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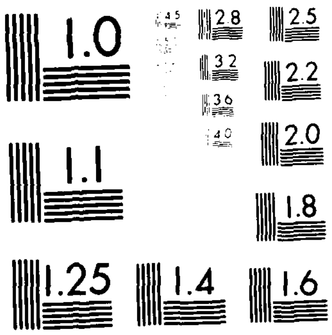


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# ON THE RELATION BETWEEN SEVERAL STATISTICS FOR TESTING FOR EXPONENTIALITY AND UNIFORMITY

BY

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## 1. Introduction.

Let $Exp(\alpha,\beta)$ denote the distribution $F(x) = 1-\exp(-(x-\alpha)/\beta)$, $x > \alpha$ where $\alpha$ and $\beta$ are constants and $\beta$ is positive. Suppose $X_1,\ldots,X_n$ is a random sample from $Exp(0,\beta)$; the $X_i$ could denote the time intervals between events at times $T_j$ in a Poisson process, so that $T_j = \Sigma_{i=1}^{j} X_i$, $j = 1,\ldots,n$. It is well known that the values $U_{(j)} = T_j/T_n$, $j = 1,\ldots,n-1$ are then the order statistics of a sample of size $n-1$ from a uniform distribution with limits 0 and 1, written $U(0,1)$. The $n$ spacings between the $U_{(j)}$ are then defined by $D_i = U_{(i)}-U_{(i-1)}$, $i = 1,\ldots,n$ with $U_{(0)} \equiv 0$ and $U_{(n)} \equiv 1$. In the present context, $D_i = X_i/T_n$, $i = 1,\ldots,n$.

Suppose now that $X_i$, $i = 1,\ldots,n$ is a random sample from a distribution $F_0(x)$, and it is desired to test either $H_0:F_0(x)$, is $Exp(0,\beta)$ with $\beta$ unknown, or the more general hypothesis $H_0^*:F_0(x)$ is $Exp(\alpha,\beta)$ with $\alpha$ and $\beta$ unknown. Many tests have been proposed for $H_0$ and $H_0^*$, some of them based on the reduction to the uniform distribution given above. Another technique is to plot the order statistics $X_{(i)}$ against $m_i$, the expected values of the order statistics of a sample from $Exp(0,1)$; test statistics can then be based on properties of the regression line calculated by Generalised Least Squares (since the $X_{(i)}$ are correlated). From these two very different approaches have emerged, for example, Greenwood's statistic based on the spacings $D_i$, and several regression statistics. In this article we show that some of the regression statistics are albegraically related to Greenwood's, so that the tests based on them are equivalent; also that the distribution of the Shapiro-Wilk statistic for exponentiality, $W_E$, is related to that of Greenwood's statistic for uniformity, so that percentage points are algebraically connected.

## 2. The Statistics.

2.1 The Greenwood spacings statistic. This statistic is usually defined for a sample $U_1,\ldots,U_n$ distributed between zero and one, and is then

$$G(n) = \sum_{i=1}^{n+1} D_i^2$$

where the $D_i$ are defined by $D_i = U_{(i)}-U_{(i-1)}$, $i = 1,\ldots,n+1$ with $U_{(0)} = 0$ and $U_{(n+1)} = 1$. In the context of testing $H_0$, the statistic derived from the $X_i$ would be $G(n-1)$, since $n$ values of $X_i$ produce $n-1$ ordered uniforms.

The null distribution of $G(n)$ was investigated by Moran (1947) and recently there has been a revival of interest; papers giving exact or approximate percentage points have been given by Burrows (1979), Hill (1979; see corrigendum 1981), Currie (1981a) and Stephens (1981). Note that when $n$ uniforms are used, $E(D_i) = \bar{D} = 1/(n+1)$, and a



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natural test statistic based on the dispersion of the $D_i$ can be defined by $G'(n) = \Sigma_{i=1}^{n+1} (D_i - 1/(n+1))^2$; however, this reduces to $G(n)-1/(n+1)$ and so is equivalent to $G(n)$. The application of $G(n)$ to test for exponentiality has been studied; e.g., by Bartholomew (1957) and by Cox and Lewis (1966, p. 163).

2.2 Regression statistics. In 1972 Shapiro and Wilk, following a principle earlier successfully applied to tests for normality, introduced a test for exponentiality based on a plot of the $X_{(i)}$ against $m_i$. If the $X_i$ were from $\mathrm{Exp}(\alpha,\beta)$ i.e. if $H_0^*$ were true, $E(X_{(i)}) = \alpha+\beta m_i$; the test statistic is based on the ratio of the two estimates of $\beta$, that given by Generalised Least Squares, and that given by the sample variance. The test statistic comes to be

$$W_E(n) = \frac{n\{\bar{X}-X_{(1)}\}^2}{(n-1)S^2}$$

where $S^2 = \Sigma X_i^2 - n\bar{X}^2$, and $\bar{X} = \Sigma X_i/n$; throughout this section all sums will run from 1 to n. Shapiro and Wilk (1972) gave percentage points for $W_E(n)$, based on Monte-Carlo studies; points based on numerical integration are given by Currie (1981b).

The statistic $W_E(n)$ was intended to test $H_0^*$, and Hahn and Shapiro (1967, p. 298) subsequently gave a modification (called $WE_0$) to test $H_0$, where we can assume that the regression line passes through the origin. For ease of notation this statistic will be called $H(n)$; it is defined by

$$H(n) = S^2/(n^2\bar{X}^2) .$$

Hahn and Shapiro provided Monte Carlo percentage points for $H(n)$.

Stephens (1978) introduced a test statistic for $H_0$, motivated by the desire to provide a test which would not require new tables. The statistic is

$$W_S(n) = \frac{n^2\bar{X}^2}{n\{(n+1)\Sigma X_i^2 - n^2\bar{X}^2\}} ,$$

and Stephens (1978) showed that $W_S(n)$ would have the same null distribution as $W_E(n+1)$; thus the Shapiro-Wilk (1972) tables could be used for $W_S(n)$.

## 3. Equivalence of Test Statistics.

The following algebraic relationships between statistics $G(n-1)$, $H(n)$ and $W_S(n)$ are easily proved but have not been previously noted:

(3.1) $$H(n) = G(n-1) - 1/n ;$$

(3.2) $$\{W_S(n)\}^{-1} = n(n+1)G(n-1) - n$$
$$= n(n+1)H(n) + 1 .$$

Thus statistics $G(n-1)$, $H(n)$ and $W_S(n)$ provide equivalent tests of $H_0$.

4. Equivalence of Distributions.

Furthermore, since $W_S(n)$ has the same distribution as $W_E(n+1)$ the distribution of $W_E(n+1)$ is related to the other statistics, and specifically to that of $G(n-1)$. Let $G(n;\alpha)$ be the percentage point at level $\alpha$, measured from the lower tail, for $G(n)$; similarly define percentage points for the other statistics. Then we have:

$$H(n;\alpha) = G(n-1;\alpha) - 1/n \; ; \tag{4.1}$$

$$\{W_S(n;1-\alpha)\}^{-1} = n(n+1)G(n-1;\alpha) - n \; ; \tag{4.2}$$

$$\{W_E(n+1;1-\alpha)\}^{-1} = n(n+1)G(n-1;\alpha) - n \; . \tag{4.3}$$

There have been numerous tables of percentage points produced for the statistics $G(n)$, $H(n)$, and $W_E(n)$ and it is of interest to assess the consistency of these tabulations. To this end we define

$$H^*(n) = H(n) + 1/n$$

$$W_E^*(n+1) = \{W_E(n+1)^{-1}+n\}/\{n(n+1)\} \; .$$

The various tabulations of the percentage points of $G(n)$, $H(n)$ and $W_E(n)$ are compared in the table.

The figures in column 1 are taken from the exact values for $G(n;\alpha)$ obtained by Burrows (1979) and Currie (1981a); in column 2 the tabulation for $G(n;\alpha)$ of Stephens (1981) using Pearson curves is used; column 3 uses the exact values for $W_E(n;\alpha)$ given by Currie (1981b); column 4 is based on the original Monte Carlo values of Shapiro and Wilk (1972) for $W_E(n;\alpha)$ and column 5 is taken from the simulation study of Hahn and Shapiro (1967, p. 334).

TABLE

Comparison of Various Tabulations

| $n$ | $\alpha$ | $G^1(n;\alpha)$ | $G^2(n;\alpha)$ | $W_E^{*1}(n+2;\alpha)$ | $W_E^{*2}(n+2;\alpha)$ | $H^*(n+1;\alpha)$ |
|---|---|---|---|---|---|---|
| 5 | 0.05 | 0.1994 | 0.2026 | 0.1994 | 0.2001 | - |
| | 0.95 | 0.4320 | 0.4330 | 0.4322 | 0.4368 | - |
| 10 | 0.05 | 0.1211 | 0.1222 | 0.1211 | 0.1209 | 0.116 |
| | 0.95 | 0.2404 | 0.2412 | - | 0.2367 | 0.257 |
| 15 | 0.05 | 0.0882 | 0.0887 | 0.0882 | 0.0881 | 0.086 |
| | 0.95 | - | 0.1641 | - | 0.1633 | 0.176 |
| 20 | 0.05 | 0.0698 | 0.0700 | 0.0698 | 0.0697 | 0.068 |
| | 0.95 | - | 0.1233 | - | 0.1233 | 0.133 |

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It is shown that Greenwood's statistic for uniformity and the Hahn-Shapiro and Stephens statistics for exponentiality with known origin are equivalent. It is also shown that the distribution of the Shapiro-Wilk statistic for testing the hypothesis of exponentiality with unknown origin is obtainable from Greenwood's distribution.

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