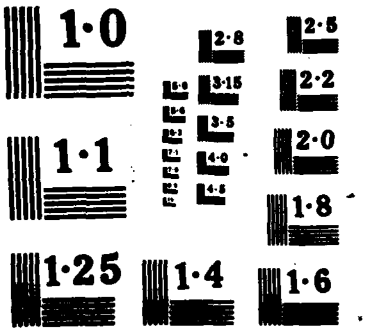
AD-A170 014 EXTREME POINT METHODS IN THE STUDY OF CLASSES OF 1/1
BIVARIATE DISTRIBUTIONS A..(U) PITTSBURGH UNIV PA
CENTER FOR MULTIVARIATE ANALYSIS K SUBRAMANYAM ET AL.
UNCLASSIFIED APR 86 TR-86-12 AFOSR-TR-86-0358 F/G 12/1 NL


1·0
2·8
2·5
3·15
2·2
3·5
2·0
1·1
4·0
4·5
1·8
1·25
1·4
1·6

2

EXTREME POINT METHODS IN THE STUDY OF CLASSES OF BIVARIATE DISTRIBUTIONS AND SOME APPLICATIONS TO CONTINGENCY TABLES

by

K. Subramanyam
Department of Mathematics and Statistics
University of Pittsburgh

M. Bhaskara Rao
Department of Probability and Statistics
University of Sheffield, UK

Center for Multivariate Analysis

University of Pittsburgh

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER
AFOSR-TR- 86-0358

2. GOVT ACCESSION NO.
A170 014

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)
EXTREME POINT METHODS IN THE STUDY OF CLASSES OF BIVARIATE DISTRIBUTIONS AND SOME APPLICATIONS TO CONTINGENCY TABLES

5. TYPE OF REPORT & PERIOD COVERED
Technical - April 1986

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)
K. Subramanyam and M. Bhaskara Rao

8. CONTRACT OR GRANT NUMBER(s)
F49620-85-C-0008

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Center for Multivariate Analysis
515 Thackeray Hall
University of Pittsburgh, Pittsburgh, PA 15260

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
61102F A5
2304

11. CONTROLLING OFFICE NAME AND ADDRESS
Air Force Office of Scientific Research/NM
Department of the Air Force
Bolling Air Force Base, DC 20332

12. REPORT DATE
April 1986

13. NUMBER OF PAGES
63 65

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

15. SECURITY CLASS. (of this report)
Unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release, distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Extreme point, convex set, compact set, bivariate distributions, positive quadrant dependence, negative quadrant dependence, power function and contingency tables.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The set of all bivariate positive quadrant distributions is neither compact nor convex. But the set of all bivariate positive quadrant distributions with fixed marginals is a convex set. These convex sets are compact in the case of discrete bivariate distributions if the marginals have finite support. A simple method to enumerate the extreme points of these convex sets is given.

In the context of contingency tables for testing the null hypothesis independence against the alternative positive quadrant dependence one can use the method of extreme point analysis to compare the performance of various tests.

DD FORM 1 JAN 73 1473

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REFERENCES

[1]. AGRESTI, A. (1984), Analysis of Ordinal Categorical Data, John Wiley, New York.

[2]. EATON, M.L. (1982), A Review of Selected Topics in Multivariate Probability Inequalities, Ann. Stat., 10, 11-43.

[3]. KEMP Jr., L.F. (1968), Construction of Joint Probability Distributions, Ann. Math. Stat., 39, 1354-1357.

[4]. LEHMANN, E.L. (1966), Some Concepts of Dependence, Ann. Math. Stat., 37, 1137-1153.

[5]. PHELPS, R.R. (1966), Lectures of Choquet's Theorem, Van Nostrand, New York.

[6]. BHASKARA RAO, M., KRISHNAIAH, P.R. and SUBRAMANYAM (1985), A Structure Theorem on Bivariate Positive Quadrant Dependent Distributions and Tests for Independence in Two-way Contingency Tables, Technical Report No. 85-48, Center for Multivariate Analysis, University of Pittsburgh, Pittsburgh.

EXTREME POINT METHODS IN THE STUDY OF CLASSES OF BIVARIATE DISTRIBUTIONS AND SOME APPLICATIONS TO CONTINGENCY TABLES

by

K. Subramanyam
Department of Mathematics and Statistics
University of Pittsburgh

M. Bhaskara Rao
Department of Probability and Statistics
University of Sheffield, UK

April 1986

Technical Report 86-12

Center for Multivariate Analysis
Fifth Floor, Thackeray Hall
University of Pittsburgh
Pittsburgh, PA 15260

*Research sponsored by the Air Force Office of Scientific Research (AFSC) under Contract F49620-85-C-0008. The United States Government is authorized to reproduce and distribute reprints for governmental purposes notwithstanding any copyright notation hereon.

EXTREME POINT METHODS IN THE STUDY OF CLASSES OF BIVARIATE DISTRIBUTIONS AND SOME APPLICATIONS TO CONTINGENCY TABLES

by

K. Subramanyam
Department of Mathematics and Statistics
University of Pittsburgh

M. Bhaskara Rao
Department of Probability and Statistics
University of Sheffield, UK

Abstract

The set of all bivariate positive quadrant distributions is neither compact nor convex. But the set of all bivariate positive quadrant distributions with fixed marginals is a convex set. These convex sets are compact in the case of discrete bivariate distributions if the marginals have finite support. A simple method to enumerate the extreme points of these convex sets is given.

In the context of contingency tables for testing the null hypothesis independence against the alternative positive quadrant dependence one can use the method of extreme point analysis to compare the performance of various tests.

AMS 1980 subject classification: Primary: 62H05 Secondary 62H17

Key words and phrases: Extreme point, convex set, compact set, bivariate distributions, positive quadrant dependence, negative quadrant dependence, power function and contingency tables.

1. <u>Introduction</u>. Let M_2 be the collection of all probability measures defined on the Borel σ-field of the two-dimensional Euclidean space, R^2. For μ in M_2; let μ_1 and μ_2 denote the marginal probability measures on the Borel σ-field of R, i.e., $\mu_1(A) = \mu(A \times R)$ and $\mu_2(A) = \mu(R \times A)$ for every Borel subset of R. A μ in M_2 is said to be positively (negatively) quadrant dependent (PQD) (NQD) if $\mu\{[c, \infty) \times [d, \infty)\} \geq (\leq) \ \mu_1\{[c, \infty)\} \ \mu_2\{[d, \infty)\}$ for all c, d in R. Let M(PQD) denote the collection of all probability measures in M_2 which are PQD and M(NQD) the collection of those in M_2 which are NQD. The main objective of this paper is to study the structure of M(PQD) and M(NQD). Using the structure theorem given in Section 3, we give some applications in the area of testing hypotheses in Contingency Tables in Section 4. For further information on properties of measures which are PQD or NQD, see Lehman [4] or Eaton [2].

A-1

2. Preliminary results. The sets $M(PQD)$ and $M(NQD)$ are not convex sets. Examples are easy to provide. We give a simple sufficient condition under which convex combinations of measures in $M(PQD)$ are again in $M(PQD)$. Let M_1 denote the collection of all probability measures on the Borel σ-field of R. For ν, η in M_1, ν is said to be stochastically smaller than η if $\nu\{[c, \infty)\} \leq \eta\{[c, \infty)\}$ for all c in R. If ν is stochastically smaller than η, we use the notation $\nu \leq st\ \eta$. The relation "$\leq st$" is a partial order on M_1. For μ, λ in M_2, we say that μ is stochastically smaller than λ if $\mu_1 \leq st\ \lambda_1$ and $\mu_2 \leq st\ \lambda_2$. In this case, we again use the same notation $\mu \leq st\ \lambda$. The relation "$\leq st$" on M_2 is only a pre-order. See Definition 2.1 of Eaton [, p.11].

Proposition 1. If $\mu, \lambda \in M(PQD)$ and either $\mu \leq st\ \lambda$ or $\lambda \leq st\ \mu$, then $\alpha\mu + (1-\alpha)\lambda \in M(PQD)$ for any $0 \leq \alpha \leq 1$.

Proof. We need to show that $[\alpha\mu + (1-\alpha)\lambda]\{[c, \infty) \times [d, \infty)\} \geq [\alpha\mu_1 + (1-\alpha)\lambda_1]\{[c,\infty)\}\ [\alpha\mu_2 + (1-\alpha)\lambda_2]\{[d,\infty)\}$ for all c, d in R. We note that $[\alpha\mu + (1-\alpha)\lambda]\{[c, \infty) \times [d, \infty)\} \geq \alpha\mu_1\{[c, \infty)\}\ \mu_2\{[d, \infty)\} + (1-\alpha)\ \lambda_1\{[c, \infty)\}\ \lambda_2\{[d, \infty)\}$ and $\alpha\mu_1\{[c, \infty)\}\ \mu_2\{[d, \infty)\} + (1-\alpha)\ \lambda_1\{[c, \infty)\}\ \lambda_2\{[d, \infty)\} - \alpha^2\ \mu_1\{[c, \infty)\}\ \mu_2\{[d, \infty)\} - \alpha(1-\alpha)\ \mu_1\{[c, \infty)\}\ \lambda_2\{[d, \infty)\} - \alpha(1-\alpha)\ \mu_2\{[d, \infty)\}\ \lambda_1\{[c, \infty)\} - (1-\alpha)^2\ \lambda_1\{[c, \infty)\}\ \lambda_2\{[d, \infty)\} = \alpha(1-\alpha)[\mu_1\{[c, \infty)\}\ \mu_2\{[d, \infty)\} + \lambda_1\{[c, \infty)\}\ \lambda_2\{[d, \infty)\} - \mu_2\{[d, \infty)\}\ \lambda_1\{[c, \infty)\} - \mu_1\{[c, \infty)\}\ \lambda_2\{[d, \infty)\}]$ $= \alpha(1-\alpha)\ [\mu_1\{[c, \infty)\} - \lambda_1\{[c, \infty)\}]\ [\mu_2\{[d, \infty)\} - \lambda_2\{[d, \infty)\}] \geq 0$, if $\mu \leq st\ \lambda$ or $\lambda \leq st\ \mu$. This completes the proof.

An analogous result holds for measures which are NQD as declared in the the following proposition.

Proposition 2. Let $\mu, \lambda \in M(NQD)$. If $\mu_1 \leq st\ \lambda_1$ and $\lambda_2 \leq st\ \mu_2$ or $\lambda_1 \leq st\ \mu_1$ and $\mu_2 \leq st\ \lambda_2$, then $\alpha\mu + (1-\alpha)\lambda \in M(NQD)$ for any $0 < \alpha < 1$.

Corollary 3. Let $\mu, \lambda \in M_2$ have the same marginals, i.e., $\mu_1 = \lambda_1$ and $\mu_2 = \lambda_2$. If $\mu, \lambda \in M(PQD)$, then $\alpha\mu + (1-\alpha)\lambda \in M(PQD)$ for any $0 < \alpha < 1$. If $\mu, \lambda \in M(NQD)$, then $\alpha\mu + (1-\alpha)\lambda \in M(NQD)$.

In order to determine the structure of $M(PQD)$ and $M(NQD)$, Corollary 3 provides a pointer. Let μ_1 and μ_2 be two given probability measures on the Borel σ-field of the real line R. Let

$$M_{PQD}(\mu_1, \mu_2) = \{\lambda \in M(PQD) ;\ \lambda_1 = \mu_1 \text{ and } \lambda_2 = \mu_2\},$$

and

$$M_{NQD}(\mu_1, \mu_2) = \{\lambda \in M(NQD) ;\ \lambda_1 = \mu_1 \text{ and } \lambda_2 = \mu_2\}.$$

$M_{PQD}(\mu_1, \mu_2)$ is the collection of all those probability measures λ which are PQD and having the same marginals μ_1 and μ_2. This set is more tractable than $M(PQD)$ as the following corollary points out.

Corollary 4. $M_{PQD}(\mu_1, \mu_2)$ and $M_{NQD}(\mu_1, \mu_2)$ are convex sets.

Our next objective is to determine the extreme points of the above convex sets. If $M_{PQD}(\mu_1, \mu_2)$ is a compact set in some relevant nice topology, then one can write every λ in $M_{PQD}(\mu_1, \mu_2)$ as a mixture of its extreme points. See Phelps [ ]. Then a knowledge of all extreme points of the set $M_{PQD}(\mu_1, \mu_2)$ provides us an insight into the structure of $M_{PQD}(\mu_1, \mu_2)$. In the sequel, we concern ourselves exclusively with the convex set $M_{PQD}(\mu_1, \mu_2)$. The convex set $M_{NQD}(\mu_1, \mu_2$

can be studied in a similar vein. The determination of all extreme points of the set $M_{PQD}(\mu_1, \mu_2)$ seems to be difficult in the general case. However, in Section 3, we give a complete description of all extreme points of $M_{PQD}(\mu_1, \mu_2)$ when μ_1 and μ_2 are some special discrete measures with finite support. The method given in Section 3 can be adapted for any discrete measures μ_1 and μ_2 with finite support. Before sailing into the next section, we identify one particular measure as an extreme point of $M_{PQD}(\mu_1, \mu_2)$ in the general case. For probability measures μ_1 and μ_2 on the Borel σ-field of the real line, let $\mu_1 \times \mu_2$ denote the product probability measure in M_2.

Proposition 5. The product probability measure $\mu_1 \times \mu_2$ is an extreme point of $M_{PQD}(\mu_1, \mu_2)$ and also of $M_{NQD}(\mu_1, \mu_2)$.

Proof. One can prove this result by using the elementary fact that if $c \geq a$, $d \geq a$, $0 < \alpha < 1$ and $\alpha c + (1 - \alpha)d = a$, then $c = a$ and $d = a$, and the fact that any measure in M_2 is determined by its values on sets of the form $[c, \infty) \times [d, \infty)$, c, d real.

Remark. The above result is in marked contrast to the following result of Kemp [3]. Let $M(\mu_1, \mu_2)$ denote the collection of all probability measures λ in M_2 satisfying $\lambda_1 = \mu_1$ and $\lambda_2 = \mu_2$. Then $M(\mu_1, \mu_2)$ is a convex set, and $\mu_1 \times \mu_2$ is an extreme point of $M(\mu_1, \mu_2)$ if and only if either μ_1 is 0-1 valued or μ_2 is 0-1 valued.

3. Extreme points. In this section, we assume that the probability measures μ_1 and μ_2 have finite support. For simplicity, assume that the support of μ_1 is $\{1,2,3,\ldots,m\}$ and that of μ_2 is $\{1,2,3,\ldots,n\}$. Any probability measure in M_2 with support contained in $\{(i,j) ; i = 1$ to m and $j = 1$ to $n\}$ and with marginals μ_1 and μ_2 can be identified as a matrix $P = \{p_{ij}\}$ of order $m \times n$ satisfying $p_{ij} \geq 0$ for all i and j, $\Sigma_{i=1}^{m} p_{ij} = \mu_2(\{j\})$ for all $j = 1$ to n, and $\Sigma_{j=1}^{n} p_{ij} = \mu_1(\{i\})$ for all $i = 1$ to m. Let $p_i = \mu_1(\{i\})$, $i = 1$ to m and $q_j = \mu_2(\{j\})$, $j = 1$ to n. The objects of interest are the convex sets $M_{PQD}(\mu_1,\mu_2)$ and $M_{NQD}(\mu_1,\mu_2)$. We denote $M_{PQD}(\mu_1,\mu_2)$ by $M_{PQD}(p_1,p_2,\ldots,p_m;q_1,q_2,\ldots,q_n)$ and $M_{NQD}(p_1,p_2,\ldots,p_m;q_1,q_2,\ldots,q_n)$. These convex sets are, obviosly, compact. Moreover, they have each a finite number of extreme points. Each of these sets is determined by a finite number of hyperplanes in R^{m+n}. Now, we describe a method of generating extreme points of $M_{PQD}(p_1,p_2,\ldots,p_m;q_1,q_2,\ldots,q_n)$ in some special cases of m and n. The method so described generalizes to for any m and n.

The case of 2 x 2 tables.

Let p, q be two numbers in the open interval (0, 1). Let $p_1 = p$, $p_2 = 1 - p$, $q_1 = q$ and $q_2 = 1 - q$. A matrix $P = \{p_{ij}\}$ of order 2×2 with the properties $p_{ij} \geq 0$ for all i and j, $p_{11} + p_{12} = p$, $p_{21} + p_{22} = 1 - p$, $p_{11} + p_{21} = q$ and $p_{12} + p_{22} = 1 - q$ is a member of $M_{PQD}(p_1,p_2;q_1,q_2)$ if and only if $p_2q_2 \leq p_{22} \leq p_2 \wedge q_2$, $a \wedge b$ denotes the minimum of the two numbers a and b. The extreme points of this convex set $M_{PQD}(p_1,p_2;q_1,q_2)$ are given by

$$\begin{bmatrix} p_1q_1 & p_1q_2 \\ p_2q_1 & p_2q_2 \end{bmatrix} \text{ and } \begin{bmatrix} q_1 & q_2 - p_2\wedge q_2 \\ 0 & p_2\wedge q_2 \end{bmatrix} \text{ when } p_2\wedge q_2 = p_2;$$

$$\begin{bmatrix} p_1q_1 & p_1q_2 \\ p_2q_1 & p_2q_2 \end{bmatrix} \text{ and } \begin{bmatrix} p_1 & 0 \\ p_2 - p_2\wedge q_2 & p_2\wedge q_2 \end{bmatrix} \text{ when } p_2\wedge q_2 = q_2.$$

If $p = q = 1/2$, then the extreme points of $M_{PQD}(1/2,1/2;1/2,1/2)$ are given by

$$\begin{bmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix} \text{ and } \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}.$$

On the other hand, the extreme points of the set $M(1/2,1/2;1/2,1/2)$, the set of all matrices $P = \{p_{ij}\}$ of order 2×2 with the properties $p_{ij} \geq 0$ for all i and j, $p_{11} + p_{12} = p_{21} + p_{22} = p_{11} + p_{21} = p_{12} + p_{22} = 1/2$, are given by

$$\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}.$$

The case of 2 x 3 tables

Let $p_1,p_2;q_1,q_2,q_3$ be five positive numbers satisfying $p_1 + p_2 = q_1 + q_2 + q_3 = 1$. We now discuss the problem of identifying the extreme points of $M_{PQD}(p_1,p_2;q_1,q_2,q_3)$. This convex set is precisely the collection of all matric

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \end{bmatrix}$$

satisfying $p_{11} + p_{12} + p_{13} = p_1$, $p_{21} + p_{22} + p_{23} = p_2$, $p_{11} + p_{21} = q_1$, $p_{12} + p_{22} = q_2$, $p_{13} + p_{23} = q_3$, p_{23} p_2q_3, $p_{22} + p_{23}$ $p_2q_2 + p_2q_3$ and each p_{ij} is non-negative.The problem of identifying the extreme points of this convex set can be solved both graphically and algebraically. The following results are helpful towards this goal.

Theorem 6. Let $P = (p_{ij})$ be a matrix of order 2 x 3. If P is a member of $M_{PQD}(p_1,p_2;q_1,q_2,q_3)$, then

(1) $p_2q_3 \leq p_{23} \leq p_2 \wedge q_3$;

(2) $(p_2q_2 + p_2q_3) \vee p_{23} \leq p_{22} + p_{23} \leq p_2 \wedge (q_2 + p_{23})$;

(3) $p_{12} = q_2 - p_{22}$, $p_{13} = q_3 - p_{23}$, $p_{21} = p_2 - p_{22} - p_{23}$ and
$p_{11} = p_1 - p_{12} - p_{13} = q_1 - p_{21}$.

Proof. Since P is positively quadrant dependent, $p_{23} \geq p_2q_3$ and $p_{22} + p_{23} \geq p_2q_2 + p_2q_3$. Since P has the prescribed marginals and non-negative entries, $p_{23} \leq p_2 \wedge q_3$, $p_{22} + p_{23} \geq p_{23}$, $p_{22} + p_{23} \leq p_2$ and $p_{22} + p_{23} \leq q_2 + p_{23}$. This proves (1) and (2). (3) is obvious. (Note. For any two real numbers a and b, $a \vee b$ means the maximum of a and b.)

In the above theorem, the inequalities (1) and (2) are very important. They help to reduce the problem of identifying the extreme points of $M_{PQD}(p_1,p_2;q_1,q_2,q_3)$ to a two-dimensional graphical problem. The following result amplifies this remark.

Theorem 7. Let p_{22} and p_{23} be two real numbers satisfying

(1) $p_2q_3 \leq p_{23} \leq p_2 \wedge q_3$ and

(2) $p_{23} \vee (p_2q_2 + p_2q_3) \leq p_{22} + p_{23} \leq p_2 \wedge (q_2 + p_{23})$.

Let $p_{12} = q_2 - p_{22}$, $p_{13} = q_3 - p_{23}$ and $p_{21} = p_2 - p_{22} - p_{23}$. Then $p_1 - p_{12} - p_{13} = q_1 - p_{21} = p_{11}$, say, and the matrix $P = (p_{ij})$ is a member of $M_{PQD}(p_1,p_2;q_1,q_2,q_3)$.

Proof. First, we show that $p_1 - p_{12} - p_{13} = q_1 - p_{21}$. Note that $p_{12} + p_{13} = q_2 + q_3 - (p_{22} + p_{23})$ and $q_1 + p_{12} + p_{13} = 1 - (p_{22} + p_{23}) = p_1 + p_2 + p_{21} - p_2 = p_1 + p_{21}$. Consequently, $p_1 - (p_{12} + p_{13}) = q_1 - p_{21}$. Next, we show that each p_{ij} is non-negative. From (1), $p_{23} > 0$ and $q_3 - p_{23} = p_{13} \geq 0$. From (2), $p_{22} \geq 0$, $p_2 - p_{22} - p_{23} = p_{21} \geq 0$ and $q_2 - p_{22} = p_{12} \geq 0$. It remains to be shown that p_{11} is non-negative. It is clear that $p_{11} + p_{12} + p_{13} + p_{21} + p_{22} + p_{23} = 1$. It suffices to show that $p_{12} + p_{13} + p_{21} + p_{22} + p_{23} \leq 1$. Note that $p_{12} + p_{13} + p_{21} + p_{22} + p_{23} = q_2 + q_3 - (p_{22} + p_{23}) + p_2 - (p_{22} + p_{23}) + (p_{22} + p_{23})$ $= q_2 + q_3 + p_2 - (p_{22} + p_{23}) \leq q_2 + q_3 + p_2 - p_2q_2 - p_2q_3$ (by (2)) $= q_2 + q_3 + p_2(1 - q_2 - q_3) = 1 - q_1 + p_2q_1 = 1 - q_1(1 - p_2) = 1 - p_1q_1 < 1$. This shows that p_{11} is always strictly positive.

It is obvious that the matrix $P = (p_{ij})$ has the prescribed marginals p_1,p_2 and q_1,q_2,q_3. From (1) and (2), it follows that P is positively quadrant dependent.

The purport of the above theorem is that it is enough to determine two numbers p_{22} and p_{23} satisfying (1) and (2) from which we can build a matrix $P = (p_{ij})$ belonging to $M_{PQD}(p_1,p_2;q_1,q_2,q_3)$. We call (1) and (2) to be the determining inequalities. This theorem also provides modus operandi to construct extreme points of $M_{PQD}(p_1,p_2;q_1,q_2,q_3)$. Intuitively, one can proceed by looking at those pairs p_{22} and p_{23} of numbers for which equality holds in (1) and (2) either on the right or on the left to yield extreme points. This method would not exhaust all the extreme points of $M_{PQD}(p_1,p_2;q_1,q_2,q_3$

The reason is that the bounds in (2) are variable and depend on the value of p_{23}. If we can partition the interval $[p_2q_3, p_2 \wedge q_3]$ into sub-intervals so that the bounds in (2) become stable over each sub-interval, then we can isolate the extreme points. We explain this method graphically and algebraically as follows.

Graphical solution

Determining inequalities

(1) $p_2q_3 \leq p_{23} \leq p_2 \wedge q_3$.

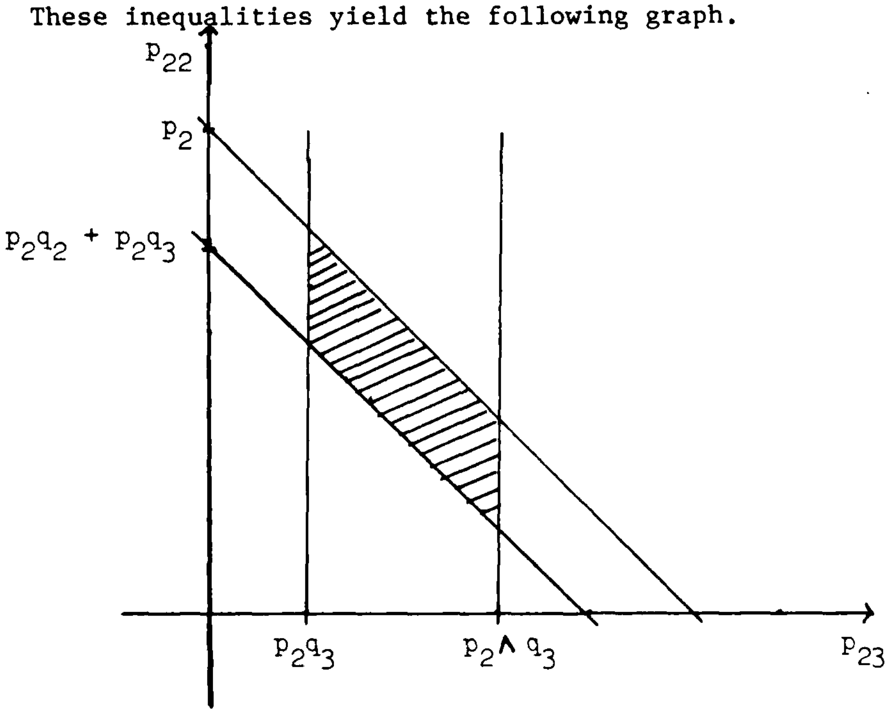
(2) $p_{23} \vee (p_2q_2 + p_2q_3) \leq p_{22} + p_{23} \leq p_2 \wedge (q_2 + p_{23})$.

Case 1 $p_2q_2 + p_2q_3 \geq p_2 \wedge q_3$ and $q_2 \geq p_2$. Then the determining inequalities become

(1) $p_2q_3 \leq p_{23} \leq p_2 \wedge q_3$ and

(2) $p_2q_2 + p_2q_3 \leq p_{22} + p_{23} \leq p_2$.

These inequalities yield the following graph.



Comments. The extreme points, in this case, are obtained by setting each expression in the centre of (1) and (2) equal to the quantities either on the left or on the right of (1) and (2). This will give us 4 extreme points.

Case 2. $p_2q_2 + p_2q_3 \geq p_2 \wedge q_3$ and $q_2 < p_2$.

Then either (A) $p_2 - q_2 \leq p_2q_3$ or (B) $p_2q_3 < p_2 - q_2 < p_2 \wedge q_3$ or (C) $p_2 - q_2 \geq p_2 \wedge q_3$ holds. The determining inequalities and the corresponding graphs then respectively work out as follows.

Case 2(A). Inequalities

(1) $p_2q_3 \leq p_{23} \leq p_2 \wedge q_3$ and

(2) $p_2q_2 + p_2q_3 \leq p_{22} + p_{23} \leq p_2$.

Graph. Same as in Case 1.

Case 2(B). Inequalities

(1) $p_2q_3 \leq p_{23} \leq p_2 - q_2$ and

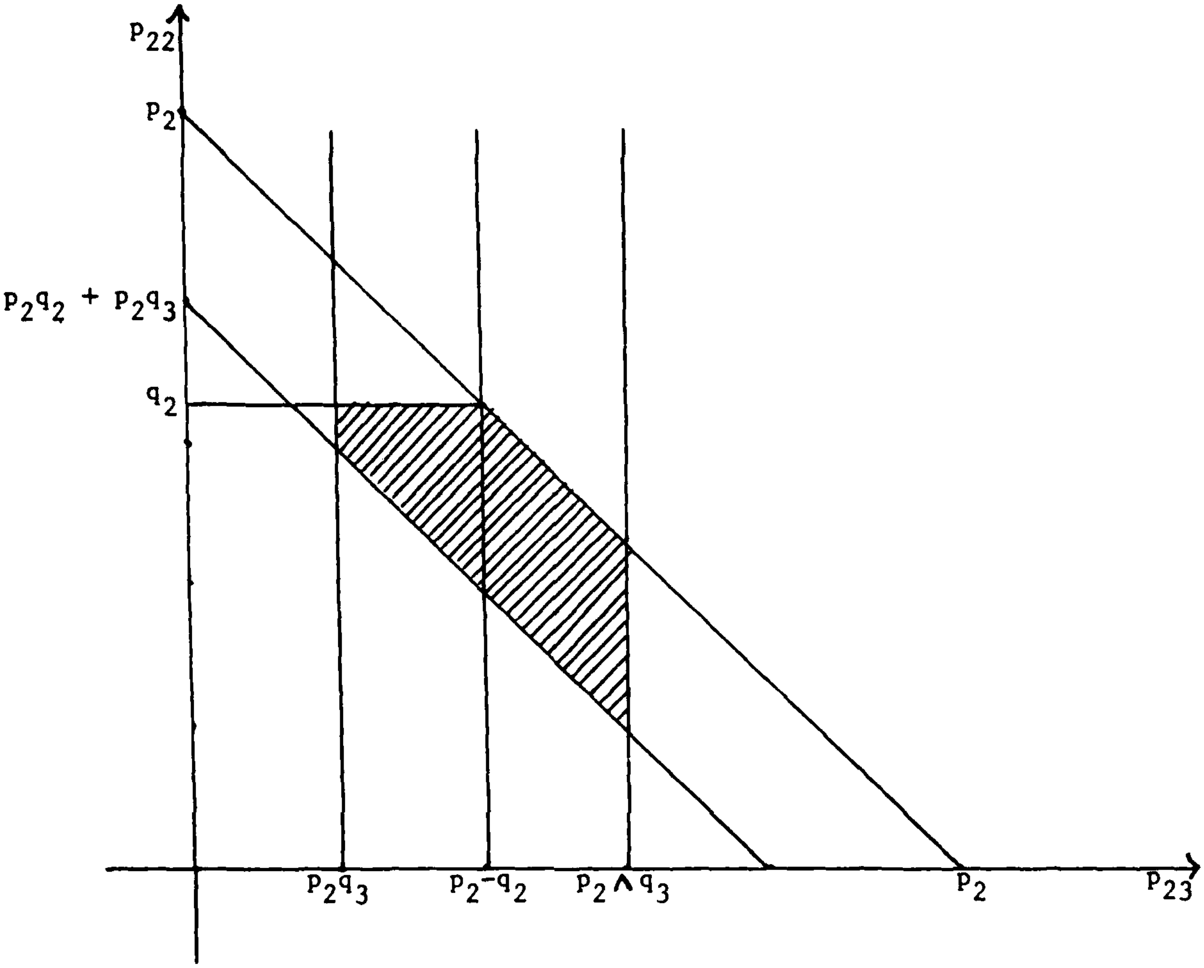
(2) $p_2q_2 + p_2q_3 \leq p_{22} + p_{23} \leq q_2 + p_{23}$

or,

(1)' $p_2 - q_2 < p_{23} \leq p_2 \wedge q_3$ and

(2)' $p_2q_2 + p_2q_3 \leq p_{22} + p_{23} \leq q_2 + p_{23}$.

Graph



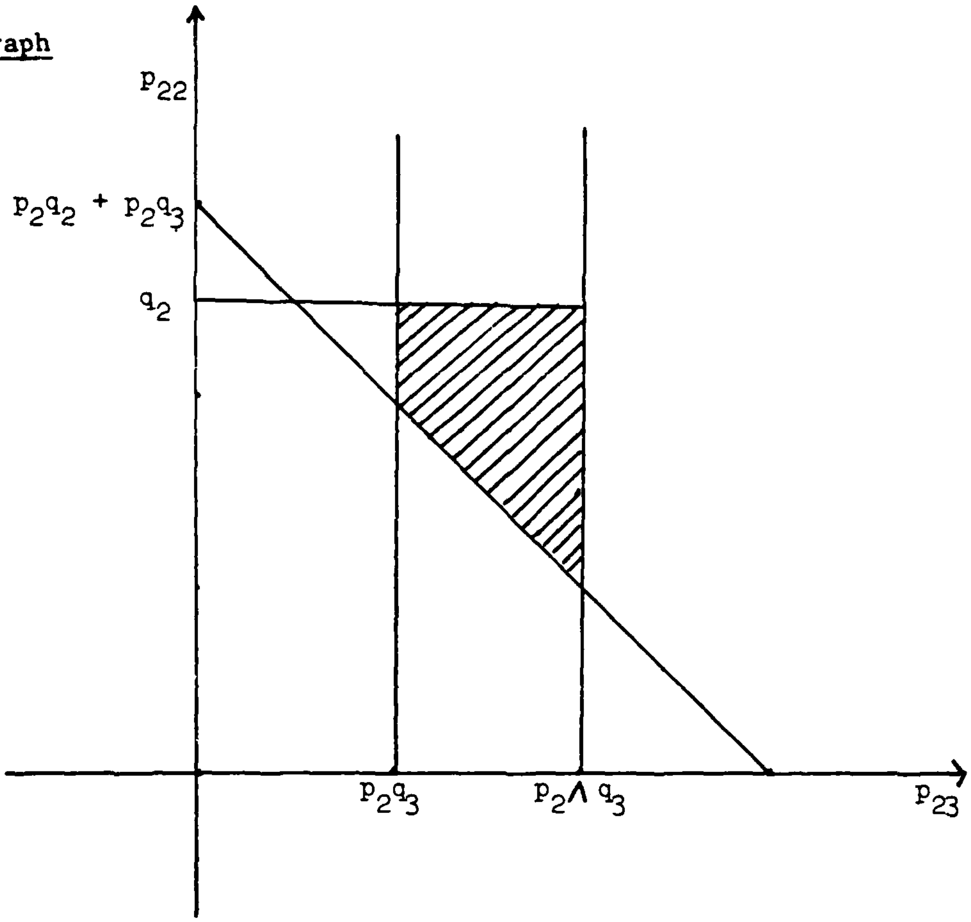
Comments. q_2 can be above $p_2q_2 + p_2q_3$. The number of extreme points is either four or five.

Case 2(C). Inequalities

(1) $p_2q_3 \leq p_{23} \leq p_2 \wedge q_3$ and

(2) $p_2q_2 + p_2q_3 \leq p_{22} + p_{23} \leq q_2 + p_{23}$.

<u>Graph</u>



<u>Comments</u>. q_2 can be above $p_2q_2 + p_2q_3$. The number of extreme points is four.

<u>Case 3</u> $p_2q_2 + p_2q_3 \leq p_2 \wedge q_3$ and $p_2 \leq q_2$.

The determining inequalities become

(1) $p_2q_3 \leq p_{23} \leq p_2q_2 + p_2q_3$ and

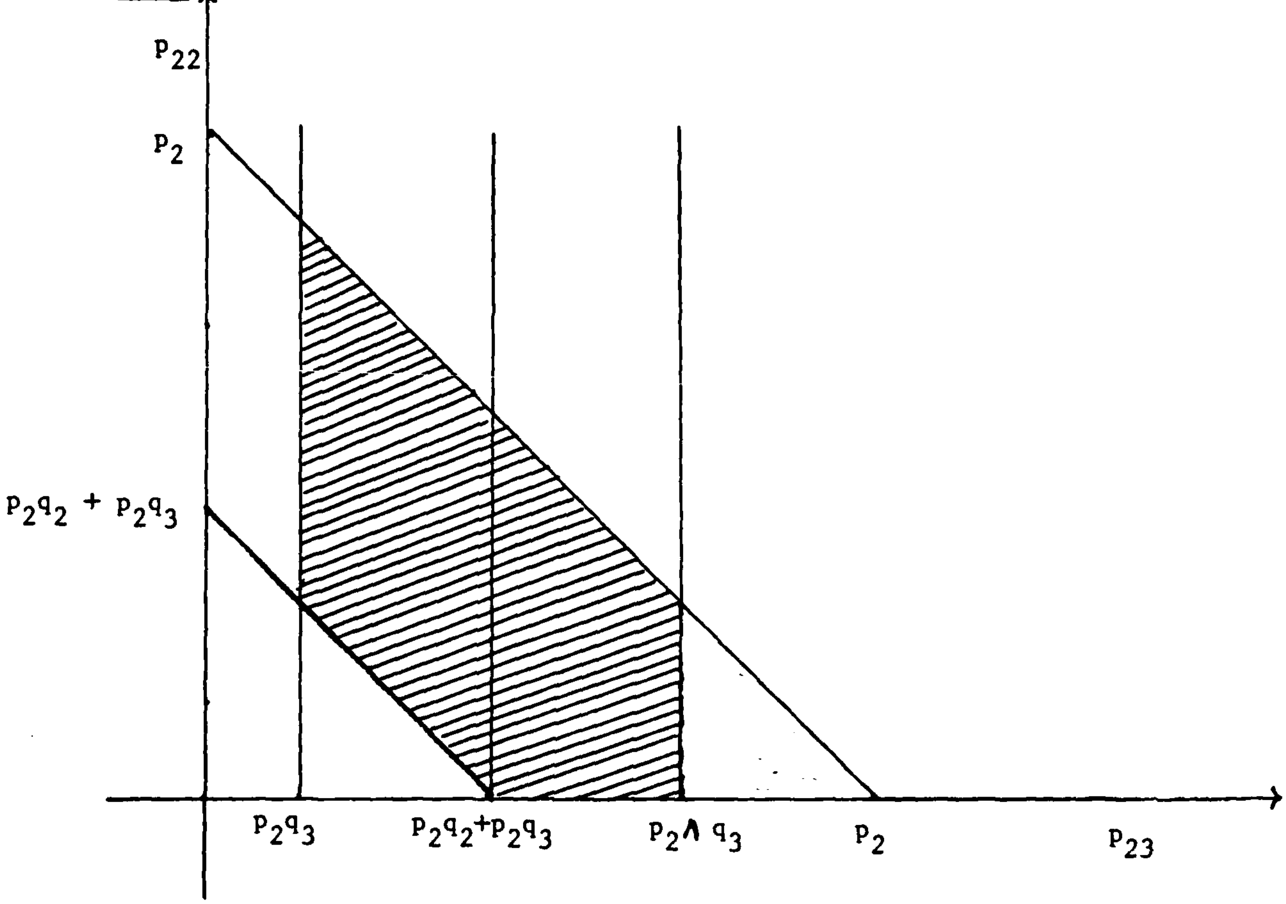
(2) $p_2q_2 + p_2q_3 \leq p_{22} + p_{23} \leq p_2$,

or,

(1) $p_2q_2 + p_2q_3 < p_{23} \leq p_2 \wedge q_3$ and

(2) $p_{23} \leq p_{22} + p_{23} \leq p_2$.

Graph



Comment. The number of extreme points is five.

Case 4 $p_2q_2 + p_2q_3 < p_2\wedge q_3$ and $p_2 > q_2$.

Then either (A) $p_2 - q_2 \geq p_2\wedge q_3$ or (B) $p_2q_3 < p_2 - q_2 < p_2\wedge q_3$ or (C) $p_2 - q_2 \leq p_2q_3$ holds. We now discuss each of these cases.

Case 4(A). Inequalities

(1) $p_2q_3 \leq p_{23} \leq p_2q_2 + p_2q_3$ and

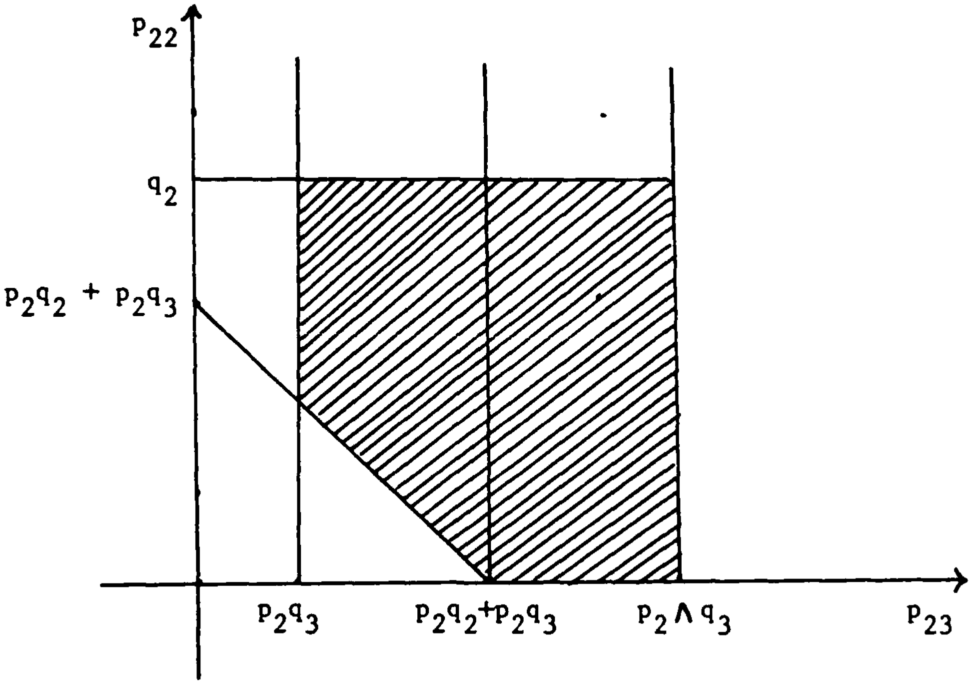
(2) $p_2q_2 + p_2q_3 \leq p_{22} + p_{23} \leq q_2 + p_{23}$,

or,

(1) $p_2q_2 + p_2q_3 < p_{23} \leq p_2\wedge q_3$ and

(2) $p_{23} \leq p_{22} + p_{23} \leq q_2 + p_{23}$.

Graph



Comments. q_2 can be below $p_2q_2 + p_2q_3$. There are five extreme points.

Case 4(B). Assume, without loss of generality, that $p_2 - q_2 \leq p_2q_2 + p_2q_3$.

Inequalities

(1) $p_2q_3 \leq p_{23} \leq p_2 - q_2$ and

(2) $p_2q_2 + p_2q_3 \leq p_{22} + p_{23} \leq q_2 + p_{23}$,

or,

(1) $p_2 - q_2 < p_{23} \leq p_2q_2 + p_2q_3$ and

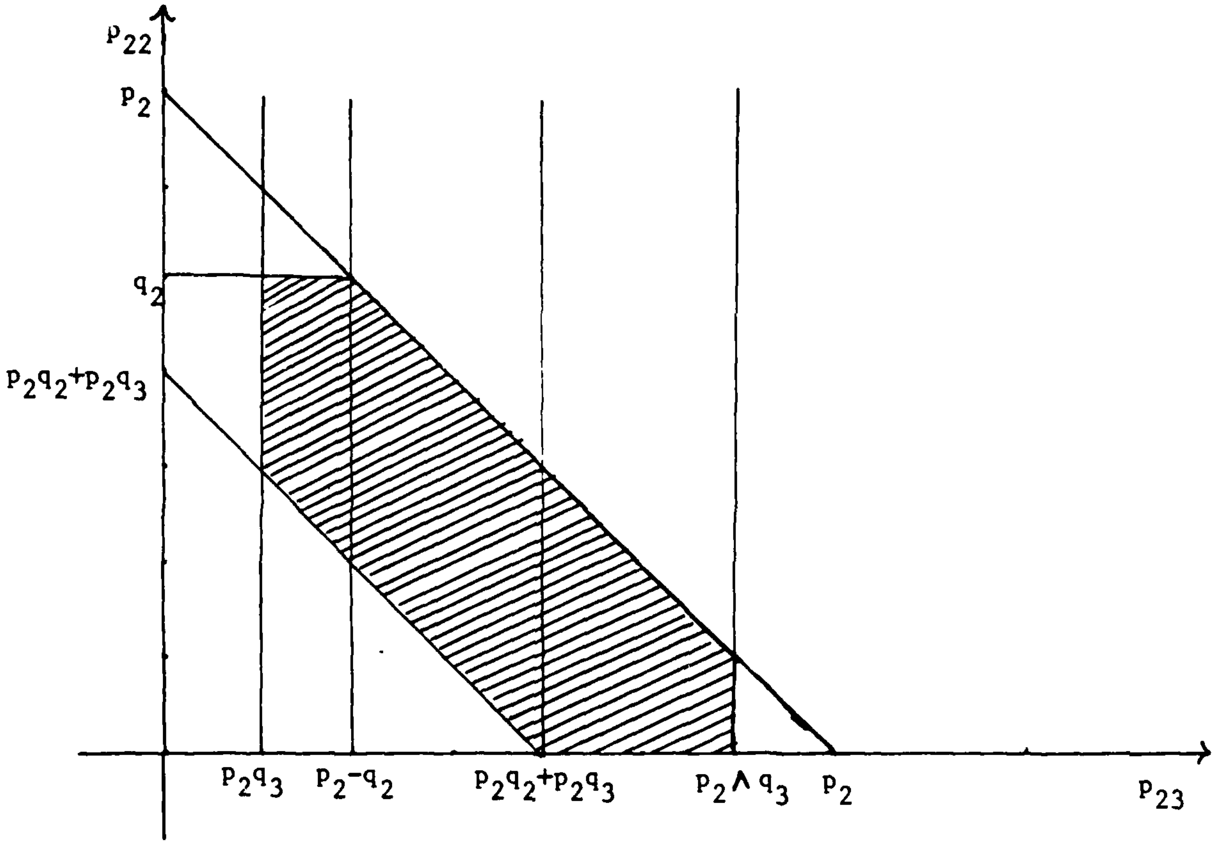
(2) $p_2q_2 + p_2q_3 \leq p_{22} + p_{23} \leq p_2$,

or,

(1) $p_2q_2 + p_2q_3 < p_{23} \leq p_2 \wedge q_3$ and

(2) $p_{23} \leq p_{22} + p_{23} \leq p_2$.

Graph



Comment. The number of extreme points is six.

Case 4(C). Inequalities

(1) $p_2q_3 \leq p_{23} \leq p_2q_2 + p_2q_3$ and

(2) $p_2q_2 + p_2q_3 \leq p_{22} + p_{23} \leq p_2$,

or,

(1) $p_2q_2 + p_2q_3 < p_{23} \leq p_2 \wedge q_3$ and

(2) $p_{23} \leq p_{22} + p_{23} \leq p_2$.

Graph. It is similar to the one given under Case 3.

Algebraic solution

The expression $(p_2q_2 + p_2q_3) \vee p_{23}$ induces a partition of the interval $[p_2q_3, p_2 \wedge q_3]$ into two sub-intervals. These intervals are given by

$$A_1 = \{p_{23} \in [p_2q_3, p_2 \wedge q_3] ; (p_2q_2 + p_2q_3) \vee p_{23} = p_2q_2 + p_2q_3\}$$

and

$$A_2 = \{p_{23} \in [p_2q_3, p_2 \wedge q_3] ; (p_2q_2 + p_2q_3) \vee p_{23} = p_{23}\}.$$

A_1 and A_2 may have a common end point. In a similar way, the expression $p_2 \wedge (q_2 + p_{23})$ induces a partition B_1 and B_2 of the interval $[p_2q_3, p_2 \wedge q_3]$ into two sub-intervals. The partitions $\{A_1, A_2\}$ and $\{B_1, B_2\}$ of $[p_2q_3, p_2 \wedge q_3]$ induce a finer partition $\{C_1, C_2, C_3\}$ of $[p_2q_3, p_2 \wedge q_3]$. Let

$$C_1 = [p_2q_3, a_1] ; \quad C_2 = [a_1, a_2] ; \quad C_3 = [a_2, p_2 \wedge q_3] .$$

We now consider three sets of inequalities.

(A) (1) $p_2q_3 \leq p_{23} \leq a_1$ and

(2) $p_{23} \vee (p_2q_2 + p_2q_3) \leq p_{22} + p_{23} \leq p_2 \wedge (q_2 + p_{23})$.

(B) (1) $a_1 \leq p_{23} \leq a_2$ and

(2) $p_{23} \vee (p_2q_2 + p_2q_3) \leq p_{22} + p_{23} \leq p_2 \wedge (q_2 + p_{23})$.

(C) (1) $a_2 \leq p_{23} \leq p_2 \wedge q_3$ and

(2) $p_{23} \vee (p_2q_2 + p_2q_3) \leq p_{22} + p_{23} \leq p_2 \wedge (q_2 + p_{23})$.

In each set, in the inequality (2), the max and min symbols $\vee$, $\wedge$ would disappear after due simplification. From each set, the central expression in each inequality is set equal to the quantity either on the right or on the left. These equations are solved for unknown values of p_{22} and p_{23}. The set of matrices so obtained after using Theorem 7 contains the set of all extreme points of $M_{PQD}(p_1, p_2; q_1, q_2, q_3)$.

A numerical illustration

Let $p_1 = 0.4$, $p_2 = 0.6$; $q_1 = 0.3$, $q_2 = 0.2$ and $q_3 = 0.5$. The inequalities (1) and (2) of Theorem 7 translate into

(1) $0.3 \leq p_{23} \leq 0.5$ and

(2) $p_{23} \vee 0.42 \leq p_{22} + p_{23} \leq 0.6 \wedge (0.2 + p_{23})$.

This set of inequalities come under Case 4(B) and is equivalent to

(1) $0.3 \leq p_{23} \leq 0.4$ and

(2) $0.42 \leq p_{22} + p_{23} \leq (0.2 + p_{23})$,

or,

(1) $0.4 < p_{23} \leq 0.42$ and

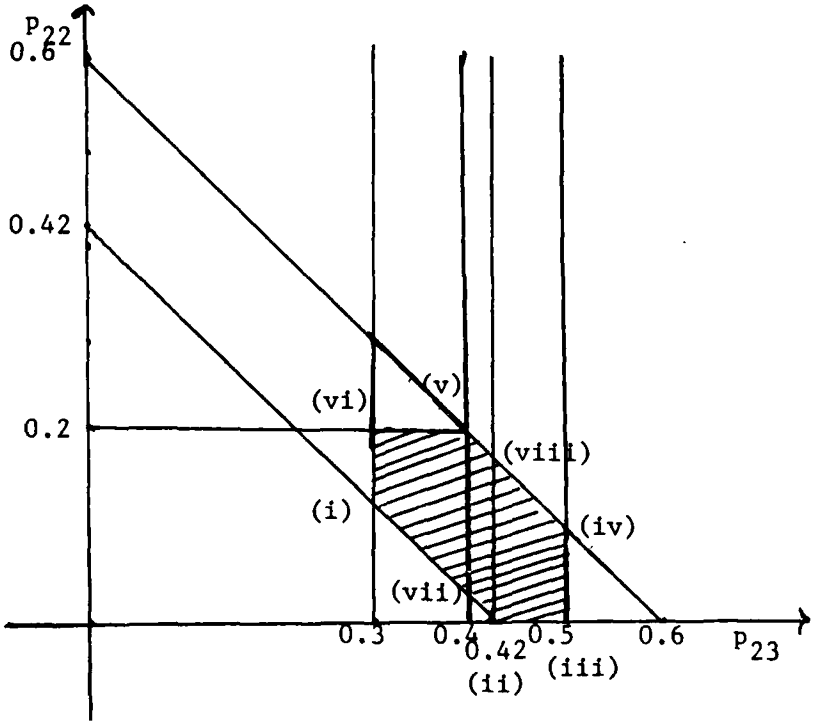
(2) $0.42 \leq p_{22} + p_{23} \leq 0.6$,

or,

(1) $0.42 < p_{23} \leq 0.5$ and

(2) $p_{23} \leq p_{22} + p_{23} \leq 0.6$.

Graph



Extreme points as marked on the graph :

(i) $\begin{bmatrix} 0.12 & 0.08 & 0.2 \\ 0.18 & 0.12 & 0.3 \end{bmatrix}$ (iv) $\begin{bmatrix} 0.3 & 0.1 & 0 \\ 0 & 0.1 & 0.5 \end{bmatrix}$

(ii) $\begin{bmatrix} 0.12 & 0.2 & 0.08 \\ 0.18 & 0 & 0.42 \end{bmatrix}$ (v) $\begin{bmatrix} 0.3 & 0 & 0.1 \\ 0 & 0.2 & 0.4 \end{bmatrix}$

(iii) $\begin{bmatrix} 0.2 & 0.2 & 0 \\ 0.1 & 0 & 0.5 \end{bmatrix}$ (vi) $\begin{bmatrix} 0.2 & 0 & 0.2 \\ 0.1 & 0.2 & 0.3 \end{bmatrix}$.

The algebraic method explained above gives $a_1 = 0.4$ and $a_2 = 0.42$. Solving equations, we obtain the following matrices in addition to those listed above.

(vii) $\begin{bmatrix} 0.12 & 0.18 & 0.1 \\ 0.18 & 0.02 & 0.4 \end{bmatrix}$ (viii) $\begin{bmatrix} 0.3 & 0.02 & 0.08 \\ 0 & 0.18 & 0.42 \end{bmatrix}$.

These matrices are not extreme points. Non-extreme points can easily be weeded out from the list of points provided by the algebraic method.

The case of 2 x n tables

Let $p_1, p_2 ; q_1, q_2, \ldots , q_n$ be positive numbers such that $p_1 + p_2 = q_1 + q_2 + \ldots + q_n = 1$. If a matrix $P = (p_{ij})$ of order 2 x n with non-negative entries and marginals $p_{i1} + p_{i2} + \ldots + p_{in} = p_i$, $i = 1,2$ and $p_{1j} + p_{2j} = q_j$, $j = 1,2,\ldots,n$, is positively quadrant dependent, then the following inequalities hold.

$$(1) \qquad p_2q_n \le p_{2n} \le p_2 \wedge q_n \ .$$

$$(2) \qquad p_{2n} \vee (p_2q_{n-1} + p_2q_n) \le p_{2n-1} + p_{2n} \le p_2 \wedge (q_{n-1} + p_{2n}) \ .$$

$$(3) \qquad (p_{2n-1} + p_{2n}) \vee (p_2q_{n-2} + p_2q_{n-1} + p_2q_n) \le p_{2n-2} + p_{2n-1} + p_{2n} \le p_2 \wedge (q_{n-2} + p_{2n-1} + p_{2n}) \ .$$

$$\vdots$$

$$(n-1) \qquad (p_{23} + p_{24} + \cdots + p_{2n}) \vee (p_2q_2 + p_2q_3 + \cdots + p_2q_n) \le p_{22} + p_{23} + \cdots + p_{2n} \le p_2 \wedge (q_2 + p_{23} + p_{24} + \cdots + p_{2n})$$

These inequalities provide a basis for generating members of $M_{PQD}(p_1,p_2;q_1,q_2,\ldots,q_n)$. The following theorem provides a method for finding members of $M_{PQD}(p_1,p_2;q_1,q_2,\ldots,q_n)$.

Theorem 8 Let $p_{22}, p_{23}, \cdots , p_{2n}$ be $(n-1)$ numbers satisfying the inequalities (1), (2), ... , (n-1) above. Let $p_{1j} = q_j - p_{2j}$, $j = 2,3,\ldots,n$ and $p_{21} = p_2 - (p_{22} + p_{23} + \ldots + p_{2n})$. Then

$$q_1 - p_{21} = p_1 - (p_{12} + p_{13} + \cdots + p_{1n}) = p_{11}, \text{ say,}$$

and the matrix $P = (p_{ij})$ of order $2 \times n$ belongs to $M_{PQD}(p_1,p_2;q_1,q_2,\ldots,q_n)$.

Proof. It is similar to the one presented for Theorem 7.

Now, we set about obtaining extreme points of $M_{PQD}(p_1,p_2;q_1,q_2,\ldots,q_n)$ algebraically.

We develop some sets of inequalities equivalent to the inequalities (1), (2),...,(n - 1) described above with the following properties.

(i) Each set contains (n - 1) equalities

(ii) The max and min symbols $\vee$, $\wedge$ disappear from each side of each inequality in each set.

Step 1. The interval $[p_2q_n, p_2\wedge q_n]$ of variation for p_{2n} is split into 3 sub-intervals using the inequality (2). The expression $p_{2n}\vee(p_2q_{n-1}) + p_2q_n)$ gives a partition of the interval $[p_2q_n, p_2\wedge q_n]$ into two sub-intervals as follows. Let

$$A_1 = \{p_{2n} \varepsilon [p_2q_n, p_2\wedge q_n]; \quad p_{2n} \vee(p_2q_{n-1} + p_2q_n) = p_{2n}\}$$

and

$$A_2 = \{p_{2n} \varepsilon [p_2q_n, p_2\wedge q_n]; \quad p_{2n} \vee(p_2q_{n-1} + p_2q_n) = p_2q_{n-1} + p_2q_n\}.$$

Note that A_1 and A_2 are closed intervals with possibly a common end point and

$$A_1 \bigcup A_2 = [p_2q_n, p_2\wedge q_n].$$

(One of the A_i's could be a null set.)

Similarly, the expression $p_2\wedge(q_{n-1} + p_{2n})$ gives a partition $\{B_1, B_2\}$ of the interval $[p_2q_n, p_2\wedge q_n]$ into two closed sub-intervals. The partitions $\{A_1, A_2\}$ and $\{B_1, B_2\}$ induce a finer partition $\{C_1, C_2, C_3\}$ of $[p_2q_n, p_2\wedge q_n]$ such that each C_i is either a closed interval or a null set and

$$C_1 \bigcup C_2 \bigcup C_3 = [p_2q_n, p_2\wedge q_n].$$

For example, if $A_1 = [a_1, a_2]$, $A_2[a_2, a_3]$, $B_1 = [b_1, b_2]$, $B_2 = [b_2, b_3]$ with $a_1 = b_1 = p_2q_n$, $a_3 = b_3 = p_2\wedge q_n$ and $a_2 < b_2$, then we can take $C_1 = [a_1, a_2]$,

$C_2 = [a_2, b_2]$ and $C_3 = [b_2, b_3]$. To maintain a conformable notation, write $a_1 = a_{[1]}$, $a_2 = a'_{[1]}$, $a_2 = a_{[2]}$, $b_2 = a'_{[2]}$, $b_2 = a_{[3]}$ and $b_3 = a'_{[3]}$. Thus we have

$$[p_2q_n,\ p_2 \wedge q_n] = [a_{[1]},\ a'_{[1]}] \bigcup [a_{[2]}, a'_{[2]}] \bigcup [a_{[3]}, a'_{[3]}],$$

where $a_{[1]} = p_2q_n$, $a'_{[1]} = a_{[2]}$, $a'_{[2]} = a_{[3]}$ and $a'_{[3]} = p_2 \wedge q_n$.

The inequalities

(1) $p_2q_n \leq p_{2n} \leq p_2q_n$ and

(2) $p_{2n} \vee (p_2q_{n-1} + p_2q_n) \leq p_{2n-1} + p_{2n}$

$$\leq p_2 \wedge (q_{n-1} + p_{2n})$$

are equivalent to

[1] $a_{[1]} \leq p_{2n} \leq a'_{[1]}$ and

[1](2) $p_{2n} \vee (p_2q_{n-1} + p_2q_n) \leq p_{2n-1} + p_{2n}$

$$\leq p_2 \wedge (q_{n-1} + p_{2n}),$$

or

[2] $a_{[2]} \leq p_{2n} \leq a'_{[2]}$ and

[2](2) $p_{2n} \vee (p_2q_{n-1} + p_2q_n) \leq p_{2n-1} + p_{2n}$

$$\leq p_2 \wedge (q_{n-1} + p_{2n}),$$

or

[3] $a_{[3]} \leq p_{2n} \leq a'_{[3]}$ and

[3](2) $p_{2n} \vee (p_2q_{n-1} + p_2q_n) \leq p_{2n-1} + p_{2n}$

$$\leq p_2 \wedge (q_{n-1} + p_{2n}).$$

In each of the inequalities [1](2), [2](2) and [3](2), the symbols $\vee$ and $\wedge$ would disappear when p_{2n} is confined to the limits imposed by [1], [2] and [3] respectively. To mark this freedom from the symbols $\vee$ and $\wedge$, let us rewrite the above inequalities as follows.

[1] $a_{[1]} \leq p_{2n} \leq a'_{[1]}$ and

[1](2) $a_{[1](2)} \leq p_{2n-1} + p_{2n} \leq a'_{[1](2)}$,

or,

[2] $a_{[2]} \leq p_{2n} \leq a'_{[2]}$ and

[2](2) $a_{[2](2)} \leq p_{2n-1} + p_{2n} \leq a'_{[2](2)}$,

or,

[3] $a_{[3]} \leq p_{2n} \leq a'_{[3]}$ and

[3](3) $a_{[3](2)} \leq p_{2n-1} + p_{2n} \leq a'_{[3](2)}$.

Step 2. Using the inequality (3), we sub-divide each of the intervals given by the inequalities [1](2), [2](2) and [3](2) onto three sub-intervals or inequalities. Let us concentrate on [1](2). The procedure outlined below is similar to the one given in Step 1. The expression $(p_{2n-1} + p_{2n}) \vee (p_2q_{n-2} + p_2q_{n-1} + p_2q_n)$ bifurcates the inequality [1](2) into two inequalities. The sets

$$D_1 = \{p_{2n-1} + p_{2n} \varepsilon [a_{[1](2)}, a'_{[1](2)}]; \\ (p_{2n-1} + p_{2n}) \vee (p_2q_{n-2} + p_2q_{n-1} + p_2q_n) \\ = p_{2n-1} + p_{2n}\} \text{ and}$$

$$D_2 = \{p_{2n-1} + p_{2n} \varepsilon [a_{[1](2)}, a'_{[1](2)}]; \\ (p_{2n-1} + p_{2n}) \vee (p_2q_{n-2} + p_2q_{n-1} + p_2q_n) \\ = p_2q_{n-2} + p_2q_{n-1} + p_2q_n\}$$

do give the necessary bifurcation of the inequality [1](2) with p_{2n} satisfying the inequality [1]. Similarly, the expression $p_2 \wedge (q_{n-2} + p_{2n-1} + p_{2n})$ bifurcates the inequality [1](2) into two inequalities. These two bifurcations would give three inequalities equivalent to [1](2). Let these inequalities be denoted by [11], [12], [13]. Let us write these inequalities including [1] as follows.

[1] $a_{[1]} \leq p_{2n} \leq a'_{[1]}$, -

[11] $a_{[11]} \leq p_{2n-1} + p_{2n} \leq a'_{[11]}$,

or

[12] $a_{[12]} \leq p_{2n-1} + p_{2n} \leq a'_{[12]}$,

or

[13] $a_{[13]} \leq p_{2n-1} + p_{2n} \leq a'_{[13]}$,

where $a'_{[11]} = a_{[12]}$ and $a'_{[12]} = a_{[13]}$. By bifurcating the inequality [1] further, necessary, ensure that $a'_{[1]} \leq a'_{[11]}$ as $p_{2n} \leq p_{2n-1} + p_{2n}$ is to be guaranteed. This procedure is applied to each of the inequalities [2](2) and [3](2).

We can now claim that the inequalities (1), (2) and (3) are equivalent to the following 9 sets of inequalities. (We describe these inequalities using a suggestive notation.)

[1] $a_{[1]} \leq p_{2n} \leq a'_{[1]}$,

[11] $a_{[11]} \leq p_{2n-1} + p_{2n} \leq a'_{[11]}$, and

[11](3)
$$(p_{2n-1} + p_{2n}) \vee (p_2 q_{n-2} + p_2 q_{n-1} + p_2 q_n)$$
$$\leq p_{2n-2} + p_{2n-1} + p_{2n}$$
$$\leq p_2 \wedge (q_{n-2} + p_{2n-1} + p_{2n}),$$

or

[1] $a_{[1]} \leq p_{2n} \leq a'_{[1]}$,

[12] $a_{[12]} \leq p_{2n-1} + p_{2n} \leq a'_{[12]}$ and

[12](3)
$$(p_{2n-1} + p_{2n}) \vee (p_2 q_{n-2} + p_2 q_{n-1} + p_2 q_n)$$
$$\leq p_{2n-2} + p_{2n-1} + p_{2n}$$
$$\leq p_2 \wedge (q_{n-2} + p_{2n-1} + p_{2n}),$$

or,

[1] $a_{[1]} \leq p_{2n} \leq a'_{[1]}$,

[13] $a_{[13]} \leq p_{2n-1} + p_{2n} \leq a'_{[13]}$ and

[13](3) $(p_{2n-1} + p_{2n}) \vee (p_2 q_{n-2} + p_2 q_{n-1} + p_2 q_n)$
$\leq p_{2n-2} + p_{2n-1} + p_{2n}$
$\leq p_2 \wedge (q_{n-2} + p_{2n-1} + p_{2n})$,

or,

[2] $a_{[2]} \leq p_{2n} \leq a'_{[2]}$,

[21] $a_{[21]} \leq p_{2n-1} + p_{2n} \leq a'_{[21]}$ and

[21](3) $(p_{2n-1} + p_{2n}) \vee (p_2 q_{n-2} + p_2 q_{n-1} + p_2 q_n)$
$\leq p_{2n-2} + p_{2n-1} + p_{2n}$
$\leq p_2 \wedge (q_{n-2} + p_{2n-1} + p_{2n})$,

or,

[2] $a_{[2]} \leq p_{2n} \leq a'_{[2]}$,

[22] $a_{[22]} \leq p_{2n-1} + p_{2n} \leq a'_{[22]}$ and

[22](3) $(p_{2n-1} + p_{2n}) \vee (p_2 q_{n-2} + p_2 q_{n-1} + p_2 q_n)$
$\leq p_{2n-2} + p_{2n-1} + p_{2n}$
$\leq p_2 \wedge (q_{n-2} + p_{2n-1} + p_{2n})$

or,

[2] $a_{[2]} \leq p_{2n} \leq a'_{[2]}$,

[23] $a_{[23]} \leq p_{2n-1} + p_{2n} \leq a'_{[23]}$ and

[23](3) $(p_{2n-1} + p_{2n}) \vee (p_2 q_{n-2} + p_2 q_{n-1} + p_2 q_n)$
$\leq p_{2n-2} + p_{2n-1} + p_{2n}$
$\leq p_2 \wedge (q_{n-2} + p_{2n-1} + p_{2n})$,

or,

[3] $a_{[3]} \leq p_{2n} \leq a'_{[3]}$

[31] $a_{[31]} \leq p_{2n-1} + p_{2n} \leq a'_{[31]}$ and

[31](3) $$(p_{2n-1} + p_{2n}) \vee (p_2 q_{n-2} + p_2 q_{n-1} + p_2 q_n)$$
$$\leq p_{2n-2} + p_{2n-1} + p_{2n}$$
$$\leq p_2 \wedge (q_{n-2} + p_{2n-1} + p_{2n}),$$

or,

[3] $a_{[3]} \leq p_{2n} \leq a'_{[3]}$

[32] $a_{[32]} \leq p_{2n-1} + p_{2n} \leq a'_{[32]}$ and

[32](3) $$(p_{2n-1} + p_{2n}) \vee (p_2 q_{n-2} + p_2 q_{n-1} + p_2 q_n)$$
$$\leq p_{2n-2} + p_{2n-1} + p_{2n}$$
$$\leq p_2 \wedge (q_{n-2} + p_{2n-1} + p_{2n}),$$

or,

[3] $a_{[3]} \leq p_{2n} \leq a'_{[3]}$

[33] $a_{[33]} \leq p_{2n-1} + p_{2n} \leq a'_{[33]}$ and

[33](3) $$(p_{2n-1} + p_{2n}) \vee (p_2 q_{n-2} + p_2 q_{n-1} + p_2 q_n)$$
$$\leq p_{2n-2} + p_{2n-1} + p_{2n}$$
$$\leq p_2 \wedge (q_{n-2} + p_2 q_{n-1} + p_{2n}).$$

The following facts should be borne in mind before proceeding to Step 3.

(i) $a'_{[11]} = a_{[12]}$; $a'_{[12]} = a_{[13]}$; $a'_{[21]} = a_{[22]}$;
$a'_{[22]} = a_{[23]}$; $a'_{[31]} = a_{[32]}$; $a'_{[32]} = a_{[33]}$.

(ii) $a'_{[1]} \leq a'_{[11]}$; $a'_{[2]} \leq a'_{[21]}$ and $a'_{[3]} \leq a'_{[31]}$.

(iii) In each of the nine inequalities [11](3), [12](3), [13](3), [21](3), [22](3), [23](3), [31](3), [32](3) and [33](3), the symbols $\vee, \wedge$ would disappear as long as p_{2n} and $p_{2n-1} + p_{2n}$ are confined to the limits of the corresponding inequalities within each set.

(iv) The inequalities [1], [2] and [3] obtained in Step 1 can further be bifurcated to accomodate (iii). This is helpful in conjunction with the inequalities [11], [12], [13], [21], [22], [23], [31], [32] and [33] in each appropriate category to get rid of the symbols $\vee$ and $\wedge$ in the inequality (3).

<u>Step 3</u>. Using the inequality (4), we split the last inequality in each set above into three inequalities. The procedure is similar to the one explained in Step 2.

. . . .

. . . .

. . . .

<u>Step (n - 2)</u>. Using the inequality (n - 1), we split the last inequality in each set into three inequalities. At the beginning of this step, each set has (n - 2) inequalities.

At the conclusion of Step (n - 2), the inequalities can be arranged pyramidically as in the following diagram, assuming that at every stage we split each inequality into three inequalities.

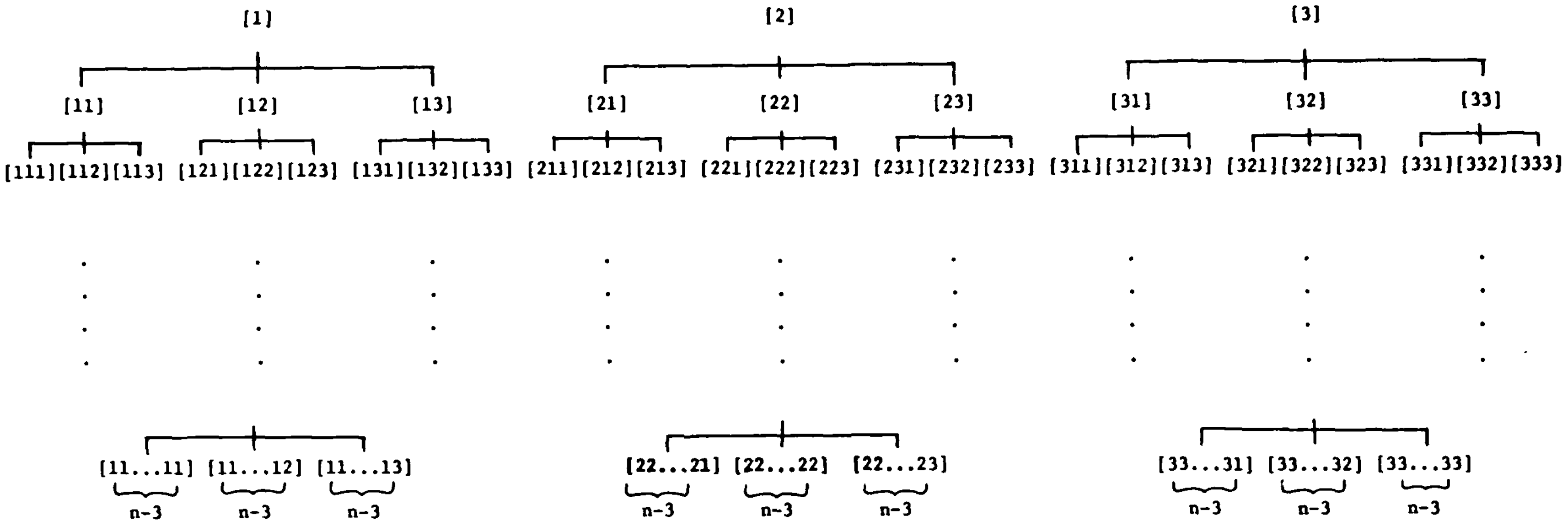
Expression involved

Inequalities

1. p_{2n}
2. $p_{2n-1}+p_{2n}$
3. $p_{2n-2}+p_{2n-1}+p_{2n}$

.
.
.
.

$(n-2).p_{23}+p_{24}+..+p_{2n}$



Each inequality at the final Step (n-2) is coupled with the inequality (n - 1). Thus we have 3^{n-2} sets of inequalities with each set containing (n - 1) inequalities including the inequality (n - 1). In the last inequality (n - 1), the symbols $\vee$ and $\wedge$ would disappear. The following are examples of two sets of inequalities when n = 6, where at every stage each inequality gave rise to exactly three inequalities.

Example 1.

[1] $a_{[1]} \leq p_{26} \leq a'_{[1]}$,

[12] $a_{[12]} \leq p_{25} + p_{26} \leq a'_{[12]}$,

[121] $a_{[121]} \leq p_{24} + p_{25} + p_{26} \leq a'_{[121]}$,

[1213] $a_{[1213]} \leq p_{23} + p_{24} + p_{25} + p_{26} \leq a'_{[1213]}$

and

(5) $$(p_{23} + p_{24} + p_{25} + p_{26}) \vee (p_2q_2 + p_2q_3 + p_2q_4 + p_2q_5 + p_2q_6) \leq p_{22} + p_{23} + p_{24} + p_{25} + p_{26} \leq p_2 \wedge (q_2 + p_{23} + p_{24} + p_{25} + p_{26}).$$

Example 2.

[3] $a_{[3]} \leq p_{26} \leq a'_{[3]}$,

[32] $a_{[32]} \leq p_{25} + p_{26} \leq a'_{[32]}$,

[321] $a_{[3211]} \leq p_{24} + p_{25} + p_{26} \leq a'_{[3211]}$,

[3212] $a_{[3212]} \leq p_{23} + p_{24} + p_{25} + p_{26} \leq a'_{[3212]}$

and

(5) $$(p_{23} + p_{24} + p_{25} + p_{26}) \vee (p_2q_2 + p_2q_3 + p_2q_4 + p_2q_5 + p_2q_6) \leq p_{22} + p_{23} + p_{24} + p_{25} + p_{26} \leq p_2 \wedge (q_2 + p_{23} + p_{24} + p_{25} + p_{26}).$$

Enumeration of extreme points of $M_{PQD}(p_1,p_2;\ q_1,q_2,\ldots,q_n)$

Let J_1 denote the collection of all those matrices $P = (p_{ij})$ in $M_{PQD}(p_1,p_2;\ q_1,q_2,\ldots,q_n)$ for which equality holds either on the left or on the right in all the inequality expressions in some set of the 3^{n-2} sets described above. Let J_i, $i = 2,3,\ldots,n$, denote the collection of all matrices $P = (p_{ij})$ in $M_{PQD}(p_1,p_2;\ q_1,q_2,q_3)$ for which strict inequality holds in $(i-1)$ inequality expressions and equality holds either on the left or on the right in the remaining $(n-1) - (i-1) = n-i$ inequality expressions in some set of the 3^{n-2} sets described above. Obviously,

$$M_{PQD}(p_1,p_2;\ q_1,q_2,\ldots q_n) = \bigcup_{i=1}^{n} J_i.$$

Theorem 9. The set of all extreme points of $M_{PQD}(p_1,p_2;\ q_1,q_2,\ldots q_n)$ is a subset of J_1.

Proof. It suffices to show that every matrix in $M_{PQD}(p_1,p_2;\ q_1,q_2,\ldots q_n)$ can be expressed as a convex combination of members of J_1. Every member of J_1, trivially, has the above property. Let $P = (p_{ij})$ be any member of J_2. We express P as a convex combination of two members of J_1. Since $P \in J_2$, strict inequality holds in exactly one inequality expression of some set of inequalities. For simplicity, let the strict inequality occur in [23], i.e.,

$$a_{[23]} < p_{2n-1} + p_{2n} < a'_{[23]}$$

and equality holds in the remaining $(n-2)$ inequality expressions either on the left or on the right. Let $Q = (q_{ij})$ be the matrix in $M_{PQD}(p_1,p_2;\ q_1,q_2,\ldots,q_n)$ for which equality holds on the left in [23] and equality holds in the remaining $(n-2)$ inequality expressions in exactly the same place as that of P in the same set of inequalities.

Let $S = (s_{ij})$ be the matrix in $M_{PQD}(p_1, p_2;\ q_1, q_2, \ldots, q_n)$ for which equality holds on the right in [23] and equality holds in the remaining (n - 2) inequality expressions in exactly the same place as that of P in the same set of inequalities. The existence of Q and S is guaranteed by Theorem 8. We also observe that $q_{2n-1} < p_{2n-1} < s_{2n-1}$. There exists $0 < \alpha < 1$ such that $p_{2n-1} = \alpha q_{2n-1} + 1 - \alpha)s_{2n-1}$. Indeed, after laborious calculations, one can check that $P = \alpha Q + (1 - \alpha)S$.

Let P be a member of J_3. We can express P as a convex combination of two members of J_2. This then would imply that P is a convex combination of members of J_1. Since $P \in J_3$, strict inequality holds in 2 inequality expressions and equality holds either on the left or on the right in each of the remaining (n - 3) inequality expressions of some set of inequalities. For simplicity, let the strict inequality hold in [123] and [23112], i.e.,

$$a_{[123]} < p_{2n-2} + p_{2n-1} + p_{2n} < a'_{[123]} \text{ and}$$

$$a_{[23112]} < p_{2n-4} + p_{2n-3} + p_{2n-2} + p_{2n-1} + p_{2n} < a'_{[23112]}.$$

Let $Q = (q_{ij})$ be the matrix in $M_{PQD}(p_1, p_2;\ q_1, q_2, \ldots, q_n)$ such that $q_{2n} = p_{2n}$, $q_{2n-1} = p_{2n-1}$, $q_{2n-2} = p_{2n-2}$, equality holds on left of [23112] and equality holds in the remaining inequality expressions in exactly the same place as that of P in the same set of inequalities. Let $S = (s_{ij})$ be the matrix in $M_{PQD}(p_1, p_2;\ q_1, q_2, \ldots, q_n)$ such that $s_{2n} = p_{2n}$, $s_{2n-1} = p_{2n-1}$, $s_{2n-2} = p_{2n-2}$, equality holds on the right of [23112] and equality holds in all the remaining inequality expressions in exactly the same place as that of P in the same set of inequalities. One can check that $P = \alpha Q + (1 - \alpha)S$ for some $0 < \alpha < 1$.

Proceeding this way, one can show that every member of J_i is a convex combination of two members of J_{i-1} for i = 2,3,...,n.

This completes the proof.

The above theorem helps to identify the extreme points of $M_{PQD}(p_1,p_2;\ q_1,q_2,\ldots,q_n)$ as follows.

Step 1. Create 3^{n-2} sets of inequalities as described above.

Step 2. For each set of inequalities, set the central expression in each inequality equal to the quantity either on the left or on the right. Solve the resultant equations for p_{2n}, $p_{2n-1},\ldots,p_{23}$ and p_{22}.

Step 3. Using Theorem 8, build a matrix $P = (p_{ij})$ in $M_{PQD}(p_1,p_2;\ q_1,q_2,\ldots,q_n)$.

By Theorem 9, every extreme point of $M_{PQD}(p_1,p_2;\ q_1,q_2,\ldots,q_n)$ arises this way. There might be a duplication of matrices on a moderate scale. Also, many non-extreme points might arise this way. We can weed them from J_1. If it is a hard task, we can keep them. Retaining them will not hamper us in discussing objectively merits of various tests for contingency tables in the next section. Once the sets of inequalities are created, solving equations can be programmed easily.

Intuitively, $M_{PQD}(p_1,p_2;\ q_1,q_2,\ldots,q_n)$ can be viewed as a multi-faceted diamond and the solutions obtained above give the points where edges meet.

A numerical example

Let $n=4$, $p_1=0.6$, $p_2=0.4$, $q_1=0.3$, $q_2=0.2$, $q_3=0.2$ and $q_4=0.3$.

Determining inequalities

(1) $p_2q_4 \le p_{24} \le p_2 \wedge q_4$.

(2) $p_{24} \vee (p_2q_3 + p_2q_4) \le p_{23} + p_{24} \le p_2 \wedge (q_3 + p_{24})$.

(3) $(p_{23} + p_{24}) \vee (p_2q_2 + p_2q_3 + p_2q_4) \le p_{22} + p_{23} + p_{24} \le p_2 \wedge (q_2 + p_{23} + p_{24})$.

Numerically, these inequalities become

(1) $0.12 \leq p_{24} \leq 0.3$,

(2) $p_{24} \vee (0.20) \leq p_{23} + p_{24} \leq 0.4 \wedge (0.2 + p_{24})$,

(3) $(p_{23} + p_{24}) \vee (0.28) \leq p_{22} + p_{23} + p_{24} \leq 0.4 \wedge (0.2 + p_{23} + p_{24})$.

The inequality (2) induces only two inequalities [1] and [2] equivalent to (1).

[1] $0.12 \leq p_{24} \leq 0.2$ and

[2] $0.20 \leq p_{24} \leq 0.3$.

The inequality (2) in conjunction with the inequalities [1] and [2] becomes

[1] $0.12 \leq p_{24} \leq 0.2$ and

[1](2) $0.20 \leq p_{23} + p_{24} \leq 0.2 + p_{24}$,

or,

[2] $0.2 \leq p_{24} \leq 0.3$ and

[2](2) $p_{24} \leq p_{23} + p_{24} \leq 0.4$.

The inequality (3) induces two inequalities equivalent to [1](2) and also two inequalities equivalent to [2](2). We have altogether 4 sets of inequalities listed below.

I Set

[1] $0.12 \leq p_{24} \leq 0.2$,

[11] $0.20 \leq p_{23} + p_{24} \leq 0.28$ and

(3) $0.28 \leq p_{22} + p_{23} + p_{24} \leq 0.4$.

II Set

[1] $0.12 \leq p_{24} \leq 0.2$,

[12] $0.28 \leq p_{23} + p_{24} \leq 0.2 + p_{24}$ and

(3) $p_{23} + p_{24} \leq p_{22} + p_{23} + p_{24} \leq 0.4$.

III Set

[2] $0.2 \leq p_{24} \leq 0.3$,

[21] $p_{24} \leq p_{23} + p_{24} \leq 0.28$ and

(3) $(p_{23} + p_{24}) \vee (0.28) \leq p_{22} + p_{23} + p_{24} \leq 0.4$.

IV Set

[2] $0.2 \leq p_{24} \leq 0.3$,

[21] $0.28 \leq p_{23} + p_{24} \leq 0.4$ and

(3) $(p_{23} + p_{24}) \vee (0.28) \leq p_{22} + p_{23} + p_{24} \leq 0.4$.

In the III set, the number on the left hand side of [2] is larger than the number on the left hand side of [21]. We, further, bifurcate [2] to set right this anomaly. After this bifurcation, we relabel the inequalities and obtain the following sets.

Set I

[1] $0.12 \leq p_{24} \leq 0.2$,

[11] $0.20 \leq p_{23} + p_{24} \leq 0.28$ and

(3) $0.28 \leq p_{22} + p_{23} + p_{24} \leq 0.4$.

Set II

[1] $0.12 \leq p_{24} \leq 0.2$,

[12] $0.28 \leq p_{23} + p_{24} \leq 0.2 + p_{24}$ and

(3) $p_{23} + p_{24} \leq p_{22} + p_{23} + p_{24} \leq 0.4$.

Set III

[2] $0.2 \leq p_{24} \leq 0.28$,

[21] $p_{24} \leq p_{23} + p_{24} \leq 0.28$ and

(3) $0.28 \leq p_{22} + p_{23} + p_{24} \leq 0.4$.

Set IV

[2] $0.20 \le p_{24} \le 0.28,$

[22] $0.28 \le p_{23} + p_{24} \le 0.4$ and

(3) $p_{23} + p_{24} \le p_{22} + p_{23} + p_{24} \le 0.4.$

Set V

[3] $0.28 \le p_{24} \le 0.3$

[31] $p_{24} \le p_{23} + p_{24} \le 0.4$ and

(3) $p_{23} + p_{24} \le p_{22} + p_{23} + p_{24} \le 0.4.$

Set the central expression in each inequality in each set to the quantity either on the right or on the left and solve the resultant equations for p_{22}, p_{23}, p_{24}. The distinct matrices in J_1 of Theorem 9 are listed below.

(1) $\begin{pmatrix} 0.18 & 0.12 & 0.12 & 0.18 \\ 0.12 & 0.08 & 0.08 & 0.12 \end{pmatrix}$

(2) $\begin{pmatrix} 0.3 & 0.0 & 0.12 & 0.18 \\ 0.0 & 0.2 & 0.08 & 0.12 \end{pmatrix}$

(3) $\begin{pmatrix} 0.18 & 0.2 & 0.04 & 0.18 \\ 0.12 & 0.0 & 0.16 & 0.12 \end{pmatrix}$

(4) $\begin{pmatrix} 0.3 & 0.08 & 0.04 & 0.18 \\ 0.0 & 0.12 & 0.16 & 0.12 \end{pmatrix}$

(5) $\begin{pmatrix} 0.18 & 0.12 & 0.2 & 0.1 \\ 0.12 & 0.08 & 0.0 & 0.2 \end{pmatrix}$

(6) $\begin{pmatrix} 0.3 & 0.0 & 0.2 & 0.1 \\ 0.0 & 0.2 & 0.0 & 0.2 \end{pmatrix}$

(7) $\begin{pmatrix} 0.18 & 0.2 & 0.12 & 0.1 \\ 0.12 & 0.0 & 0.08 & 0.2 \end{pmatrix}$

(8) $\begin{pmatrix} 0.3 & 0.08 & 0.12 & 0.1 \\ 0.0 & 0.12 & 0.08 & 0.2 \end{pmatrix}$

(9) $\begin{pmatrix} 0.22 & 0.2 & 0.0 & 0.18 \\ 0.08 & 0.0 & 0.2 & 0.12 \end{pmatrix}$

(10) $\begin{pmatrix} 0.3 & 0.12 & 0.0 & 0.18 \\ 0.0 & 0.08 & 0.2 & 0.12 \end{pmatrix}$

(11) $\begin{pmatrix} 0.3 & 0.2 & 0.0 & 0.1 \\ 0.0 & 0.0 & 0.2 & 0.2 \end{pmatrix}$

(12) $\begin{pmatrix} 0.18 & 0.12 & 0.2 & 0.1 \\ 0.12 & 0.08 & 0.0 & 0.2 \end{pmatrix}$

(13) $\begin{pmatrix} 0.18 & 0.2 & 0.2 & 0.02 \\ 0.12 & 0.0 & 0.0 & 0.28 \end{pmatrix}$ (14) $\begin{pmatrix} 0.3 & 0.08 & 0.2 & 0.02 \\ 0.0 & 0.12 & 0.0 & 0.28 \end{pmatrix}$

(15) $\begin{pmatrix} 0.3 & 0.2 & 0.08 & 0.02 \\ 0.0 & 0.0 & 0.12 & 0.28 \end{pmatrix}$ (16) $\begin{pmatrix} 0.2 & 0.2 & 0.2 & 0.0 \\ 0.1 & 0.0 & 0.0 & 0.3 \end{pmatrix}$

(17) $\begin{pmatrix} 0.3 & 0.1 & 0.2 & 0.0 \\ 0.0 & 0.1 & 0.0 & 0.3 \end{pmatrix}$ (18) $\begin{pmatrix} 0.3 & 0.2 & 0.1 & 0.0 \\ 0.0 & 0.0 & 0.1 & 0.3 \end{pmatrix}$

From the above collection of matrices belonging to J_1, we can weed out those matrices which are not extreme points of $M_{PQD}(0.6, 0.4; 0.3, 0.2, 0.2, 0.3)$. (4) is not an extreme point. For, we can write (4) = $\frac{1}{3}$(2) + $\frac{2}{3}$(10). (15) is not an extreme point. For, we can write (15) = $\frac{1}{5}$(11) + $\frac{4}{5}$(18). (7) is not an extreme point. For, we can write (7) = $\frac{1}{2}$(3) + $\frac{1}{2}$(13). (8) is not an extreme point. For, (8) = $\frac{1}{2}$(4) + $\frac{1}{2}$(14). The number of extreme points, therefore, of $M_{PQD}(0.6, 0.4; 0.3, 0.2, 0.2, 0.3)$ is 14.

Now, it is time to discuss the general m × n case. But the arguments involved and the notation used are so complex, it is not really worth the effort. Instead, we offer a thorough discussion of 3 × 3 case from which one can make a successful analysis of any m × n case. The analysis of 3 × 3 case is not exactly in tune with the 2 × n case. A study of 2 × n and 3 × 3 cases in tandem should help to map out a successful strategy to analyze m × n case in general.

<u>The case of 3 × 3 tables</u>

Let p_1, p_2, p_3, q_1, q_2, q_3, be six positive numbers such that $p_1 + p_2 + p_3 = q_1 + q_2 + q_3 = 1$. If a matrix $P = (p_{ij})$ of order 3 × 3 belongs to $M_{PQD}(p_1,p_2,p_3;\ q_1,q_2,q_3)$, then one can check that

$$(1)\ p_3q_3 \le p_{33} \le p_3 \wedge q_3,$$

$$(2)\ p_{33} \vee (p_2q_3 + p_3q_3) \le p_{23} + p_{33}$$
$$\le q_3 \wedge (p_2 + p_{33}),$$

$$(3)\ p_{33} \vee (p_3q_2 + p_3q_3) \le p_{32} + p_{33}$$
$$\le p_3 \wedge (q_2 + p_{33}),$$

$$(4)\ (p_{23} + p_{32} + p_{33}) \vee (p_2q_2 + p_2q_3 + p_3q_2 + p_3q_3)$$
$$\le p_{22} + p_{23} + p_{32} + p_{33}$$
$$\le (p_2 + p_{32} + p_{33}) \wedge (q_2 + p_{23} + p_{33})$$

hold. P is PQD yields (1) $p_{33} \ge p_3q_3$, (2) $p_{23} + p_{33} \ge p_2q_3 + p_3q_3$, (3) $p_{32} + p_{33} \ge p_3q_2 + p_3q_3$ and (4) $p_{22} + p_{23} + p_{32} + p_{33} \ge p_2q_2 + p_2q_3 + p_3q_2 + p_3q_3$. The non-negativity of the entries of P, and the marginality conditions on the entries of P coupled with P being PQD yield the above inequalities. These inequalities are important in the sense that if there are four numbers p_{22}, p_{23}, p_{32}, p_{33} satisfying (1), (2), (3), (4) above, then we can build a matrix P belonging to $M_{PQD}(p_1,p_2,p_3;\ q_1,q_2,q_3)$. We make this precise in the following theorem.

Theorem 10 Let p_{22}, p_{23}, p_{32}, p_{33} be four numbers satisfying the inequalities (1), (2), (3), (4) above. Define

$$p_{21} = p_2 - p_{22} - p_{23},$$
$$p_{31} = p_3 - p_{32} - p_{33},$$
$$p_{12} = q_2 - p_{22} - p_{32}, \text{ and}$$
$$p_{13} = q_3 - p_{23} - p_{33}.$$

Then

$$p_1 - p_{12} - p_{13} = q_1 - p_{21} - p_{31} = p_{11}, \text{ say}$$

and the matrix $P = (p_{ij}) \varepsilon M_{PQD}(p_1,p_2,p_3;\ q_1,q_2,q_3)$.

Proof. The proof is analogous to the one given for Theorem 7.

Determination of the extreme points of M_{PQD} $(p_1,p_2,p_3;\ q_1,q_2,q_3)$

As explained in the analysis of 2 × n tables, inequality (2) gives rise to at most 3 inequalities equivalent to (1). Equivalent inequalities are given below for each of the cases.

Case A $p_3 \wedge q_3 \leq p_2q_3 + p_3q_3$ and $q_3 - p_2 \leq 0$

(1) $p_3q_3 \leq p_{33} \leq p_3 \wedge q_3$ and

(2) $p_2q_3 + p_3q_3 \leq p_{23} + p_{33} \leq q_3$.

Case B $p_3 \wedge q_3 \leq p_2q_3 + p_3q_3$ and $0 < q_3 - p_2 \leq p_3q_3$

(1) $p_3q_3 \leq p_{33} \leq p_3 \wedge q_3$ and

(2) $p_2q_3 + p_3q_3 \leq p_{23} + p_{33} \leq q_3$

Case C $p_3 \wedge q_3 \leq p_2q_3 + p_3q_3$ and $p_3q_3 < q_3 - p_2 < p_3 \wedge q_3$

(1) $p_3q_3 \leq p_{33} \leq q_3 - p_2$ and

(2) $p_2q_3 + p_3q_3 \leq p_{23} + p_{33} \leq p_{33} + p_2$,

or

(1) $q_3 - p_2 \leq p_{33} \leq p_3 \wedge q_3$ and

(2) $p_2q_3 + p_3q_3 \leq p_{23} + p_{33} \leq q_3$.

Case D $p_3 \wedge q_3 \leq p_2q_3 + p_3q_3$ and $p_3 \wedge q_3 \leq q_3 - p_2$

(1) $p_3q_3 \leq p_{33} \leq p_3 \wedge q_3$ and

(2) $p_2q_3 + p_3q_3 \leq p_{23} + p_{33} \leq p_2 + p_{33}$.

Case E $p_2q_3 + p_3q_3 < p_3 \wedge q_3$ and $q_3 - p_2 \leq 0$

(1) $p_3q_3 \leq p_{33} \leq p_2q_3 + p_3q_3$ and

(2) $p_2q_3 + p_3q_3 \leq p_{23} + p_{33} \leq q_3$,

or,

(1) $p_2q_3 + p_3q_3 \leq p_{33} \leq p_3 \wedge q_3$ and

(2) $p_{33} \leq p_{23} + p_{33} \leq q_3$.

Case F $p_2q_3 + p_3q_3 < p_3 \wedge q_3$ and $0 < q_3 - p_2 \leq p_3q_3$

The inequalities here are the same as those of Case E.

Case G $p_2q_3 + p_3q_3 < p_3 \wedge q_3$ and $p_3q_3 < q_3 - p_2 < p_3 \wedge q_3$.

Assume, without loss of generality, that $q_3 - p_2 < p_2q_3 + p_3q_3$.

(1) $p_3q_3 \leq p_{33} \leq q_3 - p_2$ and

(2) $p_2q_3 + p_3q_3 \leq p_{23} + p_{33} \leq p_2 + p_{33}$,

or,

(1) $q_3 - p_2 \leq p_{33} \leq p_2q_3 + p_3q_3$ and

(2) $p_2q_3 + p_3q_3 \leq p_{23} + p_{33} \leq q_3$,

or,

(1) $p_2q_3 + p_3q_3 \leq p_{33} \leq p_3 \wedge q_3$ and

(2) $p_{33} \leq p_{23} + p_{33} \leq q_3$.

<u>Case H</u> $p_2q_3 + p_3q_3 < p_3 \wedge q_3$ and $p_3 \wedge q_3 \leq q_3 - p_2$.

(1) $p_3q_3 \leq p_{33} \leq p_2q_3 + p_3q_3$ and

(2) $p_2q_3 + p_3q_3 \leq p_{23} + p_{33} \leq p_2 + p_{33}$,

or,

(1) $p_2q_3 + p_3q_3 \leq p_{33} \leq p_3 \wedge q_3$ and

(2) $p_{33} \leq p_{23} + p_{33} \leq p_2 + p_{33}$.

Under each of the Cases A, B and D, the inequality (1) virtually remains the same, but in the Cases C, E, F and H, the inequality (1) is split into 2 inequalities. The most complicated case is Case G in which the inequality (3) is split into 3 inequalities. This splitting of (1) influences the inequality (2) in that the symbols $\vee$ and $\wedge$ disappear from each side of the inequality (2).

We assume that we have Case G to contend with. Note that we have under Case G, $p_3q_3 < q_3 - p_2 < p_2q_3 + p_3q_3 < p_3 \wedge q_3$.

Similarly, the inequality (3) gives rise to at most 3 inequalities equivalent to (1). We assume that here also we have the most complicated case of 3 inequalities equivalent to (1). Assume, without loss of generality, that

$$p_3q_3 < p_3q_2 + p_3q_3 < p_3 - q_2 < p_3 \wedge q_3.$$

Assume also that $p_3q_3 < p_2q_2 + p_2q_3 + p_3q_3 + p_3q_3 < p_3 \wedge q_3$ and

$$p_3q_3 < q_3 - p_2 < p_3q_2 + p_3q_3 < p_3 - q_2 < p_2q_3 + p_3q_3$$
$$< p_2q_2 + p_2q_3 + p_3q_2 + p_3q_3 < p_3 \wedge q_3.$$

If some other order prevails among these seven numbers, we can analyze this case with equal facility. We are dealing with the maximum number of inequalities that can arise equivalent to (1) induced by the inequalities (2) and (3) and the number $p_2q_2 + p_2q_3 + p_3q_2 + p_3q_3$.

To round up the discussion initiated by the inequalities (1), (2) and (3), we conclude that the inequalities (1), (2) and (3) are equivalent to the following sets of inequalities.

[1] $p_3q_3 \leq p_{33} \leq q_3 - p_2$,

[1](2) $p_2q_3 + p_3q_3 \leq p_{23} + p_{33} \leq p_2 + p_{33}$, and

[1](3) $p_3q_2 + p_3q_3 \leq p_{32} + p_{33} \leq q_2 + p_{33}$,

or,

[2] $q_3 - p_2 \leq p_{33} \leq p_3q_2 + p_3q_3$,

[2](2) $p_2q_3 + p_3q_3 \leq p_{23} + p_{33} \leq q_3$, and

[2](3) $p_3q_2 + p_3q_3 \leq p_{32} + p_{33} \leq q_2 + p_{33}$,

or,

[3] $p_3q_2 + p_3q_3 \leq p_{33} \leq p_3 - q_2$,

[3](2) $p_2q_3 + p_3q_3 \leq p_{23} + p_{33} \leq q_3$, and

[3](3) $p_{33} \leq p_{32} + p_{33} \leq q_2 + p_{33}$,

or,

[4] $p_3 - q_2 \leq p_{33} \leq p_2q_3 + p_3q_3$,

[4](2) $p_2q_3 + p_3q_3 \leq p_{23} + p_{33} \leq q_3$, and

[4](3) $p_{33} \leq p_{32} + p_{33} \leq p_3$,

or,

[5] $p_2q_3 + p_3q_3 \leq p_{33} \leq p_2q_2 + p_2q_3 + p_3q_2 + p_3q_3$,

[5](2) $p_{33} \leq p_{23} + p_{33} \leq q_3$, and

[5](3) $p_{33} \leq p_{32} + p_{33} \leq p_3$,

or,

[6] $p_2q_2 + p_2q_3 + p_3q_2 + p_3q_3 \leq p_{33} \leq p_3 \wedge q_3$,

[6](2) $p_{33} \leq p_{23} + p_{33} \leq q_3$, and

[6](3) $p_{33} \leq p_{32} + p_{33} \leq p_3$.

We observe that the symbols $\vee$ and $\wedge$ have disappeared from the inequalities (2) and (3).

Now, we have to fit the inequality (4) into the above scheme of inequalities equivalent to (1), (2) and (3). The primary goal is to get rid of the symbols $\wedge$ and $\vee$ from (4) before appending it to each one of the above six sets of inequalities. Neutralization of the symbol $\wedge$ on the right hand side of the inequality (4) can be effected in the same way as its done in the inequalities (2) and (3) earlier. This may involve further splitting up the inequalities (2) and (3) in each and every set of inequalities above. Or, $\wedge$ can be retained as it is in (4). To neutralize the symbol $\vee$ on the seft hand side of the inequality (4), we introduce a new inequality intermediate to (2), (3) on one side and (4) on the other side. For each set of inequalities given above, one can determine the minimum and maximum value of $p_{23} + p_{32} + p_{33}$. Let these values be a_1^* and a_2^* respectively. There are several possibilities.

Case 1 $p_2q_2 + p_2q_3 + p_3q_2 + p_3q_3 \leq a_1^*$.

In this case, there is no need to introduce an intermediate inequality.

Case 2 $a_1^* < p_2q_2 + p_2q_3 + p_3q_2 + p_3q_3 < a_2^*$.

We introduce an intermediate inequality

$$a_1^* \leq p_{23} + p_{32} + p_{33} \leq p_2q_2 + p_2q_3 + p_3q_2 + p_3q_3,$$

or,

$$p_2q_2 + p_2q_3 + p_3q_2 + p_3q_3 \leq p_{23} + p_{32} + p_{33} \leq a_2^*.$$

Case 3 $a_2^* < p_2q_2 + p_2q_3 + p_3q_2 + p_3q_3$.

In this case, there is no need to introduce an intermediate inequality.

A typical set of inequalities has the following appearance.

A (1) $a_1 \leq p_{33} \leq a_2$

(2) $b_1 \leq p_{23} + p_{33} \leq b_2$

(3) $c_1 \leq p_{32} + p_{33} \leq c_2$

(4) $d_1 \leq p_{22} + p_{32} + p_{23} + p_{33} \leq d_2$;

or,

B (1) $a_1 \leq p_{33} \leq a_2$

(2) $b_1 \leq p_{23} + p_{33} \leq b_2$

(3) $c_1 \leq p_{32} + p_{33} \leq c_2$

(4) $d_1 \leq p_{23} + p_{32} + p_{33} \leq d_2$

(5) $e_1 \leq p_{22} + p_{23} + p_{32} + p_{33} \leq e_2$.

In the first lot above, b_1, b_2, c_1 and c_2 could be either constants or linear functions of p_{33}. The same remarks apply to b_1, b_2, c_1 and c_2 in the second lot. In the first lot, $d_1 = p_{23} + p_{32} + p_{33}$ or $p_2q_2 + p_2q_3 + p_3q_2 + p_3q_3$ and $d_2 = p_2 + p_{32} + p_{33}$ or $q_2 + p_{23} + p_{33}$. In the second lot, d_1 = minimum value of $p_{23} + p_{32} + p_{33}$ subject to (1), (2) and (3) or $p_2q_2 + p_2q_3 + p_3q_2 + p_3q_3$ and $d_2 = p_2q_2 + p_2q_3 + p_3q_2 + p_3q_3$ or the maximum value of $p_{23} + p_{32} + p_{33}$ subject to (1), (2) and (3). Further, $e_1 = p_{23} + p_{32} + p_{33}$ or $p_2q_2 + p_2q_3 + p_3q_2 + p_3q_3$ and $e_2 = p_2 + p_{32} + p_{33}$ or $q_2 + p_{23} + p_{33}$.

In conclusion, we have some sets of inequalities where each set is one of the two types A or B described above. These sets of inequalities are jointly equivalent to (1), (2), (3) and (4).

In order to find the extreme points of $M_{PQD}(p_1,p_2,p_3;\ q_1,q_2,q_3)$, we look at the following problem. Let A be the collection of all vectors $(p_{22},\ p_{23},\ p_{32},\ p_{33})$ satisfying

(1) $a_1 \leq p_{33} \leq a_2$,

(2) $b_1 \leq p_{23} + p_{33} \leq b_2$,

(3) $c_1 \leq p_{32} + p_{33} \leq c_2$,

(4) $d_1 \leq p_{23} + p_{32} + p_{33} \leq d_2$ and

(5) $e_1 \leq p_{22} + p_{23} + p_{32} + p_{33} \leq e_2$,

where a_1 and a_2 are given fixed numbers; b_1, b_2, c_1, c_2 are functions of p_{33}; d_1, d_2 are fixed numbers; and e_1, e_2 are functions of p_{23}, p_{32} and p_{33}. The set A is, obviously a compact convex set with a finite number of extreme points. We now describe a method of generating extreme points of the set A. Let $F_{\delta_1,\delta_2,\delta_3,\delta_4,\delta_5}$ denote the collection of all vectors $(p_{22},\ p_{23},\ p_{32},\ p_{33})$ in A for which

equality holds in (i) on the left of (i) if $\delta_i = 1$,

equality holds in (i) on the right of (i) if $\delta_i = -1$,

strictly inequality holds in (i) if $\delta_i = 0$,

for $i = 1,2,3,4,5$. Each δ_i takes one of the values 1, -1, 0. There $3^5 = 243$ pairwise disjoint sets $F_{\delta_1,\delta_2,\delta_3,\delta_4,\delta_5}$'s whose union is A. Some of these sets could be vacuous. Assume that for each $\delta_1,\delta_2,\delta_3,\delta_4,\delta_5 = +1$ or -1 there exists δ_4 such that $F_{\delta_1,\delta_2,\delta_3,\delta_4,\delta_5} \neq \emptyset$. This assumption is satisfied in the original problem under investigation. We define the dimension of each of these sets as follows. We associate the following vectors with each of the five inequalities (1), (2), (3), (4) and (5).

Inequality	Vector
(1)	(0,0,0,1)
(2)	(0,1,0,1)
(3)	(0,0,1,1)
(4)	(0,1,1,1)
(5)	(1,1,1,1)

Let $\delta_1,\delta_2,\delta_3,\delta_4$ and δ_5 be fixed. The dimension of $F_{\delta_1,\delta_2,\delta_3,\delta_4,\delta_5}$ is defined to be the number of 4 -(the number of linearly independent vectors among (i)'s above for which $\delta_i = +1$ or -1). For example, the set $F_{1,-1,1,1,-1}$ has dimension 0. The $F_{1,-1,-1,1,0}$ has dimension 1.

Each of the sets $F_{\delta_1,\delta_2,\delta_3,\delta_4,\delta_5}$ is compact and convex. These sets have the following property.

Theorem 11 Every member of any set of dimension i is a convex combination of members of some sets of lower dimension for $i = 1,2,3,4$.

Proof The proof in all its details is very lengthy. We merely sketch the details. Take any set of dimension 1, $F_{1,1,-1,-1,0}$, say. Look at the sets $F_{1,1,-1,-1,-1}$ and $F_{1,1,-1,-1,1}$. These two sets are singleton sets. If $(p_{22},p_{23},p_{32},p_{33}) \varepsilon F_{1,1,-1,-1,0}$, $(q_{22},q_{23},q_{32},q_{33}) \varepsilon$ $F_{1,1,-1,-1,-1}$ and $(s_{22},s_{23},s_{32},s_{33}) \varepsilon F_{1,1,-1,-1,1}$, then $p_{33} = q_{33} = s_{33}$; $p_{23} = q_{23} = s_{23}$; $p_{32} = q_{32} = s_{32}$ and $q_{22} < p_{22} < s_{22}$. We can write $(p_{22},p_{23},p_{32},p_{33})$ as a convex combination of $(q_{22},q_{23},q_{32},q_{33})$ and $(s_{22},s_{23},s_{32},s_{33})$. Similarly, every member of a set of dimension 2 can be written as a convex combination of some two members of a well-chosen set of dimension 1. The same argument can be carried out for sets of higher dimension.

As a consequence of the above result, we can identify the extreme points of A.

Corollary 12 The set of extreme points of A is contained in the union of all sets of dimension 0.

Theorem 11 and Corollary 12 are useful in enumerating extreme points of $M_{PQD}(p_1,p_2,p_3;q_1,q_2,q_3)$. For each set of inequalities, using Theorems 11 and 12, enumerate the extreme points of the corresponding set A defined as above. The totality of all these extreme points pooled together from each set of inequalities constitutes the set of all extreme points of $M_{PQD}(p_1,p_2,p_3;q_1,q_2,q_3)$.

We illustrate the above procedure with some examples.

Example 13

Let $p_1 = p_2 = p_3 = q_1 = q_2 = q_3 = 1/3$

Determining inequalities are:

(1) $p_3q_3 \leq p_{33} \leq p_3 \wedge q_3$,

(2) $p_{33} \vee (p_2q_3 + p_3q_3) \leq p_{23} + p_{33} \leq q_3 \wedge (p_2 + p_{33})$,

(3) $p_{33} \vee (p_3q_2 + p_3q_3) \leq p_{32} + p_{33} \leq p_3 \wedge (q_2 + p_{33})$,

(4) $(p_{23} + p_{32} + p_{33}) \vee (p_2q_2 + p_2q_3 + p_3q_2 + p_3q_3)$

$$\leq p_{22} + p_{23} + p_{32} + p_{33}$$

$$\leq (p_2 + p_{32} + p_{33}) \wedge (q_2 + p_{23} + p_{33}).$$

After substitution, these inequalities become

(1) $\frac{1}{9} \leq p_{33} \leq \frac{1}{3}$,

(2) $p_{33} \vee (\frac{2}{9}) \leq p_{23} + p_{33} \leq \frac{1}{3} \wedge (\frac{1}{3} + p_{33})$,

(3) $p_{33} \vee (\frac{2}{9}) \leq p_{32} + p_{33} \leq \frac{1}{3} \wedge (\frac{1}{3} + p_{33})$,

(4) $(p_{23} + p_{32} + p_{33}) \vee (\frac{4}{9}) \leq p_{22} + p_{23} + p_{32} + p_{33}$

$$\leq (\frac{1}{3} + p_{32} + p_{33}) \wedge (\frac{1}{3} + p_{23} + p_{33}).$$

The inequalities (2) and (3) induce a partition of the inequality (1) into

$$\frac{1}{9} \leq p_{33} \leq \frac{2}{9} \text{ and } \frac{2}{9} \leq p_{33} \leq \frac{1}{3}.$$

The inequalities (1), (2) and (3) are equivalent to

[1](1) $\frac{1}{9} \le p_{33} \le \frac{2}{9}$

[1](2) $\frac{2}{9} \le p_{23} + p_{33} \le \frac{1}{3}$

[1](3) $\frac{2}{9} \le p_{32} + p_{33} \le \frac{1}{3}$,

and

[2](1) $\frac{2}{9} \le p_{33} \le \frac{1}{3}$

[2](2) $p_{33} \le p_{23} + p_{33} \le \frac{1}{3}$

[2](3) $p_{33} \le p_{32} + p_{33} \le \frac{1}{3}$

The maximum value of $p_{23} + p_{32} + p_{33}$ when p_{23}, p_{32}, p_{33} satisfy [1](1), [1](2) and [1](3) is $\frac{5}{9}$. The maximum value of $p_{23} + p_{32} + p_{33}$ when p_{23}, p_{32}, p_{33} satisfy [2](1), [2](2) and [2](3) is $\frac{4}{9}$. Inequality (4) can be appended to each one of the sets above giving rise to 3 sets of inequalities equivalent to (1), (2), (3) and (4).

[1](1) $\frac{1}{9} \le p_{33} \le \frac{2}{9}$

[1](2) $\frac{2}{9} \le p_{23} + p_{33} \le \frac{1}{3}$

[1](3) $\frac{2}{9} \le p_{23} + p_{33} \le \frac{1}{3}$

[1](4) $\frac{2}{9} \le p_{23} + p_{32} + p_{33} \le \frac{4}{9}$

[1](5) $\frac{4}{9} \le p_{22} + p_{23} + p_{32} + p_{33} \le (\frac{1}{3} + p_{32} + p_{33}) \wedge (\frac{1}{3} + p_{23} + p_{33})$,

or,

[2](1) $\frac{1}{9} \le p_{33} \le \frac{2}{9}$

[2](2) $\frac{2}{9} \le p_{23} + p_{33} \le \frac{1}{3}$

[2](3) $\frac{2}{9} \le p_{32} + p_{33} \le \frac{1}{3}$

[2](4) $\frac{4}{9} \le p_{23} + p_{32} + p_{33} \le \frac{5}{9}$

[2](5) $p_{23} + p_{32} + p_{33} \le p_{22} + p_{23} + p_{32} + p_{33}$

$\le (\frac{1}{3} + p_{23} + p_{33}) \wedge (\frac{1}{3} + p_{32} + p_{33})$,

or,

[3](1) $\frac{2}{9} \leq p_{33} \leq \frac{1}{3}$

[3](2) $p_{33} \leq p_{23} + p_{33} \leq \frac{1}{3}$

[3](3) $p_{33} \leq p_{32} + p_{33} \leq \frac{1}{3}$

[3](4) $\frac{4}{9} \leq p_{22} + p_{23} + p_{32} + p_{33} \leq (\frac{1}{3} + p_{23} + p_{33}) \wedge (\frac{1}{3} + p_{32} + p_{33})$.

Now, in each of the sets [1] and [2] and in each of the inequalities (1), (2), (3) and (5), we set the central expression either equal to the quantity on the right or on the left and then solve the resultant set of equations in the unknowns p_{22}, p_{23}, p_{32} and p_{33} making sure that the inequality (4) is satisfied. Using these numbers, we build a matrix belonging to $M_{PQD}(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}; \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Avoiding repititions, we obtain the following matrices.

1. $$\begin{pmatrix} \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \end{pmatrix}$$

2. $$\begin{pmatrix} \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{2}{9} & 0 & \frac{1}{9} \\ 0 & \frac{2}{9} & \frac{1}{9} \end{pmatrix}$$

3. $$\begin{pmatrix} \frac{1}{9} & \frac{2}{9} & 0 \\ \frac{1}{9} & 0 & \frac{2}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \end{pmatrix}$$

4. $$\begin{pmatrix} \frac{2}{9} & 0 & \frac{1}{9} \\ 0 & \frac{2}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \end{pmatrix}$$

5. $$\begin{pmatrix} \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{2}{9} & 0 \\ \frac{1}{9} & 0 & \frac{2}{9} \end{pmatrix}$$

6. $$\begin{pmatrix} \frac{2}{9} & 0 & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ 0 & \frac{2}{9} & \frac{1}{9} \end{pmatrix}$$

7. $$\begin{pmatrix} \frac{2}{9} & \frac{1}{9} & 0 \\ 0 & \frac{1}{9} & \frac{2}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \end{pmatrix}$$

8. $$\begin{pmatrix} \frac{2}{9} & 0 & \frac{1}{9} \\ 0 & \frac{3}{9} & 0 \\ \frac{1}{9} & 0 & \frac{2}{9} \end{pmatrix}$$

9. $$\begin{pmatrix} \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{2}{9} & \frac{1}{9} & 0 \\ 0 & \frac{1}{9} & \frac{2}{9} \end{pmatrix}$$

10. $$\begin{pmatrix} \frac{1}{9} & \frac{2}{9} & 0 \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & 0 & \frac{2}{9} \end{pmatrix}$$

11. $$\begin{pmatrix} \frac{1}{9} & \frac{2}{9} & 0 \\ \frac{2}{9} & 0 & \frac{1}{9} \\ 0 & \frac{1}{9} & \frac{2}{9} \end{pmatrix}$$

12. $$\begin{pmatrix} \frac{2}{9} & \frac{1}{9} & 0 \\ 0 & \frac{2}{9} & \frac{1}{9} \\ \frac{1}{9} & 0 & \frac{2}{9} \end{pmatrix}$$

13. $$\begin{pmatrix} \frac{2}{9} & 0 & \frac{1}{9} \\ \frac{1}{9} & \frac{2}{9} & 0 \\ 0 & \frac{1}{9} & \frac{2}{9} \end{pmatrix}$$

14. $$\begin{pmatrix} \frac{3}{9} & 0 & 0 \\ 0 & \frac{2}{9} & \frac{1}{9} \\ 0 & \frac{1}{9} & \frac{2}{9} \end{pmatrix}$$

15. $$\begin{pmatrix} \frac{2}{9} & \frac{1}{9} & 0 \\ \frac{1}{9} & 0 & \frac{2}{9} \\ 0 & \frac{2}{9} & \frac{1}{9} \end{pmatrix}$$

16. $$\begin{pmatrix} \frac{3}{9} & 0 & 0 \\ 0 & \frac{1}{9} & \frac{2}{9} \\ 0 & \frac{2}{9} & \frac{1}{9} \end{pmatrix}$$

In the third lot of inequalities, set the central expression in each inequality either equal to the quantity on the right or on the left and then solve the resultant equations in p_{22}, p_{23}, p_{32} and p_{33}. We obtain the following matrices after omitting repititions.

$$17.\quad \begin{pmatrix} \frac{1}{9} & \frac{2}{9} & 0 \\ \frac{2}{9} & \frac{1}{9} & 0 \\ 0 & 0 & \frac{3}{9} \end{pmatrix} \qquad 18.\quad \begin{pmatrix} \frac{3}{9} & 0 & 0 \\ 0 & \frac{3}{9} & 0 \\ 0 & 0 & \frac{3}{9} \end{pmatrix}$$

Note that the matrix 14 is a convex combination of the matrices 16 and 18. All these seventeen matrices with the omission of matrix 14 are the extreme points of $M_{PQD}(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}; \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

It is interesting to compare the extreme points of the set $M_{PQD}(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}; \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ with the set $M(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}; \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ of all matrices of order 3×3 with nonnegative entries, row sums $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$ and column sums $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$. The set $M(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}; \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ has only 6 extreme points given by

$$\begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \qquad \begin{pmatrix} 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

$$\begin{pmatrix} 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{pmatrix}$$

Note that the set $M_{PQD}(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}; \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is a subset of $M(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}; \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and only one extreme point of $M(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}; \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is an extreme point of $M_{PQD}(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}; \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Example 14

Let $p_1 = \frac{1}{4}$, $p_2 = \frac{1}{2}$, $p_3 = \frac{1}{4}$, $q_1 = \frac{1}{8}$, $q_2 = \frac{1}{8}$ and $q_3 = \frac{3}{4}$.

The determining inequalities are:

(1) $p_3q_3 \leq p_{33} \leq p_3 \wedge q_3$

(2) $p_{33} \vee (p_2q_3 + p_3q_3) \leq p_{23} + p_{33} \leq q_3 \wedge (p_2 + p_{33})$

(3) $p_{33} \vee (p_3q_2 + p_3q_3) \leq p_{32} + p_{33} \leq p_3 \wedge (q_2 + p_{33})$

(4) $(p_{23} + p_{32} + p_{33}) \vee (p_2q_2 + p_2q_3 + p_3q_2 + p_3q_3)$
$\leq p_{22} + p_{23} + p_{32} + p_{33} \leq (p_2 + p_{32} + p_{33}) \wedge (q_2 + p_{23} + p_{33})$.

After substitution, these inequalities become

(1) $\frac{3}{16} \leq p_{33} \leq \frac{1}{4}$

(2) $p_{33} \vee (\frac{9}{16}) \leq p_{23} + p_{33} \leq \frac{3}{4} \wedge (\frac{1}{2} + p_{33})$

(3) $p_{33} \vee (\frac{7}{32}) \leq p_{32} + p_{33} \leq \frac{1}{4} \wedge (\frac{1}{8} + p_{33})$

(4) $(p_{23} + p_{32} + p_{33}) \vee (\frac{21}{32}) \leq p_{22} + p_{23} + p_{32} + p_{33}$
$\leq (\frac{1}{2} + p_{32} + p_{33}) \wedge (\frac{1}{8} + p_{23} + p_{33})$.

The inequalities (1), (2) and (3) are equivalent to the following two sets of inequalities.

[1](1) $\frac{6}{32} \leq p_{33} \leq \frac{7}{32}$

[1](2) $\frac{9}{16} \leq p_{23} + p_{33} \leq \frac{1}{2} + p_{33}$

[1](3) $\frac{7}{32} \leq p_{32} + p_{33} \leq \frac{1}{4}$,

or,

[2](1) $\frac{7}{32} \leq p_{33} \leq \frac{8}{32}$

[2](2) $\frac{9}{16} \leq p_{23} + p_{33} \leq \frac{1}{2} + p_{33}$

[2](3) $p_{33} \leq p_{32} + p_{33} \leq \frac{1}{4}$

The maximum and minimum values of $p_{23} + p_{32} + p_{33}$ are tabulated below for each of the above two sets.

Sets of inequalities	Expression	Maximum	Minimum
[1]	$p_{23} + p_{32} + p_{33}$	$\frac{24}{32}$	$\frac{18}{32}$
[2]	$p_{23} + p_{32} + p_{33}$	$\frac{24}{32}$	$\frac{18}{32}$

We append the inequality (4) to each one of the above sets after splitting the range of $p_{23} + p_{32} + p_{33}$ into two parts $[\frac{18}{32}, \frac{21}{32}]$ and $[\frac{21}{32}, \frac{24}{32}]$. We obtain four sets of inequalities equivalent to (1), (2), (3) and (4).

[1](1) $\frac{6}{32} \leq p_{33} \leq \frac{7}{32}$

[1](2) $\frac{18}{32} \leq p_{23} + p_{33} \leq \frac{16}{32} + p_{33}$

[1](3) $\frac{7}{32} \leq p_{32} + p_{33} \leq \frac{8}{32}$

[1](4) $\frac{18}{32} \leq p_{23} + p_{32} + p_{33} \leq \frac{21}{32}$

[1](5) $\frac{21}{32} \leq p_{22} + p_{23} + p_{32} + p_{33} \leq (\frac{16}{32} + p_{32} + p_{33}) \wedge (\frac{4}{32} + p_{23} + p_{33})$;

[2](1) $\frac{6}{32} \leq p_{33} \leq \frac{7}{32}$

[2](2) $\frac{18}{32} \leq p_{23} + p_{33} \leq \frac{16}{32} + p_{33}$

[2](3) $\frac{7}{32} \leq p_{32} + p_{33} \leq \frac{8}{32}$

[2](4) $\frac{21}{32} \leq p_{23} + p_{32} + p_{33} \leq \frac{24}{32}$

[2](5) $p_{23} + p_{32} + p_{33} \leq p_{22} + p_{23} + p_{32} + p_{33}$

$$\leq (\frac{16}{32} + p_{32} + p_{33}) \wedge (\frac{4}{32} + p_{23} + p_{33});$$

[3](1) $\frac{7}{32} \leq p_{33} \leq \frac{8}{32}$

[3](2) $\frac{18}{32} \leq p_{23} + p_{33} \leq \frac{16}{32} + p_{33}$

[3](3) $p_{33} \leq p_{32} + p_{33} \leq \frac{8}{32}$

[3](4) $\frac{18}{32} \leq p_{23} + p_{32} + p_{33} \leq \frac{21}{32}$

[3](5) $\frac{21}{32} \leq p_{22} + p_{23} + p_{32} + p_{33} \leq (\frac{16}{32} + p_{32} + p_{33}) \wedge (\frac{4}{32} + p_{23} + p_{33})$;

[4](1) $\frac{7}{32} \leq p_{33} \leq \frac{8}{32}$

[4](2) $\frac{18}{32} \leq p_{23} + p_{32} \leq \frac{16}{32} + p_{33}$

[4](3) $p_{33} \leq p_{32} + p_{33} \leq \frac{8}{32}$

[4](4) $\frac{21}{32} \leq p_{23} + p_{32} + p_{33} \leq \frac{24}{32}$

[4](5) $p_{23} + p_{32} + p_{33} \leq p_{22} + p_{23} + p_{32} + p_{33}$

$$\leq (\frac{16}{32} + p_{32} + p_{33}) \wedge (\frac{4}{32} + p_{23} + p_{33}).$$

In each set, for each of the inequalities (1), (2), (3) and (5), we set the central expression equal to either the quantity on the left or on the right and then solve the resultant equations in p_{22}, p_{23}, p_{32} and p_{33} after making sure that the inequality (4) is satisfied. Using p_{22}, p_{23}, p_{32}, p_{33}, one can build P in $M_{PQD}(p_1, p_2, p_3; q_1, q_2, q_3)$ in a natural way. These matrices are given below.

$$\begin{pmatrix} \frac{1}{32} & \frac{1}{32} & \frac{6}{32} \\ \frac{2}{32} & \frac{2}{32} & \frac{12}{32} \\ \frac{1}{32} & \frac{1}{32} & \frac{6}{32} \end{pmatrix} \qquad \begin{pmatrix} \frac{2}{32} & 0 & \frac{6}{32} \\ \frac{1}{32} & \frac{3}{32} & \frac{12}{32} \\ \frac{1}{32} & \frac{1}{32} & \frac{6}{32} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{32} & \frac{1}{32} & \frac{6}{32} \\ \frac{3}{32} & \frac{1}{32} & \frac{12}{32} \\ 0 & \frac{2}{32} & \frac{6}{32} \end{pmatrix} \qquad \begin{pmatrix} \frac{2}{32} & 0 & \frac{6}{32} \\ \frac{2}{32} & \frac{3}{32} & \frac{12}{32} \\ 0 & \frac{2}{32} & \frac{6}{32} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{32} & \frac{1}{32} & \frac{6}{32} \\ \frac{2}{32} & \frac{3}{32} & \frac{11}{32} \\ \frac{1}{32} & 0 & \frac{7}{32} \end{pmatrix} \qquad \begin{pmatrix} \frac{2}{32} & 0 & \frac{6}{32} \\ \frac{1}{32} & \frac{4}{32} & \frac{11}{32} \\ \frac{1}{32} & 0 & \frac{7}{32} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{32} & \frac{1}{32} & \frac{6}{32} \\ \frac{3}{32} & \frac{2}{32} & \frac{11}{32} \\ 0 & \frac{1}{32} & \frac{7}{32} \end{pmatrix} \qquad \begin{pmatrix} \frac{2}{32} & 0 & \frac{6}{32} \\ \frac{2}{32} & \frac{3}{32} & \frac{11}{32} \\ 0 & \frac{1}{32} & \frac{7}{32} \end{pmatrix}$$

$$\begin{pmatrix} \frac{3}{32} & \frac{3}{32} & \frac{2}{32} \\ 0 & 0 & \frac{16}{32} \\ \frac{1}{32} & \frac{1}{32} & \frac{6}{32} \end{pmatrix} \qquad \begin{pmatrix} \frac{4}{32} & \frac{2}{32} & \frac{2}{32} \\ 0 & 0 & \frac{16}{32} \\ 0 & \frac{2}{32} & \frac{6}{32} \end{pmatrix}$$

$$\begin{pmatrix} \frac{4}{32} & \frac{3}{32} & \frac{1}{32} \\ 0 & 0 & \frac{16}{32} \\ 0 & \frac{1}{32} & \frac{7}{32} \end{pmatrix} \qquad \begin{pmatrix} \frac{3}{32} & \frac{4}{32} & \frac{1}{32} \\ 0 & 0 & \frac{16}{32} \\ \frac{1}{32} & 0 & \frac{7}{32} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{32} & \frac{1}{32} & \frac{6}{32} \\ \frac{3}{32} & \frac{3}{32} & \frac{10}{32} \\ 0 & 0 & \frac{8}{32} \end{pmatrix} \qquad \begin{pmatrix} \frac{4}{32} & \frac{4}{32} & 0 \\ 0 & 0 & \frac{16}{32} \\ 0 & 0 & \frac{8}{32} \end{pmatrix}$$

$$\begin{pmatrix} \frac{2}{32} & 0 & \frac{6}{32} \\ \frac{2}{32} & \frac{4}{32} & \frac{10}{32} \\ 0 & 0 & \frac{8}{32} \end{pmatrix}.$$

These are the extreme points of $M_{PQD}(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}; \frac{1}{8}, \frac{1}{8}, \frac{3}{4})$.

4. On the power of tests for testing independence against positive quadrant dependence

In this section, we apply the results of the previous sections to study power of tests in the context of contingency tables. The problem, basically, is the following. Let X and Y be two random variables and each takes finitely many numerical values. The distributions of each of X and Y are known but their joint distribution is unknown. For simplicity, assume that X takes values 1,2,...,m with probabilities $p_1, p_2, \ldots, p_m$ respectively and Y takes values 1,2,...n with probabilities $q_1, q_2, \ldots, q_n$ respectively. Let $p_{ij} = P\{x = i, y = j\}$, $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n$. We say that X and Y are *strictly positive quadrant dependent* if X and Y are positively quadrant dependent but X and Y are not independent. We want to test the validity of the null hypothesis

H_0: X and Y are independent

against

H_1: X and Y are strictly positive quadrant dependent

based on a random sample of size N on (X,Y). Let n_{ij} = number in the sample with $x = i$ and $y = j$, $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n$. The data can be tabulated in the form of a contingency table.

y x	1	2	...	n
1	n_{11}	n_{12}	...	n_{1n}
2	n_{21}	n_{22}	...	n_{2n}
⋮	⋮	⋮	...	⋮
m	n_{m1}	n_{m2}	...	n_{mn}

Here, $\sum_{i=1}^{m} \sum_{j=1}^{n} n_{ij} = N$.

Let T be any test proposed for testing H_0 against H_1 for a given level of significance, α. T is a function of the random variables n_{ij}, $i = 1,2,\ldots,m$; $j = 1,2,\ldots n$. The performance of the test T can be judged based on its power function. More precisely, let $M_{PQD}(p_1,p_2,\ldots,p_m;\ q_1,q_2,\ldots,q_n)$ be the collection of all matrices P of order m × n with non-negative entries, row marginal sums $p_1,p_2,\ldots,p_m$, column marginal sums $q_1,q_1,\ldots,q_n$ and positive quadrant dependent. Let P_0 be the matrix of order m × n whose $(i,j)^{th}$ element is given by p_iq_j, $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n$. The parameter space of the above problem is $M_{PQD}(p_1,p_2,\ldots,p_m;\ q_1,q_2,\ldots,q_n)$. The hypotheses H_0 and H_1 translate into as follows. Let P stand for the generic symbol in $M_{PQD}(p_1,p_2,\ldots,p_m;\ q_1,q_2,\ldots,q_n)$ deemed as the joint distribution of X and Y. The relevant hypotheses are:

$$H_0:\ P = P_0$$

$$H_1:\ P \neq P_0$$

The power function $\beta(.)$ of T is defined by

$$\beta_T(P) = \Pr\{\text{T rejects } H_0 \text{ / the joint distribution of X and Y is P}\},$$

$$P \in M_{PQD}(p_1,p_2,\ldots,p_m;\ q_1,q_2,\ldots,q_n).$$

Obviously, $\beta_T(P_0) = \alpha$. The following results advocate the necessity of enumerating the extreme points of $M_{PQD}(P_1,p_2,\ldots,p_m;\ q_1,q_2,\ldots,q_n)$.

Theorem 4.1 Let $P \in M_{PQD}(p_1,p_2,\ldots,p_m;\ q_1,q_2,\ldots q_n)$ be a convex combination of some matrices $P_1,P_2,\ldots,P_k$ in $M_{PQD}(p_1,p_2,\ldots,p_m;\ q_1,q_2,\ldots,q_n)$, i.e.

$$P = \sum_{i=1}^{k} \alpha_i P_i$$

for some $\alpha_1,\alpha_2,\ldots,\alpha_k \geq 0$ with $\sum_{i=1}^{k} \alpha_i = 1$. Then

$$\beta_T(P) = \sum_{i=1}^{k} \alpha_i \beta_T(P_i).$$

Proof. Obvious.

The above theorem establishes the fact that $\beta(.)$ is an affine function on the compact convex set $M_{PQD}(p_1,p_2,\ldots,p_m;\ q_1,q_2,\ldots,q_n)$.

Corollary 4.2 Let $E\ (p_1,p_2,\ldots,p_m;\ q_1,q_2,\ldots,q_n) = E$, say, be the set of all extreme points of $M_{PQD}(p_1,p_2,\ldots,p_m;\ q_1,q_2,\ldots,q_n)$. Let E consist of some k matrices $P_1,P_2,\ldots,P_k$. Then for any P in $M_{PQD}(p_1,p_2,\ldots,p_m;\ q_1,q_2,\ldots,q_n)$, $\beta_P(T)$ is a convex combination of $\beta_T(P_1)$, $\beta_T(P_2),\ldots,$ $\beta_T(P_k)$.

The above corollary provides us a criterion by which we can compare the performance of any two given tests T_1 and T_2 proposed to discriminate between H_0 and H_1. If $P_1,P_2,\ldots,P_k$ are the extreme points of $M_{PQD}(p_1,p_2,\ldots,p_m;\ q_1,q_2,\ldots q_n)$, then we compare the two sets of numbers

$$\{\ \beta_{T_1}(P_1)\ ,\ \beta_{T_1}(P_2)\ ,\ldots,\ \beta_{T_1}(P_k)\ \}$$

and

$$\{\ \beta_{T_2}(P_1)\ ,\ \beta_{T_2}(P_2)\ ,\ldots,\ \beta_{T_2}(P_k)\ \}$$

for a showdown between T_1 and T_2 as to which one is more effective in discriminating between H_0 and H_1.

In a paper by Bhaskara Rao, Krishnaiah and Subramanyam [6], a detailed comparison of performance of several tests is carried out for some cases of contingency tables.

5. Comparison of tests

In this section, we actually embark on comparing the performance of several tests for testing the null hypothesis H_o against the alternative H_1 specified in the previous section in the case of 2×2 and 2×3 contingency tables. First, we deal with the 2×2 case.

We now discuss the power function of the test based on Goodman-Kruskal's Gamma Ratio. Let

p_{11}	p_{12}	p_1
p_{21}	p_{22}	p_2
q_1	q_2	1

be the joint distribution of X and Y with specified marginal distributions p_1, p_2 and q_1, q_2 respectively. The Goodman-Kruskal's Gamma Ratio, γ, is defined by (see Agresti [1])

$$\gamma = \frac{p_{11}p_{22} - p_{12}p_{21}}{p_{11}p_{22} + p_{12}p_{21}} .$$

The following properties of γ are easy to check.

(i) $-1 \leq \gamma \leq 1$.

(ii) X and Y are independent if and only if $\gamma = 0$.

(iii) X and Y are positive quadrant dependent if and only if $\gamma \geq 0$.

(iv) X and Y are strictly positive quadrant dependent if and only if $\gamma > 0$.

From the above properties, we can assert that γ is a measure of positive quadrant dependence between X and Y. (For the validity of (iii), we have used the fact that the marginal distributions are specified.) We also observe the following. The extreme points of the convex set $M_{PQD}(p_1, p_2; q_1, q_2)$

are given by

$$P_1 = \begin{bmatrix} p_1q_1 & p_1q_2 \\ p_2q_1 & p_2q_2 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} q_1 & q_2-p_2 \\ 0 & p_2 \end{bmatrix} \tag{5.1}$$

assuming that $p_2 \wedge q_2 = p_2$. Under P_1, i.e., the joint distribution of X and Y is P_1, $\gamma = 0$ and under P_2, $\gamma = 1$.

A natural estimator of γ is given by

$$\hat{\gamma} = \frac{n_{11}n_{22} - n_{12}n_{21}}{n_{11}n_{22} + n_{12}n_{21}} . \tag{5.2}$$

<u>Test of H_o against H_1 based on $\hat{\gamma}$: T.</u>

Reject H_o in favour of H_1 if and only if $\hat{\gamma} \geq a$, where a is chosen satisfying $\Pr(\hat{\gamma} \geq a / H_o) = \alpha$, the given level of significance. One can work out the power function of this test explicitly. Let

$$\beta_T(P) = \Pr(\text{the test } T \text{ rejects } H_o / P)$$
$$= \Pr(\hat{\gamma} \geq a / P)$$

for P in $M_{PQD}(p_1,p_2;q_1,q_2)$, where the above probability is computed when X and Y have the joint distribution P. As has been pointed out in Section 4, $\beta_T(P)$ is connected to $\beta_T(P_1)$ and $\beta_T(P_2)$ as follows. Write

$$P = \lambda P_1 + (1-\lambda)P_2$$

for some $0 \leq \lambda \leq 1$. Then

$$\beta_T(P) = \lambda\beta_T(P_1) + (1-\lambda)\beta_T(P_2).$$

Obviously, $\beta_T(P_1) = \alpha$. Under P_2, $n_{12}n_{21} = 0$ almost surely and consequently $\hat{\gamma} = 1$ almost surely. This implies that $\beta_T(P_2) = 1$. Thus we have proved the following theorem.

Theorem 5.1 The power function $\beta_T(\ )$ of the test T of size α is given by

$$\beta_T(P) = \lambda\alpha + (1-\lambda)$$

for P in $M_{PQD}(p_1,p_2;q_1,q_2)$, where $P = \lambda P_1 + (1-\lambda)P_2$.

The above result yields the following result.

Theorem 5.2 Let T* be any test proposed for testing H_o against H_1 specified in the previous section. Then

$$\beta_T(P) \geq \beta_{T*}(P)$$

for every P in $M_{PQD}(p_1,p_2;q_1,q_2)$.

Thus we see that in the case of 2 x 2 contingency tables, the test based on Goodman-Kruskal's Gamma Ratio is superior to any test one can propose for testing the hypothesis of independence against the hypothesis of strict positive quadrant dependence.

Let us examine how the test based on Goodman-Kruskal's Gamma Ratio fares against another test proposed in the case of 2 x 3 contingency tables.

Let the joint distribution of X and Y be given by

p_{11}	p_{12}	p_{13}	p_1
p_{21}	p_{22}	p_{23}	p_2
q_1	q_2	q_3	1

,

where p_1, p_2 and q_1, q_2, q_3 are the marginal distributions of X and Y respectively. The Goodman-Kruskal's Gamma Ratio, γ , is defined by

$$\gamma = \frac{\Pi_c - \Pi_d}{\Pi_c + \Pi_d} ,$$

where $\Pi_c = p_{11}(p_{22} + p_{23}) + p_{12}p_{23}$ and $\Pi_d = p_{13}(p_{21} + p_{22}) + p_{12}p_{21}$. See Agresti [1, p.160]. We propose another measure of association between X and Y as follows.

$$\kappa = \frac{(p_{11}p_{22} - p_{12}p_{21})^2}{p_1p_2q_1q_2} + \frac{(p_{11}p_{23} - p_{13}p_{21})^2}{p_1p_2q_1q_3} + \frac{(p_{12}p_{23} - p_{13}p_{22})^2}{p_1p_2q_2q_3}.$$

This measure of association is motivated by the definition of Spearman's rho for 2 x 2 contingency tables. The following properties of γ and κ are easy to verify.

1. $-1 \leq \gamma \leq 1$.
2. If X and Y are independent, then $\gamma = 0$.
3. If X and Y are positive quadrant dependent, then $\kappa \geq 0$.
4. $0 \leq \kappa \leq 1$.
5. $\kappa = 0$ if and only if X and Y are independent.

One can build tests based on the Gamma Ratio and the kappa criterion.

Test based on the Gamma Ratio : T

Let C = the total number of concordant pairs in the data

$$= n_{11}(n_{22} + n_{23}) + n_{12}n_{23}$$

and D = the total number of discordant pairs in the data

$$= n_{13}(n_{21} + n_{22}) + n_{12}n_{21}.$$

An estimate of γ is given by

$$\hat{\gamma} = \frac{C - D}{C + D}.$$

Test based on $\hat{\gamma}$ is given by :

Reject the null hypothesis H_o in favour of H_1 if $\hat{\gamma} \geq a$, where a is given by $\Pr(\hat{\gamma} \geq a / H_o) = \alpha$, the given level of significance. Let $\beta_T(\ )$ be the power function of the test T.

Test based on the Kappa value : T*

An estimate of κ based on the given data is given by

$$\hat{\kappa} = \frac{(n_{11}n_{22} - n_{12}n_{21})^2}{N^4 p_1 p_2 q_1 q_2} + \frac{(n_{11}n_{23} - n_{12}n_{21})^2}{N^4 p_1 p_2 q_1 q_3} + \frac{(n_{12}n_{23} - n_{13}n_{22})^2}{N^4 p_1 p_2 q_2 q_3}.$$

Test based on $\hat{\kappa}$ is given by :

Reject H_o in favour of H_1 if and only if $\hat{\kappa} \geq a$, where a is given by $\Pr(\hat{\kappa} \geq a / H_o) = \alpha$, the given level of significance. Let $\beta_{T^*}(\ )$ be the power function of the test T*.

END

DTIC

9-86