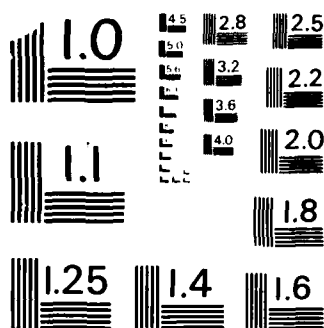MICROCOPY RESOLUTION TEST CHART

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THE UNIVERSITY OF ILLINOIS AT CHICAGO

ON BOUNDS FOR THE EFFICIENCY OF BLOCK

DESIGNS FOR COMPARING TEST

TREATMENTS WITH A CONTROL

BY

JOHN STUFKEN

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DEPARTMENT OF MATHEMATICS, STATISTICS, AND COMPUTER SCIENCE

ON BOUNDS FOR THE EFFICIENCY OF BLOCK

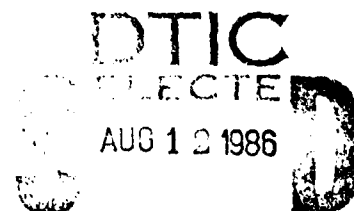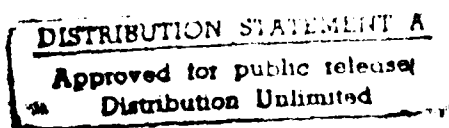DESIGNS FOR COMPARING TEST

TREATMENTS WITH A CONTROL

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In this paper we study the class of augmented balanced incomplete block designs, which are used for comparing a control treatment with a set of test treatments. Under the A-criterion we establish a condition that enables us to determine the most efficient augmented design and we suggest some methods to compute a lower bound for the efficiency of these designs. For $3 \le k \le 10$, $v \ge k$ we list the parameters of the most efficient designs with a lower bound for their efficiency or, if known, mention their optimality.

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ABSTRACT

In this paper we study the class of augmented balanced incomplete block designs, which are used for comparing a control treatment with a set of test treatments. Under the A-criterion we establish a condition that enables us to determine the most efficient augmented design and we suggest some methods to compute a lower bound for the efficiency of these designs. For $3 \leq k \leq 10$, $v \geq k$ we list the parameters of the most efficient designs with a lower bound for their efficiency or, if known, mention their optimality.

1. INTRODUCTION

Almost 30 years ago Cox [2] suggested that in comparing a control with a set
of test treatments in a proper block design, a good procedure would be to construct
a Balanced Incomplete Block Design (BIBD) based on the test treatments, followed by
augmenting each block of this BIBD with one or more replications of the control.
It is our goal in this paper to investigate this claim and study the efficiency
of designs constructed in this way for the special case that the optimality cri-
terion is the so-called A-criterion.

The remainder of this section will be used to introduce some of the termi-
nology and assumptions and give a more detailed description of our problem.

We want to compare the effect of a treatment, called the control and denoted
by 0, with those of v other treatments, called the test treatments and denoted
$1,\ldots,v$. To achieve this objective we assume that we can make bk observations,
divided over b blocks of size k each. The model that we will assume is

$$Y_{ijk} = \mu + \tau_i + \beta_j + \varepsilon_{ijk},$$

where $0 \leq i \leq v$, $1 \leq j \leq b$, $1 \leq k \leq n_{ij}$. Y_{ijk} is an observation in block j
using treatment i. The index k distinguishes between these observations if there
is more than one, say n_{ij}, replications of treatment i in block j. (If
$n_{ij} = 0$ then there is no observation of the form Y_{ijk}.) The other parameters
in the model are called the general mean (μ), the effect of treatment i (τ_i),
the effect of block j (β_j) and the error term (ε_{ijk}). The assumptions on the
error terms are that they have mean 0, variance σ^2 and that they are uncorrelated.

The objective of the experiment is to estimate the test treatment – control
contrasts $\tau_i - \tau_0$, $i = 1,\ldots,v$. We assume that this is done by using the least

square estimates, which we will denote by $\hat{\tau}_i - \hat{\tau}_0$. In order for all these

contrasts to be estimable it is necessary that the used design is connected.

We will therefore use the notation $D_0(v,b,k)$ to denote the class of all

connected block designs with b blocks of size k each and based on v test

treatments and a control. Our task is thus to select a design from $D_0(v,b,k)$

and use this to perform the experiment and obtain estimators for $\tau_i - \tau_0$ from

it. In selecting such a design we obviously need a criterion that tells us,

in terms of the estimators $\hat{\tau}_i - \hat{\tau}_0$ derived from it, whether a design is

(comparetively) good or bad. In this context two criteria have prevailed in the

literature. For one of them, MV-optimality, we refer the reader to [5]. The

other one, A-optimality, is defined as follows:

> A design $d_1 \in D_0(v,b,k)$ is said to be A-better than
>
> $d_2 \in D_0(v,b,k)$ if $Tr(M_{d_1}^{-1}) \leq Tr(M_{d_2}^{-1})$, where M_{d_1} and
>
> M_{d_2} are the information matrices for the contrasts
>
> $\tau_i - \tau_0$, $i = 1,\ldots,v$, corresponding to d_1 and d_2
>
> respectively. The design d_1 is called A-optimal if
>
> the above inequality holds for any $d_2 \in D_0(v,b,k)$.

Obviously any class $D_0(v,b,k)$ possesses at least one A-optimal design. The

only difficulty is how to find one. Several papers have made a contribution

towards a partial solution of that problem (e.g., [3], [4], [6], [7]). The

most general result in this direction is due to Majumdar and Notz [6]. To

state it we will first need two definitions.

Definition 1.1 (Bechhofer and Tamhane [1]): A design $d \in D_0(v,b,k)$ is called a Balanced Test treatment Incomplete Block Design (BTIBD) if

1. d is incomplete, i.e., no block contains all $v + 1$ treatments

2. with the notation $\lambda_{i_1 i_2} := \sum_{j=1}^{b} n_{i_1 j} n_{i_2 j}$, $0 \le i_1 \ne i_2 \le v$, it

   holds that there are constants λ_0 and λ_1 such that

   $\lambda_{0i} = \lambda_0$, $1 \le i \le v$, and $\lambda_{i_1 i_2} = \lambda_1$, $1 \le i_1 \ne i_2 \le v$.

Definition 1.2: A design $d \in D_0(v,b,k)$ is called a BTIB(v,b,k;t,s) if

1. d is a BTIBD which is binary in the test treatments

2. there are s blocks in d which contain exactly $t + 1$

   replications of the control, while the remaining $b - s$

   blocks contain exactly t replications of the control.

The result by Majumdar and Notz [6], can now be stated as:

Theorem 1.1: If $V \ge k \ge 3$ then a BTIB(v,b,k;t,s) is A-optimal in $D_0(v,b,k)$ if

$$g(t,s) = \min\{g(x,z) : (x,z) \in \Lambda\}$$

where $\quad g(x,z) := (v-1)^2(bvk(k-1) - (bx+z)(vk-v+k) + bx^2 + 2xz + z)^{-1}$

$$+ ((bx+z)k - (bx^2+2xz+z))^{-1},$$

and $\quad \Lambda := \{(x,z): x \in \{0,1,\ldots,[k/2]-1\}, z \in \{0,1,\ldots,b\}$

$$\text{and } z > 0 \text{ if } x = 0\}$$

([$\cdot$] denotes the largest integer function).

Although a very general result that can be used to find A-optimal designs in many classes of parameters, there are even more classes that remain unsolved by this result, since the needed BTIB(v,b,k;t,s) does not exist. Our apparent inability to determine an A-optimal design for the majority of parameters v, b and k leads us to the consideration that we should perhaps be less demanding and settle for a design that, though possibly not A-optimal, performs well under the A-criterion. We therefore introduce the following definition.

Definition 1.3: The efficiency $E(d^*)$ of a design $d^* \in D_0(v,b,k)$ is defined as

$$E(d^*) = \frac{\displaystyle\min_{d \in D_0(v,b,k)} Tr(M_d^{-1})}{Tr(M_{d*}^{-1})}$$

We will attempt to find designs with an efficiency close to one. One possible approach could be to search for the best design with respect to the A-criterion in the restricted class of all BTIBD's for given v, b and k and hope that this design has a high efficiency in $D_0(v,b,k)$. Although this design will, for most parameters, indeed have a high efficiency, as will follow from our results, there are at least two difficulties with this approach. The class of all BTIBD's with fixed parameters is, generally, still a large class of designs, leaving us with a difficult combinatorial problem. In addition to this, very little is known as of yet, about the existence and construction of specified BTIBD's.

Our approach is therefore to look for the best designs in a subclass of the BTIBD's, and well that of the augmented BIBD's, i.e., those designs as suggested by Cox. Existence and construction questions are in this class equivalent to those

of the corresponding BIBD's, and the literature on that subject is quite exten-
sive. Moreover since the class is significantly smaller than that of the BTIBD's
it becomes easier to compare the performance of its members and select the best
one. The only fear that one might have, though it will prove to be unfounded, is
that the efficiency of the best design in this class is not as high as desired.

In section 2 we will determine the best augmented BIBD in $D_0(v,b,k)$,
while section 3 will discuss the efficiencies of these designs.

2. DETERMINATION OF THE BEST AUGMENTED BIBD's.

Throughout the remainder of this paper we will assume that $v \geq k \geq 3$, the
conditions as in Theorem 1.1. For a discussion of the case $k = 2$ the reader is
referred to [3]. An augmented BIBD is in our notation just a BTIB(v,b,k;t,o),
and its information matrix and thus its performance under the A-criterion will,
for fixed v, b and k, be completely determined by t. Thus our problem is
to find the optimal t. Before stating the main result of this section we
introduce the following function $f(t)$, defined on the set of nonnegative integers

$$f(t) = \begin{cases} \infty & \text{if } t = 0 \\ \dfrac{1}{t(t+1)} [(k-t-1)^2 + t] & \text{if } t = 1,2,\ldots \end{cases}$$

Theorem 2.1. Let $t_0 \geq 1$. Then a BTIB$(v,b,k;t_0,\text{o})$ is A-better than any other BTIB$(v,b,k;t,\text{o})$ if

(2.1)
$$f(t_0) \leq v < f(t_0 - 1).$$

Before giving a proof we would like to make three remarks:

Remark 1: The reader should notice that condition (2.1) is completely independent of b. The only way in which b plays a role is in the question of existence of the desired BTIB$(v,b,k;t_0,\text{o})$.

Remark 2: Notice also that for any fixed v and k there exists a positive integer t_0 such that (2.1) is satisfied.

Remark 3: There are instances in which the BTIB$(v,b,k;t_0,\text{o})$ from the Theorem is actually A-optimal in $D_0(v,b,k)$. It is shown in [7] that this is the case if

$$\frac{1}{t_0^2}[(k-t_0-1)^2 + 1] \leq v \leq \frac{1}{t_0^2}(k-t_0)^2.$$

A useful lemma for the proof of Theorem 2.1 is the following:

Lemma 2.1: Let $g(x,z)$ be the function as defined in Theorem 1.1, with the convention that $g(0,0) = \infty$. If

$$g(t_0,0) < \min\{g(t_0-1,0), g(t_0+1,0)\}$$

then

$$g(t_0,0) = \min\{g(t,0) : t = 1,\ldots,k-1\}.$$

Proof: A simple evaluation gives

$$g(t,0) = \frac{1}{b(k-t)} \left(\frac{(v-1)^2}{v(k-1) - t} + \frac{1}{t} \right) .$$

Hence $\lim_{t \downarrow 0} g(t,0) = \lim_{t \uparrow k} g(t,0) = \infty$, which implies that

(2.2)
$$\frac{d}{dt} g(t,0) = 0$$

has a solution on the interval $(0,k)$.

Similarly, since $\lim_{t \downarrow k} g(t,0) = \lim_{t \uparrow v(k-1)} g(t,0) = -\infty$, it follows that (2.2) has a solution on $(k, v(k-1))$.

Finally, since $\lim_{t \uparrow 0} g(t,0) = -\infty$ and $\lim_{t \to -\infty} g(t,0) = 0$ while $g(t,0) > 0$ if $t < -\frac{k-1}{v-2}$, it follows that (2.2) has a solution on $(-\infty, 0)$. It is obvious that (2.2) has at most 3 solutions, so that we can conclude that there is a unique solution on $(0,k)$. If we call this solution t_1 we know that $g(t,0)$ is decreasing for $0 < t < t_1$ and increasing for $t_1 < t < k$. From $g(t_0,0) < g(t_0-1,0)$ it follows that $t_1 > t_0 - 1$, while $g(t_0,0) < g(t_0+1,0)$ implies that $t_1 < t_0 + 1$. The conclusion of the lemma is now obvious. □

There are two other lemmas which are useful for the proof of Theorem 2.1.

Lemma 2.2: If $f(t) > k$ for $t \geq 1$, then $v \geq f(t)$ if and only if $g(t,0) < g(t+1,0)$.

Proof: From $f(t) > k$ we obtain

$$t(t+1)k < (k-t-1)^2 + t,$$

or equivalently

$$k^2 - (t+1)(t+2)k + (t+1)^2 + t > 0.$$

As a quadratic equation in k the roots of the left-hand side are $k = 1$ and $k = t^2 + 3t + 1$. Since by assumption $k \geq 3$, it follows that

$$(2.3) \qquad\qquad k \geq t^2 + 3t + 2.$$

The statement $g(t,0) < g(t+1,0)$ is equivalent to

$$\frac{1}{b(k-t)} \left(\frac{(v-1)^2}{v(k-1) - t} + \frac{1}{t} \right) < \frac{1}{b(k-t-1)} \left(\frac{(v-1)^2}{v(k-1) - t - 1} + \frac{1}{t+1} \right)$$

or $\qquad\qquad\qquad\qquad q(v) > 0,$

where

$$q(v) = t(t+1)(k-1)v^2 - ((k-1)((k-2t-1)^2 + 2t(k-t)) - (k-2t-1)t(t+1))v$$
$$- (k-2t-3)(k-t)t^2 + (k-2t-1)(k-t-1)(t+1)^2.$$

Since q is a convex function of v, to prove the lemma it suffices to show that

1. $q(1) < 0$,

2. $q\left(\frac{1}{t(t+1)} ((k-t-1)^2 + t-1) \right) < 0$,

3. $q\left(\frac{1}{t(t+1)} ((k-t-1)^2 + t) \right) > 0$,

under the assumption that $k \geq t^2 + 3t + 2$. Evaluating $q(1)$ and making some simplifications gives

$$q(1) = -(k-1)^3 + (4t+1)(k-1)^2 - (5t^2+3t)(k-1) + 2t^2(t+1)$$
$$\leq -(t^2-t)(k-1) - (5t^4+16t^3+12t^2+3t) < 0,$$

where we used (2.3). This shows 1. Also

$$q\left(-\frac{1}{t(t+1)}((k-t-1)^2 + t - 1)\right) = \frac{1}{t(t+1)} \, (-(k-1)(k-2t-1)(k-t^2-t-1)$$

$$+ \, k - 1 - (k-2t-1)t(t+1)(t^2+t+1)) < 0,$$

again using (2.3). Hence 2. follows.

Finally $q\left(-\frac{1}{t(t+1)}((k-t-1)^2 + t)\right) = (k-2t-1)(k-t^2-t-1) > 0$ by (2.3). This shows 3. and concludes the proof of Lemma 2.2.

Lemma 2.3: If t is the smallest positive integer such that $f(t) \leq k$ then $g(t,0) < g(t+1,0)$.

Proof: Notice first that the existence of a t as in this lemma is obvious. We have the inequality

$$f(t) \leq k < f(t-1),$$

which is equivalent to

$$(2.4) \qquad\qquad t^2 + t \leq k \leq t^2 + 3t + 1.$$

With $q(v)$ as in the proof of Lemma 2.2 we have to show that, assuming (2.4), $q(v) > 0$ for any $v \geq k$. Thus, again using the convexity of $q(v)$, it suffices to show that

1. $q(k) > 0$,

2. $q(1) \leq q(k)$.

To show this we will distinguish between two cases.

Case 1: $t = 1$.

Case 2: $t \geq 2$.

Case 1 is easy, since (2.4) tells us that $3 \leq k \leq 5$ and

$q(k) = -k^4 + 7k^3 - 8k^2 - 13k + 19$, which is indeed positive for $3 \leq k \leq 5$. This

shows 1 for this case. Also $q(1) \leq 0 \leq q(k)$ is easy to verify for this case.

So let us turn to Case 2, $t \geq 2$.

After some simplification we obtain

$$q(k) = -k^4 + (t^2+3t+3)k^3 - 2(t+1)^2k^2 - (2t^3+6t^2+4t+1)k$$
$$+ 4t^3 + 9t^2 + 5t + 1.$$

It can be shown that considering this as a function of k on the interval

$[t^2+t, \ t^2+3t+1]$ we have a concave function (use $t \geq 2$), implying that the

minimum value occurs at one of the endpoints $k = t^2 + t$ or $k = t^2 + 3t + 1$.

Evaluating at these two points gives the values

$$2t^7 + 7t^6 + 5t^5 - 9t^4 - 11t^3 + 2t^2 + 4t + 1$$

and

$$2t^2(t+1)$$

respectively. Since the latter one is the smaller of the two it follows that

$q(k) \geq 2t^2(t+1) > 0$ under the assumption of (2.4). This shows 1.

For 2. we see

$$q(1) = -(k-1)^3 + (4t+1)(k-1)^2 - (5t^2+3t)(k-1) + 2t^2(t+1)$$
$$\leq -(k-1)((k-1-(2t+1))^2 + t^2 - t - 1) + 2t^2(t+1)$$
$$\leq 2t^2(t+1) \leq q(k).$$

This concludes the proof of Lemma 2.3.   ☐

Weaponed with these three lemmas we are ready to prove Theorem 2.1.

Proof of Theorem 2.1: First observe that if d is a BTIB(v,b,k;t,o), then

$$\sigma^{-2} \, Tr(M_d^{-1}) = kv \, g(t,0).$$

Hence we have to show that

$$g(t_0,0) = \min\{g(t,0), \, t = 1,\ldots,k-1\}.$$

By Lemma 2.1 it suffices to show that

(2.5)
$$g(t_0,0) < \min\{g(t_0-1,0), \, g(t_0+1,0)\}.$$

We know that

(2.6)
$$f(t_0) \leq v < f(t_0-1).$$

If $f(t_0) > k$, (2.5) follows immediately from (2.6) and Lemma 2.2.

If $f(t_0) \leq k$, then since $v \geq k$ it follows from (2.6) that t_0 is the smallest positive integer t with $f(t) \leq k$. Hence by Lemma 2.3 we obtain $g(t_0,0) < g(t_0+1,0)$. Since $v < f(t_0-1)$, Lemma 2.2 gives us that $g(t_0,0) < g(t_0-1,0)$. Combining these two gives that (2.5) holds and concludes the proof of Theorem 2.1.   □

There are two different approaches in which Theorem 2.1 can be used. The first would be to specify the class $D_0(v,b,k)$ from which we like to choose a design. Then we have to find the unique value of t for which

$$f(t) \leq v < f(t-1).$$

The desired augmented BIBD will exist if a BIBD with b blocks of size $k - t$ based on v treatments exists. A disadvantage of this approach is that the existence question is answered at the very end.

This can be avoided if we start with a BIBD, say with b blocks of size k each based on v treatments, and try to augment this by t replications of the control. This would be a design in $D_0(v,b,k+t)$, and it is the best augmented BIBD in this class if

(2.7) $$\frac{1}{t(t+1)} ((k-1)^2 + t) \le v < \frac{1}{t(t-1)} (k^2 + t - 1)$$

where for $t = 1$ the upper bound is ∞ .

It is easy to show that there is always at least one value of t that satisfies this inequality, and thus that any BIBD can be embedded in an augmented BIBD which is the best augmented BIBD in its class.

Example 2.1: If we start with a symmetric BIBD based on 11 treatments and block size 5, it is easy to verify that (2.7) is satisfied for $t = 1$ and $t = 2$. Hence a BTIB(11,11,6;1,0) is the best augmented BIBD in $D_0(11,11,6)$, while a BTIB(11,11,7;2,0) is the best augmented BIBD in $D_0(11,11,7)$. In addition it can be shown that .991 and .985 are respective lower bounds for their efficiencies. We return to this in the next section.

A disadvantage of this second approach is that we do not know a priori what the block size is of the design that we will eventually wind up with.

3. BEST AUGMENTED BIBD's AND LOWER BOUNDS FOR THEIR EFFICIENCY FOR $3 \leq k \leq 10$.

The efficiency of a design in its class as given in Definition 1.3 would not be very useful if we could not obtain a good lower bound for the expression in the numerator.

One such lower bound can easily be shown to be the following:

$$\min_{d \in D_0(v,b,k)} \sigma^{-2} \mathrm{Tr}(M_d^{-1}) \geq kv \min\{g(x,z) : (x,z) \in \Lambda\},$$

where $g(x,z)$ and Λ are as defined in Theorem 1.1. This bound could indeed be used to obtain lower bounds for the efficiency of a design. It is however slightly easier for computational purposes, to use the following inequalities, which can be found in [7].

(3.1)   If $v \geq (k-1)^2 + 1$ then $\min\{g(x,z) : (x,z) \in \Lambda\} \geq g(0,z_0)$

where $z_0 = bk \dfrac{(v-1)(v+1)^{1/2} - (v+1)}{(v+1)(v-3)}$.

(3.2)   If $\dfrac{1}{4}((k-2)^2 + 3) \leq v \leq (k-2)^2$ then

$\min\{g(x,z) : (x,z) \in \Lambda\} \geq g(1,z_1)$, where

$$z_1 = b(k-1) \frac{(k-1)(k-2)(v-1)((k-3)((v+1)(k-1)-2))^{1/2} - (k-3)(v+k-3)((v+1)(k-1)-2)}{(k-3)((v+1)(k-1)-2)((v-3)(k-3)-2)}$$

Using these bounds we compiled the following list of best augmented BIBD's with their efficiencies. In each case a BTIB(v,b,k;t,o) is the best augmented BIBD provided that a BIBD with b blocks of size k - t based on v treatments exists. A lower bound for the efficiency is given, except in those cases where the design is known to be A-optimal.

THE BEST BTIB(v,b,k;t,o) AND A LOWER BOUND FOR ITS EFFICIENCY E.

$\underline{k = 3}$:   Take $t = 1$ for any $v \geq 3$.

      (i)   If $v = 3$ or   4   the design is A-optimal.

      (ii)   If $v \geq 5$, $E \geq \dfrac{(2v-1)(v-1 + (v+1)^{1/2})^2}{3v^3}$

             (for $v = 5$, $b = 10$ the design is A-optimal).

$\underline{k = 4}$:   Take $t = 1$ for any $v \geq 4$.

      (i)   If $v = 4$, $E \geq .9999$

             (for $v = 4$, $b = 4s$, $s = 1,\ldots,5$ the design is A-optimal).

      (ii)   If $5 \leq v \leq 9$ the design is A-optimal.

      (iii)   If $v \geq 10$, $E \geq \dfrac{(3v-1)(v-1 + (v+1)^{1/2})^2}{4v^2(v+1)}$.

$\underline{k = 5}$:   Take $t = 1$ for any $v \geq 5$.

      (i)   If $5 \leq v \leq 9$, $E \geq \dfrac{g(1,z_1)}{g(1,0)}$, where

$$g(x,z) = (v-1)^2(20v - (x+z)(4v+5) + x^2 + 2xz + z)^{-1}$$
$$+ (5(x+z) - (x^2 + 2xz + z))^{-1}, \text{ and}$$
$$z_1 = \frac{12(v-1)(2v+1)^{1/2} - (v+2)(4v+2)}{(2v+1)(v-4)}$$

             (for $v = 9$, $b = 18s$, $s = 1,\ldots,5$ the design is A-optimal).

      (ii)   If $10 \leq v \leq 16$ the design is A-optimal.

      (iii)   If $v \geq 17$, $E \geq \dfrac{(4v-1)(v-1 + (v+1)^{1/2})^2}{5v^2(v+2)}$.

$\underline{k = 6}$: Take $t = 1$ if $v \geq 9$, take $t = 2$ if $6 \leq v \leq 8$.

(i) If $6 \leq v \leq 8$, $E \geq \dfrac{g(1,z_1)}{g(2,0)}$, where

$$g(x,z) = (v-1)^2(30v - (x+z)(5v+6) + x^2 + 2xz + z)^{-1}$$

$$+ (6(x+z) - (x^2+2xz+z))^{-1}, \text{ and}$$

$$z_1 = 5\,\frac{20(v-1)(3(5v+3))^{1/2} - 3(v+3)(5v+3)}{3(5v+3)(3v-11)} \quad .$$

(ii) If $9 \leq v \leq 16$, $E \geq \dfrac{g(1,z_1)}{g(1,0)}$, where g and z_1 are as defined in (i) above.

(for $v = 16$, $b = 48s$, $s = 1,\ldots,6$, the design is A-optimal).

(iii) If $17 \leq v \leq 25$ the design is A-optimal.

(iv) If $v \geq 26$, $E \geq \dfrac{(5v-1)(v-1 + (v+1)^{1/2})^2}{6v^2(v+3)}$

$\underline{k = 7}$: Take $t = 1$ if $v \geq 13$, take $t = 2$ if $7 \leq v \leq 12$.

(i) If $7 \leq v \leq 12$, $E \geq \dfrac{g(1,z_1)}{g(2,0)}$, where

$$g(x,z) = (v-1)^2(42v - (x+z)(6v+7) + x^2 + 2xz + z)^{-1}$$

$$+ (7(x+z) - (x^2 + 2xz + z))^{-1}, \text{ and}$$

$$z_1 = 3\,\frac{15(v-1)(6v+4)^{1/2} - (v+4)(6v+4)}{(6v+4)(2v-7)} \quad .$$

(ii) If $13 \leq v \leq 25$, $E \geq \dfrac{g(1,z_1)}{g(1,0)}$, where g and z_1 are as defined in (i) above.

(for $v = 25$, $b = 100s$, $s = 1,\ldots,7$, the design is A-optimal).

(iii) If $26 \leq v \leq 36$ the design is A-optimal.

(iv) If $v \geq 37$, $E \geq \dfrac{(6v-1)(v-1 + (v+1)^{1/2})^2}{7v^2(v+4)}$.

$\underline{k = 8}$: Take $t = 1$ if $v \geq 19$, take $t = 2$ if $8 \leq v \leq 18$.

(i)   If $v = 8$ or 9 the design is A-optimal.

(ii)   If $10 \leq v \leq 18$, $E \geq \dfrac{g(1,z_1)}{g(2,0)}$, where

$$g(x,z) = (v-1)^2(56v - (x+z)(7v+8) + x^2 + 2xz + z)^{-1}$$
$$+ (8(x+z) - (x^2 + 2xz + z))^{-1}, \text{ and}$$

$$z_1 = 7\ \frac{42(v-1)(5(7v+5))^{1/2} - 5(v+5)(7v+5)}{5(7v+5)(5v-17)}\ .$$

(iii) If $19 \leq v \leq 36$, $E \geq \dfrac{g(1,z_1)}{g(1,0)}$, where g and z_1, are as

defined in (ii) above.

(for $v = 36$, $b = 180s$, $s = 1,\ldots,8$, the design is A-optimal).

(iv)   If $37 \leq v \leq 49$ the design is A-optimal.

(v)   If $v \geq 50$, $E \geq \dfrac{(7v-1)(v-1 + (v+1)^{1/2})^2}{8v^2(v+5)}$.

$\underline{k = 9}$: Take $t = 1$ if $v \geq 25$, take $t = 2$ if $9 \leq v \leq 24$.

(i)   If $v = 9$, $E \geq .9999$

(for $b = 36$ the design is A-optimal).

(ii)   If $10 \leq v \leq 12$ the design is A-optimal.

(iii) If $13 \leq v \leq 24$, $E \geq \dfrac{g(1,z_1)}{g(2,0)}$, where

$$g(x,z) = (v-1)^2(72v - (x+z)(8v+9) + x^2 + 2xz + z)^{-1}$$
$$+ (9(x+z) - (x^2 + 2xz + z))^{-1}, \text{ and}$$

$$z_1 = \frac{112(v-1)(3(4v+3))^{1/2} - 12(v+6)(4v+3)}{3(4v+3)(3v-10)}$$

(for $v = 13$, $b = 26$ the design is A-optimal).

(iv)   If $25 \leq v \leq 49$, $E \geq \dfrac{g(1,z_1)}{g(1,0)}$, where g and z_1 are

as defined in (iii) above.

(for $v = 49$, $b = 294s$, $s = 1,\ldots,9$, the design is A-optimal).

(v)   If $50 \leq v \leq 64$ the design is A-optimal.

(vi)   If $v \geq 65$, $E \geq \dfrac{(8v-1)(v-1 + (v+1)^{1/2})^2}{9v^2(v+6)}$

$\underline{k = 10}$:   Take $t = 1$ if $v \geq 33$, take $t = 2$ if $10 \leq v \leq 32$.

(i)   If $10 \leq v \leq 12$, $E \geq \dfrac{g(2,z_2)}{g(2,0)}$, where

$$g(x,z) = (v-1)^2 (90v - (x+z)(9v+10) + x^2 + 2xz + z)^{-1}$$

$$+ (10(x+z) - (x^2 + 2xz + z))^{-1}, \text{ and}$$

$$z_2 = \frac{504(v-1)(5(9v+5))^{1/2} - 40(9v+5)(2v+5)}{5(9v+5)(5v-19)}$$

(ii)   If $13 \leq v \leq 16$ the design is A-optimal.

(iii)   If $17 \leq v \leq 32$, $E \geq \dfrac{g(1,z_1)}{g(2,0)}$, where g is as

defined in (i) above, and

$$z_1 = 9 \frac{72(v-1)(7(9v+7))^{1/2} - 7(v+7)(9v+7)}{7(9v+7)(7v-23)} .$$

(iv)   If $33 \leq v \leq 64$, $E \geq \dfrac{g(1,z_1)}{g(1,0)}$, where g and z_1 are

defined as in (iii) above.

(for $v = 64$, $b = 448s$, $s = 1,\ldots,10$, the design is A-optimal).

(v)   If $65 \leq v \leq 81$ the design is A-optimal.

(vi)   If $v \geq 82$, $E \geq \dfrac{(9v-1)(v-1 + (v+1)^{1/2})^2}{10v^2(v+7)}$

The bounds for the efficiencies are generally extremely good. The only exception to this is for extreme large values of v, which should not come as a surprise. But even there the values seem to be acceptable, unless k is very small. To illustrate this we give some of the values for large v explicitly.

	v = 100	v = 500	v = 1000
k = 3	E $\geq$ .7888	E $\geq$ .7241	E $\geq$ .7077
k = 6	E $\geq$ .9602	E $\geq$ .9003	E $\geq$ .8823
k = 10	E $\geq$ .9991	E $\geq$ .9648	E $\geq$ 9492

4. DISCUSSION AND CONCLUSION.

Theorem 2.1 provides us with a tool to determine the parameters of the best augmented BIBD, either in a fixed class $D_0(v,b,k)$ or, when starting with a BIBD, in one or more classes of the form $D_0(v,b,k+t)$. These designs are very appealing for at least two reasons. As members of the class of BTIB designs they possess all the desirable features of designs in this class. In addition, much is known about questions concerning their existence and construction, since these questions are equivalent to those of the corresponding BIBD's, a subject that has received its share of attention in the literature.

Since using the best augmented BIBD gives, except for some extreme cases, a very high efficiency (in fact many of the listed designs in section 3, although not marked as such, may very well be A-optimal) we strongly recommend the use of these designs.

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