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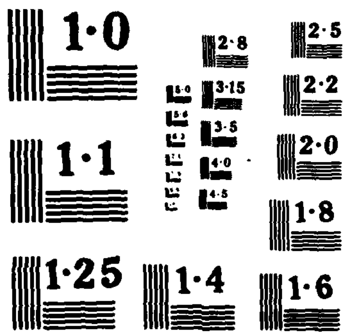
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Limiting Behavior of Linearly Damped Hyperbolic Equations

by

Jack K. Hale and Nicholas Stavrakakis

January 1986 LCDS #86-6

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Limiting Behavior of Linearly Damped Hyperbolic Equations

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Abstract

For a linearly damped wave equation in a bounded domain in R^n, it is shown that there is a compact attractor in $H_0^1 \times L^2$ as well as in $(H^2 \cap H_0^1) \times H_0^1$. Similar results are given for the linearly damped beam equation.

I. Introduction

Let Y be a Banach space, A be a sectorial operator on Y and let Y^α be the fractional power spaces of A, $Y^0 = Y$. Suppose $f: Y^{1/2} \to Y$ is locally Lipschitzian, $\beta > 0$ is a constant and consider the equation:

$$u_{tt} + 2\beta\, u_t + Au = f(u) \tag{1.1}$$

with the initial data $(u,u_t) = (\varphi,\psi) \in X_2 \stackrel{def}{=} Y^{1/2} x Y$ at $t = 0$. If the solution (u,u_t) is defined for $t \geq 0$ and $T(t)(\varphi,\psi) = (u,u_t)$, suppose $T(t)$: $X_2 \to X_2$ is a C^0-semigroup.

A set $J \subset X_2$ is said to be *invariant* under $T(t)$ if $T(t)J = J$, for $t \geq 0$. A set is *maximal compact invariant* if it is compact, invariant and maximal with respect to these properties. An invariant set J in X_2 is a *compact attractor* for $T(t)$ in X_2, if J is maximal compact invariant and *attracts* the bounded sets of X_2; that is, for any bounded set $B \subset X_2$, and any $\epsilon > 0$, there is a $t_0 = t_0(\epsilon,B,J)$ such that $dist_{X_2}(T(t)B,J) < \epsilon$, if $t \geq t_0$. Orbits of points in B approach J uniformly with respect to B.

The semigroup $T(t)$ is *point* (*compact*) (*bounded*) *dissipative* if there is a bounded set B in X_2, that attracts each point (compact set) (bounded set) of X_2 under T.

Our objective in this paper is to give conditions under which (1.1) has a compact attractor J in X_2 and also prove that J belongs to $X_1 = (Y^1 \cap Y^{1/2}) \times Y^{1/2}$ and is a compact attractor in X_1. The definition of a compact attractor in X_1 is given in the same way as the one above in X_2.

Applications are given for the linearly damped wave equation and the linearly damped beam equation.

The following hypotheses are needed:

(H1) $A = B^2$, B^{-1} compact, $\pm iB$ generates a C^0-semigroup on Y and $Y^{1/2}$, $|e^{\pm iB}| \leq k\, e^{\omega t}$, $t \geq 0$, on each space and $\omega < \beta$.

(H2) The nonhomogeneous linear problem:

$$u_{tt} + 2\beta u_t + Au = g(t)$$
$$(u,u_t)|_{t=0} = 0$$

has the property that

(i) $g \in W^{1,1}(0,T;Y)$ implies $(u,u_t) \in X_1$ and is continuous in (t,g)

(ii) $g \in W^{1,1}(0,T;Y^{1/2})$ implies (u,u_t) belongs to a compact set of X_1

(H3) Equation (1.1) defines a C^0-semigroup on X_1 and X_2

Theorem 1.1. *Suppose hypotheses* (H1) - (H3) *are satisfied. If* $T(t)$ *is point dissipative in* X_2 *and orbits of bounded sets in* X_2 *are bounded in* X_2, *then there is a compact connected attractor* J *in* X_2. *Furthermore,* $J \subset X_1$ and J *is a compact attractor in* X_1.

It will be clear from the proof of Theorem 1.1 that Hypotheses (H1) - (H3) imply that any invariant set J in X_2 belongs to X_1.

In particular, if $(\varphi,\psi) \in J$, then $(\varphi,\psi) \in D(C)$, the domain of C, where C is the generator of the group on X_2 defined by the linear system

$$\{u_t = v,\ v_t = -Au - 2\beta v\}$$

Thus, $J \subset$ the domain of the generator of $T(t)$. So we have that $T(t)|J$ is a C^1-function in t. This implies, for example, that any periodic orbit of (1.1) must be a C^1-manifold. We do not exploit this fact here since the special cases of (1.1) to be considered in this paper are actually gradient systems and no periodic orbits exist.

Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$, $\beta > 0$ be a constant, Δ be the Laplacian, $g \in L^2(\Omega)$, $f \in C^2(\mathbb{R},\mathbb{R})$ and suppose there is a positive constant $c > 0$ such that

(1.2) $$|f''(u)| \leq c(|u|+1), \quad \text{for} \quad u \in \mathbb{R}$$

(1.3) $$\overline{\lim_{|u|\to\infty}}\ f(u)/u \leq 0\ .$$

Consider the *wave equation*

(1.4) $$\begin{cases} u_{tt} + 2\beta\, u_t - \Delta u = f(u) - g\ , & \text{in} \quad \Omega \\ u = 0 & , \text{on} \quad \partial\Omega \end{cases}$$

As an application of Theorem 1.1, we prove

Theorem 1.2. *If* $X_2 = H_0^1(\Omega) \times L^2(\Omega)$, $X_1 = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ *then equation* (1.4) *defines a* $C^{j-1,1}$-semigroup *on* X_j, $j = 1,2$. *Also, there is a compact connected attractor* J *in* X_2, $J \subset X_1$, *and* J *is a compact attractor in* X_1.

For equation (1.4), Babin and Vishik [2], [3] proved that the set J is a compact invariant set in X_2. Since B is bounded in X_1 implies $c\ell B$ is compact in X_2, the assertion of Babin and Vishik is that J attracts these special compact sets in X_2. The assertion in Theorem 1.2 is J attracts in X_2 all bounded sets of X_2 and, also J attracts in X_1 all bounded sets of X_1. Babin and Vishik show also that $J \subset X_1$, by a method different from the one below. They make no remarks about convergence of orbits to J in X_1.

If $f \in C^1(\mathbb{R},\mathbb{R})$ and there are constants $c > 0$, $\gamma > 0$ such that

$$|f'(u)| \leq c(|u|^{2-\gamma} + 1)$$

then Haraux [13], Hale [10] proved part (i) of Theorem 1.2. Part (ii) of Theorem 1.2 seems to be completely new.

Since equation (1.4) will be shown to be a gradient system, one can say more about the attractor J, as has been observed by Babin and Vishik [2] and Hale [10]. In fact, if E is the set of *equilibrium* solutions of (1.4); that is, E is the set of $(\varphi,0) \in X_2$ such that

$$\begin{aligned} &\Delta\varphi + f(\varphi) - g = 0, \text{ in } \Omega \\ &u = 0 \qquad\qquad \text{on } \partial\Omega \end{aligned}$$

then $J = W^u(E)$, the *unstable set of* E; that is, $W^u(E) = \{(\varphi,\psi) \in X_2$: $T(t)(\varphi,\psi)$ can be defined for $t \leq 0$ and $T(t)(\varphi,\psi) \to E$, as $t \to -\infty\}$.

If, in addition, each equilibrium point $(\varphi,0) \in E$ is hyperbolic, then

$$J = \bigcup_{(\varphi,0)\in E} W^u(\varphi,0) \quad ,$$

where $W^u(\varphi,0)$ is the unstable set for $(\varphi,0)$. An equilibrium point $(\varphi,0)$ is *hyperbolic* if $\mathrm{Re}\,\lambda \neq 0$ for every λ for which the equation

$$\begin{aligned} &\Delta u + f'(\varphi)u = (\lambda^2 + 2\beta\lambda)u, \text{ in } \Omega \\ &u = 0 \qquad\qquad\qquad\qquad\qquad\;\, , \text{ on } \partial\Omega \end{aligned}$$

has a nontrivial solution. Since $\beta > 0$, this is equivalent to saying that no eigenvalue of $\Delta + f'(\varphi)$ on X_2 is zero.

For the case where $\Omega \subset \mathbb{R}^2$, Theorem 1.2 is also valid with condition (1.2) replaced by:

There are constants $c > 0$, $\tau > 0$ such that

$$(1.2)' \qquad |f''(u)| \leq c(|u|^\tau + 1)$$

If $\Omega \subset \mathbb{R}^1$, Theorem 1.2 is valid with no restrictions of the form (1.2). If $\Omega \subset \mathbb{R}^n$, $n \geq 4$, condition (1.2) should be replaced by: there is a constant $c > 0$ such that

$$(1.2)'' \qquad |f^{(j)}(u)| \leq c\,, \quad \text{for} \quad u \in \mathbb{R}\,,\ j = 1,2.$$

The proof of the theorem in these cases is essentially the same as the one for $\Omega \subset \mathbb{R}^3$.

Another application of Theorem 1.1, is the *Beam equation.* Let $\Omega = [0,\ell]$, $\ell > 0$, and $a,\delta,\kappa \in \mathbb{R}^+ = (0,\infty)$, $\beta \in \mathbb{R}$

We consider the equation:

$$(1.5) \qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \delta \dfrac{\partial u}{\partial t} + a\dfrac{\partial^4 u}{\partial x^4} = \left(\beta + \kappa \displaystyle\int_0^\ell \left[\dfrac{\partial u(\xi,t)}{\partial \xi}\right]^2 d\xi\right) \dfrac{\partial^2 u}{\partial x^2} \\ u(0) = u_0 \, , \; u_t(0) = u_1 \end{cases}$$

with either *clamped ends*

$$(1.6) \qquad u(0,t) = u(\ell,t) = u_x(0,t) = u_x(\ell,t) = 0$$

or *hinged ends*

$$(1.7) \qquad u(0,t) = u(\ell,t) = u_{xx}(0,t) = u_{xx}(\ell,t) = 0 \, .$$

Theorem 1.3. *If* $X_2^c = H_0^2(\Omega) \times L^2(\Omega)$, $X_1^c = (H^4(\Omega) \cap H_0^2(\Omega)) \times H_0^2(\Omega)$, $X_2^h = (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega)$, $X_1^h = (H^4(\Omega) \cap H_0^1(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega))$, *then*

(i) problem (1.5), (1.6) *defines a* $C^{j-1,1}$*-semigroup* $T^c(t)$ on X_j^c, $j=1,2$, *there is a compact, connected, attractor* J^c *for* $T^c(t)$ *in* X_2^c, $J^c \subset X_1^c$ *and* J^c *is a compact attractor for* $T^c(t)$ *in* X_1^c.

(ii) problem (1.5), (1.7) *defines a* $C^{j-1,1}$*-semigroup* $T^h(t)$ *on* X_j^h, $j=1,2$, *there is a compact, connected attractor* J^h *for* $T^h(t)$ *in* X_2^h, $J^h \subset X_1^h$ *and* J^h *is a compact attractor for* $T^h(t)$ *in* X_1^h

In each case, the attractor is the union of the unstable manifolds of the equilibrium points, if they are all hyperbolic.

Ball [4], [5], [6] has discussed the existence of the semigroup defined by the beam equation in the spaces X_2^c, X_2^h and proved that every solution approaches the set of equilibrium points. The concept of weakly invariant sets of Dafermos [8] played an important role.

Arstein and Slemrod [1] show that, in the weak topology, each stable equilibrium point is connected by an orbit to some other equilibrium point. The connectedness of the attractor implies this result. Our results in the strong topology of X_2^c (resp. X_2^h) show that the use of the weak topology is unnecesary.

The proof of Theorem 1.1 is based on some abstract theorems of Massatt [20], [21], [22] on dissipative processes. These later results were inspired by much earlier works of Billotti and LaSalle [7] and Hale, LaSalle and Slemrod [11] in the early 1970's.

The proofs of Theorems 1.2 and 1.3 will involve very few technical estimates on the partial differential equations. The estimates that are necessary involve energy estimates to obtain global existence in X_2 and to show that orbits of bounded sets in X_2 are bounded in X_2. We must also show that the equilibrium set E is bounded.

The other parts of the proofs use elementary properties of linear hyperbolic equations.

2. Summary of results on dissipative processes.

Suppose $T(t): X \to X$ is a C^0-semigroup on the Banach space X. Also assume that

(2.1) $$T(t) = S(t) + U(t)$$

(2.2) $$\begin{cases} S(t): X \to X \quad \text{linear, } C^0\text{-semigroup} \\ \exists\ \kappa > 0,\ \delta > 0 \quad \text{such that} \quad |S(t)|_X \leq \kappa\, e^{-\delta t},\ t \geq 0 \end{cases}$$

(2.3) $U(t): X \to X$ is continuous.

The following theorems are adapted from Massatt [20, 21, 22] for the special case (2.1) - (2.3).

Theorem 2.1. *Suppose* $T(t): X \to X$ *satisfies* (2.1) - (2.3). *If*

(i) $U(t)$ *is completely continuous for* $t \geq 0$.

(ii) $T(t)$ *is point dissipative*

(iii) orbits of bounded sets in X *are bounded.*

Then there is a compact attractor J *for* $T(t)$ *in* X, *which is connected.*

The next results deal with the case in which a semigroup may be defined on two different spaces X_1 and X_2 as, for example, equations (1.1), (1.4), (1.5).

We suppose abstractly that X_1, X_2 are Banach spaces and

(2.4) i: $X_1 \to X_2$ is a compact embedding

(2.5) X_1 is dense in X_2

(2.6) T(t): $X_j \to X_j$, j=1,2 has the decomposition (2.1) and (2.2), (2.3) are satisfied with $X = X_j$, j=1,2.

(2.7) U(t): $X_2 \to X_1$ is continuous and, for any $\tau \geq 0$ and any $B \subset X_2$, for which $\{U(t)B, 0 \leq t \leq \tau\}$ is bounded in X_2, the set $\{U(t)B, 0 \leq t \leq \tau\}$ is bounded in X_1.

The map U(t): $X_j \to X_j$ is said to be *conditionally completely continuous* if, for any bounded set B in X_j for which $\{U(s)B; 0 \leq s \leq t\}$ is bounded, it follows that U(t)B is precompact in X_j, j=1,2.

Theorem 2.2. *Suppose* (2.4) - (2.7) *are satisfied. Then*

(i) U(t) *is conditionally completely continuous in* X_2

(ii) If T(t) *is point dissipative in* X_2, *then* T(t) *is bounded dissipative in* X_1

(iii) If U(t): $X_1 \to X_1$ *is conditionally completely continuous on* X_1, *then any closed bounded invariant set in* X_2 *is a compact invariant set in* X_1.

Hypothesis (2.7) together with the fact that X_1 is compactly embedded in X_2 implies (i) of Theorem 2.2. Conclusions (ii) and (iii) require rather lengthy proofs. Notice that the conclusions are very strong.

Conclusion (ii) says that orbits of bounded sets in X_1 are bounded in X_1 and are uniformly attracted in X_1 to a fixed bounded set. Thus, if

$U(t)\colon X_1 \to X_1$ is completely continuous, Theorem 2.1 implies that there is a compact attractor in X_1. This result is a consequence only of the point dissipativeness in X_2!

Conclusion (iii) of Theorem 2.2 is a regularity result for l nded invariant sets; namely, being in X_2, is enough to imply they are in X_1!

3. Proof of Theorem 1.1.

We need the following lemma which is essentially contained in Pazy [23].

Lemma 3.1. *If* (H1) *is satisfied and*

$$C_\beta = \begin{bmatrix} 0 & I \\ -B^2 & -2\beta I \end{bmatrix} \tag{3.1}$$

then C_β *generates a* C^0-group $e^{C_\beta t}$ *on* $X_2 = Y^{1/2} \times Y$ *and* $X_1 = (Y^1 \cap Y^{1/2}) \times Y^{1/2}$. *Furthermore, if*

$$|e^{\pm iB}| \leq \kappa\, e^{\omega t},\ t \geq 0 \quad \text{in} \quad Y \quad \text{and} \quad Y^{1/2}$$

then, for any $\epsilon > 0$, *there is a constant* $\kappa = \kappa(\epsilon)$ *such that*

$$e^{C_\beta t} = S(t) + U_1(t) \tag{3.2}$$

where $S(t) = e^{C_0 t} W e^{-\beta t}$, $W = \begin{bmatrix} I & 0 \\ -\beta I & I \end{bmatrix}$

$$|S(t)| \leq \kappa\, e^{(\omega-\beta+\epsilon)t}, \quad t \geq 0 \tag{3.3}$$

and $U_1(t)$ *is completely continuous for* $t \geq 0$.

Proof. If

$$\frac{dU}{dt} = c\ U\ ,\ U = (u_1,u_2),$$

and

$$u_1 = v_1\ e^{-\beta t},\quad u_2 = (v_2 - \beta v_1)\ e^{-\beta t},\quad v = (v_1,v_2)$$

then

$$\frac{dv}{dt} = D_\beta v\ ,\quad D_\beta = \begin{bmatrix} 0 & I \\ -B^2+\beta I & 0 \end{bmatrix},\quad D_0 = C_0\ .$$

For the time being, let us suppose that $\beta = 0$ and consider the space $X_2 = Y^{1/2} \times Y$. Consider the transformation of variables from $Y^{1/2} \times Y$ to $Y \times Y$ given by

$$w_1 = v_1 + iB^{-1}\ v_2\ ,\ w_2 = v_1 - iB^{-1}v_2$$

with the inverse transformation

$$v_1 = \frac{w_1+w_2}{2}\quad ,\quad v_2 = \frac{iB(-w_1+w_2)}{2}$$

Define

$$e^{D_0 t}(v_1,v_2) = \left[\frac{e^{-iBt}w_1+e^{iBt}w_2}{2}\ ,\ iB\ \frac{(-e^{-iBt}w_1+e^{iBt}w_2)}{2}\right]$$

This is a C_0-semigroup on $Y^{1/2} \times Y$ with generator D_0. Furthermore, $(e^{D_0 t})^{-1}$ exists for each $t \geq 0$ and is a C^0-semigroup of bounded linear operators with infinitesimal generator $-D_0$. Therefore,

$$V(t) = e^{D_0 t},\ t \geq 0,\ V(t) = \left[e^{-D_0 t}\right]^{-1},\ t \leq 0$$

defines a C^0-group on $Y^{1/2} \times Y$ (see Pazy [23], p.62). This notation is cumbersome and we let $e^{D_0 t} = V(t)$, for all $t \in \mathbb{R}$. If we define

$$S(t) = V(t)\, W\, e^{-\beta t},$$

then the estimate (3.3) holds.

Since $D_\beta - D_0: Y^{1/2} \times Y \to Y^{1/2} \times Y$ is compact, it follows from S.G. Krein [17] that D_β defines a C_0-group on $Y^{1/2} \times Y$. If we apply the variation of constants formula to the equation

$$\frac{dw}{dt} = D_0 w + (D_\beta - D_0)w,$$

then

$$e^{D_\beta t} v_0 = e^{D_0 t} v_0 + \int_0^t e^{D_0 (t-s)} (D_\beta - D_0)\, e^{D_\beta s} v_0\, ds$$

Since $D_\beta - D_0$ is compact, the later integral is compact. Using the fact that $e^{C_\beta t} = e^{D_\beta t} W e^{-\beta t}$, one completes the proof of the Lemma for the space X_2. The proof for X_1, is similar and, therefore, omitted.

Equation (1.1) can be written as a system

$$\frac{dw}{dt} = C_\beta w + F(w)$$

$$w = \begin{bmatrix} u \\ v \end{bmatrix},\ C_\beta = \begin{bmatrix} 0 & I \\ -B^2 & -2\beta I \end{bmatrix},\ F(w) = \begin{bmatrix} 0 \\ f(u) \end{bmatrix}$$

or, in integral form

$$w(t) = e^{C_\beta t} w_0 + \int_0^t e^{C_\beta (t-s)} F(w(s))ds$$

For given w_0, let $U_2(t)w_0 = \int_0^t e^{C_\beta (t-s)} F(w(s))ds$, suppose that B is a bounded set in X_2 such that $\{w(s),\ 0 \leq s \leq t,\ w(0) = w_0 \in B\}$ is bounded. Then, for each $w_0 \in B$, the function $u(\cdot, w_0): [0,t] \to Y^{1/2}$ is continuous and $\{u(s,w_0)\ ,\ 0 \leq s \leq t,\ w(0) \in B\}$ is bounded. Thus, $g: [0,t] \times B \to Y$, $g(s,w_0) = f(u(s,w_0))$ belongs to $W^{1,1}([0,t];Y)$. The function $U_2(t)w_0$ is the solution of the differential equation

$$\frac{dz}{dt} = C_\beta z + \begin{bmatrix} 0 \\ g(t,\cdot) \end{bmatrix}$$

$$z(0) = 0$$

From Hypothesis (H2)(i), it follows that the set

$$\{U_2(s)B;\ 0 \leq s \leq t\}$$

belongs to a bounded set in X_1 and is therefore precompact in X_2. From Lemma 3.1, this implies that

$$T(t) = S(t) + U(t).$$

where $S(t)$ satisfies (3.3) and $U(t)$ is conditionally completely continuous.

Since $\omega < \beta$, we can choose $\epsilon > 0$ so that $\omega - \beta + \epsilon < 0$. If we repeat the same argument as above on the space X_1, then $T(t)$ satisfies (2.1) - (2.7) and conditions (i) and (iii) of Theorem 2.1. Now Theorems 2.1. and 2.2 complete the proof of Theorem 1.1.

4. Proof of Theorem 1.2. (*wave equation*).

Equation (1.4) is a special case of (1.1) with $Y = L^2(\Omega)$, $A = -\Delta$. If $A = B^2$, λ_n the eigenvalues of A and φ_n are the normalized eigenfunctions, then each $\lambda_n > 0$ and

$$\varphi = \sum_{n=1}^{\infty} (\varphi,\varphi_n)\varphi_n\,, \quad \varphi \in L^2$$

$$A\varphi = \sum_{n=1}^{\infty} \lambda_n^2 (\varphi,\varphi_n)\varphi_n, \quad \varphi \in D(A) = H^2 \cap H_0^1$$

$$B\varphi = \sum_{n=1}^{\infty} \lambda_n(\varphi,\varphi_n)\varphi_n, \quad \varphi \in D(B) = H_0^1$$

$$B^{-1}\psi = \sum_{n=1}^{\infty} \lambda_n^{-1} (\psi,\varphi_n)\varphi_n, \quad \psi \in L^2$$

If $\lambda \in \rho(B)$, the resolvent set of B, then

$$(\pm iB - \lambda I)^{-n}\psi = \sum_{n=1}^{\infty} (\pm i\lambda_n - \lambda)^{-n} (\psi,\varphi_n)\varphi_n, \ \psi \in L^2 .$$

Using this expressions, one easily sees that B^{-1} is compact on L^2 and, for any $\omega > 0$, $\lambda < -\omega$, we have $|(\pm iB - \lambda I)^{-n}|_{L^2} \leq |\lambda + \omega|^{-n}$. Thus, $\pm iB$ generate C^0-semigroups on L^2 and, for any $\omega > 0$, $|e^{\pm iBt}| \leq e^{\omega t}$, $t \geq 0$.

Analogous reasoning shows that $\pm iB$ generate C^0-semigroups on H_0^1 with the same bound on $e^{\pm iBt}$.

If we choose $\omega < \beta$, then

$$C = \begin{bmatrix} 0 & I \\ \Delta & -2\beta I \end{bmatrix}$$

generates a C^0-group e^{Ct} on $X_2 = H_0^1 \times L^2$ and $X_1 = (H^2 \cap H_0^1) \times H_0^1$ and there are constants $\kappa > 0$, $\delta > 0$, such that

(4.1) $$|e^{Ct}|_{X_j} \leq \kappa\, e^{-\delta t}, \quad t \geq 0, \quad j=1,2 .$$

Let us write (1.4) abstractly as

(4.2)
$$\frac{dw}{dt} = Cw + \tilde{f}^e(w) - \tilde{g}$$
$$w = \begin{bmatrix} u \\ v \end{bmatrix}, \quad \tilde{f}^e(w) = \begin{bmatrix} 0 \\ f^e(u) \end{bmatrix}, \quad \tilde{g} = \begin{bmatrix} 0 \\ g \end{bmatrix}$$

with $f^e(\varphi)(x) = f(\varphi(x))$. The variation of constants formula for the initial value problem for (4.2) is

(4.3) $$w(t) = e^{Ct} w_0 + U(t) w_0$$

(4.4) $$U(t)w_0 = \int_0^t e^{C(t-s)} [\tilde{f}^e(w(s)) - \tilde{g}]ds$$

To prove Theorem 1.2, we need the following lemma whose proof can be found in O.A. Ladyzhenskaya [15], p. 156-165.

Lemma 4.1.

(i) For any $h \in L^1(0,T;L^2)$, there is a unique solution $W(t,h)$, $0 \leq t \leq T$, in X_2 of the initial value problem:

$$\text{(4.5)} \qquad \frac{dw}{dt} = Cw + \tilde{h}, \quad \tilde{h} = \begin{bmatrix} 0 \\ h \end{bmatrix}$$

$$w_0 = 0$$

Furthermore

(ii) all of the following maps are continuous:

$$W : [0,T] \times L^1(0,T;L^2) \to X_2 = H_0^1 \times L^2$$

$$W : [0,T] \times W^{1,1}(0,T;L^2) \to X_1 = (H^2 \cap H_0^1) \times H_0^1$$

$$W : [0,T] \times W^{1,1}(0,T;H_0^1) \to (H^3 \cap H_0^1) \times (H^2 \cap H_0^1)$$

By standard application of the Sobolev embedding theorems we can prove

Lemma 4.2.

If (1.2), (1.3) *are satisfied, then*

$$f^e : H_0^1 \to L^2 \quad \textit{is a local} \quad C^{1,1}\textit{-map}$$

$$f^e : H^2 \cap H_0^1 \to H_0^1 \quad \textit{is a local} \quad C^{0,1}\textit{-map}.$$

Lemma 4.3.

If (1.2) *is satisfied and*

$$\overline{\lim_{|u|\to\infty}} f(u)/u \leq 0$$

then (1.4) *defines a* $C^{j-1,1}$*-semigroup on* X_1 *and* X_2. *Also, orbits of bounded sets in* X_2 *are bounded in* X_2.

Proof. The proof is an application of energy estimates and follows Babin and Vishik [2]. Under the above assumption on f, one can show (see for example Henry [14], p. 119) that, for any $\epsilon > 0$ there is a constant c_ϵ such that

$$F(u) \leq \epsilon\, u^2 + c_\epsilon$$

$$F(u) = \int_0^u f(s)ds$$

If (1.2) is satisfied, then there is a constant $c_0 > 0$ such that

$$|F(u)| \leq c_0(|u|^4 + 1)\ , \quad \text{for all} \quad u$$

For any $(\varphi,\psi) \in X_2$, let

$$V(\varphi,\psi) = \int_\Omega [1/2|\nabla\varphi(x)|^2 + 1/2\psi(x)^2 - F(\varphi(x)) + g(x)\varphi(x)]dx$$

Then there is a constant $c > 0$ such that

$$V(\varphi,\psi) \geq c[|\varphi|^2_{H^1_0} + |\psi|^2_{L^2} - 1]$$

$$V(\varphi,\psi) \leq c[|\varphi|^2_{H^1_0} + |\psi|^2_{L^2} + 1]$$

Babin and Vishik [2] show hat a solution $w(t,\cdot) = (u(t,\cdot),u_t(t,\cdot))$ satisfies:

$$(4.6)\qquad V(u(t,\cdot),\ u_t(t,\cdot)) - V(u(\tau\cdot),\ u_t(\tau,\cdot)) = -2\alpha\int_\tau^t \int_\Omega u_t^2(s,x)dxds.$$

From the inequalities on $V(\varphi,\psi)$, one easily obtains the global existence of solutions of (1.4) in X_2 and that orbits of bounded sets in X_2 are bounded in X_2. Thus, $T(t)$: $X_2 \to X_2$ is a $C^{1,1}$-semigroup. If $(\varphi,\psi) \in X_1$ then the solution remains in X_1 as long as it exists. But, $X_1 \subset X_2$ implies the solution exists for all $t \geq 0$ in X_2. Hence, it exists in X_1 for all $t \geq 0$ and $T(t)$: $X_1 \to X_1$ is a $C^{0,1}$-semigroup.

Lemma 4.4. *If* f *satisfies* (1.2), (1.3) *and* $U(t)$ *in* (4.4) *is completely continuous in* X_2, *then* $T(t)$ *is point dissipative in* X_2 *and there is a compact connected attractor* J *in* X_2.

Proof. For any $(\varphi,\psi) \in X_2$, we know that $\gamma^+(\varphi,\psi)$ is bounded in X_2. Since $T(t) = e^{Ct} + U(t)$, e^{Ct} satisfies (4.1) and $U(t)$ is completely continuous, $\gamma^+(\varphi,\psi)$ is precompact (see, for example, Hale [10]). Furthermore, if $(u(t), v(t)) = T(t)(\varphi,\psi)$ and $V(u(t), v(t)) = V(u(0), v(0))$, for all $t \in \mathbb{R}$ then (4.6) implies that $v(t) = 0$, for all $t \in R$. Since $v(t) = u_t(t)$ this implies that u is an equilibrium point. Thus (1.4) is a gradient system (see Hale [10], Babin and Vishik [2]). Therefore, $\omega(\varphi,\psi) \subset E$, for every $(\varphi,\psi) \in X_2$ where $\omega(\varphi,\psi)$ is the ω-limit set of (φ,ψ). To show that $T(t)$ is point dissipative in X_2, we show that the set E of equilibrium points is bounded if f satisfies (1.2), (1.3). A point $(\varphi,0) \in E$ if and only if $\varphi \in H_0^1$ and φ is an extreme value of the functional

$$I(\varphi) = \int_\Omega [1/2|\nabla\varphi|^2 - F(\varphi)]dx;$$

that is,

$$\int_\Omega [\nabla\varphi\ \nabla\psi - f(\varphi)\psi] = 0, \quad \text{for all} \quad \psi \in H_0^1. \tag{4.7}$$

Since f satisfies (1.3), for any $\epsilon > 0$, there is an $M > 0$ such that $|u| \geqslant M$ implies $f(u)/u \leqslant \epsilon$. If $\varphi \in E$, then choosing ψ in (4.7) to be φ, we have

$$\int_\Omega |\nabla\varphi|^2 dx = \int_\Omega f(\varphi(x))\varphi(x)dx$$

$$= \int_{\Omega_1} f(\varphi(x))\varphi(x)dx + \int_{\Omega_2} f(\varphi(x))\varphi(x)dx$$

where $\Omega_1 = \Omega \cap \{x:|\varphi(x)| \geqslant M\}$, $\Omega_2 = \Omega \cap \{x:|\varphi(x)| < M\}$. The first integral is bounded by $\epsilon C(M,\Omega)|\varphi|^2_{H_2^1}$ and the second is bounded by a constant $C(M,\Omega)$. Thus, $|\varphi|_{H_0^1} \leqslant C(M,\Omega)$ and the set E is bounded.

Therefore, we have proved that $T(t)$ is point dissipative. So by Theorem 2.1 we get that there is a compact attractor J in X_2.

Remark 4.5. If $f \in C^1(\mathbb{R})$ and there are constants $c > 0$, $\gamma > 0$, such that $|f'(u)| \leqslant c(|u|^{2-\gamma} + 1)$ for all u, then the Sobolev embedding theorems imply that f^e: $H_0^1 \to L_2$ is compact. This implies that $U(t)$ is completely continuous. Thus, the conclusions of Lemma 4.4 are valid. This coincides with the result obtained by Haraux [13], Hale [10]. The above proof is the same as the one in Hale [10].

Lemma 4.6. *If* (1.2) *is satisfied, then, for any* $T > 0$, $0 \leq t \leq T$, *we have*:

(i) $U(t): X_2 \to X_1$, *is continuous*

(ii) If B *and* $U(t)B$, $0 \leq t \leq T$ *are bounded in* X_2, *then* $U(t)B$, $0 \leq t \leq T$, *are bounded in* X_1.

Proof. Suppose $B \subset X_2$ is bounded. Then, by Lemma 4.3, $\{T(t)B, t \geq 0\}$ is bounded in X_2. Let $T(t)(\varphi,\psi) = (u(t,\varphi,\psi), u_t(t,\varphi,\psi))$, $(\varphi,\psi) \in B$, and let $g(t,\varphi,\psi) = f^e(u(t,\varphi,\psi))$. Then by Lemma 4.2

$$g(\cdot,\varphi,\psi) \in W^{1,1}(0,T;L^2)$$

and $g(\cdot,\varphi,\psi)$ is uniformly bounded for $(\varphi,\psi) \in B$. Lemma 4.1 implies that $U(t)(\varphi,\psi)$ is in X_1, is continuous in (t,φ,ψ) and is uniformly bounded for $0 \leq t \leq \tau$, $(\varphi,\psi) \in B$.

Proof of Theorem 1.2. We know that (2.4) - (2.6) are satisfied. Furthermore, Lemma 4.6 implies that (2.7) is satisfied. Theorem 2.2(i) implies that $U(t)$ is completely continuous in X_2. Lemma 4.4 implies that there is a compact attractor J in X_2.

To prove J is in X_1 and also an attractor in X_1, let's first observe that Theorem 2.2 part (ii) implies that $T(t)$ is bounded dissipative in X_1. We next observe that $U(t)$ is completely continuous in X_1. To show this, there is no loss in generality in supposing that $f(0) = 0$ since we can replace $f(u)$ by $f(u) - f(0)$ and g by $g - f(0)$ in (1.4). Then $U(t)$ is completely continuous in X_1 if and only if:

$$\tilde{U}(t)(\varphi,\psi) = \int_0^t e^{C(t-s)} \begin{bmatrix} 0 \\ f^e(u(s)) \end{bmatrix} ds$$

is completely continuous in X_1. Let

$$g(t,\varphi,\psi) = f^e(u(t,\varphi,\psi)) \; , \; 0 \leq t \leq T.$$

Since $T(t)$ is a semigroup on X_1 and takes bounded sets into bounded sets, the function

$$g(\cdot,\varphi,\psi) \in W^{1,1}(0,T;H_0^1)$$

Lemma 4.1 (ii) implies that $\tilde{U}(t)$ takes a bounded set V in X_1 into a bounded set in $(H^3 \cap H_0^1) \times (H^2 \cap H_0^1)$. Thus $\tilde{U}(t)V$ is precompact in X_1. Now since $U(t)$ is completely continuous in X_1 and $T(t)$ is also bounded dissipative in X_1, theorem 2.1 implies there is a compact attractor $\tilde{J}$ in X_1.

Since $U(t)$ is completely continuous in X_1 and X_2, Theorem 2.2(ii) implies the attractor J in X_2 belongs to X_1 and is a precompact invariant set in X_1. Thus, $J \subset \tilde{J}$. But obviously $\tilde{J} \subset J$; that is $J = \tilde{J}$.

Remark 4.7. Since the solution operator $T(t)$ of (1.4) as well as $DT(t)(\varphi,\psi)$ are α-contractions the results of Mallet-Paret [18] and Mañé [19] imply that the limit capacity (and thus the Hansdorf dimension) of J is finite. Similar results have been given by Ghidaglia and Temam [9].

Remark 4.8. Since e^{Ct} is a group on X, it follows that T(t) is also a group on J. The results in Hale and Scheurle [12] imply that the flow restricted to the local unstable sets $W^u_{loc}(\varphi,\psi)$ is as smooth in t as the function $\tilde{f}^e$, even up to analyticity. Since T(t) is a group on J and these sets are finite dimensional, it follows that $T(t)|J$ is as smooth in t as $\tilde{f}^e$.

Remark 4.9. From results of Sola-Morales [26] one can show that J coincides with the globally defined bounded solutions of (1.3). In fact, replacing t by -t, one has the radius of the essential spectrum of the corresponding semigroup outside the unit circle for any $t > 0$.

5. Proof of Theorem 1.3.

The beam equation (1.5) may be written as an abstract evolutionary equation

$$\frac{dw}{dt} = Cw + \tilde{g}^e(w) \tag{5.1}$$

$$w = \begin{bmatrix} u \\ v \end{bmatrix}, \; C = \begin{bmatrix} 0 & I \\ -aA & -\delta I \end{bmatrix} \quad A = \frac{\partial^4}{\partial x^4}, \; \tilde{g}^e(w) = \begin{bmatrix} 0 \\ g^e(u) \end{bmatrix}$$

$$g^e(u)(x) = g\big(u(x)\big) = \left\{\beta + k \int_0^{\ell} \left(\frac{\partial u}{\partial \xi}\right)^2 d\xi\right\} \frac{\partial^2 u(x)}{\partial x^2}$$

or, in integral form,

$$w(t) = e^{Ct} w_0 + \int_0^t e^{C(t-s)} \tilde{g}^e\big[w(s)\big] ds$$

For a given w_0 in either of the spaces X_j^c or X_j^h, j=1,2, let

$$U(t)w_0 = \int_0^t e^{C(t-s)} \tilde{g}^e\big[w(s)\big] ds \tag{5.2}$$

To prove Theorem 1.3, we need the following lemmas.

Lemma 5.1. *The operator* C *generates a linear* C^0*-group* e^{Ct} *on* X_j^c *or* X_j^h, j=1,2, *and there are constants* $k > 0$, $\sigma > 0$ *such that*

$$|e^{Ct}| \leq k e^{-\sigma t}, \quad t \geq 0$$

The proof follows along the lines of the proof of inequality (4.1) using the results of Ball [4, Sections 3-5], [5, Theorems 3 and 11].

Lemma 5.2.

(i) For any $g \in L^1(0,T;L^2(\Omega))$ *there is a unique solution* $w_c(t,g)$ *(resp.* $w_h(t,g)$*)* $0 \le t \le T$ *in* X_2^c *(resp.* X_2^h*) of the initial value problem*

$$(5.3) \quad \begin{cases} \dfrac{dw}{dt} = C\,w + \tilde{g} \qquad \tilde{g} = \begin{bmatrix} 0 \\ g \end{bmatrix} \\ w_0 = 0 \end{cases}$$

(ii) Furthermore, all of the following maps are continuous

α. *clamped ends:*

α_1. w_c: $[0,T] \times L^1(0,T;L^2) \to H_0^2 \times L^2 = X_2^c$

α_2. w_c: $[0,T] \times W^{1,1}(0,T;L^2) \to (H_0^2 \cap H^4) \times H_0^2 = X_1^c$

α_3. w_c: $[0,T] \times W^{1,1}(0,T;H_0^2) \to (H_0^2 \cap H^5) \times (H_0^2 \cap H^3)$

β. *hinged ends:*

β_1. w_h: $[0,T] \times L^1(0,T;L^2) \to (H_0^1 \cap H^2) \times H^2 = X_2^h$

β_2. w_h: $[0,T] \times W^{1,1}(0,T;L^2) \to (H_0^1 \cap H^4) \times (H_0^1 \cap H^2) = X_1^h$

β_3. w_h: $[0,T] \times W^{1,1}(0,T;H^2 \cap H_0^1) \to (H_0^1 \cap H^5) \times (H_0^1 \cap H^3)$.

Assertion (i) follows by ([15], Theorem 3.1, 3.2 p.157-161) and ([4], Theorem 4.1, p.119).

Let us now prove (ii) under the condition of clamped ends. Let "·" = d/dt, "'" = d/dx and let w_c be the solution of the linear nonhomogeneous equation

$$\ddot{w}_c + \delta \dot{w}_c + a w_c'''' = g(t,x). \tag{5.4}$$

Suppose that $g \in L^1(0,T;L^2)$. Multiplying (5.3) by $\dot{w}_c$, we obtain

$$|\dot{w}_c|_{L^2}^2 + \frac{\delta}{2}\frac{d}{dt}|w_c|_{L^2}^2 + a|w_c''|_{L^2}^2 = (g(t,x),w_c)_{L^2}\,.$$

Integrating over t in $[0,\tau]$, $\tau \in [0,T]$, we obtain

$$\int_0^\tau (|\dot{w}_c|_{L^2}^2 + a|w_c''|_{L^2}^2)dt + \frac{\delta}{2}|w_c|_{L^2}^2$$

$$\leq \frac{\epsilon}{2}|g|_{L^2}^2 + \frac{1}{2\epsilon}|w_c|_{L^2}^2 + \frac{\delta}{2}|w_c(0)|_{L^2}^2\,.$$

Thus,

$$\int_0^\tau \left(|\dot{w}_c|_{L^2}^2 + a|w_c''|_{L^2}^2\right)dt + k_1|w_c|_{L^2}^2 \leq \epsilon_1|g|_{L^2}^2 + \epsilon_2,$$

where k_1, ϵ_1, ϵ_2 depend only on T. From here, it follows that $w_c \in H_0^2$, $\dot{w}_c \in L^2$ and the mapping is continuous.

To prove the assertion in (ii)(α_2), differentiate (5.4) in t and substitute $u_c = \dot{w}_c$ to obtain

$$\ddot{u} + \delta\dot{u}_c + au_c'''' = \dot{g}(t,x)$$

From the proof of (ii)(α_1) $(u_c,\dot{u}_c) \in H_0^2 \times L^2$. Thus, $(\dot{w}_c,\ddot{w}_c) \in H_0^2 \times L^2$. From (5.4), $w_c \in H^4$. Since part (ii)(α_1) implies $w_c \in H_0^2$, we have $(w_c,\dot{w}_c) \in (H_0^2 \cap H^4) \times H_0^2$. The continuity of w_c is obtained as before.

To prove (ii)(α_3), let $w = w_c$ and first differentiate with respect to x to obtain

$$w_{ttx} + \delta w_{tx} + aw^{(5)} = g_x \ . \tag{5.5}$$

If $u = w_x$, then

$$u_{tt} + \delta u_t + au^{(4)} = g_x \ ,$$

From the proof of (ii)(α_1), we have $(u,\dot{u}) \in H_0^2 \times L^2$ and so $(w_x,w_{tx}) \in H_0^2 \times L^2$; that is, $w \in H^3 \cap H_0^2$ and $w_{tx} \in L^2$ with these functions being continuous in g.

For the next step, we differentiate (5.4) with respect to x and t, let $u = w_{xt}$ and observe that u satisfies

$$u_{tt} + \delta u_t + au^{(4)} = g_{xt}$$

Thus, as before, $(u,\dot{u}) \in H_0^2 \times L^2$ and $(w_{xt},w_{xtt}) \in H_0^2 \times L^2$. This implies that $w_t \in H^3 \cap H_0^2$. From (5.5) and the fact that $w_{xtt} \in L^2$, we have $w^{(5)} \in L^2$. From (ii)(α_2), we know that $w \in H^4 \cap H_0^2$. Thus, $w \in H^5 \cap H_0^2$. This completes the proof of part (ii). An analogous proof can be given for (ii)(β) and is omitted.

Lemma 5.3. *For the map*

$$g^e(u) = (\beta + k|u'|^2)u''$$

we have the following:

(a) (*clamped ends*)

a_1) g^e: $H_0^2 \to L^2$ *is a local* $C^{1,1}$-*map.*

a_2) g^e: $H^4 \cap H_0^2 \to H_0^2$ *is a local* $C^{0,1}$-*map.*

(b) (*hinged ends*)

g^e: $H^2 \cap H_0^1 \to L^2$ *is a local* $C^{1,1}$-*map*

g^e: $H^4 \cap H_0^1 \to H^2 \cap H_0^1$ *is a local* $C^{0,1}$-*map.*

Proof. We'll prove the lemma for case (a). The case (b) is similar.

a_1) (i) We prove first that g^e: $H_0^2 \to L^2$.

Since $u \in H_0^2$, we have

$$|u|^2 < c_1, \ |u'|^2 < c_2, \ |u''|^2 < c_3.$$

So

$$|g^e(u)|^2 = |(\beta + k|u'|^2)u''|^2$$

$$\leq 2\,\beta^2 c_3 + 2kc_2^2 c_3 \Rightarrow g^e(u) \in L^2.$$

(ii) We now prove g^e is continuous:

Let u_n, $u \in H_0^2$, with $u_n \to u$ in H_0^2. i.e.

$$\|u_n - u\|_{H_0^2} \xrightarrow[n\to\infty]{} 0$$

We have:

$$|g^e(u_n) - g^e(u)|_{L^2}$$

$$= |(\beta + k|u_n'|^2)u_n'' - (\beta + k|u'|^2)u''|$$

$$= |\beta(u_n'' - u'') + k(|u_n'|^2 u_n'' - |u'|^2 u'')|$$

$$\leq |\beta| \ |u_n'' - u''| + k \ \Big||u_n'|^2 - |u'|^2\Big| \ |u''| + |u_n'' - u''||u_n'|^2$$

$$\leq k_1|u_n - u|_{H_0^2}$$

for some constant k_1. This proves the continuity.

(iii) We next prove that g^e is differentiable.

$$g'(u)v = \lim_{\epsilon \to 0} \frac{1}{\epsilon} [(\beta + k|(u + \epsilon v)'|^2) (u + \epsilon v)'' - (\beta + k|u'|^2)u'']$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} [\beta u'' + \epsilon\beta v'' + k|u'|^2u'' + k \epsilon^2|v'|^2u''$$

$$+ 2\epsilon k|u'v'|u'' + k \epsilon^3|v'|^2 v'' + \epsilon k|u'|^2v'' + 2k\epsilon^2|u'v'|v''$$

$$- \beta u'' - k|u'|^2u''] = \beta v'' + 2 k|u'v'|u'' + k|u'|^2v''$$

$$= (\beta + k|u'|^2) v'' + 2k|u'v'|u''.$$

(iv) Finally, we show that $(g^e)'$ is Lipschitz.

$$|g'(u_1)v - g'(u_2)v|$$

$$= k\Big|2|u_1'v'|u_1'' + |u_1'|^2v'' - 2 |u_2'v'|u_2'' - |u_2'|^2v''\Big|$$

$$\leq k|v''|\,\big||u_1'|^2 - |u_2'|^2\big| + 2k\big||u_1|\cdot|v''|u_1'' - |u_2||v''|u_2''\big|$$

$$\leq k|\ v''|\left[|u_1' - u_2'|^2 + 2\big||u_1|u_1'' - |u_2|u_2''|\right]$$

$$\leq k|v''|\left[|u_1' - u_2'|^2 + 2|u_1||u_1''| + 2|u_2||u_2''|^2\right]$$

$$\leq k|v''|\ \left[|u_1' - u_2'|^2 + |u_1|^2 + |u_1''|^2 + |u_2|^2 + |u_2''|^2\right]$$

$$\leq k|v''|\ \left[|u_1 - u_2|^2 + |u_1' - u_2'|^2 + |u_1'' - u_2''|^2\right]$$

This implies that $(g^e)'$ is Lipschitz.

a_2) It's an immediate consequence of Ball ([4], p. 136, Lemma 6.2).

Lemma 5.4. *The equation* (1.5), (1.6) (*resp.* (1.5, (1.7)) *defines a* $C^{j-1,1}$*-semigroup* $T(t)$ *on* X_1^c, X_2^c *(resp.* X_1^h, X_2^h). *Also, orbits of bounded sets in* X_2^c *(resp. in* X_2^h) *are bounded in* X_2^c *(resp.* X_2^h).

Proof. The existence of a local $C^{1,1}$-semigroup $T(t)$ on X_2^c (resp. X_2^h) follows from Lemma 5.3 and Segal [25] (see also Ball [4] p.119), where $T(t)(\varphi,\psi) = w(t,\varphi,\psi)$, the solution through (φ,ψ). Moreover, the solution $w(t) = w(t,\varphi,\psi)$ in both cases, satisfies the energy equation:

$$V(w(t)) + \delta\int_0^t |\dot{u}(s)|^2 ds = V(0) \tag{5.6}$$

where

$$V(w(t)) = \frac{1}{2}|\dot{u}(t)|^2 + \frac{a}{2}|u''(t)|^2 + \frac{\beta}{2}|u'(t)|^2 + \frac{k}{4}|u'(t)|^4$$

is the energy function.

A formal calculation shows that the following inequalities are true:

(5.7) $$V(w(t)) \geq \tfrac{1}{2}(\|w(t)\|^2 - \beta^2/2k)$$

(5.8) $$V(w(t)) \leq \tfrac{1}{2}(1 + \frac{\ell^2\beta}{a\pi^2})\ \|w\|^2 + \frac{k}{4}\frac{\ell^4}{\pi^4 a^2}\|w\|^4.$$

Using (5.6), (5.7) and (5.8), one easily obtains

$$\|w(t)\|^2 \leq \frac{\beta^2}{2k} + (1 + \frac{\ell^2\beta}{a\pi^2})\ \|w_0\|^2 + \frac{k}{2}\frac{\ell^4}{\pi^4 a^2}\ \|w_0\|^4\ .$$

From here we deduce the global existence of solutions of (1.5) in X_2^c (resp. X_2^h) and that orbits of bounded sets in X_2^c (resp. X_2^h) are bounded in X_2^c (resp. X_2^h).

Hence $T(t) : X_2^c \to X_2^c$ (resp. $X_2^h \to X_2^h$) is a $C^{1,1}$-semigroup. If $(u_0,u_1) \in X_1^c$ (resp. X_1^h), then the solution remains in X_1^c (resp. X_1^h) as long as it exists. But $X_1^c \subset X_2^c$ (resp. $X_1^h \subset X_2^h$) implies the solution exists for all $t \geq 0$ in X_2^c (resp. X_2^h). Thus it exists in X_1^c (resp. X_1^h) for all $t \geq 0$. Therefore $T(t) : X_1^c \to X_1^c$ (resp. $X_1^h \to X_1^h$) is a $C^{0,1}$-semigroup.

For a given $(\varphi,\psi) \in X_j^c$ (resp. X_j^h), j=1,2, let $T(t)(\varphi,\psi) = (u(t), v(t))$ where $(u(t),v(t))$ is the solution of (5.1) through (φ,ψ). Also, define $U(t)(\varphi,\psi)$ by (5.2).

Lemma 5.5. *If* $U(t)$ *is completely continuous in* X_2^c *(resp.* X_2^h*), then* $T(t)$ *is point dissipative and there is a compact connected attractor in* X_2^c *(resp.* X_2^h*).*

Proof. The proof is similar to the proof of Lemma 4.4 and is therefore omitted.

Lemma 5.6. *For any* $T > 0$, $0 \leq t \leq T$, *we have*

(i) $U(t) : X_2^c \to X_1^c$ (*resp.* $X_2^h \to X_1^h$)

(ii) if B *and* $U(t)B$, $0 \leq t \leq T$, *are bounded in* X_2^c (*resp.* X_2^h), *then* $U(t)B$, $0 \leq t \leq T$, *are bounded in* X_1^c (*resp.* X_1^h).

The proof is essentially the same as the proof of Lemma 4.6 and is therefore omitted. The same remark applies to the remainder of the proof of the first two parts of Theorem 1.3 on the existence of the compact attractor.

To show that the attractor has the form stated in part (iii) of Theorem 1.3, we need to show only that the energy function $V(w(t))$ defines a Liapunov functional in the space X_2^c (resp. X_2^h) (for a definition, see [10]). From (5.6) and, for a sufficiently smooth dense set of initial data, we obtain

$$\dot{V}(w(t)) = -\delta\int_0^\ell \dot{u}^2 dx \leq 0 \tag{5.9}$$

This implies that $V(w(t)x)$ is nonincreasing in t for each x in X_2^c (resp. X_2^h).

From (5.7), we have $V(x) \to \infty$, as $x \to +\infty$. Also, $V(x)$ is bounded below. Furthermore, if $V(w(t)x) = V(x) = V(w(0)x)$ for all t in R, then, by (5.9), we have $\delta\int_0^\ell \dot{u}(t,x)^2 dx = 0$, for $t \in R$. Thus, $\dot{u}(t,x) \equiv 0$ for $t \in R$ and $u(t,x) = u(0,x)$ for $t \in R$. That is, u is an equilibrium point of (5.1). Hence, V is a Liapunov function for (5.1) in the space X_2^c (resp. X_2^h).

Remark 5.7. The equilibrium states of the beam equation have been studied by, for example (Reiss [24], Ball [4], [5]). The set E^c (resp. E^h) of equilibrium points of (1.5), (1.6) (resp. (1.5), (1.7)) consists of the points $(u,0) \in X_2^c$ (resp. X_2^h) such that:

$$a\, u'''' = (\beta + k|u'|^2)u'' \tag{5.10}$$

subject to the clamped (resp. hinged) boundary conditions. Any non zero equilibrium point v_j is an eigenfunction satisfying:

$$a\, v_j'''' + \lambda_j\, v_j'' = 0$$

subject to the relevant boundary conditions, where,

$$|v_j''|^2 = -\frac{(\beta + \lambda_j)}{k}$$

The positive sequence $\{\lambda_j\}$ is strictly increasing and has no finite accumulation point.

So, if $-\beta \leq \lambda_1$, the only equilibrium point is $v = 0$, while if $\lambda_n < -\beta \leq \lambda_{n+1}$ there are $2n + 1$ equilibrium points given by

$$v = 0 \quad \text{and} \quad v = \pm v_m, \quad 1 \leq m \leq n.$$

Hence the set of equilibrium points for either boundary condition is finite.

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Postscript

After this paper was written, the authors became aware of some related and, in some cases, more general, results of O. Lopes and S.S. Ceron. [Existence of forced periodic solutions of dissipative hyperbolic equations and systems. Annali di Mat. Pura Applicata, submitted]. In this paper, Lopes and Ceron were concerned with nonautonomous evolutionary equations which were periodic in time. We summarize their results for the autonomous case and relate them to those stated above. Let Ω be a bounded subset of R^3. Consider the equation

$$(1)\qquad \begin{cases} u_{tt} + h(u_t) - \Delta u = f(u), & \text{in } \Omega \\ u = 0 & \text{, on } \partial\Omega, \end{cases}$$

where $h(v)$ satisfies $h \in C^1(R,R)$, $h(0) = 0$ and there exist positive constants $\beta \geq \alpha > 0$ such that

$$0 < \alpha \leq h'(v) \leq \beta.$$

Also, suppose that $f \in C^1(R,R)$, $|f'(u)| \leq c(|u|^{2-\gamma} + 1)$ for some positive constants c,γ and $\int_0^u f$, $uf(u)$ are bounded below for $u \in R$. Lopes and Ceron proved that the solution operator $T(t)$: $H_0^1 \times L^2 \to H_0^1 \times L^2$ is an α-contraction and the system is bounded dissipative. Thus, there is a compact attractor for (1) in $H_0^1 \times L^2$ from Theorem 2.1. This improves on the result of Haraux [13], Hale [10] by allowing a nonlinear damping term $h(u_t)$. The result of Lopes and Ceron does not include part (i) of Theorem 1.2 since he assumes the stronger growth rate on $f(u)$.

For the beam equation and the damping term δu_t replaced by $h(u_t)$ with h satisfying the conditions above, the results of Lopes and Ceron imply part (i) of Theorem 1.3.

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