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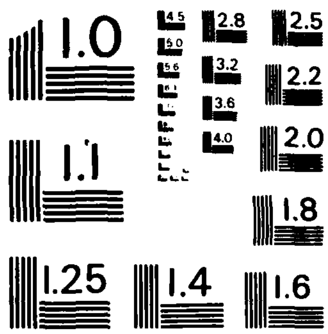

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This paper is the first in a series devoted to the analysis of the regularity of the solution of elliptic partial differential equations with piecewise analytic data. The present paper analyzes the case of linear, second order partial differential of elliptic type. These results are essential for the design and the analysis of the h-p version of the finite element method for solving the elliptic differential equations of structural engineering.

INSTITUTE FOR PHYSICAL SCIENCE
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REGULARITY OF THE SOLUTION OF ELLIPTIC PROBLEMS
WITH PIECEWISE ANALYTIC DATA
PART I: BOUNDARY VALUE PROBLEMS FOR LINEAR ELLIPTIC
EQUATION OF SECOND ORDER

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May 1986

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REGULARITY OF THE SOLUTION OF ELLIPTIC PROBLEMS WITH PIECEWISE ANALYTIC DATA

PART I. BOUNDARY VALUE PROBLEMS FOR LINEAR ELLIPTIC EQUATION OF SECOND ORDER

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ABSTRACT

This paper is the first in a series devoted to the analysis of the regularity of the solution of elliptic partial differential equations with piecewise analytic data. The present paper analyzes the case of linear, second order partial differential of elliptic type. It concentrates on the case when the domain $\Omega \subset \mathbf{R}^2$ is a polygon, boundary condition are of changing type and coefficients are analytic on $\bar{\Omega}$. The main result states that the solution belongs to a countably normed space based on weighted Sobolev spaces of all orders with weights located in the vertices of the domain and points where the type of boundary conditions changes.

These results are essential for the design and the analysis of the h-p version of the finite element method for solving the elliptic differential equations of structural engineering. See [6], [11].

1. THE PRELIMINARIES

1.1. INTRODUCTION

In applications, as for example in structural mechanics, the problems of elliptic partial differential equations are typically characterized by piecewise analytic input data. The boundary of the domain is piecewise analytic with corners and edges, the coefficients of the equation are piecewise analytic with interfaces having corners and edges. The type of boundary condition is abruptly changing but they are piecewise analytic, etc.

The regularity theory is typically developed in the framework of Sobolev spaces. We refer here e.g. to the survey [15] and to the monography [10], [14] addressing the problem of unsmooth boundary. We refer to [2] for more classical results. Results mentioned above do not characterize sufficiently accurately the class of solutions of problems of applications. The detailed knowledge of the properties of the solutions of problems of applications is essential for the design and analysis of effective numerical methods for solving these problems. We mention here e.g. the h-p version of the finite element method which was recently developed and is very successfully used in practice.* For more about the theory and practice of the h-p version we refer to [6], [17] [18].

The solution of the problem with piecewise analytic data is analytic with the exception of special areas of the domain, where the solution has singular character. Typically it happens in the neighborhood of the

*Program PROBE of Noetic Technologies, St. Louis.

corners of the domain, place where the type of the boundary condition changes, etc.

This paper, which is the first one in a series of papers, deals with the problem of characterizing the regularity of the solution of the linear partial differential equation of elliptic type on a polygonal domain. It addresses the case of constant and analytic coefficients. The main tool of the characterization of the solution is the theory of countably normed spaces based on weighted Sobolev spaces of all orders, when the weights are placed in the vertices of the domain. The main result is that the solution is from the set $B^2_\beta(\Omega)$ of functions which belong to the weighted Sobolev spaces $H^{k,2}_\beta(\Omega)$ for $k = 2,\ldots$ and $\|u\|_{H^{k,2}_\beta(\Omega)} \le Cd^{k-2}(k-2)!$ with C and d independent of k. The main theorem of the paper is Theorem 2.1 addressing the case of the Poisson equation and its generalization for the general equation with analytic coefficients is given in Theorem 3.1. Theorem 3.1. can be further generalized for the case when the coefficients have singular behavior in the neighborhood of the corners too. (Problem of this type is important in appolications when nonlinear equations are considered.) Chapter 1 gives basic notions and preliminaries. Chapter 2 deals with the regularity of the solution of the Poisson problem. Chapter 3 deals with the general equation and Chapter 4, Appendix proves some technical lemmas used in the paper.

1.2. THE NOTATIONS

Throughout this paper we shall denote integer by i, j, k, ℓ, m, n. By $\mathbf{R}^1$ and $\mathbf{R}^2$ we shall denote the one and two dimensional

Euclidean space. If $Q \subset \mathbf{R}^1$ repectively $Q \subset \mathbf{R}^2$, then $\bar{Q}$ denotes the closure of Q in $\mathbf{R}^1$, respectively in $\mathbf{R}^2$.

By Ω we denote the polygonal domain in $\mathbf{R}^2$ with boundary $\partial\Omega = \Gamma$, the vertices A_i, $i = 1,\dots,M$, and Γ_i, $i = 1,\dots,M$ the open edges of $\partial\Omega$ connecting A_{i-1} on A_i $(A_0 = A_M)$. Obviously we have $\partial\Omega = \bigcup_{i=1}^{M} \bar{\Gamma}_i$. By ω_i we denote the measure of the interior angle of Ω at A_i. We allow also $\omega_i = 2\pi$, and $\omega_i = \pi$ and the polygon Ω has hence to be understood in this generalized sense. Let further $\Gamma = \bar{\Gamma}^0 + \bar{\Gamma}^1$, $\Gamma^0 = \bigcup_{i \in \mathcal{D}} \bar{\Gamma}_i$, $\Gamma^1 = \Gamma - \Gamma^0$ where $\mathcal{D}$ is some subset of set $\{1,2,\dots,M\}$. Γ^0 will be sometimes referred to as Dirichlet boundary and Γ^1 as Neumann boundary.

By $H^m(\Omega)$ (resp. $H^m(Q)$), $m \geq 0$, m integer, we denote the Sobolev space of functions with square integrable derivatives of order $\leq m$ on Ω (resp. Q) furnished with the norm:

$$\|u\|^2_{H^m(\Omega)} = \sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|^2_{L_2(\Omega)}$$

$$\alpha = (\alpha_1, \alpha_2), \quad \alpha_i \geq 0, \quad \text{integers}, \quad i = 1,2,$$

$$|\alpha| = \alpha_1 + \alpha_2,$$

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{x_2}} = u_{x_1^{\alpha_1} x_2^{\alpha_2}}$$

As usual $H^0(\Omega) = L_2(\Omega)$. Further let

$$H^1_0(\Omega) = \{u \in H^1(\Omega) \mid u = 0 \text{ on } \Gamma^0\}$$

and

$$|u|^2_{H^m(\Omega)} = \sum_{|\alpha|=m} \|D^\alpha u\|^2_{H^0(\Omega)}$$

$$|D^m u|^2 = \sum_{|\alpha|=m} |D^\alpha u|^2.$$

By $r_i(x) = |x-A_i|$, $i = 1,\ldots,M$, we shall denote the Euclidean distance between x and the vertex A_i of Ω. Let $\beta = (\beta_1,\beta_2,\ldots,\beta_M)$ be an M-tuple of real numbers, $0 < \beta_i < 1$, $i = 1,\ldots,M$, $|\beta| = \sum_{i=1}^{M} \beta_i$. For any integer k let $\beta \pm k = (\beta_1 \pm k,\ldots,\beta_M \pm k)$. Further we denote $\Phi_\beta(x) = \prod_{i=1}^{M} r_i^{\beta_i}(x)$ and $\Phi_{\beta\pm k} = \prod_{i=1}^{M} r_i^{\beta_i \pm k}(x)$.

By $H_\beta^{m,\ell}(\Omega)$, $m \geq \ell \geq 0$, ℓ an integer $(H_\beta^{m,0}(\Omega) = H_\beta^m(\Omega))$ we denote the completion of the set of all infinitely differentiable functions under the norm

$$\|u\|^2_{H_\beta^{m,\ell}(\Omega)} = \|u\|^2_{H^{\ell-1}(\Omega)} + \sum_{|\alpha|=\ell}^{m} \||D^\alpha u|\Phi_{\beta+k-\ell}\|^2_{L_2(\Omega)}, \qquad \ell \geq 1$$

$$\|u\|^2_{H_\beta^{m,0}(\Omega)} = \|u\|^2_{H_\beta^m(\Omega)} = \sum_{|\alpha|=0}^{m} \||D^\alpha u|\Phi_{\beta+k}\|^2_{L_2(\Omega)}, \qquad \ell = 0.$$

For $m = \ell = 0$ we shall write $H_\beta^{0,0}(\Omega) = {}_\beta(\Omega)$. The space $H_\beta^{m,2}(\Omega)$ was introduced and widely used in [5].

For $0 < \delta \leq \infty$, $0 < \omega \leq 2\pi$ let

$$S = S_\delta^\omega = \{(r,\theta) | 0 < r < \delta,\ 0 < \theta < \omega\},$$

$$\mathcal{D}^\alpha u = \frac{\partial^{|\alpha|} u}{\partial r^{\alpha_1} \partial\theta^{\alpha_2}} = u_{r^{\alpha_1}\theta^{\alpha_2}}$$

and

$$|\mathcal{D}^m u|^2 = \sum_{|\alpha|=m} |r^{-\alpha_2}\mathcal{D}^\alpha u|^2.$$

For $0 < \beta < 1$, $m \geq \ell > 1$

$$\mathcal{H}_\beta^{m,\ell}(S) = \{u \mid \|u\|^2_{H^{\ell-1}(S)} + \sum_{\ell \leq |\alpha| \leq m} \|r^{\alpha_1-\ell+\beta}\mathcal{D}^\alpha u\|^2_{L_2(S)} = \|u\|^2_{\mathcal{H}_\beta^{m,\ell}(S)} < \infty\}$$

and $m \geq 0$

$$\mathcal{H}_\beta^{m,0}(S) = \{ u \mid \sum_{0 \leq |\alpha| \leq m} \|r^{\alpha_1+\beta}\mathcal{D}^\alpha u\|^2_{L_2(S)} = \|u\|^2_{\mathcal{H}_\beta^{m,0}(S)} < \infty\}$$

Obviously

$$\mathcal{H}_\beta^{0,0}(S) = H_\beta^{0,0}(S) = L_\beta(S)$$

and

$$\mathcal{H}_\beta^{1,1}(S) = H_\beta^{1,1}(S).$$

We will show now that $\mathcal{H}_\beta^{2,2}(S) = H_\beta^{2,2}(S)$.

Lemma 1.1. Let $0 < \delta < \infty$. Then the spaces $\mathcal{H}_\beta^{2,2}(S)$ and $H_\beta^{2,2}(S)$ with $\Phi_\beta = r^\beta$ are equivalent.

Proof. Observe

$$u_{x_1} = u_r \cos\theta = u_\theta \frac{\sin\theta}{r}$$

$$u_{x_1x_1} = u_{rr}\cos^2\theta - u_{r\theta}\frac{\sin 2\theta}{r} + \frac{1}{r^2}u_{\theta^2}\sin^2\theta$$

$$- \frac{1}{r}u_r \sin^2\theta - \frac{1}{r^2}u_\theta \sin 2\theta.$$

Hence

$$\|u_{x_1x_1}\|_{L_\beta(S)} < \sum_{|\alpha|=2} \|r^{\alpha_1-2}\mathcal{D}^\alpha u\|_{L_\beta(S)} + \sum_{|\alpha|=1} \|r^{\alpha_1-2}\mathcal{D}^\alpha u\|_{\beta(S)}.$$

By Lemma A2 (see Appendix) we have for $|\alpha| = 1$

$$\|r^{\alpha_1-2}\mathcal{D}^\alpha u\|_{L_\beta(S)} \leq C(\delta)[\sum_{|\alpha|=2} \|r^{\alpha_1-2}\mathcal{D}^\alpha u\|_{L_\beta(S)} + \|u\|_{H^1(S)}]$$

and hence

$$\|u_{x_1x_1}\|_{L_\beta(S)} \leq C\|u\|_{H^{2,2}_\beta(S)}.$$

Similarly, we have the same relation for $u_{x_1x_2}$ and $u_{x_2x_2}$, and hence $H^{2,2}_\beta(S) \subset H^{2,2}_\beta(S)$. The other direction follows directly. ∎

Later we will investigate the case S^ω_δ when $\delta = \infty$. In this case we will write Q instead of S^ω_δ.

Lemma 1.2. Let Ω be the polygon, then for $j = 0,1$ we have

(1.1a)
$$\int_\Omega \Phi^2_{-1+\beta} |u_{x_1^{1-j}x_2^j}|^2 dx_1dx_2 \leq C\|u\|_{H^{2,2}_\beta(\Omega)}$$

(1.1b)
$$\int_\Omega \Phi^2_{-1+\beta} r_i^{-2j} |u_{r_i^{1-j}\theta_i^j}|^2 r_i dr_i d\theta_i \leq C\|u\|_{H^{2,2}_\beta(\Omega)}.$$

where (r_i,θ_i) are polar coordinates with respect to A_i, $1 \leq i \leq M$.

Proof. We can write

$$\Omega = \sum_{\ell=1}^m S^{\omega_i}_{\delta_i}(A_i) + R$$

$$R = \Omega - \sum_{\ell-1}^{m} S_{\delta_i}^{\omega_i}(A_i)$$

where $S_{\delta_i}^{\omega_i}(A_i) \subset \Omega$ are sectors with the origin in A_i such that $S_{\delta_i}(A_i) \cap S_{\delta_j}(A_j) = \emptyset$ for $i \neq j$. ω_i is the interior angle at A_i. Obviously (1.1a,b) hold on R. Lemma A3 yields (1.1a) on $S_{\delta_i}(A_i)$ and Lemma 1.1 yields (1.1b) on $S_{\delta_i}(A_i)$.

We also recall the spaces $W_\beta^k(S)$ introduced by Kondrat'ev (see [14], [15])

$$W_\beta^k(S) = \{u \mid \sum_{0\le|\alpha|\le k} \| r^{\beta-k+\alpha_1} \mathcal{D}^\alpha u\|_{L_2(S)} = \|u\|^2_{W_\beta^k(S)} < \infty.$$

Finally let

$$D = \{\tau,\theta \mid -\infty < \tau < \infty, \quad 0 < \theta < \omega\}$$

and for $h > 0$ and $k \geq 0$ an integer define

$$H_h^k(D) = \{u \mid \sum_{0\le|\alpha|\le k} e^{2h\tau}|D^\alpha u|^2 d\tau d\theta = \|u\|^2_{H_h^k(D)} < \infty\}.$$

We will write also $H_h^0(F) = L_h(D)$.

1.3. THE SPACES $\Psi_\beta^\ell(\Omega)$ AND $B_\beta^\ell(\Omega)$

For ℓ an integer $0 \leq \ell \leq 2$ let

(1.2) $$\psi_\beta^\ell(\Omega) = \{u(x) \mid u \in H_\beta^{m,\ell}(\Omega), \quad m \geq \ell\}$$

and

(1.3) $$B_\beta^\ell(\Omega) = \{u(x) \mid u \in \psi_\beta^\ell(\Omega), \ \| |D^\alpha u| \Phi_{\beta+k-\ell}\|_{L^2(\Omega)} \leq Cd^{k-\ell}(k-\ell)!$$

for $|\alpha| = k = \ell, \ \ell + 1, \ldots, \ d \geq 1$, C independent of $k\}$.

For $\ell = 0$ we shall write $B_\beta(\Omega)$ instead $B_\beta^0(\Omega)$. Constants C and d in (1.3) depend on u.

The space $B_\beta^\ell(\Omega)$ was defined in Cartesian coordinates. There is also an equivalent definition of $B_\beta^\ell(\Omega)$ in polar coordinates.

Let (r_i, θ_i) be the polar coordinates with respect to A_i, $i = 1, \ldots, M$ as before.

Theorem 1.1. Let $0 \leq \ell \leq 2$. Then

$$(1.4) \qquad \| |D^\alpha u| \Phi_{k-\ell+\beta} \|_{L_2(\Omega)} \leq C d^{k-\ell}(k-\ell)! \qquad |\alpha| = k, \quad k \geq \ell$$

then and only then if

$$(1.5) \qquad \left(\int_\Omega |\mathcal{D}_i^{\alpha'} u|^2 \, r_i^{-2\alpha_2'} \Phi_{k'-\ell+\beta}^2 r_i \, dr_i \, d\theta_i\right)^{1/2} \leq C_i d_i^{k'-\ell}(k'-\ell)!$$

hold for all $\ell \leq |\alpha'| = k' \leq k$, $\alpha' = (\alpha_1', \alpha_2')$ and $i = 1, \ldots, M$. By $\mathcal{D}_i^\alpha u$ we denoted differentiation with respect to the polar coordinates (r_i, θ_i).

Proof. We prove first that if (1.4) holds then (1.5) holds for every $i = 1, \ldots, M$. To this end we fix i and will omit to write the index i. Then

$$(1.6) \qquad u_{r^k} = \sum_{j=0}^{k} \binom{k}{j} u_{x_1^{k-j} x_2^j} \cos^{k-j}\theta \sin^j \theta.$$

Hence

$$(1.7) \qquad \left(\int_\Omega |u_{r^k}|^2 \Phi_{k-\ell+\beta}^2 \, r \, dr \, d\theta\right)^{1/2}$$

$$\leq \sum_{j=0}^{k} \binom{k}{j} \| u_{x_1^{k-j} x_2^{j}} \Phi_{k-\ell+\beta} \|_{L_2(\Omega)}$$

$$\leq C_1 2^k d^{k-\ell}(k-\ell)!$$

$$= C_2 2^{\ell}(2d)^{k-\ell}(k-\ell)!$$

and (1.5) is proven for $\alpha_2' = 0$.

We show now by induction that for any $k \geq 1$:

$$\text{(1.8a)} \qquad u_{\theta^k} = \sum_{m=1}^{k} r^m \sum_{j=0}^{m} \sum_{\substack{\ell_1, \ell_2 \geq 0 \\ \ell_1 + \ell_2 = m}} a^{(k)}_{m,j,\ell_1,\ell_2} \sin^{\ell_1}\theta \cos^{\ell_2}\theta \, u_{x_1^{m-j} x_2^{j}}$$

$$\text{(1.8b)} \qquad A^{(k)}_m = \sum_{j=0}^{m} \sum_{\substack{\ell_1, \ell_2 \geq 0 \\ \ell_1 + \ell_2 = m}} |a^{(k)}_{m,j,\ell_1,\ell_2}| \leq 3^k \frac{k!}{m!}.$$

Suppose now that (1.8) holds for $k = n - 1$. Then

$$u_{\theta^n} = \sum_{m=1}^{n-1} r^m \sum_{j=0}^{m} \sum_{\substack{\ell_1, \ell_2 \geq 0 \\ \ell_1 + \ell_2 = m}} a^{(n-1)}_{m,j,\ell_1,\ell_2} [-r \sin^{\ell_1+1}\theta \cos^{\ell_2}\theta \, u_{x_1^{m-j+1} x_2^{j}}$$

$$+ r \sin^{\ell_1}\theta \cos^{\ell_2+1}\theta \, u_{x_1^{m-j} x_2^{j+1}}$$

$$+ (\ell_1 \sin^{\ell_1-1}\theta \cos^{\ell_2+1}\theta - \ell_2 \sin^{\ell_1+1}\theta \cos^{\ell_2-1}\theta) u_{x_1^{m-j} x_2^{j}}].$$

Comparing the coefficients we get

$$a^{(n)}_{m,j,\ell_1,\ell_2} = -a^{(n-1)}_{m-1,j,\ell_1-1,\ell_2} + a^{(n-1)}_{m-1,j-1,\ell_1,\ell_2-1}$$

$$+ \ell_1 a^{(n-1)}_{m,j,\ell_1+\cdot,\ell_2-1} - \ell_2 a^{(n-1)}_{m,j,\ell_1-1,\ell_2+1} .$$

Thus

$$A^{(n)}_m \leq m\, A^{(n-1)}_m + 2A^{(n-1)}_{m-1} .$$

Using the induction assumption we get for $n = k$ and $m \leq k$

$$A^k_m \leq 3^k \frac{k!}{m!}$$

and (1.8b) is proven.

Let now $D = \max(1, \operatorname{diam} \Omega)$. Then for $k \geq 1$, $\ell \leq 1$

$$(1.9) \quad \Big(\int_\Omega r^{-2k} |u_{\theta^k}|^2 \Phi^2_{k-\ell+\beta}\, r\, dr\, d\theta\Big)^{1/2}$$

$$\leq \sum_{m=1}^{k} \sum_{j=0} \sum_{\substack{\ell_1,\ell_2 \geq 0 \\ \ell_1+\ell_2=m}} |a^{(k)}_{m,j,\ell_1,\ell_2}| D^{(k+1-\ell)(M-1)} \| \Phi_{m-\ell+\beta} u_{x_1^{m-j} x_2^j} \|_{L_2(\Omega)}$$

$$\leq C D_1^{k-\ell} \sum_{m=1}^{k} A^{(k)}_m d^{m-\ell} (m-\ell)!$$

$$\leq C D_2^{k-\ell} \sum_{m=1}^{k} 3^k \frac{k!}{m!} (m-\ell)!$$

$$\leq C D_3^{k-\ell} (k-\ell)!$$

where C and D_3 are independent of k.

By Lemma 1.2 we have for $j = 0,1$

$$(1.10) \qquad \int_\Omega \Phi^2_{-1+\beta} |u_{x_1^{1-j} x_2^j}|^2 dx \le C \|u\|^2_{H^{2,2}_\beta(\Omega)} \le \tilde{C}.$$

Hence (1.9) holds for all $k \ge \ell$. $\ell \le 2$ and (1.5) holds for $\alpha_1' = 0$. Combining the arguments we have used above we get (1.5) in the full generality.

2) We will show now that if (1.5) holds for every i, then (1.4) holds too. First we will show that for any $k \ge 1$

$$(1.11a) \qquad u_{x_1^k} = \sum_{m-1}^{k} \sum_{j=0}^{m} \sum_{\substack{\ell_1,\ell_2 \ge 0 \\ \ell_1+\ell_2=k}} b^{(k)}_{m,j,\ell_1,\ell_2} \sin^{\ell_1}\theta \cos^{\ell_2}\theta \, r^{-(k-m+j)} u_{r^{m-j}\theta^j}$$

$$(1.11b) \qquad B^{(k)}_m = \sum_{j=0} \sum_{\substack{\ell_1,\ell_2 \ge 0 \\ \ell_1+\ell_2=k}} |b^{(k)}_{m,j,\ell_1,\ell_2}| \le 4^k \frac{k!}{m!}.$$

It is easy to check that (1.11) holds for $k = 0,1$. Analogously as in the first part we get

$$b^{(k)}_{m,j,\ell_1,\ell_2} = b^{(k-1)}_{m-1,j,\ell_1,\ell_2-1} - b^{(k-1)}_{m-1,j-1,\ell_1-1,\ell_2}$$

$$- \ell_1 b^{(k-1)}_{m,j,\ell_1,\ell_2-1} + \ell_2 b^{(k-1)}_{m,j,\ell_1-1,\ell_2} - (k-m+j) b^{(k-1)}_{m,j,\ell_1,\ell_2-1}$$

and hence

$$B^{(k)}_m \le 2B^{(k-1)}_{m-1} + kB^{(k-1)}_m + kB^{(k)}_m \le 2B^{(k-1)}_{m-1} + 2kB^{(k-1)}_m.$$

Using the induction hypothesis we get (1.11b). Using (1.11) we get for $k \ge \ell$ and $\ell \ge 1$

$$(1.12)\qquad \left(\int |u_{x_1^k}|^2 \Phi^2_{k-\ell+\beta}dx\right)^{1/2} \le CD^{(k-\ell)} \sum_{m=1}^{k} B_m^{(k)} d^{m-\ell}(m-\ell)!$$

$$\le CD_2^{(k-\ell)}(k-\ell)!$$

where D_2 and C are independent of k. For $j = 0,1$ we have by Lemma 1.2

$$\int_\Omega \Phi^2_{-1+\beta} r_i^{-2j} |u_{r_i^{1-j}\theta_i^j}|^2 r_i dr_i d\theta_i \le C\|u\|^2_{H^{2,2}_\beta(\Omega)} \le \tilde{C}.$$

Hence (1.12) holds for $0 \le \ell \le 2$, $k \ge \ell$ and (1.4) holds for $\alpha_2 = 0$. The general case can be proven quite analogously. ∎

Theorem 1.1 yields an equivalent definition of $B^\ell_\beta(\Omega)$, $0 \le \ell \le 2$

$$(1.13)\qquad B^\ell_\beta(\Omega) = \{u \in \psi^\ell_\beta(\Omega) | (\int r_i^{-2\alpha_2} \Phi^2_{k-\ell+\beta} |\mathcal{D}^\alpha_i u|^2 r_i dr_i d\theta_i)^{1/2}$$

$$\le Cd^{k-\ell}(k-\ell)! \text{ for any } k \ge \ell \text{ and}$$

$$|\alpha| = k;\ C \text{ and } d \text{ independent of } k,\ i = ,\dots,M\}$$

where (r_i,θ_i) are polar coordinates with the origin in A_i and $\mathcal{D}^\alpha_i u = u_{r_i^{\alpha_1}\theta_i^{\alpha_2}}$. In what will follow both definitions will be used interchangeably.

1.4. THE SPACES $H^{m-1/2}(\gamma)$ AND $H_\beta^{m-1/2,\ell-1/2}(\gamma)$.

Let $Q \subset \mathbb{R}^2$ be an open, bounded set with piecewise analytic boundary ∂Q and let γ be part or the whole boundary ∂Q. We define $H^{m-1/2}(\gamma)$, $m \ge 1$ as the set of all functions φ on γ such that there exists $f \in H^m(\gamma)$, with $\varphi = f|_\gamma$. The norm is defined by

$$\|\varphi\|_{H^{m-1/2}(\gamma)} = \inf \|f\|_{H^m(Q)}$$

where the infimum is taken over all functions $f \in H^m(Q)$ with $f = \varphi$ on γ.

Suppose that $A_i \in \partial Q$ or $A_i \notin \bar{Q}$, $i = 1,2,\dots,M$, then we define the spaces $H_\beta^{m,\ell}(Q)$, $\ell \geq 0$ as in Section 1.2. Let $H_\beta^{m,1/2,\ell+1/2}(\gamma)$, $m \geq 1$, $\ell \geq 0$ be the set of all functions φ on γ such that there exists $f \in H_\beta^{m,\ell}(Q)$ with $\varphi = f|_\gamma$ and

$$\|\varphi\|_{H_\beta^{m-1/2,\ell-1/2}(\gamma)} = \inf \|f\|_{H_\beta^{m,\ell}(Q)}.$$

The infimum is taken over all functions $f \in H_\beta^{m,\ell}(Q)$ such that $f|_\gamma = \varphi$.

By $L_2(\gamma)$ we denote the space of the square integrable functions on γ. We also define the space $B_\beta^\ell(Q)$, $0 \leq \ell \leq 2$ analogously as in (1.2) replacing Ω by Q. Finally $B_\beta^{\ell-1/2}(\gamma)$, $0 \leq \ell \leq 2$, be the space of all functions φ for which there exists $f \in B_\beta^\ell(Q)$ such that $f = \varphi$ on γ.

Remark 1. Although $B_\beta^\ell(Q)$, $0 \leq \ell \leq 1$ is not a subspace of $H^1(Q)$ the trace of $f \in B_\beta^\ell(Q)$ on γ obviously exists.

Remark 2. The norms $\|\cdot\|_{H^{m-1/2}(\gamma)}$ and $\|\cdot\|_{H^{m-1/2,\ell-1/2}(\gamma)}$ obviously depend on Q.

Remark 3. In what will follow Q will often be the polygonal domain and γ some of its edges. Although the set $H_\beta^{m-1/2,\ell-1/2}(\gamma)$ is characterized only by β_i associated to the vertices of the edges γ, we are defining the space $H_\beta^{m-1/2,\ell-1/2}(\gamma)$ in dependence on $\beta = (\beta_1,\ldots,\beta_M)$.

2. REGULARITY OF THE SOLUTION OF THE POISSON PROBLEM ON A POLYGONAL DOMAIN

In this chapter we will discuss the regularity of the problem

$$(2.1)\qquad \begin{aligned} -\Delta u &= f \quad \text{on} \quad \Omega \\ u &= g^0 \quad \text{on} \quad \Gamma^0 \\ \frac{\partial u}{\partial n} &= g^1 \quad \text{on} \quad \Gamma^1 \end{aligned}$$

where

$$\Gamma^0 = \bigcup_{i \in \mathcal{D}} \bar{\Gamma}_i, \qquad \Gamma^1 = \Gamma - \Gamma^0.$$

Γ^0 will be called the Dirichlet boundary, Γ^1 the Neumann boundary. If $\Gamma^0 = \Gamma$ (respectively $\Gamma^1 = \Gamma$), then we will speak about *Dirichlet* (respectively Neumann) problem. If $\Gamma^0 \neq \Gamma$ and $\Gamma^1 \neq \Gamma$, then we will speak about the mixed problem. The main theorem of this chapter is:

Theorem 2.1. Let $f \in B^0_\beta(\Omega)$, $g^i \in B_\beta^{\frac{3}{2}-i}(\Gamma^i)$, $i = 0,1$, $\beta = (\beta_1,\ldots,\beta_M)$, $0 < \beta_i < 1$, $\beta_i > 1 - \frac{\pi}{\omega_i}$ *(respectively* $\beta_i > 1 - \frac{\pi}{2\omega_i}$ if Dirichlet or Neumann boundary conditions are imposed on the edges Γ_i, Γ_{i+1}, $\bar{\Gamma}_i \cap \bar{\Gamma}_{i+1} = A_i$) and let $\Gamma^0 \neq \emptyset$. Then the problem (2.1) has a unique solution in $H^1(\Omega)$ and $u \in B^2_\beta(\Omega)$. ∎

Remark 1. *If $\Gamma^0 = \emptyset$ then the theorem still holds provided that* f and g satisfy the condition (2.38) and the uniqueness is understood modulo constant function.

Remark 2. g^1 should be understood as the vector $g^1 = (g^1_1, g^1_2, \cdots, g^1_p)$;

p is an integer $\le M$ such that $g^1_\ell = G^1_\ell\big|_{\Gamma_{i_\ell}}$, $\bigcup_{\ell=1}^{p} \Gamma_{i_\ell} = \Gamma^1$, $G^1_\ell \in B^1_\beta(\Omega)$, and $\|G^1\|^2_{H^{k,1}_\beta(\Omega)} = \sum_{\ell=1}^{p} \|G^1_\ell\|^2_{H^{k,1}_\beta(\Omega)}$.

Remark 3. It can be seen from the proof of the theorem that if $f \in H^{k,0}_\beta(\Omega)$, $G^j \in H^{k+2-j,2}_\beta(\Omega)$, $j = 0,1$, $\beta_i > 1 - \frac{\pi}{\omega_i}$, (respectively $\beta_i > 1 - \frac{\pi}{2\omega_i}$) and $k \ge 2$, then the solution of (2.1) exists in $H^{k+2,2}_\beta(\Omega)$ and

$$\|u\|_{H^{k+2,2}_\beta(\Omega)} \le C(k)\Big(\|f\|_{H^{k,0}_\beta(\Omega)} + \sum_{j=0,1} \|G^j\|_{H^{k+2-j,2}_\beta(\Omega)}\Big)$$

which is a kind of the "shift" theorem. Usually the shift theorem is expressed in the terms of usual Sobolev spaces so that

$$u = w + \sum_{i=1}^{m(k)} C_i\varphi_i$$

with φ_i are singular functions and for w there is the same shift theorem as for the domain with smooth boundary and without specific estimates of various constants in dependence on k. Theorem 2.1 is in the same way related to the known results but the authors were unable to find the theorem characterizing the solution in the framework of the countable normed space $B^2_\beta(\Omega)$ which is essential for applications.

Remark 4. The proof of the theorem utilizes simple expansions of the solution, although this reasoning is very special. This approach is used to illuminate the main idea which will be used in Part 2 in an abstract form without using explicitly the mentioned expansion argument.

2.1. AUXILIARY PROBLEMS ON THE CONE AND THE STRIP

Let

$$Q = S_\infty^\omega = \{r,\theta | 0 < r < \infty, \quad 0 < \theta < \omega\}$$

$$\Gamma_1 = \{r,\theta \mid 0 < r < \infty, \ \theta = 0\}$$

$$\Gamma_2 = \{r,\theta \mid 0 < r < \infty, \quad \theta = \omega\}$$

and

$$D = \{\tau,\theta \mid -\infty < \tau < \infty, \quad 0 < \theta < \omega\}$$

$$\tilde{\Gamma}_1 = \{\tau,\theta \mid -\infty < \tau < \infty, \quad \theta = 0\}$$

$$\tilde{\Gamma}_2 = \{\tau,\theta \mid -\infty < \tau < \infty, \quad \theta = \omega\}$$

The spaces $H_\beta^{k,\ell}(Q)$, $\ell \leq 2$ and $H_h^k(D)$ be defined as in Section 1.2. Let $C_\#^\infty(Q)$ be the collection of infinitely differentiable functions on $\bar{Q}$ such that:

> for any $u \in C_\#^\infty(Q)$ there exists a positive number $A = A(u)$ such that u vanishes on $Q - Q_A$ where $Q_A = \{(r,\theta) \mid \frac{1}{A} < r < A, \quad 0 < \theta < \omega\}$.

Analogously we denote by $C_\#^\infty(D)$ the collection of infinitely differentiable functions such that for any $u \in C_\#^\infty(D)$ there exists $A = A(u) > 0$ such that u vanishes on $D - D_A$ where $D_A = \{(\tau,\theta) \mid -A < \tau < A, \quad 0 < \theta < \omega\}$. It is not difficult to show (see [12]):

Lemma 2.1. $C_\#^\infty(Q)$ (respectively $C_\#^\infty(D)$) is dense in $H_\beta^{k,\ell}(Q)$ (respectively $H_h^k(D)$), $k \geq \ell \geq 0$. ■

Lemma 2.2. The spaces $H^{k,\ell}_{\beta}(Q)$ and $H^{k}_{h}(D)$ are complete.

Consider now the following problem on Q,

(2.2) $$-\Delta u = -\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) = f \quad \text{on} \quad Q,$$

$$u|_{\theta=0} = g^0 = G^0|_{\theta=0},$$

$$\left.\frac{\partial u}{\partial n}\right|_{\theta=\omega} = g^1 = G^1|_{\theta=\omega}$$

where g^0 and g^1 are the traces of functions G^0 and G^1 defined on Q. By the change of the variable

$$\tau = \ln\frac{1}{r}$$

we transform the problem (2.2) into the problem on D

(2.3a) $$-\left(\frac{\partial^2 \tilde{u}}{\partial \tau^2} + \frac{\partial^2 \tilde{u}}{\partial \theta^2}\right) = \tilde{f}(\tau,\theta)$$

(2.3b) $$\tilde{u}|_{\theta=0} = \tilde{g}^0 = \tilde{G}^0|_{\theta=0}$$

$$\left.\frac{\partial \tilde{u}}{\partial \theta}\right|_{\theta=\omega} = \tilde{g}^1 = \tilde{G}^1\Big|_{\theta=\omega}$$

where

$$\tilde{u}(\tau,\theta) = u(e^{-\tau},\theta), \quad \tilde{f}(\tau,\theta) = e^{-2\tau}f(e^{-\tau},\theta)$$

$$\tilde{G}^{\ell}(\tau,\theta) = e^{-\ell\tau}G^{\ell}(e^{-\tau},\theta), \qquad \ell = 0,1.$$

Lemma 2.3. Let $\tilde{f} \in L_h(D)$, $\tilde{G}^i \in H_h^{2-i}(D)$, $i = 0,1$, $0 < h < \frac{\pi}{2\omega}$, then the solution $\tilde{u}$ of (2.3) exists in $H_h^2(D)$, is unique and for $0 \le |\alpha| \le 2$:

$$(2.4) \qquad \|D^\alpha \tilde{u}\|^2_{L_h(D)} = \int_D e^{2h\tau} |D^\alpha \tilde{u}|^2 d\tau\, d\theta$$

$$\le C[\|\tilde{f}\|^2_{L_h(D)} + \sum_{i=0}^{1} \|\tilde{G}^i\|^2_{H_h^{2-i}(D)}],$$

where

$$D^\alpha \tilde{u} = \frac{\partial^{|\alpha|} \tilde{u}}{\partial \tau^{\alpha_1} \partial \theta^{\alpha_2}}$$

and C is independent of $\tilde{f}$ and $\tilde{G}^i$.

<u>Proof</u>. Because of Lemma 2.1 and 2.2, we may assume that $\tilde{f}, \tilde{G}^i \in C_{\#}^\infty(D)$. Denote by $\hat{f}(\lambda,\theta) = F(\tilde{f}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\lambda\tau} f(\tau,\theta) d\tau$, $\hat{G}^i(\lambda,\theta) = F(\tilde{G}^i)$ the Fourier transform (in τ) of $\tilde{f}$ and $\tilde{G}^i$.

Because $\tilde{f}$, $\tilde{G}^i \in C_{\#}^\infty(D)$ the Fourier transform exists for all λ. By the basic properties of the Fourier transform we get with $\lambda = \xi + ih$, $-\infty < \xi < \infty$:

$$-\frac{\partial \hat{u}}{\partial \theta^2}(\lambda,\theta) + \lambda^2 \hat{u}(\lambda,\theta) = \hat{f}(\lambda,\theta) \qquad \text{for } \theta \in I = (0,\omega),$$

$$(2.5) \qquad \hat{u}(\lambda,\theta)|_{\theta=0} = \hat{g}^0 = \hat{G}^0(\lambda,\theta)|_{\theta=0},$$

$$\frac{\partial \hat{u}}{\partial \theta}(\lambda,\theta)|_{\theta=0} = \hat{g}^1 = \hat{G}^1(\lambda,\theta)|_{\theta=\omega}.$$

The ordinary (homogeneous) boundary value problem for the equation

$$-\hat{u}'' + \lambda^2\hat{u} = 0$$

$$\hat{u}\big|_{\theta=0} = \frac{\partial\hat{u}}{\partial\theta}\Big|_{\theta=\omega} = 0$$

has the eigenvalues $\lambda_k = i\frac{\pi}{\omega}(k-\frac{1}{2})$, $k = 1,2,\ldots$ and corresponding eigenfunctions $u_k = \sin\frac{\pi}{\omega}(k-\frac{1}{2})\theta$. Hence for $0 < h < \frac{\pi}{2\omega}$ (2.5) has always unique solution and by [3] [12] (formula 1.14)

$$\|\hat{u}\|^2_{H^2(I)} + |\lambda|^4\|\hat{u}\|^2_{L_2(I)} \tag{2.6}$$

$$\leq C[\|\hat{f}\|^2_{L_2(I)} + \|\hat{G}^0\|^2_{H^2(I)} + \|\hat{G}^1\|^2_{H^1(I)}$$

$$+ |\lambda|^3|\hat{G}^0(\lambda,0)|^2 + |\lambda||\hat{G}^1(\lambda,\omega)|^2].$$

It follows from the basic property of the Fourier transform that for any integer $\leq k$, $s \leq k$ and any F in the set of admissible functions

$$\int_0^{\Omega}\int_{-\infty}^{+\infty}\left|\frac{\partial^k F(\tau,\theta)}{\partial\tau^s\partial\theta^{k-s}}\right|^2 C^{2h\tau}d\tau\, d\theta \tag{2.7}$$

$$= \int_0^{\omega}\left(\int_{-\infty}^{\infty}\left|e^{h\tau}\frac{\partial^k F(\tau,\theta)}{\partial\tau^s\partial\theta^{k-s}}\right|^2 d\tau\right)d\theta$$

$$= \int_0^{\omega}\int_{-\infty+ih}^{\infty+ih}|\lambda|^{2s}\left|\frac{\partial^{k-2}\hat{F}(\lambda,\theta)}{\partial\theta^{k-2}}\right|^2 d\lambda\, d\theta.$$

Hence for $\tilde{u} = F^{-1}(\hat{u})$ we get

$$(2.8)\qquad \int_D e^{2h\tau}\left|\frac{\partial^2\tilde{u}}{\partial\theta^2}\right|^2 d\tau\, d\theta = \int_{-\infty+ih}^{\infty+ih} \left\|\frac{\partial^2\hat{u}}{\partial\theta^2}\right\|^2_{L_2(I)} d\lambda$$

$$(2.9)\qquad \int_D e^{2h\tau}\left|\frac{\partial^2\tilde{u}}{\partial\tau^2}\right|^2 d\tau\, d\theta = \int_{-\infty+ih}^{\infty+ih} |\lambda|^4 \|\hat{u}\|^2_{L_2(I)} d\lambda.$$

By the interpolation space theorem [8]

$$|\lambda|^2\left\|\frac{\partial\hat{u}}{\partial\theta}\right\|^2_{L_2(I)} \le |\lambda|^2\|\hat{u}\|^2_{H^1(I)} \le C|\lambda|^2\|\hat{u}\|_{L_2(I)}\|\hat{u}\|_{H^2(I)}$$

$$\le C|\lambda|^2\|\hat{u}\|_{L_2(I)}\left(\|\hat{u}\|_{L_2(I)} + \left\|\frac{\partial\hat{u}}{\partial\theta}\right\|_{L_2(I)} + \left\|\frac{\partial^2\hat{u}}{\partial\theta^2}\right\|_{L_2(I)}\right)$$

$$\le C\Big(\big((1+\tfrac{C}{2})|\lambda|^1 + \tfrac{1}{2}|\lambda|^4\big)\|\hat{u}\|^2$$

$$+ \frac{1}{2C}|\lambda|^2\left\|\frac{\partial\hat{u}}{\partial\theta}\right\|^2_{L_2(I)} + \frac{1}{2}\left\|\frac{\partial^2\hat{u}}{\partial\theta^2}\right\|^2_{L_2(I)}\Big)$$

$$\le C_1\left(|\lambda|^4\|\hat{u}\|^2_{L_2(I)} + \left\|\frac{\partial^2\hat{u}}{\partial\theta^2}\right\|^2_{L_2(I)}\right)$$

where C_1 depends on h but not on λ and $\hat{u}$. Hence

$$(2.10)\qquad \int_D e^{2h\tau}\left|\frac{\partial^2\tilde{u}}{\partial\tau\partial\theta}\right|^2 d\tau\, d\theta = \int_{-\infty+ih}^{\infty+ih} |\lambda|^2 \left\|\frac{\partial\hat{u}}{\partial\theta}\right\|^2_{L_2(I)} d\lambda$$

$$\le C_1 \int_{-\infty+ih}^{\infty+ih}\left(|\lambda|^4\|\hat{u}\|^2_{L_2(I)} + \left\|\frac{\partial^2\hat{u}}{\partial\theta^2}\right\|^2_{L_2(I)}\right)d\lambda.$$

We have also

$$(2.11)\qquad \int_{-\infty+ih}^{\infty+ih}\left(\|\hat{f}\|^2_{L_2(I)} + \|\hat{G}^0\|^2_{H^2(I)} + \|\hat{G}^1\|^2_{H^1(I)}\right)d\lambda$$

$$= \int_D e^{2h\tau}(|\tilde{f}|^2 + \sum_{\ell=0}^{2} |\frac{\partial^\ell \tilde{G}^0}{\partial\theta^\ell}|^2 + \sum_{\ell=0}^{1} |\frac{\partial^\ell \tilde{G}^1}{\partial\theta^\ell}|^2) d\tau\, d\theta$$

(2.12) $$\int_{-\infty+ih}^{\infty+ih} |\lambda|^3 |\hat{G}^0(\lambda,0)|^2 d\lambda \leq C\|e^{h\tau}\tilde{G}^0(\lambda,0)\|^2_{H^{3/2}(\mathbf{R}_1)}$$

$$\leq C\|e^{h\tau}\tilde{G}^0(\lambda,\theta)\|^2_{H^2(D)}$$

(2.13) $$\int_{-\infty+ih}^{\infty+ih} |\lambda||\hat{G}^1(\lambda,\omega)|^2 d\lambda \leq \|e^{h\tau}\tilde{G}^1(\lambda,\omega)\|^2_{H^{1/2}(\mathbf{R}_1)}$$

$$\leq C\|e^{h\tau}\tilde{G}^1(\lambda,\theta)\|^2_{H^1(D)}.$$

Hence from (2.6) using (2.8)-(2.13) we get for $|\alpha| = 2$:

(2.14) $$\int_D e^{2h\tau}|D^\alpha \tilde{u}|^2_{L_h(D)} + \sum_{i=0}^{1} \|\tilde{G}^i\|^2_{H_h^{2-i}(D)}].$$

For $\ell = 0,1$ we have

(2.15) $$\int_D e^{2h\tau}|\frac{\partial^\ell \hat{u}}{\partial\tau^\ell}|^2 d\tau\, d\theta = \int_{-\infty+ih}^{\infty+ih} |\lambda|^{2\ell}\|\hat{u}\|^2_{L^2(I)} d\lambda$$

$$\leq h^{-2\ell} \int_{-\infty+ih}^{\infty+ih} |\lambda|^4 \|\hat{u}\|^2_{L_2(I)} d\lambda$$

$$\leq Ch^{-2\ell}(\|\tilde{f}\|^2_{L_h(D)} + \sum_{i=0}^{1} \|\tilde{G}^i\|_{H_h^{2-i}(D)})$$

(2.16) $$\int_D e^{2h\tau}|\frac{\partial\tilde{u}}{\partial\theta}|^2 d\tau\, s\theta = \int_{\infty-ih}^{\infty+ih} |\hat{u}|^2_{H^1(I)} d\lambda$$

$$\leq h^{-2} \int_{-\infty+ih}^{\infty+ih} |\lambda|^2 |\hat{u}|^2_{H^1(I)} \, d\lambda$$

$$\leq Ch^{-2}\left(\|\tilde{f}\|^2_{L_h(D)} + \sum_{i=0}^{1} \|\tilde{G}^i\|^2_{H_h^{2-i}(D)}\right)$$

and (2.4) is proven.

2) Let $\tilde{u} \in H_h^2(D)$ and

$$\Delta\tilde{u} = 0 \text{ in } D$$

$$\tilde{u}|_{\theta=0} = \frac{\partial\tilde{u}}{\partial\theta}\bigg|_{\theta=\omega} = 0.$$

Hence we can write

$$\tilde{u}(\tau,\theta) = \sum_{j=1}^{\infty} a_j(\tau) \sin \frac{\pi\theta}{\omega} (j-\tfrac{1}{2}) \tag{2.17}$$

$$a_j(\tau) = \frac{2}{\omega} \int_0^{\omega} \tilde{u}(\tau,\theta) \sin \frac{\pi\theta}{\omega} (j-\tfrac{1}{2}) d\theta \tag{2.18}$$

$$a_j''(\tau) = \frac{2}{\omega} \int_0^{\omega} \frac{\partial^2\tilde{u}(\tau,\theta)}{\partial\tau^2} \sin \frac{\pi\theta}{\omega} (j-\tfrac{1}{2}) d\theta$$

$$= \left(\frac{\pi}{\omega} (j-\tfrac{1}{2})\right)^2 a_j(\tau).$$

Therefore for each j, $j = 1,2,\ldots,a_j(\tau)$ satisfies

$$a_j''(\tau) - \left(\frac{\pi}{\omega} (j-\tfrac{1}{2})\right)^2 a_j(\tau) = 0$$

and

$$a_j(\tau) = c_j e^{-\frac{\pi}{\omega}(j-\frac{1}{2})\tau} + d_j e^{\frac{\pi}{\omega}(j-\frac{1}{2})\tau}. \tag{2.19}$$

Let now $A > 0$ arbitrary, then

$$\infty > \int_{D_A} e^{2h\tau}|\tilde{u}|^2\, d\tau\, d\theta$$

$$= \int_{-A}^{A} \left(\int_0^{\omega} \left(\sum_{j=1}^{\infty} a_j(\tau) \sin \frac{\pi\theta}{\omega}(j-\tfrac{1}{2})\right)^2 d\theta\right) e^{2h\tau}\, d\tau$$

$$= \int_{-A}^{+A} \frac{\omega}{2} \sum_{j=1}^{\infty} |a_j(\tau)|^2\, e^{2h\tau}\, d\tau$$

$$= \frac{\omega}{2} \sum_{j=1}^{\infty} \int_{-A}^{+A} |a_j(\tau)|^2\, e^{2h\tau} d\tau.$$

For $j = 1$

$$|a_1(\tau)|^2 = c_1^2 e^{-\frac{\pi}{\omega}\tau} + d_1^2 e^{\frac{\pi}{\omega}\tau} + 2c_1 d_1$$

and hence

$$\int_{-A}^{A} |a_1(\tau)|^2 e^{2h\tau} d\tau = 2c_1 d_1 \frac{1}{2h}\left(e^{2hA} - e^{-2hA}\right)$$

$$+ \frac{c_1^2}{(2h-\frac{\pi}{\omega})}\left(e^{(2h-\frac{\pi}{\omega})A} - e^{-(2h-\frac{\pi}{\omega})A}\right)$$

$$+ \frac{d_1^2}{(2h+\frac{\pi}{\omega})}\left(e^{(2h+\frac{\pi}{\omega})A} - e^{-(2h+\frac{\pi}{\omega})A}\right)$$

and

$$\infty > \lim_{A\to\infty} \int_{-A}^{+A} |a_1(\tau)|^2 e^{2h\tau} d\tau$$

$$= \lim_{A\to\infty} \left(\frac{c_1 d_1}{h} e^{2hA} + \frac{c_1^2}{(\frac{\pi}{\omega} - 2h)} e^{|(\frac{\pi}{\omega} - 2h)|A} \right.$$

$$\left. + \frac{d_1^2}{(2h + \frac{\pi}{\omega})} e^{(2h - \frac{\pi}{\omega})A} \right).$$

Hence $c_1, d_1 = 0$. Similarly we get $c_j = d_j = 0$ for all $1 \leq j < \infty$. ∎

We have not explicitly used in the proof of Lemma 2.3 the assumption $h < \frac{\pi}{2\omega}$. We have used only $h > 0$ and $h \neq \frac{\pi}{2\omega}(k-\frac{1}{2})$, $k = 1,2,\ldots.$ Assumption $h < \frac{\pi}{2\omega}$ is essential in the next lemma.

Lemma 2.4. Let the assumptions of Lemma 2.3 hold. Let in addition $\tilde{f}(\tau,\theta) = 0$, $\tilde{G}^i(\tau,\theta) = 0$ for $\tau < 0$. Then for $\varepsilon > 0$ and $0 \leq \Upsilon = h + \frac{\pi}{2\omega} - \varepsilon$, $\tilde{D} = \{\tau,\theta \mid -\infty < \tau < 0,\ 0 < \theta < \omega\}$ one has

$$\int_{\tilde{D}} |D^\alpha \tilde{u}|^2 e^{2(h-\Upsilon)\tau} d\tau\, d\theta \tag{2.20}$$

$$\leq C(\varepsilon) \int_{\tilde{D}} |D^\alpha \tilde{u}|^2 e^{2h\tau} d\tau\, d\theta, \quad |\alpha| \leq 2.$$

<u>Proof</u>. Equations (2.17) (2.18) (2.19) hold, and

$$\infty > \int_{\tilde{D}} |\tilde{u}|^2 e^{2h\tau} d\tau\, d\theta = \frac{\omega}{2} \int_{-\infty}^{0} \sum_{j=1}^{\infty} |a_j(\tau)|^2 e^{2h\tau}\, d\tau.$$

Hence for $A > 0$ arbitrary

$$\int_{-A}^{0} |a_j(\tau)|^2 e^{2h\tau}\, d\tau = 2c_j d_j (1 - e^{-2hA})$$

$$+ \frac{c_j^2}{2(h-\frac{\pi}{\omega}(j-\frac{1}{2}))}(1-e^{-2(h-\frac{\pi}{\omega}(j-\frac{1}{2}))A})$$

$$+ \frac{d_j^2}{2(h+\frac{\pi}{\omega}(j-\frac{1}{2}))}(1-e^{-2(h+\frac{\pi}{\omega}(j-\frac{1}{2})A)}).$$

Since $0 < h < \frac{\pi}{2\omega}$ and $A > 0$ is arbitrary we have $c_j = 0$, $j = 1,2,\ldots$ and

$$\int_{-A}^{0} |a_j(\tau)|^2 e^{2h\tau} d\tau = \frac{d_j^2}{2(h+\frac{\pi}{\omega}(j-\frac{1}{2}))}(1-e^{-2(h+\frac{\pi}{\omega}(j-\frac{1}{2}))A})$$

$$\int_{-A}^{0} |a_j(\tau)|^2 e^{2(h-\gamma)\tau} d\tau = \frac{d_j^2}{2(h-\gamma+\frac{\pi}{\omega}(j-\frac{1}{2}))}(1-e^{-2(h-\gamma+\frac{\pi}{\omega}(j-\frac{1}{2}))A}).$$

If $0 \le \gamma = h + \frac{\pi}{2\omega} - \varepsilon$, $\varepsilon > 0$ then

$$\int_{\tilde{D}} |\tilde{u}(\tau,\theta)|^2 e^{2(h-\gamma)\tau} d\tau\, d\theta = \int_{\tilde{D}} |\tilde{u}(\tau,\theta)|^2 e^{2(-\frac{\pi}{2\omega}+\varepsilon)\tau} d\tau\, d\theta$$

$$= \frac{\omega}{2} \lim_{A\to\infty} \sum_{j=1}^{\infty} \frac{d_j^2(1-e^{-2(\frac{\pi}{\omega}(j-1)+\varepsilon)A})}{2(\varepsilon+\frac{\pi}{\omega}(j-1))} \le C\frac{\omega}{2} \sum_{j=1}^{\infty} \frac{d_j^2}{2(h+\frac{\pi}{\omega}(j-\frac{1}{2}))}$$

$$= C \int_{\tilde{D}} |\tilde{u}|^2 e^{2h\tau} d\tau\, d\theta.$$

Similarly we have for $|\alpha| \le 2$

$$\int_{\tilde{D}} |D^\alpha \tilde{u}|^2 e^{2(h-\gamma)\tau} d\tau\, d\theta \le C \int_{\tilde{D}} |D^\alpha \tilde{u}|^2 e^{2h\tau}\, d\tau\, d\theta.$$ ■

Lemma 2.3 and 2.4 address the regularity of the problem (2.3) when on Γ_1, respectively Γ_2, the Dirichlet with respect to the Neumann condition has been given. The same statement holds if on Γ_1 and Γ_2 the Dirichlet or Neumann condition are given.

Lemma 2.5. Let $\tilde{f} \in L_h(D)$, $\tilde{G}^0 \in H_h^2(D)$ (respectively $\tilde{G}^1 \in H_h^1(D)$), $0 < h < \frac{\pi}{\omega}$. Then the Dirichlet (respectively Neumann) problem

$$-\left(\frac{\partial^2 \tilde{u}}{\partial \tau^2} + \frac{\partial^2 u}{\partial \theta^2}\right) = \tilde{f}(\tau,\theta) \tag{2.21a}$$

$$\tilde{u}\Big|_{\substack{\theta=0\\ \theta=\omega}} = \tilde{g}^0 = \tilde{G}^0\Big|_{\substack{\theta=0\\ \theta=\omega}} \tag{2.21b}$$

(respectively)

$$\frac{\partial \tilde{u}}{\partial n}\Big|_{\substack{\theta=0\\ \theta=\omega}} = \tilde{g}^1 = \tilde{G}^1\Big|_{\substack{\theta=0\\ \theta=\omega}}, \tag{2.21c}$$

has unique solution in $H_h^2(D)$ and for $0 \le |\alpha| \le 2$

$$\|D^\alpha \tilde{u}\|^2_{L_h(D)} \le C[\|\tilde{f}\|^2_{L_h(D)} + \|\tilde{G}^0\|^2_{H_h^2(D)} \text{ (respectively } \|\tilde{G}^1\|^2_{H_h^1(D)})] \tag{2.22}$$

If in addition $\tilde{f} = 0$, $\tilde{G}^0$ (respectively $\tilde{G}^1$) $= 0$ for $\tau < 0$ then for $0 \le \gamma = h + \frac{\pi}{\omega} - \varepsilon$, $\varepsilon > 0$, $0 \le |\alpha| \le 2$

$$\int_{\tilde{D}} |D^\alpha \tilde{u}^*|^2 e^{2(h-\gamma)\tau} d\tau\, d\theta \le C \int_{\tilde{D}} |D^\alpha \tilde{u}|^2 e^{2h\tau} d\tau\, s\theta. \tag{2.23}$$

where $\tilde{u}^*(\tau,\theta) = \tilde{u}(\tau,\theta)$ for the Dirichlet problem and

$$\tilde{u}^*(\tau,\theta) = \tilde{u}(\tau,\theta) - \frac{1}{\omega}\int_0^\omega \tilde{u}(\tau,\theta)d\theta$$

for the Neumann problem.

The proof is quite the same. In the case of Neumann conditions it is enough to realize that the summation in (2.17) if for $j = 1,2,\ldots.$

Lemma 2.6. Let $f \in L_\beta(Q)$, $G^i \in H_\beta^{2-i,2-i}(Q)$, $i = 0,1$, $0 < \beta < 1$, $\beta > 1 - \frac{\pi}{2\omega}$ and let $f = G^i = 0$ for $r \geq 1$, then the mixed problem

(2.24a) $$-\Delta u = f(r,\theta)$$

$$u|_{\theta=0} = g^0 = G^0|_{\Gamma_1}$$

(2.24b) $$\left.\frac{\partial u}{\partial n}\right|_{\theta=0} = \left.\frac{1}{r}\frac{\partial u}{\partial \theta}\right|_{\theta=\omega} = g^1 = G^1|_\Gamma$$

has a solution such that

i) $(u - G^0) \in W_\beta^2(Q)$, $u \in H_\beta^{2,2}(S_1)$

ii) $\|\mathcal{D}^1 u\|_{H^0(Q)} < \infty$

iii) There exists a constant C independent of u, f, G^i such that for $|\alpha| \leq 2$

(2.25) $$\|u\|^2_{H_\beta^{2,2}(S_1)} \leq C[\|F\|^2_{L_\beta(Q)} + \sum_{i=0}^{1} \|G^i\|_{H_\beta^{2-i,2-i}(Q)}]$$

where $Q = \{r,\theta \mid 0 < r < \infty,\ 0 < \theta < \omega\}$

and $S_1 = \{r,\theta \mid 0 < r < 1,\ 0 < \theta < \omega\}$.

<u>Proof</u>. Assume first that $G^0 = 0$. Let $0 < h = 1 - \beta < \frac{\pi}{2\omega}$ and $\tau = \ln \frac{1}{r}$. Then we have for $\tilde{f}(\tau,\theta) = e^{2\tau} f(e^{-\tau},\theta)$ and $\tilde{G}^1(\tau,\theta) = e^{-\tau} G^1(e^{-\tau},\theta)$

(2.26a)
$$\int_D e^{2h\tau} |\tilde{f}(\tau,\theta)|^2 d\tau\, d\theta$$

$$= \int_D e^{-2(2-h)\tau} |f(e^{-\tau},\theta)|^2 d\tau\, d\theta$$

$$= \int_Q r^{2\beta} |f(r,\theta)|^2 r\, dr\, d\theta$$

$$= \|f\|^2_{L_\beta(Q)}$$

(2.26b)
$$\int_D e^{2h\tau} |\tilde{G}^1(\tau,\theta)|^2 d\tau\, d\theta$$

$$= \int_Q r^{2h} |G^1(r,\theta)|^2 r\, dr\, d\theta$$

$$= \int_{S_1 \cup Q_2} r^{2(\beta-1)} |G^1(r,\theta)|^2 r\, dr\, d\theta$$

where we denoted

$$Q_2 = \{(r,\theta) \mid 1 \le r < \infty, \ 0 < \theta < \omega\}.$$

By Lemma A2 (see Appendix)

$$\int_{S_1} r^{2(\beta-1)} |G^1|^2 r\, dr\, d\theta$$

$$\le C[\sum_{|\alpha|=1} \int_{S_1} r^{2(\beta+\alpha_1-1)} |D^\alpha G^1|^2 r\, dr\, d\theta$$

$$+ \int_{S_1} r^{2\beta} |G^1|^2 r \, dr \, d\theta]$$

and

$$\int_{Q_2} r^{2(\beta-1)} |G^1|^2 r \, dr \, d\theta$$

$$\leq \int_{Q_2} |G^1|^2 r \, dr \, d\theta.$$

Hence

$$\|\tilde{G}\|_{L_h(D)} \leq C\|G^1\|_{H^{1,1}_\beta(Q)}$$

and for $\alpha_1 + \alpha_2 = 1$

$$\int_D e^{2h\tau} |\tilde{G}^1_{\tau^{\alpha_1}\theta^{\alpha_2}}|^2 d\tau \, d\theta = \int_Q r^{2(\alpha_1 - 1 + \beta)} |G^1_{r^{\alpha_1}\theta^{\alpha_2}}|^2 r \, dr \, d\theta.$$

Therefore

$$\|\tilde{G}^1\|_{H^1_h(D)} \leq C\|G^1\|_{H^{1,1}_\beta(Q)}. \tag{2.27}$$

Using (2.26) and (2.27) and Lemma 2.3 we see that the equation

$$-(\tilde{u}_{\tau\tau} + \tilde{u}_{\theta\theta}) = \tilde{f} \text{ in } D$$

$$\tilde{u}|_{\theta=0} = 0 \tag{2.28}$$

$$\frac{\partial \tilde{u}}{\partial \theta}\Big|_{\theta=\omega} = \tilde{G}^1|_{\theta=\omega}$$

has unique solution $\tilde{u} \in H^2_h(D)$ and (2.4) holds. Let $u = \tilde{u}(\ln \frac{1}{r}, \theta)$, then u satisfies (2.24) and for $|\alpha| \leq 2$

$$(2.29) \qquad \int_Q r^{2(\alpha_1-2+\beta)} |u_{r^{\alpha_1}\theta^{\alpha_2}}|^2 r\,dr\,d\theta$$

$$= \int_D e^{2h\tau} |\tilde{u}_{\tau^{\alpha_1}\theta^{\alpha_2}}|^2 d\tau\,d\theta$$

$$\leq C[\|\tilde{f}\|^2_{L_h(D)} + \|\tilde{G}^1\|^2_{H^1_h(D)}]$$

$$= C[\|f\|^2_{L_\beta(Q)} + \|G^1\|^2_{H^{1,1}_\beta(Q)}].$$

Hence $u \in W^2_\beta(Q)$ and (2.25) is proven for $G^0 = 0$. For $G^0 \neq 0$ we define $w = u - G^0$; then

$$-\Delta w = f + \Delta G^0 = \bar{f}$$

$$w|_{\theta=\omega} = 0$$

$$\frac{1}{r}\frac{\partial w}{\partial \theta}\Big|_{\theta=\omega} = (G^1 - \frac{1}{r}\frac{\partial G^0}{\partial \theta})\Big|_{\Gamma_2} = \bar{G}^1|_{\Gamma_2}.$$

Applying now (2.29) (respectively (2.25)) to this case we get $w \in W^2_\beta(Q)$ and

$$\|w\|^2_{H^{2,2}_\beta(S_1)} \leq C[\|\bar{f}\|^2_{L_\beta(Q)} + \|\bar{G}^1\|^2_{H^{1,1}_\beta(Q)}]$$

$$\leq C[\|f\|^2_{L_\beta(Q)} + \sum_{i=0,1} \|G^i\|^2_{H^{2-i,2-i}_\beta(Q)}]$$

which proves (2.25) in full generality.

Let us prove now that $|\mathcal{D}^1 u|_{H^0(Q)} < \infty$. (2.25) shows that $|\mathcal{D}^1 u|_{H^0(S_1)} < \infty$, hence we have to prove only that

$$|\mathcal{D}^1 u|_{H^0(Q_2)}, \quad Q_2 = Q - \bar{S}_1.$$

We have by Lemma 2.4 for $h = 1 - \beta$ and $0 \leq \gamma = \frac{\pi}{2\omega} + h - \varepsilon, \quad \varepsilon > 0$

$$(2.30) \qquad \int_{Q_2} |\mathcal{D}^\alpha u|^2 \, r^{2(\alpha_1 - 2 + \beta) + 2\gamma} r \, dr \, d\theta = \int_{\tilde{D}} |D^\alpha \tilde{u}|^2 e^{2(h-\gamma)\tau} d\tau \, d\theta \quad < \quad \infty.$$

Particularly for $\alpha_1 + \alpha_2 = 1$ we have for $0 < \varepsilon < \frac{\pi}{2\omega}$

$$(2.31) \qquad \int_{Q_2} |u_r|^2 r \, dr \, d\theta \leq \int_{Q_2} |u_r| r^{2(-1+\beta+1-\beta+\frac{\pi}{2\omega}-\varepsilon)} rd \, rd\theta$$

$$\leq \int_{Q_2} |u_r|^2 r^{2(\alpha_1 - 2 + \beta) + 2\gamma} r \, dr \, d\theta \quad < \infty$$

and

$$(2.32) \qquad \int_{Q_2} \frac{1}{r^2} |u_\theta|^2 r \, dr \, d\theta \leq \int_{Q_2} |u_\theta|^2 r^{2(-2+\beta+(1-\beta)+\frac{\pi}{2\omega}-\varepsilon)}$$

$$\leq \int_{Q_2} |u_\theta|^2 r^{2(-2+\beta)+2\gamma} r \, dr \, d\theta \quad < \quad \infty$$

and (ii) of Lemma 2.6) is proven. ∎

Analogously we have

Lemma 2.7. Let $f \in L_\beta(Q)$, $G^i \in H_\beta^{2-i,2-i}(Q)$, $i = 0,1$, $0 < \beta < 1$, $\beta > 1 - \frac{\pi}{\omega}$ then the Dirichlet (respectively Neumann) problem

(2.33a) $$-\Delta u = f(r,\theta)$$

(2.33b) $$u|_{\Gamma_1 \cup \Gamma_2} = g^0 = G^0|_{\Gamma_1 \cup \Gamma_2}$$

$$\left(\text{respectively } \frac{\partial u}{\partial n}\Big|_{\Gamma_1 \cup \Gamma_2} = g^1 = G^1|_{\Gamma_1 \cup \Gamma_2}\right)$$

has a solution such that

i) $(u - G^0) \in W^2_\beta(Q)$ (respectively $u \in W^2_\beta(Q)$), $u \in H^{2,2}_\beta(S_1)$,

ii) $\|\mathcal{D}^1 u^*\|_{H^0(Q)} < \infty$

where $u^*(r,\theta) = u(r,\theta) - \frac{1}{\omega}\int_0^\omega u(r,\theta)d\theta$ for the Neumann problem,

iii) (2.25) holds with $G^1 = 0$ (respectively $G^0 = 0$).

Let us prove now

Lemma 2.8. Let $u \in \tilde{H}^1(Q) = \{u | \int_Q |\mathcal{D}^1 u|^2 r \, dr \, d\theta < \infty, \; u|_{\Gamma^0} = 0\}$ and $u = 0$ for $r > 1$ be the solution of the problem

$$-\Delta u = f$$

$$u|_{\Gamma^0} = G^0|_{\Gamma^0}$$

$$\frac{\partial u}{\partial n}\Big|_{\Gamma^1} = G^1|_{\Gamma^1}$$

with $f \in L_\beta(Q)$, $G^i \in H^{2-i,2-i}_\beta(Q)$, $i = 0,1$, $0 < \beta < 1$, $\beta > 1 - \frac{\pi}{2\omega}$ for the mixed problem and $\beta > 1 - \frac{\pi}{\omega}$ for the Dirichlet and Neumann problem.

Further let $w \in W^2_\beta(Q)$ be the solution of the same problem given in Lemma 2.6 and 2.7. Then $u = w$ if $\Gamma^0 \neq \emptyset$ (i.e. for the Dirichlet or mixed problem) and $u = w + c$ if $\Gamma^0 = \emptyset$ (i.e. Neumann problem).

Proof. We first prove the lemma for the Dirichlet problem. We may assume $G^0 = 0$. Since $\|\mathcal{D}^1 w\|_{H^0(Q)} < \infty$ by Lemma 2.7, we have for every $v \in \tilde{H}^1_0(Q) = \{v \mid \int_Q |{}^1 v|^2 r\, dr\, d\theta < \infty,\ v|_{\Gamma^0} = 0,\ v = 0 \text{ for } r > A(v)\}$

$$\int_Q (w_r v_r + \frac{1}{r^2} v_\theta w_\theta) r\, dr\, d\theta = \int_Q (u_r v_r + \frac{1}{r} u_\theta v_\theta) r\, dr\, d\theta$$

and hence

$$\int_Q ((w_r - u_r) v_r + \frac{1}{r^2} (w_\theta - u_\theta) v_\theta) r\, dr\, d\theta = 0.$$

Because $\tilde{H}^1_0(Q)$ is dense in $\tilde{H}^1(Q)$ the equality holds also for $v = w - u$. This immediately gives $w = u + C$ and obviously $C = 0$. Now we prove the lemma for the Neumann problem.

Let $u^* = u - \frac{1}{\omega}\int_0^\omega u(r_1,\theta) d\theta = u - b_0(\tau)$ and $w^* = w - \int_0^\omega \frac{1}{\omega} w(r,\theta) d\theta = w - a_0(\tau)$. Then by Lemma 2.7 $\|\mathcal{D}^1 w^*\|_{H^0(Q)} < \infty$. Let

$$\tilde{\tilde{H}}^1_0(Q) = \{u \mid \int_Q |\mathcal{D}^1 u|^2 r\, dr\, d\theta < \infty, \int_0^\omega u(r,\theta) d\theta = 0\}.$$

Then for any $v \in \tilde{\tilde{H}}^1_0$ having bounded support

$$\int_Q (u_r v_r + \frac{1}{r^2} u_\theta v_\theta) r\, dr\, d\theta = \int_Q (u^*_r v_r + \frac{1}{r^2} u^*_\theta v_\theta) r\, dr\, d\theta$$

$$= \int_Q (w_r v_r + \frac{1}{r^2} w_\theta v_\theta) r \, dr \, d\theta \quad = \quad \int_Q (w_r^* v_r + \frac{1}{r} w_\theta^* v_\theta) r \, dr \, d\theta$$

and hence

$$\int_Q ((u_r^* - w_r^*) v_r + \frac{1}{r^2} (u^*_\theta - w_\theta^*) v_\theta) r \, dr \, d\theta \quad = \quad 0.$$

Since the set of $v \in \tilde{\tilde{H}}_0^1(Q)$ having bounded support is dense in $\tilde{\tilde{H}}^1(Q)$ we get $u^* - w^* = C$. Thus

$$u - w \quad = \quad u^* - w^* \quad = a_0(\tau) - b_0(r) \quad = \quad C(r)$$

and because u, w solve the same problem and obviously $\dfrac{\partial C_0}{\partial \theta} = 0$, we get

$$\int_Q C_r v_r r \, dr \, d\theta \quad = \quad 0$$

for any $v \in H = \{v \mid \int_Q |D^1 v|^2 \, dr \, d\theta \ < \infty$. Hence $C(r) = C_1 + C_2 \log \frac{1}{r}$; but $C(r) = (u - w) \in H^1(S_1)$ by Lemma 2.7, *and hence* $C_2 = 0$ and $u - w = C_1$. ∎

2.2. THE REGULARITY OF THE SOLUTION ON A POLYGONAL DOMAIN Ω

Lemma 2.9. Let $g = G|_\gamma \in H^{1/2,1/2}(\gamma)$ where γ is the edge of $\partial\Omega$. Then $(\Phi_{\beta/2} G)|_\gamma \in L_2(\gamma)$.

Proof. Let $\Gamma_i = \gamma$ is the edge connecting the vertices A_{i+1} and A_i, let A_i be placed in the origin and Γ_i lies on the x_1-axis. Assume that $S_\delta^{\omega_i} \subset \Omega$. It is sufficient to prove that on $I_\delta = (0,\delta)$ $\int_0^\delta r^\beta |G(x_1,0)|^2 dx_1 < \infty$ with $\beta = \beta_i$.

Let $F = r^\beta G$. Then

$$F_{x_i} \quad = \quad r^\beta G_{x_i} + \beta r^{\beta-1} \frac{x_i}{r} G.$$

By Lemma A3 of the Appendix

$$\|F_{x_i}\|_{L_\beta(S_\delta)} \leq \|G\|_{H^{1,1}_\beta(\Omega)}$$

and hence

$$\|F\|_{H^1(S_\delta^{\omega_i})} \leq C\|G\|_{H^{1,1}_\beta(\Omega)}.$$

By the imbedding theorem $F \in L_p(I_\delta)$ for any $p > 1$ (see [1]) and

$$\|F\|_{L_p(I_\delta)} \leq C\|F\|_{H^1(S_\delta^{\omega_i})} \leq C\|G\|_{H^{1,1}_\beta(\Omega)}.$$

Hence

$$\|r^{\frac{\beta}{2}}G\|_{L_2(I_\delta)} = \int_{I_\delta} r^{-\beta}|F|^2 dx_1$$

$$\leq C(\int_{I_\delta} r^{-\beta q}\, dx_1)^{1/q} (\int_{I_\delta} |F|^{2p})^{1/p}$$

$$\leq C\|F\|^2_{L_{2p}(I_\delta)} \leq C\|F\|_{H^1(S_\delta^{\omega_i})} \leq C\|G\|_{H^{1,1}_\beta(\Omega)}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\beta q < 1$, $p > 1$, $q > 1$. ■

Lemma 2.10. Let $f \in L_\beta(\Omega)$. Then $\int_\Omega fv\, dx$ is a linear continuous functional on $H^1(\Omega)$ and $\|f\|_{(H^1(\Omega))'} \leq C\|f\|_{L_\beta(\Omega)}$.

The proof follows easily from the Schwarz inequality. See also [12]. ■

Lemma 2.11. Let $f \in L_\beta(\Omega) = H^{0,0}_\beta(\Omega)$, $G^i \in H^{2-i,2-i}_\beta(\Omega)$ and $|\Gamma_0| \neq 0$.

Then

$$(2.34)\qquad \begin{cases} -\Delta u = f \\ u|_{\Gamma^0} = g^0 = G^0|_{\Gamma^0} \\ \frac{\partial u}{\partial n}|_{\Gamma^1} = g^1 = G^1|_{\Gamma^1} \end{cases}$$

has unique solution $u \in H^1(\Omega)$ (in the weak sense) and

$$(2.35)\qquad \|u\|_{H^1(\Omega)} \leq C[\|f\|_{L_\beta(\Omega)} + \sum_{i=0,1} \|G^i\|_{H_\beta^{2-i,2-i}(\Omega)}].$$

Proof. Without loss of generality we can assume that $G^0 = 0$ because $\Delta G^0 \in L_\beta(\Omega)$ and $\frac{\partial G^0}{\partial x_i} \in H_\beta^{1,1}(\Omega)$. Applying Lemma 2.10 it suffices to show that

$$\int_{\Gamma_1} g^1 v\, dx$$

is a linear functional on $H^1(\Omega)$.

We have

$$(2.36)\qquad \int_{\Gamma^1} g^1 v\, ds = \int_{\Gamma^1} r^{\frac{\beta}{2}} g^1 r^{-\frac{\beta}{2}} v\, ds$$

$$(2.37)\qquad \int_{\Gamma^1} r^{-\beta} v^2 ds \leq (\int_{\Gamma^1} r^{-\beta p} ds)^{\frac{1}{p}} (\int_{\Gamma^1} |v|^{2q} dx)^{\frac{1}{q}} \leq C\|v\|_{H^1(\Omega)}$$

and (2.36) together with Lemma 2.9 and (2.37) shows that

$$|\int_{\Gamma^1} g^1 v\, ds| \leq C\|G\|^1_{H^{1,1}(\Omega)} \|v\|_{H^1(\Omega)}.$$

The Lax-Milgram lemma yields (2.35) and the uniqueness.

Remark. If $|\Gamma^0| = 0$ and

$$\int_\Omega f\,dx + \int_{\Gamma^1} g^1\,ds = 0 \tag{2.38}$$

then Lemma 2.11 holds in the factor space modulo a constant.

Proof of Theorem 2.1. Consider the polygonal domain shown in Fig. 2.1. Let

$$S_{i,\delta_i} = \{r_i,\theta_i \mid 0 < r_i < \theta_i,\ 0 < \theta_i < \omega_i\} \quad \Omega$$

where (r_i,θ_i) are the polar coordinates with respect to the vertex A_i. (See Fig. 2.2b.)

Let $\delta_i < 1$, be such that $S_{i,2\delta_i} \cap S_{j,2\delta_j} = \emptyset$ for $i \neq j$, $i,j = 1,\ldots M$. By Lemma 2.11 there is a unique solution of (2.1) in $H^1(\Omega)$. By Theorem 5.71, 5.71' and 6.61 of [16], u is analytic in Ω and on Γ_i, $1 \leq i \leq M$ (because f and g^i are analytic functions by our assumption). Hence Theorem 2.1 holds on $\Omega - S_{i,\delta_i/4}$, $i = 1,\ldots,M$ and in particular we have for $|\alpha| = k$, $k \geq 2$

$$\|r_i^{\alpha_i} D^\alpha u\|_{L_{\beta_i}(S_{i,\delta_i} - S_{i,\delta_i/2})} \leq C_{i,0}\, d_{i,0}^{k-2}(k-2)! \tag{2.39}$$

Hence, it is sufficient to prove that in each sector $S_{i,\delta_i/2}$ $1 \leq i \leq M$ and $|\alpha| = k$, $h \geq 2$ we have

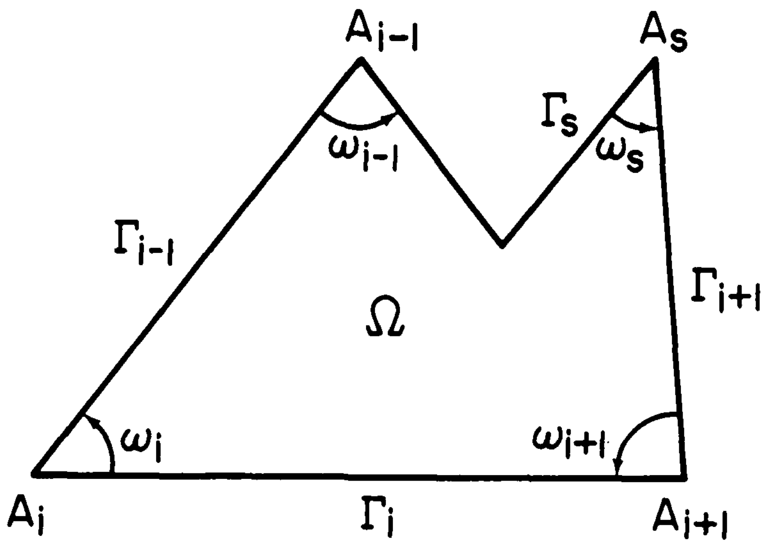

Fig. 2.1. The polygonal domain.

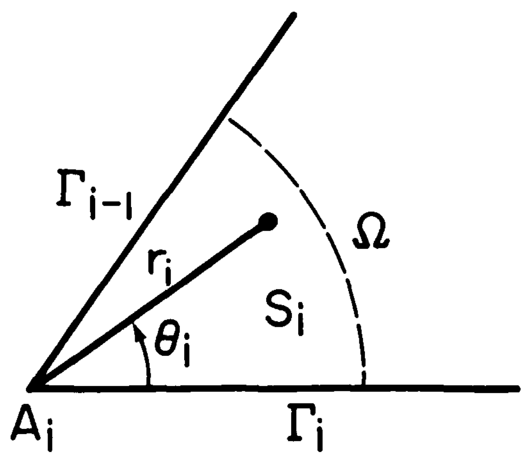

Fig. 2.2. The scheme of coordinates (r_i, θ_i).

$$(2.40) \qquad \| r_i^{\alpha_1 - 2} \mathcal{D}^\alpha u \|_{L_{\beta_i}(S_{i,\delta_i/2})} \leq L_i D_i^{k-2} P_i^{\alpha_2}(k-2)!$$

with L_i, D_i, P_i independent of k (see also Theorem 1.1). There are three cases to be considered

i) $\Gamma_i \subset \Gamma^0$, $\Gamma_{i-1} \subset \Gamma^1$

ii) $\Gamma_i, \Gamma_{i-1} \subset \Gamma^0$

iii) Γ_i, $\Gamma_{i-1} \subset \Gamma^1$.

We may assume that A_i is located in the origin and Γ_i lies on x_1-axes. To simplify the notation we will write $S_{i,\delta_i} = S_\delta$ and $\beta_i = \beta$, etc.

We will prove case i) only. The proof for the other two cases is analogous.

Obviously the solution of problem 2.1 satisfies

$$(2.41) \qquad \begin{aligned} -\Delta u &= f \\ u|_{\tilde{\Gamma}_i} &= G^0|_{\tilde{\Gamma}_i} \\ \frac{\partial u}{\partial n}\Big|_{\tilde{\Gamma}_{i-1}} &= G^1|_{\tilde{\Gamma}_{i-1}} \end{aligned}$$

where

$$\tilde{\Gamma}_\ell = \Gamma_\ell \cap S_\delta, \qquad \ell = i-1, i.$$

Let

$$\varphi_0 \in C^\infty(R^1_+)$$

$$\varphi_0(x) = 1 \quad \text{for} \quad 0 \le x \le \tfrac{1}{2}$$

$$\varphi_0(x) = 0 \quad \text{for} \quad x \ge 1$$

$$\varphi_\delta(r) = \varphi_0\left(\frac{r}{\delta}\right) = \varphi(r).$$

Denote

$$v = \varphi u.$$

Then, by zero extension outside S_δ, function v is defined on the infinite sector $Q = \{(r,\delta) \mid 0 < r < \infty, \ 0 < \theta < \omega\}$ and v satisfies

$$-\Delta v = f + w\nabla\varphi\nabla u + u\Delta\varphi = \tilde{f}$$

(2.42)
$$v\big|_{\theta=0} = \varphi G^0\big|_{\theta=0} = \tilde{G}^0\big|_{\theta=0}$$

$$\frac{\partial v}{\partial n}\bigg|_{\theta=\omega} = \frac{1}{r}\frac{\partial v}{\partial\theta}\bigg|_{\theta=\omega} = \varphi G^1\big|_{\theta=\omega} = \tilde{G}^1\big|_{\theta=\omega}.$$

Obviously $v \in H^1(Q)$ and $\tilde{f}$, $\tilde{G}^0$, $\tilde{G}^1 = 0$ for $r > 1$. Denote by w the solution of (2.24) mentioned in Lemma 2.6. The using Lemma 2.8 we see that $v = w$ and hence by (2.25)

(2.43)
$$\|v\|_{H^{2,2}_\beta(S_\delta)} \le C[\|\tilde{f}\|_{L_\beta(Q)} + \|\tilde{G}^0\|_{H^{2,2}_\beta(Q)} + \|\tilde{G}^1\|_{H^{1,1}_\beta(Q)}]$$

$$\le C[\|\tilde{f}\|_{L_\beta(S_\delta)} + \sum_{\ell=0}^{1} \|\tilde{G}^\ell\|_{H^{2-\ell,2-\ell}_\beta(S_\delta)}].$$

In (2.43) we have used the fact that $\varphi = 0$ for $r > \delta$. Because $\nabla\varphi = \Delta\varphi = 0$ for $0 < r < \frac{\delta}{2}$ and $r > \delta$ we have

$$\|\nabla\varphi\nabla u\|_{L_\beta(S_\delta)} \le C\|u\|_{H^1_\beta(S_\delta - S_{\delta/2})} \le C_1$$

$$\|u\Delta\varphi\|_{L_\beta(S_\delta)} \le C\|u\|_{H^0_\beta(S_\delta - S_{\delta/2})} \le C_1.$$

Because $f \in B^0_\beta(\Omega)$, $G^i \in B^{2-i}_\beta(\Omega)$ we get immediately from (2.43)

(2.44) $$\|u\|_{H^{2,2}_\beta(S_{\frac{\delta}{2}})} = \|v\|_{H^{2,2}_\beta(S_{\frac{\delta}{2}})} \le \|v\|_{H^{2,2}_\beta(S_\delta)}$$

$$\le C\{\|f\|_{L_\beta(S_\delta)} + \sum_{\ell=0}^{1} \|G^\ell\|_{H^{2-\ell,2-\ell}_\beta(S_\delta)} + \|u\|_{H^1(S_\delta - S_{\delta/2})}\}$$

with C_2 dependent on δ and u. Hence (2.40) hold for $|\alpha| = 2$. Let

(2.45) $$v_k = r^k u_{r^k}, \quad k \ge 2.$$

Then

(2.46) $$\begin{cases} -\Delta v_k = r^{k-2}(r^2 f)_{r^k} \quad \text{in } S_\delta \\ v_k\big|_{\theta=0} = r^k G^0_{r^k}\big|_{\theta=0} \\ \dfrac{\partial v_k}{\partial n}\Big|_{\theta=\omega} = \dfrac{1}{r}\dfrac{\partial v_k}{\partial \theta}\Big|_{\theta=\omega} = (r^k G^1_{r^k} + k r^{k-1} G^1_{r^{k-1}})\big|_{\theta=\omega}. \end{cases}$$

Let $w_k = \varphi v_k$. Then

$$-\Delta w_k = -\varphi\Delta v_k - \nabla\varphi\nabla v_k - v_k\Delta\varphi$$

and

$$w_k\big|_{\theta=0} = \varphi r^k G^0_{r^k}\big|_{\theta=0}$$

$$\left.\frac{\partial w_k}{\partial n}\right|_{\theta=\omega} = \left.\frac{1}{r}\frac{\partial w_k}{\partial \theta}\right|_{\theta=\omega} = \left.\varphi(r^k G^1_{r^k} + kr^{k-1}G^1_{r^{k-1}})\right|_{\theta=\omega}.$$

Hence analogously as before

$$\|v_k\|_{H^{2,2}_\beta(S_{\frac{\delta}{2}})} \le C[\|r^{k-2}(r^2f)_{r^k}\|_{L_\beta(S_\delta)}$$

$$+ \|v_k\|_{H^{1,1}_\beta(S_\delta - S_{\frac{\delta}{2}})} + \|v_k\|_{H^0_\beta(S_\delta - S_{\frac{\delta}{2}})}$$

$$+ \|r^k G^0_{r^k}\|_{H^{2,2}_\beta(S_\delta)} + \|r^k G^1_{r^k}\|_{H^{1,1}_\beta(S_\delta)}$$

$$+ k\|r^{k-1}G^1_{r^{k-1}}\|_{H^{1,1}_\beta(S_\delta)}).$$

Because $f \in B^0_\beta(\Omega)$, $G^\ell \in B^{2-i}_\beta(\Omega)$ we have

$$\|r^{k-2}(r^2f)_{r^k}\|_{L_\beta(S_\delta)} \le C_3 d_3^k k! \tag{2.48a}$$

$$\|r^k G^i_{r^k}\|_{H^{2-i,2-i}_\beta(S_\delta)} \le C_4 d_4^k k! \tag{2.48b}$$

$$k\|r^{k-1}G^1_{r^{k-1}}\|_{H^{1,1}_\beta(S_\delta)} \le C_5 d_5^k k! \tag{2.48c}$$

Using (2.39) we get

$$\|v_k\|_{H^{1,1}_\beta(S_\delta - S_{\delta_2})} \le Cc_0 d_0^{k-1}(k-1)!$$

$$|v_k|_{H^0_\beta(S_\delta - S_{\frac{\delta}{2}})} \leq C c_0 d_0^{k-2}(k-2)!$$

with C independent of k (depending on δ). Hence

$$|v_k|_{H^{2,2}_\beta(S_{\delta/2})} \leq C_6 d_6^k k! \tag{2.49}$$

with C_6, d_6 independent of k and L, D, P. Assume now by induction that (2.40) holds for $k' < k$. Then we get using (2.49)

(2.50a)

$$|r^k u_{r^{k+2}}|_{L_\beta(S_{i,\delta_i/2})} \leq C_6 d_6^k k! + 2kLD^{k-1}(k-1)! + k(k-1)LD^{k-2}(k-2)!$$

$$\leq LD^k k!$$

provided that D and L are large enough. Further with $P > 1$, e.g. $P = 2$

$$|r^{k-1} u_{r^{k+1}\theta}|_{\beta(S_{i,\delta_i})} \leq C_6 d_6^k k + k(k-1)!LD^{k-1}P \leq LD^k Pk! \tag{2.50b}$$

$$|r^{k-2} u_{r^k\theta^2}|_{\beta(S_{i,\delta_i/2})} \leq C_6 d_6^6 k! \leq LK^k P^2 k! \tag{2.50c}$$

Inequalities (2.50) yield (2.40) with $\alpha_1 \geq 2$ and $\alpha_2 \leq 2$.

Let us now prove (2.40) by induction with respect to α_2. Let $v = r^{\alpha_1} u_{r^{\alpha_1}\theta^{\alpha_2 - 2}}$, $\alpha_2 \geq 2$, $\alpha_1 + \alpha_2 = k$, $\alpha_1 \geq 2$, then

$$-\Delta v = r^{\alpha_1-2}\left(r^2 f_{\theta^{\alpha_2-2}}\right)_{r^{\alpha_1}}$$

and also

$$-\Delta v = -r^{\alpha_1+2} u_{r^{\alpha_1}\theta^{\alpha_2}} - (2\alpha_1+1) r^{\alpha_1-1} u_{r^{\alpha_1+1}\theta^{\alpha_2-2}}$$

$$- \alpha_1^2 r^{\alpha_1-2} u_{r^{\alpha_1}\theta^{\alpha_2-2}} - r\alpha_1 u_{r^{\alpha_1+2}\theta^{\alpha_2-2}}.$$

Hence

$$\text{(2.51)} \qquad \| r^{\alpha_1-2} u_{r^{\alpha_1}\theta^{\alpha_2}} \|_{L_\beta(S_{\delta/2})}$$

$$\leq \| r^{\alpha_1-2}\left(r^2 f_{\theta^{\alpha_2-2}}\right)_{r^{\alpha_1}} \|_{L_\beta(S_{\delta/2})} + (2\alpha_1+1) \| r^{\alpha_1-1} u_{r^{\alpha_1+1}\theta^{\alpha_2-2}} \|_{L_\beta(S_{\delta/2})}$$

$$+ \alpha_1^2 \| r^{\alpha_1-2} u_{r^{\alpha_1}\theta^{\alpha_2-2}} \|_{L_\beta(S_{\delta/2})} + \| r^{\alpha_1} u_{r^{\alpha_1+2}\theta^{\alpha_2-2}} \|_{L_\beta(S_{\delta/2})}.$$

Because $f \in B^0_\beta(\Omega)$ we have

$$\text{(2.52)} \qquad \| r^{\alpha_1-2}\left(r^2 f_{\theta^{\alpha_2-2}}\right)_{r^{\alpha_1}} \|_{L_\beta(S_{\delta/2})} \leq C_3 d_3^{k-2}(k-2)!$$

By induction assumption

$$\text{(2.53a)} \qquad \| r^{\alpha_1-1} u_{r^{\alpha_1+1}\theta^{\alpha_2-2}} \|_{L_\beta(S_{\delta/2})} \leq LD^{k-3} P^{\alpha_2-2}(k-3)!$$

$$\text{(2.53b)} \qquad \| r^{\alpha_1} u_{r^{\alpha_1+2}\theta^{\alpha_2-2}} \|_{L_\beta(S_{\delta/2})} \leq LD^{k-2} P^{\alpha_2-2}(k-2)!$$

$$(2.53c) \qquad \| r^{\alpha_1 - 2} u_{r^{\alpha_1}\theta^{\alpha_2 - 2}} \|_{L_\beta(S_{\delta/2})} \leq LD^{k-4} P^{\alpha_2 - 2} (k-4)!$$

Hence from (2.47)

$$(2.54) \quad \| r^{\alpha_1 - 2} u_{r^{\alpha_1}\theta^{\alpha_2}} \|_{L_\beta(S_{\delta/2})}$$

$$\leq (k-2)! [C_3 d_3^{k-2} + LD^{k-2} P^{\alpha_1 - 2} + LD^{k-3} P^{\alpha_2 - 2} (2\alpha_1 + 1) + LD^{k-4} P^{\alpha_2 - 2} \alpha_1^2]$$

$$\leq LD^{k-2} P^{\alpha_2} (k-2)!$$

provided that L, D and P are sufficiently large. Similarly we can prove (2.40) for $\alpha_2 \geq 2$, $\alpha_1 = 0,1$.

Theorem 2.1 is proven for the case i). The other cases are analogous.

Combining our results for every vertex we easily complete the proof. ∎

3. REGULARITY OF THE SOLUTION OF THE ELLIPTIC EQUATION IN A POLYGONAL DOMAIN Ω

In Chapter 2 we analyzed the problem of the regularity of the solution of the Poisson problem on a polygonal domain. In this chapter we will consider the general case of the elliptic equation of second order with analytic coefficients.

3.1. THE PROBLEM AND ITS BASIC PROPERTIES

Let

$$L(u) = -\sum_{i,j=1}^{2} (a_{i,j}u_{x_i})_{x_j} + \sum_{i=1}^{2} b_i u_{x_i} + cu. \tag{3.1}$$

Let us consider the problem

$$L(u) = f \quad \text{in} \quad \Omega$$

$$u\big|_{\Gamma^0} = g^0 = G^0\big|_{\Gamma^0} \tag{3.2}$$

$$\frac{\partial u}{\partial n_c}\bigg|_{\Gamma^1} = g^1 = G^2\big|_{\Gamma^1} \quad \text{on} \quad \Gamma^1$$

where

$$\Gamma^0 = \bigcup_{i=\mathcal{D}} \bar{\Gamma}_i, \qquad \Gamma^1 = \Gamma - \Gamma^0,$$

and n_c is the conormal.

Let Ω be the polygonal domain in $\mathbf{R}^2$ and Γ_i be the open edge of $\partial\Omega$ (see Section 2.1).

About f and g^i, $i = 0,1$ we will make the same assumptions as in Theorem 2.1 but replacing ω_i by $0 < \omega_i^* \leq 2\delta$ which will be defined later. About the operator L we will assume

i) $a_{i,j} = a_{j,i}$, b_i, c are analytic function on $\bar{\Omega}$.

ii)

$$\sum_{\ell,j=1}^{2} a_{i,j}\xi_i\xi_j \geq \mu_0(\xi_1^2 + \xi_2^2), \quad \mu_0 > 0, \tag{3.3}$$

i.e. the operator is strongly elliptic.

iii) Denote (see Section 1.2)

$$H_0^1(\Omega) = \{u \in H^1(\Omega) \mid u = 0 \text{ on } \Gamma^0\}$$

and

$$B(u,v), \quad u \in H_0^1(\Omega), \quad v \in H_0^1(\Omega)$$

be the bilinear form

$$B(u,v) = \int_\Omega (a_{i,j}u_{x_i}v_{x_j} + b_i u_{x_i} v + cuv)dx. \tag{3.4}$$

We assume that

$$\inf_{\substack{\|u\|_{H^1(\Omega)}=1 \\ u \in H_0^1(\Omega)}} \sup_{\substack{\|v\|_{H^1(\Omega)}=1 \\ v \in H_0^1(\Omega)}} |B(u,v)| \geq \gamma > 0, \tag{3.5a}$$

(3.5b) for any $v \in H_0^1(\Omega)$, $v \neq 0$

$$\sup_{\substack{\|u\|_{H^1(\Omega)}=1 \\ u \in H_0^1}} |B(u,v)| > 0.$$

Conditions i), ii) guarantee the existence and uniqueness of $u \in H_0^1(\Omega)$ such that

(3.6) $$B(u,v) = F(v)$$

holds for any $v \in H_0^1(\Omega)$ for any $F(v) \in (H_0^1(\Omega))'$, i.e. $F(v)$ being a linear functional on $H_0(\Omega)$. In addition we have

(3.7) $$\|u\|_{H^1(\Omega)} \leq C\|F\|_{(H_0^1(\Omega))'}$$

with C independent of F.

For the proof see e.g. [4], p. 112. Hence we have

Lemma 3.1. Let $f \in L_\beta(\Omega) = H_\beta^{0,0}(\Omega)$, $G^i \in H_\beta^{2-i,2-i}(\Omega)$ and $|\Gamma^0| \neq 0$, $\beta = (\beta_1,\dots,\beta_M)$, $0 \leq \beta_i < 1$. Hence (3.2) has unique solution $u \in H^1(\Omega)$ (in the week sense) and

$$\|u\|_{H^1(\Omega)} \leq C[\|f\|_{L_\beta(\Omega)} + \sum_{i=0,1} \|G^i\|_{H_\beta^{2-i,2-i}(\Omega)}].$$

The proof is completely analogous to the proof of Lemma 2.11, only replacing Lax-Milgram lemma by its generalized form based on (3.5), (3.6), (3.7). ■

Remark. Condition (3.5a) and (3.5b) exclude the case when $\Gamma^0 = 0$. Nevertheless this case which occurs e.g. in the case of Neumann problem and $b_i = C = 0$ can be treated in the usual way by restriction of $H_0^1(\Omega)$ to a modulo space.

Lemma 3.2. Let L^0 be the operator (3.1) with $a_{i,j}^0 = a_{j,i}^0$ constants and $b_i = c = 0$. Let M be the linear transformation

$$\xi_1 = (a^0_{12}x_1 - a^0_{1,1}x_2)/\sqrt{a^0_{1,1}}\,A, \quad A = (a^0_{1,1}a^0_{2,2} - |a^0_{1,2}|^2)^{1/2}.$$

(3.8)

$$\xi_2 = x_1/\sqrt{a^0_{1,1}}$$

and $\tilde{u}(\xi_1,\xi_2) = u(x_1(\xi_1,\xi_2),\ x_2(\xi_1,\xi_2))$. Then

$$L^0u = (-\Delta\tilde{u}).$$

and the conormal n_c in (x_1,x_2) transforms into the normal n in (ξ_1,ξ_2). ■

The lemma follows easily by simple computation.

The transformation M maps the polygonal domain Ω into the polygonal domain Ω^* with interior angle ω_i^*, $\omega_i^* = M(\omega_i)$.

Let now L be the general operator (3.1). By assumption the coefficient $a_{i,j}$ are analytic on $\bar{\Omega}$. Hence we can define mappings M_k associated to the vertices A_k with $a^0_{i,j} = a_{i,j}(A_k)$ and set $\omega_k^* = M(\omega_k)$.

3.2. THE REGULARITY OF THE SOLUTION

The main theorem of this chapter is

Theorem 3.1. Let $f \in B^0_\beta(\Omega)$, $g^\ell \in B_\beta^{\frac{3}{2}-i}(\Gamma_i)$, $i = 0,1$, $\beta = (\beta_1,\dots,\beta_M)$, $0 < \beta_i < 1$, $\beta_i > 1 - \pi/\omega_i^*$ (respectively $\beta_i > 1 - \pi/2\omega_i^*$ if Dirichlet or Neumann boundary conditions are imposed on the edges Γ_i and Γ_{i-1}, $\bar{\Gamma}$ $\bar{\Gamma}_i \cap \bar{\Gamma}_{i-1} = A_i$) and $\Gamma^0 \neq \emptyset$. Then problem (3.1) has a unique solution in $H^1(\Omega)$ and $u \in B^2_\beta(\Omega)$.

<u>Proof</u>. The main idea of the proof is the same as in Theorem 2.1, namely that in the neighborhood of every vertex A_i the inequality (2.40) holds. By Theorem 5.7.1, 5.6.1' and 6.6.1 of [16] u is analytic in Ω and on (open) Γ_i, $1 \leq i \leq M$.

Let the mapping M_ℓ maps Ω into Ω^* with the vertex A_ℓ mapped into origin and the edge Γ_ℓ being mapped into Γ_ℓ^* lying on the axes ξ_1. Defining $\tilde{u}(\xi_1,\xi_2) = u(M^{-1}(\xi_1,\xi_2))$ we have

$$(3.9) \qquad L^*(\tilde{u}) = \tilde{f}$$

where

$$(3.10) \qquad L^*(\tilde{u}) = -\Delta\tilde{u} - \sum_{i,j=1}^{2} \tilde{a}_{i,j}\tilde{u}_{\xi_1\xi_2} + \sum_{j=1}^{2} \tilde{b}_j\tilde{u}_{\xi_j} + \tilde{c}\tilde{u}$$

with $\tilde{a}_{i,j}(0,0) = 0$, and $\tilde{a}_{i,j}$, $\tilde{b}_j$ and $\tilde{c}$ are analytic functions in $\bar{\Omega}^*$.

Let $S = \{r,\theta \mid 0 < r < \delta_\ell,\ 0 < \theta < \omega_\ell^*\}$, $\delta_\ell = \delta < 1$ and S Ω^*. Let us analyze in detail the case $\Gamma_{\ell-1}^*$, Γ_ℓ^* Γ^{0*}. (Let us write further Γ_ℓ instead of Γ_ℓ^*.) We have

$$(3.11) \qquad L^*(\tilde{u}) = \tilde{f} \ \text{ on } S$$

$$\tilde{u}\big|_{\tilde{\Gamma}_\ell \cup \tilde{\Gamma}_{\ell-1}} = G^0\big|_{\tilde{\Gamma}_\ell \cup \tilde{\Gamma}_{\ell-1}}$$

where $\tilde{\Gamma}_\ell = \Gamma_\ell \cap S$. Without a loss of generality we can assume $G^0 = 0$ (if not we set $v = u - G^0$). We rewrite (3.10) by replacing $\tilde{u}$, $\tilde{a}_{ij}$, $\tilde{b}_j$, $\tilde{c}$, $\tilde{f}$ by u, a_{ij}, b_j, c,f, and (ξ_1,ξ_2) by (x_1,x_2). Then

$$(3.12) -\Delta u = f_1 = f + \sum_{i,j=1}^{2} a_{i,j}u_{x_ix_j} - \sum_{j=1}^{2} b_j u_{x_j} - cu, \qquad u\big|_{\tilde{\Gamma}_\ell \cup \tilde{\Gamma}_{\ell-1}} = 0.$$

By Theorem 2.1 (see (2.44)) we have for $\beta_\ell > 1 - \frac{\pi}{\omega_\ell^*}$ and $\delta_1 < \frac{\delta}{2}$

$$(3.13) \qquad \|u\|_{H_\beta^{2,2}(S_{\delta_1})} \le C_0(\|f_1\|_{L_\beta(S_\delta)} + \|u\|_{H^1(S_\delta - S_{\delta/2})})$$

where for simplicity we set $\beta = \beta_\ell$. Since $a_{i,j}(0,0) = 0$ and $a_{i,j}$ are analytic in $\bar{\Omega}^*$ we have $|a_{i,j}| \le C_1 r$ in $\bar{S}_\delta$ and hence

$$(3.14) \qquad \|a_{i,j} u_{x_i x_j}\|_{L_\beta(S_{\delta_1})} \le C_1 \delta_1 \|u_{x_i x_j}\|_{L_\beta(S_{\delta_1})}.$$

One has

$$u_{x_1} = u_r \cos\theta - u_\theta \frac{\sin\theta}{r}$$

$$u_{x_1^2} = u_{r^2} \cos^2\theta - u_{r\theta} \frac{\sin 2\theta}{r} + \frac{1}{r^2} u_{\theta^2} \sin^2\theta$$

$$- \frac{1}{r} u_r \sin^2\theta - \frac{1}{r^2} u_\theta \sin 2\theta$$

and similar expressions for $u_{x_1 x_2}$ and $u_{x_2^2}$. Using Lemma A2 scaled to the sector S_{δ_1} we get for $|\alpha| = 1$

$$\|r^{\alpha_1 - 2} D^\alpha u\|_{L_\beta(S_{\delta_1})} \le C_2\Big(\sum_{|\alpha'|=2} \|r^{\alpha_1' - 2} D^\alpha u\|_{L_\beta(S_{\delta_1})}$$

$$+ \sum_{|\alpha'| \le 1} \|r^{\alpha_1' - 1} D^\alpha u\|_{L_2(S_{\delta_1})}\Big)$$

with $C_2 \ge 1$ independent of δ_1.

Hence

(3.15)

$$\|u_{x_1^2}\|_{L_\beta(S_{\delta_1})} \le C_2\Big(\sum_{|\alpha|=2} \|r^{\alpha_1-2}\mathcal{D}^\alpha u\|_{L_\beta(S_{\delta_1})} + \sum_{|\alpha|\le 1} \|r^{\alpha_1-1}\mathcal{D}^\alpha u\|_{L_2(S_{\delta_1})}\Big)$$

$$\le C_2\|u\|_{H^{2,2}_\beta(S_{\delta_1})}.$$

Analogously it can be readily proven that (3.15) holds for $u_{x_i x_j}$, $i = 1,2$.

Using (3.14) in (3.15) we get

$$\text{(3.16)} \qquad \sum_{i,j=1}^{2} \|a_{i,j}u_{x_i x_j}\|_{L_\beta(S_{\delta_1})} \le C_2 C_1 \delta_1 \|u\|_{H^{2,2}_\beta(S_{\delta_1})}$$

$$= C_3\delta_1\|u\|_{H^{2,2}_\beta(S_{\delta_1})}$$

where C_3 is independent of u and δ_1. Let us select δ_1 so that $C_0C_3\delta_1 < \frac{1}{2}$. Then we get from (3.13)

$$\text{(3.17)} \qquad \|u\|_{H^{2,2}_\beta(S_{\delta_1})} \le C_0[\|f\|_{L_\beta(\delta)} + \|u\|_{H^1(S_\delta - S_{\delta/2})} + \|u\|_{H^2(S_\delta - S_{\delta_1})}]$$

$$+ C_0C_3\delta_1\|u\|_{H^{2,2}_\beta(S_{\delta_1})}.$$

Hence

$$\text{(3.18)} \qquad \|u\|_{H^{2,2}_\beta(S_{\delta_1})} \le C_4[\|f\|_{L_\beta(\delta)} + \|u\|_{H^1(S_\delta)} + \|u\|_{H^2(S_\delta - S_{\delta_1})}].$$

Because u is analytic we have for any $|\alpha| = k$

$$(3.19) \qquad \|D^{\alpha}u\|_{L_2(S_\delta - S_{\delta_1})} \le C_5 d_5^k k!$$

and we have also $u \in H^1(S_\delta)$. Hence $u \in H_\beta^{2,2}(S_{\delta_1})$.

Let now $v_k = r^k u_{r^k}$, $k \ge 1$. Then

$$(3.20) \qquad -\Delta v_k = f_k = f_k^{(1)} + f_k^{(2)} \quad \text{on} \quad S_{\delta_1}$$

$$v_k\big|_{\tilde{\Gamma}_\ell \cup \tilde{\Gamma}_{\ell-1}} = 0$$

where

$$(3.21a) \quad f_k^{(1)} = r^{k-2}(r^2 f)_{r^k}$$

$$(3.21b) \quad f_k^{(2)} = r^{k-2}\Big(r^2 \sum_{\ell,j=1}^{2} a_{i,j} u_{x_i x_j} - r^2 \sum_{j=1}^{2} b_j u_{x_j} - r^2 cu\Big)_{r^k}.$$

As before we have

$$(3.22) \quad \|v_k\|_{H_\beta^{2,2}(S_{\delta_1})} \le C_0\big(\|f_k\|_{L_\beta(S_\delta)} + \|v_k\|_{H^1(S_\delta - S_{\delta/2})}\big).$$

Since by assumption $f \in B_\beta^{0}(\Omega)$ there exists constants C_f, d_f such that

$$(3.23) \qquad \|f_k^{(1)}\|_{L_\beta(S_\delta)} \le C_f d_f^k d!$$

Because the coefficients $a_{i,j}$, b_j, c are analytic in $\bar{\Omega}$ there exist constants C_5, d_5 such that for $|\alpha| = k > 0$

$$|D^\alpha a_{i,j}| \le C_6 d_6^k k!,$$

(3.24) $$|D^\alpha b_j| \le C_6 d_6^k k!,$$

$$|D^\alpha c| \le C_6 d_6^k k!.$$

Hence for $0 \le k - m \le 2$

$$|(r^2 a_{i,j})_{r^{k-m}}| \le \tilde{C}_6 r^{(2-k+m)} d_6^{(2-k+m)}$$

and for $k - m \ge 2$

$$|(r^2 a_{i,j})_{r^{k-m}}| \le \tilde{C}_6 d_6^{k-m}(k-m)!.$$

Therefore for $i,j = 1,2$

(3.24) $$\| r^{k-2}(r^1 a_{i,j} u_{x_i x_j})_{r^k} \|_{L_\beta(S_{\delta_1})}$$

$$\le \sum_{m=0}^{k} \binom{k}{m} \| r^{k-2}(r^2 a_{i,j})_{r^{k-m}} (u_{x_i x_j})_{r^m} \|_{L_\beta(S_{\delta_1})}$$

$$\le C_1 \delta_1 \| r^k u_{x_i x_j r^k} \|_{L_\beta(S_{\delta_1})}$$

$$+ \sum_{m=0}^{k-1} \binom{k}{m} \tilde{C}_6 d_6^{k-m}(k-m)! \| r^{k-2+\xi(m,k)} u_{x_i x_j r^m} \|_{L_\beta(S_{\delta_1})}$$

where

$$\xi(m,k) = 2 - k + m \quad \text{for} \quad 0 < k - m \le 2$$

$$\xi(m,k) = 0 \qquad \text{for} \quad k - m > 2.$$

Obviously

$$(3.26)\qquad |D^2 u_{r^k}| \leq r^k|\ ^2 v_k| + 2k|r^{k-1}u_{r^{k+1}}| + k(k-1)|r^k u_{r^k}|$$

and by (A.10) of Lemma A.4 we get

$$(3.27)\qquad \| r^k u_{x_i x_j r^k} \|_{L_\beta(S_{\delta_1})} \leq \| v_k \|_{H_\beta^{2,2}(S_{\delta_1})}$$

$$+ C_7 k! \Big(\sum_{2\leq|\alpha|\leq k+1} \frac{k-|\alpha|+3}{(|\alpha|-2)!} \| r^{\alpha_1 - 2} D^\alpha u \|_{L_\beta(S_{\delta_1})} + \| u \|_{H^1(S_{\delta_1})} \Big).$$

For $m \leq k - 1$, by (A.11) of Lemma A.4

$$(3.28)\qquad \| r^{k-2+\xi(m,k)} u_{x_i x_j r^m} \|_{L_\beta(S_{\delta_1})}$$

$$\leq C_7 m! \Big(\sum_{\substack{2\leq|\alpha|\leq m+2 \\ 0\leq\alpha_2\leq 2}} \frac{(m-|\alpha|+3)}{(|\alpha|-2)!} \| r^{\alpha_1 - 2} D^\alpha u \|_{L_\beta(S_{\delta_1})} + \| u \|_{H^1(S_{\delta_1})} \Big).$$

Hence

$$(3.29)\qquad \sum_{m=0}^{k-1} \binom{k}{m} \| r^{k-2} (r^2 a_{ij})_{r^{k-m}} (u_{x_i x_j})_{r^m} \|_{L_\beta(S_{\delta_1})}$$

$$\leq C_7 k! \sum_{m=0}^{k-1} d_7^{k-m} \Big(\sum_{2\leq|\alpha|\leq m+2} \frac{(m-|\alpha|+3)}{(|\alpha|-2)!} \| r^{\alpha_1 - 2} D^\alpha u \|_{L_\beta(S_{\delta_1})} + \| u \|_{H^1(S_{\delta_1})} \Big).$$

Similarly

$$(3.30)\qquad \| r^{k-2} (r^2 \sum_{j=1}^{2} b_j u_{x_j} + r^2 cu)_{r^k} \|_{L_\beta(S_{\delta_1})}$$

$$\leq \; C_7 k! \Big(\sum_{m=0}^{k} d_7^k \sum_{\substack{2\leq|\alpha|\leq m+1 \\ \alpha_2\leq 1}} \frac{1}{(|\alpha|-1)!} \| r^{\alpha_1 - 2} \mathcal{D}^\alpha u \|_{L_\beta(S_{\delta_1})} + \|u\|_{H^1(S_{\delta_1})} \Big).$$

Thus

$$(3.31) \quad \|v_k\|_{H^{2,2}_\beta(S_{\delta_1})} \leq C_0 C_1 \delta \|v^k\|_{H^{2,2}_\beta(S_{\delta_1})}$$

$$+ C(\delta)k! \Big[\Big(\sum_{\substack{2\leq|\alpha|\leq k+1 \\ \alpha_2\leq 2}} \frac{k-|\alpha|+3}{(|\alpha|-2)!} \|r_1^{\alpha_1 - 2} \mathcal{D}^\alpha u\|_{L_\beta(S_{\delta_1})}$$

$$+ \|u\|_{H^1(S_{\delta_1})} + \sum_{m=0}^{k-1} d_7^{k-m} \sum_{\substack{2\leq|\alpha|\leq m+2 \\ 0\leq\alpha_2\leq 2}} \frac{(m-|\alpha|+3)}{(|\alpha|-2)!} \|r^{\alpha_1 - 2} \mathcal{D}^\alpha u\|_{L_\beta(S_{\delta_1})}$$

$$+ \sum_{m=0}^{k} d_7^{k-m} \sum_{\substack{2\leq|\alpha|\leq m+1 \\ 0\leq\alpha_2\leq 1}} \frac{1}{(|\alpha|-1)!} \|r^{\alpha_1 - 2} \mathcal{D}^\alpha u\|_{L_\beta(S_{\delta_1})}$$

$$+ C_f d_f^k k! + \|u\|_{H^{k+2}(S_\delta - S_{\delta_1})} \Big].$$

Assuming by induction that (2.40) holds for $\alpha_2 \leq 2$ and $|\alpha| \leq k-1$ and realizing that for $C_0 C \delta_1 < \frac{1}{2}$ we get (2.40) for $|\alpha| = k$, $\alpha_2 \leq 2$ provided that L and D are sufficiently large.

The same argument as has been used in Section 2 yields (2.40) for $\alpha_2 > 2$. So far we have assumed that $(\Gamma_{\ell-1} \cup \Gamma_\ell) \subset \Gamma^0$. In the case $\Gamma_\ell \subset \Gamma^1$, $\Gamma_{\ell-1} \subset \Gamma^0$ we proceed analogously. We have

$$-\Delta u = f + \sum_{i,j=1}^{2} a_{i,j} u_{x_i x_j} - \sum_{j=1}^{2} b_j i_{x_j} - cu = f_1 \quad \text{on} \quad S_{\delta_1}, \qquad u|_{\tilde{\Gamma}_{\ell-1}} = 0$$

$$\frac{\partial u}{\partial n}\Big|_{\tilde{\Gamma}_\ell} = \Big[\frac{\partial u}{\partial n_c} - (a_{1,2} u_{x_1} + a_{2,2} u_{x_2})\Big]\Big|_{\tilde{\Gamma}_\ell}$$

$$= G^1|_{\tilde{\Gamma}_\ell} - (a_{1,2} u_{x_1} + a_{2,2} u_{x_2})|_{\tilde{\Gamma}_\ell} = \tilde{G}^1|_{\tilde{\Gamma}_\ell}$$

with $a_{i,j}(0,0) = 0$. By (2.44) we have

$$\|u\|_{H^{2,2}_\beta(S_{\delta_1})} \leq C(\|f_1\|_{L_\beta(S_\delta)} + \|G^1\|_{H^{1,1}_\beta(S_\delta)} + \delta\|u\|_{H^{2,2}_\beta(S_{\delta_2})}$$

$$+ \|u\|_{H^1(S_\delta)} + \|u\|_{H^2(S_\delta - S_{\delta_1})})$$

and the proof is very similar as before. The same arguments hold for the case $(\Gamma_\ell \quad \Gamma_{\ell-1}) \subset \Gamma^1$. Combining the results for every vertex we get our theorem. ■

Remark. In the proof that $u \in B^2_\beta(\Omega)$ we have only assumed that the solution exists. The other conditions, namely (3.4) and (3.5), only guarantee this existence.

We have assumed that the coefficients $a_{i,j}$, b_j, c are analytic on $\bar{\Omega}$. This assumption can be weakened. For example, we can assume that $a_{i,j}$, b_j, c are analytic on $\bar{\Omega} - \bigcup_{j=1}^{M} A_j$ and in the neighborhood of A_ℓ, $\ell = 1,\ldots,M$

$$|D^\alpha(a_{i,j} - a_{i,j}(0,0))| \leq Cd^k k! r_\ell^{\varepsilon_\ell^a - k}$$

$$|D^\alpha b_j| \leq Cd^k k! r_\ell^{\varepsilon_\ell^b - k - 1}$$

$$|D^\alpha c| \leq Cd^k k! r_\ell^{\varepsilon_\ell^c - k - 2},$$

with ε_ℓ^a, $\varepsilon_\ell^b > 0$ arbitrarily, $\varepsilon_\ell^c > 0$, $\varepsilon_\ell^c + \beta_\ell > 1$, and $k = |\alpha|$. Nevertheless we will not go here in further details although this case plays an important role when nonlinear problem is studied.

4. APPENDIX

Lemma A.1. One has the inequality

$$(A.1) \qquad \int_0^1 t^{\alpha-2}[z(t)-a]^2 dt \le C(\alpha) \int_0^1 t^{\alpha}\left(\frac{dz}{dt}\right)^2 dt, \qquad \alpha \neq 1$$

where $a = z(0)$ if $\alpha < 1$, $a = z(1)$ if $\alpha > 1$.

Proof. For $\alpha < 1$ we have by Theorem 2.54 of [13]

$$\int_0^1 s^{-2}|w(s)|^2 dx \le C\left[\int_0^1 |w'(s)|^2 ds + w(1)\right], \qquad w(0) = 0.$$

Because $|w(1)|^2 \le \int_0^1 |w'd(s)|^2 ds$ we have

$$\int_0^1 s^2|w(s)|^2 ds \le C \int_0^1 |w'(s)|^2 ds$$

Setting $w(t) = z(t) - z(0)$ and $s = t^{\ell-\alpha}$ we get

$$\int_0^1 t^{\alpha-2}(z(t)-z(0))^2 dt \le C \int_0^1 t^{\alpha}|z'(t)|^2 dt.$$

For $\alpha > 1$ we use Theorem 2.55 of [13]

$$\int_0^\infty s^{-2}|w(s)|^2 ds \le 4 \int_0^\infty |w'(s)|^2 ds, \qquad w(0) = 0.$$

Setting $t = s^{\frac{1}{1-\alpha}}$, $w(s) = z(s^{\frac{1}{1-\alpha}}) - z(1)$ for $s > 1$ and $w(s) = 0$ for $s \le 1$. Then we get A1.

Lemma A.2. Let $S_\delta = \{r,\theta \mid 0 < r < \delta,\ 0 < \theta < \omega\}$. Then for $0 < \beta < 1$:

i)

$$\|r^{-1}u\|^2_{L_\beta(S_1)} \le C\Big[\sum_{|\alpha|=1} \|r^{\alpha_1-1}\mathcal{D}^\alpha u\|^2_{L_\beta(S_1)} + \|u\|^2_{L_2(S_1-S_{1/2})}\Big] \tag{A.2}$$

ii) for $|\alpha| = 1$

$$\|r^{\alpha_1-2}\mathcal{D}^\alpha u\|^2_{L_\beta(S_1)} \le C\Big[\sum_{|\alpha'|=2} \|r^{\alpha_1'-2}\mathcal{D}u\|^2_{L_\beta(S_1)} + \|u\|^2_{H^1(S_1-S_{1/2})}\Big]. \tag{A.3}$$

<u>Proof</u>. The proof is similar to the one of [5].

1) Let

$$\bar{u}(r) = \frac{1}{\omega}\int_0^\omega u(r,\theta)d\theta$$

then

$$\bar{u}_r(r) = \frac{1}{\omega}\int_0^\omega u_r(r,\theta)d\theta$$

and hence

$$\int_0^1 r^{2\beta+1}|\bar{u}_r(r)|^2 dr \le C\|u_r\|^2_{L_\beta(S_1)}.$$

Using (A.1) we get

$$\int_0^1 r^{2\beta-1}\ |\bar{u}(r) - a|^2 dr \le C\|u_r\|^2_{L_\beta(S_1)}$$

where $a = \bar{u}(1)$, and by the imbedding theorem

$$|a| \le C\int_{1/2}^1 (\bar{u}^2 + \bar{u}_r^2)dr.$$

Hence

$$\int_0^1 r^{2\beta-1}|\bar{u}(r)|^3 dr \quad \le \quad C[\,\|u_r\|^2_{L_\beta(S_1)} + \|u\|^2_{L_2(S_1-S_{1/2})}]$$

and

(A.4) $$\|r^{-1}\bar{u}\|^2_{L_\beta(S_1)} \;\le\; C[\,\|u_r\|^2_{L_\beta(S_1)} + \|u\|^2_{L_2(S_1-S_{1/2})}].$$

Further for almost all r, φ we get

$$u(r,\varphi) - u(r,\psi) \;=\; \int_\psi u_\theta(r,\theta)d\theta$$

and therefore

$$|u(r,\varphi) - \bar{u}(r)| \;=\; |\omega^{-1}\int_0^\omega d\psi \int_\psi^\varphi u_0(r,\theta)d\theta|$$

$$\le\; C[\int_0^\omega |u_\theta(r,\theta)|^2 d\theta]^{1/2}$$

and

$$\int_0^\omega |u(r,\ ) - \bar{u}(r)|^2\, d\theta \;\le\; C\int_0^\omega |u_\theta(r,\theta)|^2 d\theta.$$

Hence

(A.5) $$\int r^{2\beta-2}|u(r,\varphi)-\bar{u}(r)|^2 r\, dr\, d\varphi$$

$$\le\; C\int r^{2\beta-2}|u_\theta(r,\theta)|^2 r\, dr\, d\theta.$$

Combining (A.4) and (A.5) we get (A.2).

2) Let $v = u_r$. Then using (A.2) we have

$$\|r^{-1}u_r\|^2_{L_\beta(S_1)} \le C[\|u_{rr}\|^2_{L_\beta(S_1)} + \|r^{-1}u_{r\theta}\|^2_{L_\beta(S_1)} + \|u_r\|^2_{L_2(S_1-S_{1/2})}]$$

which is (A.3) for $\alpha_1 = 1$, $\alpha_2 = 0$. Let now denote $v = \frac{u_\theta}{r}$ and repeat our argument. Let

$$\bar{v}(r) = \frac{1}{\omega}\int v(r,\theta)d\theta$$

then analogously

$$\text{(A.6)} \qquad \|r^{-1}\bar{v}\|_{L_\beta(S_1)} \le C[\|v_r\|^2_{L_\beta(S_1)} + \|v\|^2_{L_2(S_1-S_{1/2})}]$$

$$= C[\|r^{-1}u_{\theta r}\|_{L_\beta(S_1)} + \|r^{-1}u_\theta\|^2_{L_2(S_1-S_{1/2})}].$$

Now

$$v(r,\varphi) - v(r,\psi) = \int_\psi^\varphi \frac{u_{\theta\theta}}{r}\,d\theta$$

and hence

$$\int_0^\omega |v(r,\varphi) - \bar{v}(r)|^2 d\varphi \le C\int_0^1 \left(\frac{u_{\theta\theta}}{r}\right)^2 d\varphi.$$

Multiplying this equation by $r^{2\beta-1}$ and integrating with respect to r we get

$$\text{(A.7)} \qquad \int_0^1\int_0^\omega r^{2\beta-2}|v(r,\varphi) - \bar{v}(r)|^2 r\,dr\,d\varphi$$

$$\le \int_0^1\int_0^\omega \frac{|u_{\theta\theta}|^2}{r^2}\,r^{2\beta-2}\cdot r\,dr\,d\varphi.$$

Combining (A.6) and (A.7) we get (A.3). ■

Lemma A.3. For $0 < \beta < 1$ one has

$$\text{(A.8)} \qquad \|r^{-1}u\|^2_{L_\beta(S_1)} \leq C[\sum_{|\alpha|=1} \|D^\alpha u\|^2_{L_\beta(S_1)} + \|u\|^2_{L_2(S_1-S_{1/2})}].$$

For $|\alpha| = 1$

$$\text{(A.9)} \qquad \|r^{-1}D^\alpha u\|_{L_\beta(S_1)} \leq C[\sum_{|\alpha|=2} \|D^\alpha u\|^2_{L_\beta(S_1)} + \|u\|^2_{H^1(S_1-S_{1/2})}].$$

Proof. Because

$$\sum_{|\alpha|=1} \|D^\alpha u\|^2_{L_\beta(S_1)} = \sum_{|\alpha|=1} \|r^{\alpha_1-1}\mathcal{D}^\alpha u\|_{L_\beta(S_1)}$$

(A.8) follows immediately from (A.2). Let $v = D^\alpha u$. Then (A.9) follows immediately from A.8. ∎

Lemma A.4. Let $S = \{r,\theta \mid 0 < r < \delta,\ 0 < \theta < \omega\}$. Then for $k < 0$, $i,j = 1,2$

$$\text{(A.10)} \qquad \|r^k u_{x_i x_j r^k}\|_{L_\beta(S)} \leq \|r^k |\mathcal{D}^2 u_{r^k}|\|_{L_\beta(S)}$$

$$+ Ck![\sum_{\substack{2\leq|\alpha|\leq k+1 \\ 0\leq\alpha_2\leq 2}} \frac{(k-|\alpha|+3)}{(|\alpha|-2)!} \|r^{\alpha_1-2}\mathcal{D}^\alpha u\|_{L_\beta(S)} + \|u\|_{H^1(S)}]$$

and for $m \leq k-1$, $k \geq 1$, $i,j = 1,2$

$$\text{(A.11)} \qquad \|r^{k-2+\xi(m,k)} u_{x_i x_j r^m}\|_{L_\beta(S)}$$

$$\leq Cm!(\sum_{\substack{1\leq|\alpha|\leq m+2 \\ 0\leq\alpha_2\leq 2}} \frac{m-|\alpha|+3}{(|\alpha|-2)!} \|r^{\alpha_1-2}\mathcal{D}^\alpha u\|_{L_\beta(S)} + \|u\|_{H^1(S)})$$

and

$$\xi(m,k) = 2 - k + m \quad \text{for} \quad 0 < k - m \leq 2$$

$$\xi(m,k) = 0 \qquad\qquad \text{for} \quad k - m > 2$$

and C is independent of u, but depends on δ and ω.

Proof. We will prove A(10) only for $i = 1$, $j = 0$. Proof of the other two cases is completely analogous. We have

$$u_{x_1^2} = u_{r^2} \cos^2 \theta - u_{r\theta} \frac{\sin 2\theta}{r} + \frac{1}{r^2} u_{\theta^2} \sin^2\theta - \frac{1}{r} u_r \sin^2\theta - \frac{1}{r^2} u_\theta \sin 2\theta.$$

Hence

$$\text{(A.12)} \quad r^k u_{x_1^2 r^k} = r^k(u_{r^{2+k}} \cos^2\theta - u_{r^{k+1}\theta} \frac{\sin 2\theta}{r}$$
$$+ \frac{1}{r^2} u_{r^k\theta^2} \sin^2\theta) - \sin 2\theta \sum_{\ell=0}^{k-1} (-1)^{k-\ell}\binom{k}{\ell}(k-\ell)!r^{\ell+1}u_{r^{\ell+1}\theta}$$
$$+ \sin^2\theta \sum_{\ell=0}^{k-1} (-1)^{k-\ell}\binom{k}{\ell}(k-\ell+1)!r^{\ell-2}u_{r^\ell\theta^2}$$
$$- \sin^2\theta \sum_{\ell=0}^{k} (-1)^{k-\ell}\binom{k}{\ell}(k-\ell)!r^{\ell-1}u_{r^{\ell+1}}$$
$$- \sin\theta \sum_{\ell=0}^{k} (-1)^{k/\ell}\binom{k}{\ell}(k-\ell+1)!r^{\ell-2}u_{r^\ell\theta}$$

which yields

$$\text{(A.13)} \quad \| r^k u_{x_1^2 r^k} \|_{L_\beta(S)} \leq \| r^k |D^2 u_{r^k}| \|_{L_\beta(S)}$$

$$+ k!\Big(\sum_{\ell=0}^{k-1}\frac{1}{\ell!}\|r^{\ell-1}u_{r^{\ell+1}\theta}\|_{L_\beta(S)} + \sum_{\ell=0}^{k-1}\frac{(k-\ell+1)}{\ell!}\|r^{\ell-2}u_{r^\ell\theta^2}\|_{L_\beta(S)}$$

$$+ \sum_{\ell=0}^{k-1}\frac{1}{\ell!}\|r^{\ell-1}u_{r^{\ell+1}}\|_{L_\beta(S)} + \sum_{\ell=0}^{k-1}\Big(\frac{k-\ell+1}{\ell!}\Big)\|r^{\ell-2}u_{r^\ell\theta}\|_{L_\beta(S)}\Big)$$

$$\leq \|r^2|\mathcal{D}^2u_{r^k}|\|_{L_\beta(S)} + Ck!\Big(\sum_{\substack{2\leq|\alpha|\leq k+1\\ \alpha_2\leq 2}}\frac{(k-|\alpha|+3)}{(|\alpha|-2)!}\|r^{\alpha_1-2}\mathcal{D}^\alpha u\|_{L_\beta(S)}$$

$$+ \|r^{-1}u_r\|_{L_\beta(S)} + \|r^{-2}u_\theta\|_{L_\beta(S)}\Big).$$

(A.13) combined with Lemma 2 yields (A.10).

The proof of (A.11) is quite analogous as above. ■

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