = \{s^2 \leq a\}$, then

$$E\left(\frac{1}{\gamma^2} - \frac{1}{\max(x^2,a)}\right)^2\Sigma(x_i-\bar{x})^2\ I_A \tag{2.26}$$

$$\leq cn\ P[s^2 \leq a]$$

for some positive constant c. And

$$P[s^2 \le a] \le P[\Sigma((x_i-\mu)^2-\gamma^2) \le (a-\gamma^2)n+n(\bar{x}-\mu)^2] \tag{2.27}$$

$$\le P[\Sigma(x_i-\mu)^2-\gamma^2 \le (a-\gamma^2)n + n(\bar{x}-\mu)^2,\ (\bar{x}-\mu)^2 < \delta]$$

$$+ P[(\bar{x}-\mu)^2 \ge \delta].$$

Choose $\delta > 0$ such that $a-\gamma^2+\delta = -\varepsilon < 0$. Let B be

$$\{\Sigma((x_i-\mu)^2 - \gamma^2) \le -\varepsilon n\},$$

then

$$nP[\Sigma((x_i-\mu)^2 - \gamma^2) \le -\varepsilon n] \tag{2.28}$$

$$\le \frac{V_n}{\varepsilon^2 n} \int_B \frac{(\Sigma((x_i-\mu)^2-\gamma^2))^2}{V_n}\, dP$$

which goes to zero as n tends to infinity by conditions (iv) and (v), and Brown's Theorem (see Chow and Teicher (1978), p. 398). Next, consider

$$nP[(\bar{x}-\mu)^2 \ge \delta] \tag{2.29}$$

$$\le \frac{n}{\delta^2} E(\bar{x}-\mu)^4 = \frac{1}{\delta^2 n^3} (\Sigma \mathrm{Var}(x_i-\mu)^2 + (3n^2-2n)\gamma^4)$$

which goes to zero as n tends to infinity by condition (v). On the complement of A, and for any event D

$$E\left(\frac{1}{\gamma^2} - \frac{1}{\max(s^2,a)}\right)^2 \Sigma(x_i-\bar{x})^2\ I_{A^c} I_D \tag{2.30}$$

$$= E\left(\frac{\Sigma(x_i-\bar{x})^2 - (n-1)\gamma^2}{\gamma^2\Sigma(x_i-\bar{x})^2}\right)^2 \Sigma(x_i-\bar{x})^2\ I_{A^c} I_D$$

$$\le \frac{V_n}{(n-1)a\gamma^4} E \frac{(\Sigma(x_i-\bar{x})^2 - (n-1)\gamma^2)^2}{V_n} I_{A^c} I_D$$

which may be made very small uniformly in n if $P[D]$ is small for the same reason as in (2.28). Therefore the family

$$\{(\frac{1}{\gamma^2} - \frac{1}{\max(s^2,a)})^2 \Sigma(x_i-\bar{x})^2,\ n \geq 2\} \tag{2.31}$$

is uniformly integrable. And by condition (v), as n tends to infinity s^2 tends to γ^2 and $\bar{x}$ tends to μ in probability. By condition (iv), as n tends to infinity

$$\frac{\Sigma(x_i-\bar{x})^2 - (n-1)\gamma^2}{\sqrt{V_n}} \cdot \to N(0,1) \text{ in distribution.} \tag{2.32}$$

Hence

$$E(\frac{1}{\gamma^2} - \frac{1}{\max(s^2,a)})^2 \Sigma(x_i-\bar{x})^2 \to \frac{1}{\gamma^6} \lim_{n\to\infty} \frac{V_n}{n}\ ; \tag{2.33}$$

that is

$$\lim_{n\to\infty} R_n = \frac{a^2}{\gamma^2} + \frac{a^2}{\gamma^6} \lim_{n\to\infty} \frac{V_n}{n}\ . \tag{2.34}$$

Corollary: Let (θ_1,x_1), (θ_2,x_2), ... be independent and identically distributed random vectors satisfying condition (A) in the Theorem. If $Ex^4 < \infty$, R_n and $\tilde{\theta}_i$ are defined in the same way as in (1.13) and (1.14), then

$$\lim_{n\to\infty} R_n = \frac{a^2}{\gamma^2} E(\frac{x-\mu}{\gamma})^4. \tag{2.35}$$

Example 1. Suppose θ has a common normal distribution with mean μ and variance τ^2 and given θ, x has a normal distribution with mean θ and variance a. Then x has a normal distribution with mean μ and variance $\gamma^2 = a+\tau^2$. Obviously the conditions of the corollary hold; hence the regret R_n is $3a^2/(a+\tau^2)+o(1)$, as n tends to infinity.

$\tilde{\theta}_i$ in this normal case is a variant of the James-Stein estimator, which has been studied extensively in the literature. See Efron and Morris (1973).

Example 2. Suppose θ has an inverted gamma distribution with density function

$$g(\theta) = \begin{cases} (\frac{\beta}{\theta})^{\alpha+i} \frac{e^{-\beta/\theta}}{\Gamma(\alpha)} \frac{1}{\beta} & \text{if } \theta > 0 \\ 0 & \text{if } \theta \le 0, \end{cases}$$

where $\beta > 0$ and $\alpha > 6$, and given θ, x has an exponential distribution with mean θ and variance θ^2. The conditions (i), (ii) and (iii) in Theorem 1 hold with $a=b=0$ and $c=1$. It can be computed that, for any $0 \le p \le 6$;

$$Ex^p = \frac{\beta^p \Gamma(p+1)\Gamma(\alpha-p)}{\Gamma(\alpha)}$$

If the linear empirical Bayes estimators are used, then the cumulative regret R_n will satisfy

$$\lim_{n\to\infty} R_n = \frac{2(\alpha-1)(\alpha^2-4\alpha+6)\beta^2}{\alpha^2(\alpha-2)(\alpha-3)(\alpha-4)} .$$

Example 3. Suppose θ has a gamma distribution with the density function

$$g(\theta) = \begin{cases} 0 & \text{if } \theta \le 0 \\ \frac{\beta^\alpha \theta^{\alpha-1} e^{-\beta\theta}}{\Gamma(\alpha)} & \text{if } \theta > 0, \end{cases}$$

where α and β are positive constants, and given θ, x has a Poisson distribution with mean θ. In this case, the conditions (i), (ii) and (iii) in Theorem 1 hold with $a=c=0$ and $b=1$. If the linear empirical Bayes estimators are used, then the regret R_n will satisfy

$$\lim_{n\to\infty} R_n = \frac{3\alpha(\beta+1) + 2\beta+3}{(\beta+1)^2} .$$

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