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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER Technical Report No. 26	2. GOVT ACCESSION NO. 12	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) CHEMICAL APPLICATIONS OF TOPOLOGY AND GROUP THEORY. 21. CHIRALITY IN TRANSITIVE SKELETONS AND QUALITATIVE COMPLETENESS [1]		5. TYPE OF REPORT & PERIOD COVERED Technical Report
7. AUTHOR(s) R.B. KING		6. PERFORMING ORG. REPORT NUMBER
9. PERFORMING ORGANIZATION NAME AND ADDRESS University of Georgia Department of Chemistry Athens, GA 30602		8. CONTRACT OR GRANT NUMBER(s) N00014-84-K-0365
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Department of the Navy Arlington, VA 22217		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR051-861
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		12. REPORT DATE 8/18/86
		13. NUMBER OF PAGES 40
		15. SECURITY CLASS. (of this report)
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) This document has been approved for public release and sale; its distribution is unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES To be published in <u>Journal of Mathematical Chemistry</u>		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Chirality, Chirality algebra, Group Theory, Permutation Groups, Chirality polynomials		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) This paper unifies the following ideas for the study of chirality polynomials in transitive skeletons: (1) Generalization of chirality to permutation groups not corresponding to three-dimensional symmetry point groups leading to the concepts of signed permutation groups and their signed subgroups; (2) Determination of the total dimension of the chiral ligand partitions through the Frobenius reciprocity theorem; (3) Determination of signed permutation groups, not necessarily corresponding to three-dimensional point groups, of which a given ligand partition is a maximum symmetry chiral ligand partition (over)		

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by the Ruch-Schönhofer partial ordering thereby allowing the determination of corresponding chirality polynomials depending only upon differences between ligand parameters; such permutation groups having the point group as a signed subgroup relate to qualitative completeness. In the case of transitive permutation groups on four sites, the tetrahedron and polarized square each have only one chiral partition but the allene and polarized rectangle skeletons each have two chiral ligand partitions related to their being signed subgroups of the tetrahedron and polarized square, respectively. The single transitive permutation group on five sites, the polarized pentagon, has a degenerate chiral ligand partition related to its being a signed subgroup of a metacyclic group with 20 elements. The octahedron has two chiral ligand partitions both of degree six; a qualitatively complete chirality polynomial is therefore homogeneous of degree six. The cyclopropane (or trigonal prism or trigonal antiprism) skeleton is a signed subgroup of both the octahedron and a twist group of order 36; two of its six chiral ligand partitions come from the octahedron and two more from the twist group. The polarized hexagon is a signed subgroup of the same twist group but not of the octahedron and thus has a different set of six chiral ligand partitions than the cyclopropane skeleton. Two of its six chiral ligand partitions come from the above twist group of order 36 and two more from a signed permutation group of order 48 derived from the $P_3[P_2]$ wreath product group with a different assignment of positive and negative operations than the octahedron.

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Control N00014-84-K-0365

TECHNICAL REPORT NO. 26

Chemical Applications of Topology and Group Theory. 21. Chirality
in Transitive Skeletons and Qualitative Completeness

by

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Prepared for publication in

Journal of Mathematical Chemistry

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August 18, 1986

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Abstract

→ This paper unifies the following ideas for the study of chirality polynomials in transitive skeletons: (1) Generalization of chirality to permutation groups not corresponding to three-dimensional symmetry point groups leading to the concepts of signed permutation groups and their signed subgroups; (2) Determination of the total dimension of the chiral ligand partitions through the Frobenius reciprocity theorem; (3) Determination of signed permutation groups, not necessarily corresponding to three-dimensional point groups, of which a given ligand partition is a maximum symmetry chiral ligand partition by the Ruch-Schönhofer partial ordering thereby allowing the determination of corresponding chirality polynomials depending only upon differences between ligand parameters; such permutation groups having the point group as a signed subgroup relate to qualitative completeness. In the case of transitive permutation groups on four sites, the tetrahedron and polarized square each have only one chiral ligand partition but the allene and polarized rectangle skeletons each have two chiral ligand partitions related to their being signed subgroups of the tetrahedron and polarized square, respectively. The single transitive permutation group on five sites, the polarized pentagon, has a degenerate chiral ligand partition related to its being a signed subgroup of a metacyclic group with 20 elements. The octahedron has two chiral ligand partitions both of degree six; a qualitatively complete chirality polynomial is therefore homogeneous of degree six. The cyclopropane (or trigonal prism or trigonal antiprism) skeleton is a signed subgroup of both the octahedron and a twist group of order 36; two of its six chiral ligand partitions come from the octahedron and two more from the twist group. The polarized hexagon is a signed subgroup of the same twist group but not of the octahedron and thus has a different set of six chiral ligand partitions than the cyclopropane skeleton. Two of its six chiral ligand partitions come from the

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1. Introduction

A number of chemically significant phenomena such as optical rotation and circular dichroism relate to pseudoscalar measurements, which give real numbers having the the following two properties:

- (1) They depend upon the measured object but not its spatial orientation;
- (2) They have opposite signs for mirror images such as the two enantiomers of a chiral molecule.

This interest in pseudoscalar measurements has led to the geometrical and algebraic study of chirality [2,3,4,5,6,7,8]. Such studies have the following objectives:

- (1) Determination of the ligand partitions for a given molecular skeleton which lead to chiral systems, namely how unsymmetrical must a ligand partition be before all improper rotation symmetry elements S_n (including reflection planes $S_1 \equiv \sigma$ and inversion centers $S_2 \equiv i$) of an achiral skeleton are destroyed.
- (2) Determination of mathematical functions (chirality functions) by which the magnitude and sign of a given pseudoscalar property (the dependent variable) can be calculated for a given skeleton using parameters which depend only upon the ligands located at specific sites on the skeleton (the independent variables).

Chirality algebra dissects a molecule into a collection of ligands and an underlying skeleton. A molecule with a monocentric skeleton may be represented as ML_n in which M is a metal or other central atom and the n ligands L may or may not be equivalent. Polycentric skeletons such as allene and cyclopropane are also of interest and can be analogously represented. In this paper as well as in my previous paper [8] the ligands L are assumed to be achiral.

My previous paper on chirality [8] relates chirality functions to framework groups [9], a method for describing the symmetry of skeletons having a finite

numbers of sites. In this connection the sites in a given skeleton may be divided into orbits where sites in a given orbit are symmetry equivalent, namely interchangeable by a symmetry operation of the skeleton. A skeleton in which all sites are symmetry equivalent thus has only one orbit and is called a transitive skeleton. The chirality functions for transitive skeletons are the basic building blocks for chirality functions of all skeletons since the chirality function of a given intransitive skeleton is the product of a chirality functions of its individual orbits which correspond to transitive subskeletons having a smaller number of sites [8].

This paper presents a number of details which are important in understanding various aspects of chirality algebra in the limited number of transitive skeletons having six or less sites. These include the most important skeletons for building chiral molecules such as methane, allene, and cyclopropane derivatives; cyclobutadiene, cyclopentadienyl, and benzene metal complexes; and octahedral metal complexes. In addition signed permutation groups and their signed subgroups are defined which relate to qualitatively complete chirality polynomials.

2. Groups and Generators

Certain elements of a given discrete group G form a set of its generators if every element of G is expressible as a finite product of their (positive or negative) powers [10]. A set of relations satisfied by the generators of G is called a presentation of G if every relation satisfied by the generators is an algebraic consequence of the particular relations of the presentation. The presentations of the permutation groups of the transitive skeletons of interest are given in Table 1. More extensive listings of the presentations of all possible distinct permutation groups of up to seven objects are given by Miller [11]; a similar

listing of presentations of all possible distinct permutation groups on eight objects is given by Burns [12]. A much more recent listing of all transitive groups on up to 11 objects and their generators is given by Butler and McKay [13].

In order to extend the concept of chirality to the presentations of permutation groups, a sign (positive or negative) must be assigned to each group element so that the identity operation is positive and the following symbolic relationships are satisfied:

$$(+)(+) = (+) \tag{1a}$$

$$(+)(-) = (-) \tag{1b}$$

$$(-)(-) = (+) \tag{1c}$$

The process of assigning signs to group elements satisfying 1a-1c describes a homomorphism from G to the cyclic multiplicative group on the two elements (1, -1), namely the Z_2 group. A permutation group having such sign assignments for each element is a signed permutation group. In signed permutation groups corresponding to three-dimensional point groups the positive group elements are the proper rotations including the identity and the negative group elements are the improper rotations including an inversion center and reflection planes. In general a given presentation of a permutation group may admit more than one self-consistent sign assignment satisfying equation 1a-1c. Thus the permutation groups of the polarized square and allene skeletons in Table 1 have the same presentation but different sign assignments relating to the fact that the operation of period four (ab in Table 1) is a proper rotation (C_4) in the polarized square but an improper rotation (S_4) in the allene skeleton. Also alternative sign assignments are possible for the presentation of the octahedron (e.g., a negative and b positive in Table 1) but these do not correspond to actual point

groups.

The signed permutation group of an achiral skeleton contains equal numbers of positive and negative elements. The positive elements form a chiral subgroup of index two. Introduction of non-equivalent ligands at the sites of the achiral skeleton in general reduces the symmetry of the system by destroying some of the symmetry elements of the unlabeled skeleton. The resulting smaller signed permutation group can still contain equal quantities of positive and negative elements (an achiral group) or only positive elements (a chiral group). In this abstract manner chirality can be studied without using the concepts of proper and improper rotations. This allows the extension of the concept of chirality to permutation groups not isomorphic to three-dimensional point groups. An example of such a permutation group is the order 72 $P_2[P_3]$ wreath product automorphism group of the non-planar $K_{3,3}$ bipartite graph.

Permutation groups may also be characterized by their cycle indices [14] $Z(G)$, which are polynomials of the form

$$Z(G) = \frac{1}{|G|} \sum_{i=1}^{i=c} a_i x_1^{c_{i1}} x_2^{c_{i2}} \dots x_n^{c_{in}} \quad (2)$$

In equation 2 the terms and indices have the following significance:

$|G|$ = number of elements in the permutation group

n = number of sites in the permutation group

c = number of terms in the cycle index

a_i = number of elements in the permutation group having the indicated cycle structure

x_k = dummy variable referring to cycles of length k

c_{ik} = exponent indicating the number of cycles of length k in term i

Signed permutation groups may be characterized analogously by their signed

cycle indices in which each term is preceded by a plus or minus sign depending upon whether the term corresponds to a positive or negative element of the corresponding permutation group. The signed cycle index $Z^\pm(G)$ thus may be expressed as follows:

$$Z^\pm(G) = \frac{1}{|G|} \sum_{i=1}^{i=c} (\text{sgn}) a_i x_1^{c_{i1}} x_2^{c_{i2}} \dots x_n^{c_{in}} \quad (3)$$

where

(sgn) = + if the group element is positive

(sgn) = - if the group element is negative

By way of illustration the signed cycle indices for the polarized square and allene permutation groups are listed below:

(A) Polarized Square

$$Z^\pm(G) = \frac{1}{8} (x_1^4 + 2x_4 + x_2^2 - 2x_1^2 x_2 - 2x_2^2) \quad (4)$$

(B) Allene Skeleton

$$Z^\pm(G) = \frac{1}{8} (x_1^4 - 2x_4 + x_2^2 - 2x_1^2 x_2 + 2x_2^2) \quad (5)$$

Note that the signed cycle indices ($Z^\pm(G)$ from equation 3) for these two distinct skeletons are different whereas the regular cycle indices ($Z(G)$ from equation 2) are identical. Thus signed cycle indices $Z^\pm(G)$ provide more information than ordinary cycle indices $Z(G)$ and this additional information is essential for the study of chirality. Furthermore, in the case of achiral point groups the signed cycle index as defined by equation 3 corresponds to the generalized character cycle index [15,16,17] for the antisymmetric irreducible representation A^- having +1 characters for all proper rotations and -1 characters for all improper rotations.

For brevity the factor $1/|G|$ will be omitted from cycle indices and signed cycle indices in this paper.

An element of a permutation group may be assigned a sign (positive or negative) as outlined above but must have a parity (even or odd). Thus the parity of a permutation is odd or even depending upon whether the total number of cycles of even length in the permutation is odd or even, respectively. Equivalently, the parity of a permutation is odd or even depending on whether the number of two-site transpositions in the permutation is odd or even, respectively [18]. Such an assignment of odd and even parities to elements of a permutation group G describes another homomorphism from G to Z_2 .

A permutation group must contain either only even permutations or equal numbers of even and odd permutations. In an ordinary signed permutation group all positive elements (namely, proper rotations in three-dimensional point groups) are even permutations and all negative elements (namely, improper rotations in three-dimensional point groups) are odd permutations; if this is not the case the signed permutation group is extraordinary. (In a previous paper [8] ordinary groups were called normal but this terminology can be confusing since a normal subgroup, discussed below, refers to a totally different concept.) The signed permutation groups of the tetrahedron and allene skeletons are ordinary and those of the polarized square, polarized rectangle, polarized pentagon, and octahedron skeletons are extraordinary. Extraordinary signed permutation groups may be characterized by the anomalous terms in their signed cycle indices where an anomalous term is a positive term corresponding to an odd permutation or a negative term corresponding to an even permutation.

The assignment of signs to each element of a permutation group to give a signed permutation group provides a method for defining a new type of subgroup relationship. Thus a signed permutation group H is a signed subgroup of a signed

permutation group G if all of the operations of H are contained in G with the same signs. In order for a group H to be a signed subgroup of group G the signed cycle index of G , $Z^\pm(G)$, must contain with the proper sign (but not necessarily the same numerical coefficient) each term in the signed cycle index of H , $Z^\pm(H)$. Thus any anomalous terms in the signed cycle index of H must be found in the signed cycle index of any group G of which H is a signed subgroup. For this reason the anomalous terms of the signed cycle indices of signed permutation groups are convenient indicators of possible signed subgroup relationships important for the study of qualitative completeness in chirality functions.

It is instructive to compare the properties of signed subgroups with those of normal subgroups. A signed permutation group K is a normal subgroup of a signed permutation group G if K consists only of complete conjugacy classes of G [19]. In order for a group K to be a normal subgroup of group G the signed cycle index of G , $Z^\pm(G)$, must contain each term in the signed cycle index of K , $Z^\pm(K)$, with the proper sign and generally the same numerical coefficient. An exception to the latter condition can occur when a single signed cycle index term of G corresponds to two or more classes of elements having the same cycle structure. Note that all normal subgroups of G are signed subgroups. However, not all subgroups of G are signed subgroups and not all signed subgroups of G are normal subgroups.

The following groups are significant in the study of the chirality algebra of an achiral skeleton having n sites:

P_n : the symmetric group on n sites having $n!$ elements.

G : the three-dimensional symmetry point group of the skeleton.

R : the subgroup of index 2 in the achiral G consisting of only its proper rotations (positive elements).

N : the normalizer of G in P_n , namely the largest subgroup of P_n containing

G as a normal subgroup. This defines the total permutations of a set of isomers whose symmetry point groups contain exactly the same set of ligand permutations and has led to the mathematically interesting [6,20] but physically dubious [21,22,23] concept of "hyperchirality."

M: a maximum symmetry signed permutation group on n objects which has the least symmetrical chiral ligand partition of G as its most symmetrical chiral ligand partition using the Ruch-Schonhöfer partial ordering relating to the degree of the corresponding chirality polynomial. This relates to the concept of qualitative completeness [4,5] and a similar group can be defined for the concept of qualitative supercompleteness [7].

Since G is a normal subgroup of N, N can be expressed as the semidirect product [24,25]

$$N = G \Delta N/G = G \Delta H \quad (6).$$

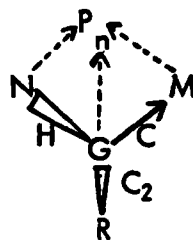
In a few but not all cases the signed subgroup G is also a normal subgroup of M so that M can also be expressed as semidirect product

$$M = G \Delta M/G = G \Delta C \quad (7).$$

The group H contains one element corresponding to each diastereomer ("member of a hyperchiral family" [6]) having the same set of ligand permutations in its symmetry point group and therefore the same chirality polynomial. The group C, when it exists, contains one element corresponding to each diastereomer, which, when present in equal quantities in the set of diastereomers represented by C, leads to a mixture for which the lowest degree chirality polynomial vanishes identically [4,5]. A major portion of the confusion in the literature arises from

the fact that N and M and thus H and C are defined very differently and thus are not necessarily the same. Thus G must be a normal subgroup of N but may be only a signed subgroup of M .

The relationships between the groups P_n , G , R , N , M , H , and C for an achiral skeleton can be summarized by the following scheme:



In this scheme a dotted arrow represents a subgroup relationship which cannot be normal for $n \geq 5$, a solid arrow represents a signed subgroup relationship which can or cannot be normal, a wedge represents a subgroup relationship which must be normal, and the groups indicated next to the wedges and the solid arrow are the corresponding quotient groups (see equations 6 and 7).

3. Isomer Count and Group Representation Theory

Consider a skeleton having n sites. The number of permutations of these n sites is $n!$. These permutations may be partitioned into $n!/|R|$ sets of $|R|$ permutations each where $|R|$ is the order of the proper rotation subgroup of the skeleton. The number $n!/|R|$ is, of course, the isomer count, I , of the skeleton [26,27,28]. This section uses group representation theory [25,29] to relate the isomer count to the sum of the dimensions of the chiral representations which, in turn, are closely connected with chiral ligand partitions and chirality functions.

Some related ideas are presented rather differently elsewhere [5,25]. In this section subgroup relationships are not signed.

A chirality function has the property that it is unaffected by ligand permutations corresponding to proper rotations (i.e., positive permutation group elements) but undergoes a change in sign with no change in absolute value with ligand permutations corresponding to improper rotations (i.e., negative permutation group elements). As a consequence of this, the representation of the chirality function (i.e., the chiral representation) must contain the antisymmetric representation A^- which is the one-dimensional irreducible representation of all achiral point groups having +1 characters for all proper rotations, namely $\chi(G, A^-, C_n) = +1$, and -1 characters for all improper rotations, namely $\chi(G, A^-, S_n) = -1$.

Consider the point group G of a skeleton having n sites as a subgroup of the corresponding symmetric group P_n . In order to determine which of the irreducible representations Γ_r of P_n correspond to chiral ligand partitions the standard procedure [5] involves subduction [25,29] of these representations of P_n by its subgroup G . The characters of the subduced representation Γ_r , namely $\chi(P_n, \Gamma_r, g)|_G$ for the operations g in G contained in P_n , are equal to the characters of the corresponding representation and operation in P_n using the cycle partitions of the symmetry operations in G and permutations in P_n as a basis for comparison. In general irreducible representations of P_n may lead to reducible representations when subduced by G . Such subduced representations may contain the antisymmetric representation A^- which indicates the chirality of the ligand partition corresponding to the irreducible representation of P_n in question.

In order to determine whether the subduced representation contains the antisymmetric representation A^- , standard orthogonality methods [25,29,30] are applied using the following scalar product of two character vectors:

$$\#(A^-)_r = \frac{1}{|G|} [\vec{\chi}(G, A^-)] \cdot [\vec{\chi}(P_n, \Gamma_r)|_G] \quad (8)$$

In equation 8 the symbols have the following significance:

$\#(A^-)_r$ = the number of times that the antisymmetric representation A^- appears in the representation of G subduced by the r^{th} irreducible representation Γ_r of P_n .

$|G|$ = the number of operations in G .

$\vec{\chi}(G, A^-)$ = a $|G|$ -dimensional vector having +1 for the coordinates corresponding to proper rotations of G and -1 for the coordinates corresponding to the improper rotations of G corresponding to the definition of the antisymmetric representation A^- .

$\vec{\chi}(P_n, \Gamma_k)|_G$ = another $|G|$ -dimensional vector having the coordinates $[\chi(P_n, \Gamma_r, e)|_G, \chi(P_n, \Gamma_r, g_2)|_G, \dots, \chi(P_n, \Gamma_r, g_{|G|})|_G]$ where e is the identity operation of G and $g_2, g_3, \dots, g_{|G|}$ are the remaining (non-identity) operations of G listed in the same order as in $\vec{\chi}(G, A^-)$.

In order to obtain all of the chiral ligand partitions of the skeleton in question, equation 8 must be applied successively to all irreducible representations of P_n as indexed by r . This separately tests all possible ligand partitions for chirality.

This method is actually the most convenient method if specific information on chiral ligand partitions is sought, such as might be required for construction of specific chirality functions. However, if the objective is only to obtain some insight regarding the variety of chiral representations (chiral ligand partitions) through a parameter that might be called the chiral dimensionality, χ_d , then a further mathematical analysis of the problem suggests a more direct approach.

The characters $\chi(P_n, \Gamma_r, g)|_G$ appearing as components in the character vector $\vec{\chi}(P_n, \Gamma_r)|_G$ used in equation 8 arise from subduction of an irreducible representation Γ_r of the symmetric group P_n by its subgroup G . The opposite

process is possible, namely induction of a representation Λ of G into the larger group P_n of which G is a subgroup. Gorenstein [29] provides a useful summary of this induction process. Let $a_i, 1 \leq i \leq |G|$, be a complete set of coset representatives of G in P_n . Let Λ be a representation of G and let its dimension be d . The definition of Λ can be extended to all of P_n by setting p equal to the $d \times d$ zero matrix for all permutations p found in P_n but not in G . The following equation is then used to define a map Λ^* :

$$p \Lambda^* = ((a_i p a_j^{-1}) \Lambda) \quad p \in P_n \quad (9)$$

Thus $p \Lambda^*$ is a $|G| \times |G|$ matrix of blocks whose (i,j) th entry is the $d \times d$ matrix $(a_i p a_j^{-1}) \Lambda$. In general we call Λ^* the representation of P_n induced by the representation Λ of G . The characters of the induced representation can be found using the following formula [25,29] after setting $\chi(G, \Lambda, p) = 0$ for permutations of p of P_n not found in G :

$$\chi(P_n, \Lambda^*, p) = \frac{1}{|G|} \sum_{u \in P_n} \chi(G, \Lambda, upu^{-1}) \text{ for all } p \text{ in } P_n \quad (10)$$

Furthermore the dimension of a representation Λ^* of P_n induced by a representation Λ of G can be found using the following equation since $|P_n| = n!$:

$$\dim \Lambda^* = \frac{n!}{|G|} \dim \Lambda \quad (11)$$

Scalar products of character vectors are useful for determining the irreducible representations contained in a reducible representation according to the orthogonality relationships as exemplified by equation 8. The Frobenius reciprocity

theorem [25,29] relates such character vector products involving subduced and induced characters as represented by the following equation using the notation introduced above and referring to the case of interest in this work in which symmetry point group G is a subgroup of the symmetric group P_n :

$$\{[\vec{\chi}(G, \Lambda)] \cdot [\vec{\chi}(P_n, \Gamma_r) | G]\}_G = \{[\vec{\chi}(P_n, \Lambda^*)] \cdot [\vec{\chi}(P_n, \Gamma_r)]\}_{P_n} \quad (12)$$

Now let Λ be the antisymmetric representation A^- . Then combining equations 8 and 12 shows the equivalence of the following procedures:

(1) Subduction: Determination of the number of times that the antisymmetric irreducible representation A^- of the point group G appears in the representation Γ_r of P_n subduced by G as represented in equation 8. Note that in general the irreducible representation Γ_r of P_n becomes reducible after subduction by G .

(2) Induction: Determination of the irreducible representations Γ_r of P_n which occur in the representation A^{-*} of P_n induced by the irreducible representation A^- of the point group G .

Now consider the dimensionality of the induced representation A^{-*} . Since the antisymmetric representation A^- is always one-dimensional, equation 11 indicates that the dimensionality of A^{-*} is $n!/|G|$. This means that the sum of the dimensions of the irreducible representations Γ_r of P_n contained in A^{-*} (i.e., the representations Γ_a corresponding to chiral ligand partitions) must also be $n!/|G|$. Note the close relationship of this quantity to the isomer count $\mathcal{I} = n!/|R|$ discussed above. For an achiral framework group $n!/|G| = \mathcal{I}/2 = n!/|R|$ since $|G| = 2|R|$. The quantity $n!/|G|$ thus corresponds to the number of enantiomer pairs when all n sites of the skeleton have different ligands and may be called the chiral dimensionality, χ_d . The chiral dimensionality rapidly increases as the number of sites n increases indicating that the set of chiral ligand partitions

also rapidly becomes large and therefore complicated. For example, in the cube, which has 8 sites and the O_h point group of order 48, the chiral dimensionality is $8!/48 = 840$ indicating a complicated set of chiral ligand partitions even for this very symmetrical polyhedron. This complicated set of chiral ligand partitions for the cube is listed in one of Ruch and Schonhöfer's papers [4]. For this reason the detailed discussion of the chiral ligand partitions and chirality polynomials in this paper will be limited to transitive skeletons having no more than six sites.

4. Chirality Polynomials

The subduction procedure outlined above and described in detail elsewhere [5,8] can be used to determine the irreducible representations of the symmetric group corresponding to the chiral ligand partitions for a given skeleton. The sum of the dimensions of these chiral representations must equal the chiral dimensionality, $\chi_d = n! / |G|$; this relationship can be used to check the subduction calculations. Furthermore, a partial ordering of the chiral ligand partitions is possible based on the corresponding Young diagrams [4,5]. This partial ordering corresponds to the degrees of the corresponding chirality polynomials as calculated by a group theoretical algorithm [6,8] and is used to define the relative symmetries of different ligand partitions throughout this paper. In its most convenient forms [6,8] the group-theoretical algorithm uses the permutations of all operations of the symmetry point group on the skeletal sites, i.e., the permutations of the corresponding framework group [8,9].

Chirality polynomials are most useful if they can express the magnitudes and signs of pseudoscalar properties as functions which depend only upon the differences between the parameters assigned to ligands at the various sites of the skeleton in question. By using algebraic invariant theory Meinköhn [31,32]

has shown that only the lowest degree chirality polynomials for a given skeleton (framework group) are required to have this desirable property of depending only upon the differences between the ligand parameters.

Previous studies of chirality algebra [4,5,8] have focused on identifying the Ruch-Schönhofer maximum symmetry ligand partition making chiral a given achiral skeleton by destroying all of the improper rotation symmetry elements. This relates to calculating the lowest degree chirality polynomial, namely a chirality polynomial which must depend only upon the differences between the ligand parameters [31,32]. This problem can be "inverted" to give the following problem: Find the permutation group on n objects for which a given partition of n ligands is a maximum symmetry chiral ligand partition by the Ruch-Schönhofer partial ordering [4,5] thereby allowing the determination of a chirality polynomial for the ligand partition in question which is required to depend only upon the differences between ligand parameters. This problem has the following aspects of particular interest:

(1) Solution of this problem provides a group theoretical approach to the search for qualitatively complete chirality polynomials [4,5]. In this context a chirality polynomial is said to be qualitatively complete if there is no nonracemic isomer mixture for which it vanishes identically [5]. The permutation groups M for the least symmetrical chiral ligand partitions may define quotient groups (e.g., C of equation 7) which in turn define permutations leading to nonracemic isomer mixtures for which lower degree chirality polynomials derived from more symmetrical chiral ligand partitions vanish. In these cases a chirality polynomial derived from a less symmetrical chiral ligand partition must be added to the lowest degree chirality polynomial in order to obtain a qualitatively complete chirality polynomial.

(2) The permutation group M may not always correspond to an actual

three-dimensional point group, i.e., a framework group [9] having n sites. Therefore all transitive permutation groups of degree n (i.e., those permuting n objects) [11,12,13] are potentially important to the study of qualitatively complete chirality functions, not just those realizable as three-dimensional point or framework groups.

(3) Since the treatment of qualitatively complete chirality functions may require consideration of permutation groups which are not realizable as three-dimensional point or framework groups, the generalization of chirality to signed permutation groups having characteristic signed cycle indices (equation 3) is required.

(4) Signed permutation groups on n objects of which the framework group G is a signed subgroup are important to this treatment as exemplified by the group M defined above. In order for G to be a signed subgroup of M , the same anomalous terms in the signed cycle index of G must be present in the signed cycle index of M . This facilitates greatly the search for signed permutation groups M that can correspond to ligand permutations leading to non-racemic isomer mixtures for which some chirality polynomials corresponding to G may vanish.

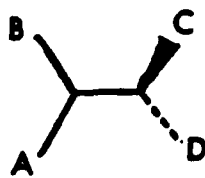
(5) There are ligand partitions for which there is no possible transitive signed permutation group having as its lowest degree chirality polynomial a polynomial corresponding to the ligand partition in question. The simplest such cases are the (21^3) and (2^21) ligand partitions on five sites, which lead to the lowest degree chirality polynomials for the intransitive framework groups [9] $T_d[O(L),4C_3(L)]$ (tetrahedron + center) and $D_{3h}[C_3(L_2),3C_2(L)]$ (trigonal bipyramid), respectively, but not for any transitive signed permutation groups of degree five (Table 3).

The general ideas outlined in this section are illustrated in the relatively simple cases of the transitive skeletons with four sites and are then extended to the more interesting transitive skeletons with five and six sites.

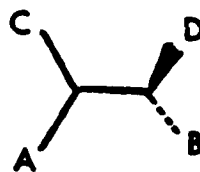
5. Transitive Skeletons Having Four Sites

The four transitive signed permutation groups on four objects have the four non-trivial possible partitions of four ligands as their highest symmetry chiral ligand partitions by the Ruch-Schönhofer partial ordering [4,5]. Thus the ligand partitions (1^4) , (21^2) , (2^2) , and (31) corresponding to irreducible representations of the symmetric group $P_4 (\approx T_d)$ of dimensions 1, 3, 2, and 3, respectively, are the highest symmetry chiral ligand partitions for the T_d tetrahedron, C_{4v} polarized square, D_{2d} allene, and C_{2v} polarized rectangle, respectively. The ligand partitions (1^4) and (21^2) are the only chiral ligand partitions for the tetrahedron and polarized square in accord with their chiral dimensionalities $\chi_d = n!/|G|$ of $24/24 = 1$ and $24/8 = 3$, respectively. However, both the allene and polarized rectangle skeletons have two chiral representations, namely $(1^4) + (2^2)$ of total dimension 3 for the allene skeleton ($24/|D_{2d}| = 3$) and $(21^2) + (31)$ of total dimension 6 for the polarized rectangle skeleton ($24/|C_{2v}| = 6$). Thus a single chirality polynomial is qualitatively complete for the tetrahedron or polarized square whereas two-component chirality polynomials are required in the cases of allene and the polarized rectangle for qualitative completeness. This fundamental point is already recognized in the original papers by Schönhofer and Ruch [4] and Mead [5] on chirality functions. However, the full permutation group theoretical significance of this situation has not been clearly stated.

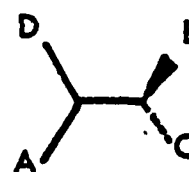
Consider first the following three permutation isomers having the allene skeleton with the framework group $D_{2d}[2\sigma_d(L_2)]$:



I (bcd)



I (cdb)



I (dbc)

These isomers are related by keeping ligand A fixed and permuting ligands B, C, and D in a three-fold "symmetry" operation. The set of three isomers I(bcd), I(cdb), I(dbc) thus forms a group isomorphic to the cyclic group C_3 based on the ligand permutations required to form a given member of the set from a reference isomer, say I(bcd), taken as the identity element of the group.

The lowest degree chirality polynomial corresponding to the (2^2) ligand partition for one of the above isomers, say I(bcd), is the following polynomial of degree 2:

$$(2^2)[I(bcd)] = (B-A)(D-C) \quad (13)$$

In equation 13 and in the following similar equations the letters representing the ligands are also taken to mean the corresponding chirality polynomial parameters for brevity and to avoid too many confusing indices.

The chirality polynomial in equation 13 has a major flaw, namely it vanishes identically for a non-racemic mixture of equal quantities of the three isomers I(bcd), I(cdb), and I(dbc), i.e.

$$X(2^2)[I(bcd) + I(dcb) + I(dbc)] \equiv 0 \quad (14)$$

In order to obtain a chirality polynomial without this flaw, it is necessary to use a polynomial of degree 6 derived from the other chiral ligand partition of the allene skeleton, namely (1^4) . However, a chirality polynomial obtained from this chiral ligand partition and the permutations of the D_{2d} symmetry point group of the allene skeleton using the standard group theoretical algorithm [6,8] will not be a function only of the differences between the ligand parameters. In order to obtain a chirality polynomial from the chiral ligand partition (1^4) which

must be a function only of the differences between the ligand parameters, it is necessary to use the permutations from a group of degree 4 for which the ligand partition (1⁴) is the lowest degree chiral ligand partition. This permutation group (M in equation 7) is, of course, that of the tetrahedron $T_d \cong P_4$ (Table 1). From the ligand partition (1⁴) and the permutation group T_d (not D_{2d}) the following chirality polynomial of degree 6 is obtained using the standard group-theoretical procedure [6,8]:

$$\chi(1^4)[I(bcd)] = (s_2-s_1)(s_3-s_1)(s_4-s_1)(s_3-s_2)(s_4-s_2)(s_4-s_3) \quad (15)$$

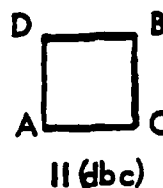
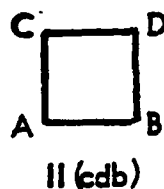
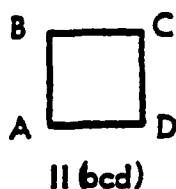
In equation 15 and in similar following equations a variable s_k represents the parameter of the ligand located at site k of the skeleton.

The chirality polynomial of equation 15 is familiar as the lowest degree chirality polynomial for a tetrahedral skeleton such as methane. Furthermore, this chirality polynomial does not vanish for a non-racemic mixture of equal quantities of the three isomers I(bcd), I(cdb), and I(dbc).

The observations outlined above suggest that T_d plays the role of M and C_3 plays the role of C (see equation 7). The group D_{2d} is a signed subgroup of T_d and represents the symmetry of the allene skeleton. The group C_3 represents the permutations forming the isomers of the non-racemic mixture (i.e., I(bcd) + I(cdb) + I(dbc)) for which the lowest degree chirality polynomial of the allene skeleton vanishes. The non-racemic mixture of equal quantities of I(bcd), I(cdb), and I(dbc) has the higher effective permutational symmetry T_d relative to the D_{2d} permutational symmetry of a single isomer and this higher symmetry is reflected in the form of the chirality polynomial required in order to avoid its vanishing for a non-racemic mixture. In other words, the vanishing of the degree 2 polynomial of equation 13 for the non-racemic mixture of equal quantities

of $l(bcd)$, $l(cdb)$, and $l(dbc)$ is diagnostic of a higher effective symmetry for this system.

It is instructive to compare this property of the allene skeleton with that of the polarized square skeleton, which has (21^2) as its only chiral ligand partition. A set of polarized square isomers analogous to $l(bcd)$, $l(cdb)$, and $l(dbc)$ is the following:



The chirality polynomial corresponding to the (21^2) ligand partition has degree 3 and can be represented as follows:

$$\chi(21^2)[l(bcd)] = (s_2-s_1)(s_4-s_3)[(s_1-s_3)+(s_2-s_4)] \quad (16)$$

For a non-racemic mixture of equal quantities of the three isomers $l(bcd)$, $l(cdb)$, and $l(dbc)$ this polynomial has the following value:

$$\chi(21^2)[l(bcd) + l(cdb) + l(dbc)] = 2[(AC+DB)(B-D)+(AD+BC)(C-B)+(AB+CD)(D-C)] \neq 0 \quad (17)$$

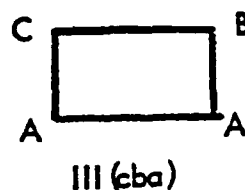
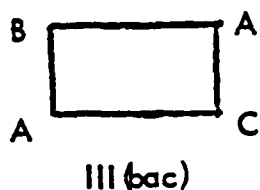
The added $[(s_1-s_3)+(s_2-s_4)]$ factor of the degree 3 polynomial (17) prevents equation 17 from vanishing for the non-racemic mixture of equal quantities of $l(bcd)$, $l(cdb)$, and $l(dbc)$. The single chirality polynomial of equation 16 is thus qualitatively complete. Note also that the C_{4v} point group of the polarized

square is not a signed subgroup of the tetrahedron (T_d) since the cycle index term x_4 corresponds to a positive operation (C_4) in C_{4v} but a negative operation (S_4) in T_d .

The remaining transitive skeleton with four sites is the polarized rectangle which has two chiral ligand partitions, namely the degree 1 (31) partition and the degree 3 (21^2) partition. The lowest degree chirality polynomial is therefore the following degree 1 (linear) polynomial:

$$\chi(31) = (s_1 - s_3) + (s_2 - s_4) \quad (18)$$

This polynomial, however, vanishes for the non-racemic mixture of equal quantities of the following two isomers:



However, the following degree 3 polynomial from the (21^2) chiral ligand partition (compare equation 16) does not vanish for this isomer mixture:

$$\begin{aligned} X(21^2)[III(bac) + III(cba)] &= (B-A)(C-A)[A-A+B-C] + (C-A)(A-B)[A-B+C-A] \\ &= 2(B-A)(C-A)(B-C) \end{aligned} \quad (19)$$

In this connection note the following:

- (1) The ligand partition in the non-racemic isomer pair $III(bac) + III(cba)$ is (21^2)

in accord with the ligand partition leading to the polynomial in equation 19.

(2) C_{2v} is a normal subgroup of C_{4v} .

(3) C_{4v} can be represented by the following semidirect product [24,25]:

$$C_{4v} = C_{2v} \wedge C_2 \quad (20)$$

Thus the groups C_{4v} , C_{2v} , and C_2 correspond to the groups M, G, and C, respectively, in equation 7.

6. Transitive Skeletons Having Five Sites

There are only four transitive signed permutation groups on five sites (Table 2). Furthermore, three of these signed permutation groups do not correspond to framework groups [9] having five sites; these are the following:

(1) The symmetric group P_5 having 120 operations.

(2) The ordinary metacyclic group M_5 having 20 operations and the following signed cycle index:

$$x_1^5 + 5x_1x_2^2 + 4x_5 - 10x_1x_4 \quad (21)$$

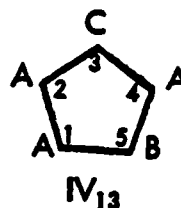
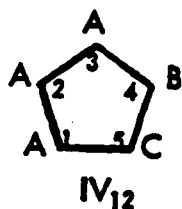
(3) The extraordinary metacyclic group M_5' having 20 operations and the following signed cycle index:

$$x_1^5 + 4x_5 + 5x_1x_4 - 5x_1x_2^2 - 5x_1x_4 \quad (22)$$

Signed permutation groups on five sites thus represent the simplest examples of the extension of the concept of chirality to permutation groups not

corresponding to three-dimensional point groups or framework groups. The two signed metacyclic groups M_5 and M_5' are derived from the same underlying unsigned group but with different assignments of positive and negative operations. They thus have the same cycle index (equation 2) but different signed cycle indices (equation 3). One of these metacyclic groups, M_5 , is ordinary and its chiral ligand partitions are the one-dimensional (1^5) partition of the symmetric group P_5 and the five-dimensional (32) ligand partition. The other metacyclic group, M_5' , is extraordinary with the anomalous signed cycle index terms $+5x_1x_4$ and $-5x_1x_2^2$ and has a single chiral ligand partition, namely the six-dimensional (31^2) . This latter metacyclic group is important in understanding the chirality polynomials of the single type of transitive skeleton having five sites, namely the polarized pentagon. In general extraordinary signed permutation groups appear to arise more frequently in chirality algebra than related ordinary signed permutation groups.

The only transitive signed permutation group on five sites which corresponds to a three-dimensional point-group is the polarized pentagon, framework group C_{5V} [$5 \sigma_V(L)$], isomorphic to the D_5 dihedral group and corresponding, for example, to the cyclopentadienyl-manganese tricarbonyl (cymantrene) skeleton. This skeleton has been treated in some detail both in my previous paper [8] and in an earlier paper by Dugundji, Marquarding, and Ugi [6] but also exhibits some other interesting features in the context of the present paper. The polarized pentagon has only one chiral ligand partition, namely the six-dimensional (31^2) partition also found in the extraordinary metacyclic group M_5' . However, the (31^2) chiral ligand partition appears twice in the polarized pentagon. This degeneracy is in accord with its chiral dimensionality of $5!/|C_{5V}| = 12$ and corresponds to two distinctive chiral isomers having the (31^2) ligand partition and depicted as follows:



Using the standard group theoretical algorithm [6,8] the following chirality polynomials of degree 3 can be obtained from isomers IV₁₂ and IV₁₃, respectively:

$$X(31^2)_{12}(C_{5V}) = (s_5-s_1)^3 + (s_1-s_2)^3 + (s_2-s_3)^3 + (s_3-s_4)^3 + (s_4-s_5)^3 \quad (23a)$$

$$X(31^2)_{13}(C_{5V}) = (s_5-s_2)^3 + (s_1-s_3)^3 + (s_2-s_4)^3 + (s_3-s_5)^3 + (s_4-s_1)^3 \quad (23b)$$

However, the polynomial $X(31^2)_{12}(C_{5V})$ (equation 23a) vanishes for the chiral isomer IV₁₃ and the polynomial $X(31^2)_{13}(C_{5V})$ (equation 23b) vanishes for the chiral isomer IV₁₂. Therefore neither of these chirality polynomials is qualitatively complete. Their sum is qualitatively complete and is equal to the degree 3 polynomial obtained by applying the standard group theoretical algorithm to either IV₁₂ and IV₁₃ using the extraordinary metacyclic group, M_5' , which is the signed permutation group generated by the ten symmetry permutations of one C_{5V} isomer and the x_1x_4 cycle index permutation necessary to convert one isomer into the other. The chirality polynomial of the extraordinary metacyclic group M_5' can thus be represented as the following sum:

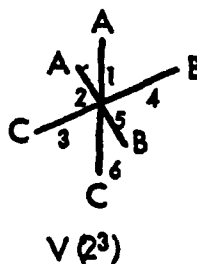
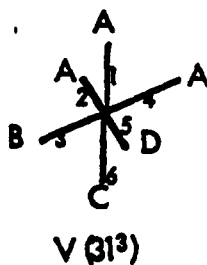
$$X(31^2)(M_5') = (31^2)(C_{5V})_{12} + (31^2)(C_{5V})_{13} \quad (24)$$

Some of these observations on transitive signed permutation groups on five sites are summarized in Table 3, particularly the transitive signed permutation groups (Table 2) having a given chiral ligand partition as a maximum symmetry chiral ligand partition by the Ruch-Schönhofer partial ordering [4,5].

7. Transitive Skeletons Having Six Sites

There are three transitive skeletons having six sites, the octahedron $O_h[3C_4(L_2)]$ having 48 symmetry operations and a chiral dimensionality of $6!/|O_h| = 720/48 = 15$ and the polarized hexagon $C_{6v}[3\sigma_v(L_2)]$ and cyclopropane skeleton $D_{3h}[3C_2(L_2)]$ (trigonal prism) having 12 symmetry operations and chiral dimensionalities of $720/12 = 60$. The permutation groups of the polarized hexagon and cyclopropane skeletons have the same cycle indices (equation 2) but differ in the assignments of positive and negative operations (Table 1) and therefore have different signed cycle indices (equation 3 and Table 2). All three of these skeletons are extraordinary and therefore cannot contain the fully asymmetric (1^6) chiral ligand partition. Other transitive permutation groups on six sites (Table 2) do not correspond to actual point or framework groups.

The octahedron, like the polarized pentagon discussed above, has two chiral ligand partitions of the same degree, namely the degree 6 ligand partitions (31^3) and (2^3) of dimensions 10 and 5, respectively, leading to the required chiral dimensionality of 15, and corresponding to the following two ligand distributions:



The degree 6 chirality polynomials for both of the chiral ligand partitions of the octahedron can be expressed in terms of the same three degree 3 polynomials $f(s_n)$, $g(s_n)$, and $h(s_n)$ as follows ($1 \leq n \leq 6$):

$$\chi(2^3) = f(s_n)[g(s_n) + h(s_n)] \quad (25a)$$

$$\chi(31^3) = f(s_n)[g(s_n) - h(s_n)] \quad (25b)$$

where

$$f(s_n) = (s_1 - s_6)(s_2 - s_5)(s_3 - s_4) \quad (26a)$$

$$g(s_n) = (s_1 - s_2)(s_2 - s_3)(s_3 - s_1) + (s_3 - s_6)(s_6 - s_5)(s_5 - s_3) + (s_1 - s_5)(s_5 - s_4)(s_4 - s_1) + (s_2 - s_4)(s_4 - s_6)(s_6 - s_2) \quad (26b)$$

$$h(s_n) = (s_1 - s_2 + s_6 - s_5)(s_1 - s_4 + s_6 - s_3)(s_2 - s_4 + s_5 - s_3) \quad (26c)$$

The degree 3 polynomials $f(s_n)$, $g(s_n)$, and $h(s_n)$ measure the effects of the separating [8] reflection planes (σ_h), the three-fold rotation axes (C_3), and the four-fold rotation axes (C_4), respectively, of the octahedron on the pseudoscalar properties evaluated by the chirality polynomial. Because of the same degrees of both chiral ligand partitions of the octahedron, a homogeneous qualitatively complete chirality polynomial $\chi_{QC}[O_h]$ can be expressed as follows:

$$\chi_{QC}[O_h] = f(s_n)[w_1 g(s_n) + w_2 h(s_n)] \quad (27)$$

In equation 29 the coefficients w_1 and w_2 are weighting coefficients that control

the proportions of $\chi(31^3)$ and $\chi(2^3)$ in the homogeneous $\chi_{QC}[O_h]$. Also the following relationships are found involving $f(s_n)$, $g(s_n)$, and $h(s_n)$ where $O_h^*(a^m b^n c^p \dots)$ schematically indicates the permutations of a monomial of the form $a^m b^n c^p \dots$ under the 48 operations of the octahedral group as when applying the group-theoretical algorithm [6,8]:

$$f(s_n)h(s_n) = -f(s_n)g(s_n) + f(s_n)p(s_n) \quad (28)$$

$$f(s_n)g(s_n) = O_h^*(a^3 b^2 c) + O_h^*(a^2 b^2 c d) \quad (29)$$

$$f(s_n)h(s_n) = -O_h^*(a^3 b^2 c) + O_h^*(a^2 b^2 c d) \quad (30)$$

$$f(s_n)p(s_n) = (2)O_h^*(a^2 b^2 c d) \quad (31)$$

This leads to the following results in accord with the full group-theoretical algorithm for the calculation of the chirality polynomials for the (2^3) and (31^3) chiral ligand partitions of the octahedron:

$$\chi(2^3) = f(s_n)g(s_n) + f(s_n)h(s_n) = (2)O_h^*(a^2 b^2 c d) \quad (32)$$

$$\chi(31^3) = f(s_n)g(s_n) - f(s_n)h(s_n) = (2)O_h^*(a^3 b^2 c) \quad (33)$$

The remaining two transitive skeletons having six sites have only 12 symmetry operations and therefore have chiral dimensionalities of 60. Because of these large chiral dimensionalities the sets of chiral ligand partitions for these skeletons are larger and more complicated than the examples discussed above. Understanding these systems is therefore facilitated by searching for larger

signed permutation groups of degree 6 for which the signed permutation groups of these transitive skeletons are signed subgroups. The anomalous terms of the signed permutation groups can serve as a guide for this purpose.

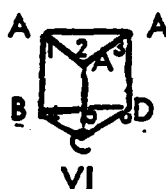
Consider first the skeleton represented by the framework group $D_{3h}[3C_2(L_2)]$ or the permutationally equivalent $D_{3d}[3\sigma_d(L_2)]$ corresponding, for example, to the cyclopropane skeleton as well as the trigonal prism or antiprism [8]. The anomalous terms in the signed cycle index of the corresponding signed permutation group are $+x_2^3$ and $-x_1^2x_2^2$ (Table 2). The largest signed permutation groups having both of these anomalous terms in their signed cycle indices are the octahedron O_h of order 48, chiral dimensionality $720/48 = 15$, and the chiral ligand partitions (31^3) and (2^3) (see above) and a group designated in Table 2 as $T9'$ of order 36, chiral dimensionality $720/36 = 20$, and the ten-dimensional chiral ligand partitions (31^3) and (41^2) . The trigonal antiprism or the permutationally equivalent trigonal prism arises by removal of the C_4 axis from the octahedron and is a signed subgroup of the octahedron. The $T9'$ signed permutation group of order 36 as a "twist group" derived from a trigonal prism by adding a three-fold twist operation T_3 of cycle structure $x_1^3x_3$ in which one of the parallel triangular faces of the trigonal prism is twisted by 120° while leaving the other triangular face fixed.

The trigonal prism signed permutation group $D_{3h}[3C_2(L_2)]$ contains no less than the six chiral ligand partitions $(2^3) + 2(31^3) + (321) + (41^2) + (42)$ in accord with its chiral dimensionality of 60. The chiral ligand partitions $(2^3) + (31^3)$ of degrees 6 relate to the generation of this signed permutation group by removal of the C_4 axis from the octahedron whereas the chiral ligand partitions (31^3) and (41^2) of degrees 6 and 3, respectively, relate to the generation of the $D_{3h}[3C_2(L_2)]$ signed permutation group from the twist group $T9'$ by removal of the T_3 twist operation defined above. Thus reduction of the symmetry to $D_{3h}[3C_2(L_2)]$ adds only two new chiral ligand partitions, namely (321) and (42) ;

the latter leads to the following lowest degree chirality polynomial of degree 2:

$$X(42)(D_{3h}) = (s_1-s_2)(s_4-s_6) - (s_1-s_3)(s_4-s_5) \quad (34)$$

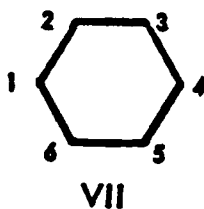
This chirality polynomial vanishes for the following chiral isomer (VI) having a (31^3) ligand partition indicating that it is not qualitatively complete:



This chiral isomer effectively has the twist operation T_3 discussed above as a "symmetry operation" since leaving the bottom BCD face fixed and rotating the top AAA face by 120° around the C_3 axis leads to a configuration indistinguishable from the original configuration. This indicates that a chirality polynomial derived from the twist group T_3' must be added to the polynomial 36 in order to obtain a qualitatively complete chirality function for the trigonal prism (cyclopropane) skeleton.

Now consider the polarized hexagon skeleton $C_{6v}[3\sigma_v(L_2)]$ which also has chiral dimensionality 60 and six chiral ligand partitions, namely $(2^21^2) + (31^3) + (321) + (3^2) + 2(41^2)$. In this case the anomalous signed cycle index terms are $+x_2^3$, $+x_6$, and $-x_1^2x_2^2$. The anomalous signed cycle index term $+x_6$ found in $C_{6v}[3\sigma_v(L_2)]$ prevents it from being a signed subgroup of the octahedron for which the x_6 signed cycle index term is not anomalous. However, all three of the anomalous signed cycle index terms in C_{6v} are also present in the twist group

T₉' (Table 2) and indeed the chiral ligand partitions of T₉', namely (31³) and (41²), are among those found in C_{6v}[3 σ_v(L₂)]. In addition the polarized hexagon is a signed subgroup of the extraordinary signed wreath product group T11" or P₃[P₂]" (Table 2) which has the chiral ligand partitions (41²) and (3²). The group T11" is derived from the same unsigned wreath product group P₃[P₂] as the octahedron but has a different assignment of positive and negative operations (Table 2). The reduction of the symmetry to C_{6v}[3 σ_v(L₂)] adds only two new chiral ligand partitions, namely (321) and (2²1²). The action of the T₃ twist operation of the twist group T₉' on a polarized hexagon can be visualized as leaving three alternate vertices (e.g., 1, 3, and 5 in VII below) fixed while rotating the other three vertices through a C₃ operation thereby leading to the required x₁³x₃ cycle index term for the twist operation. The polarized hexagon may also be obtained from the



extraordinary wreath product group P₃[P₂]" (T11") by removal of the period four operation x₁²x₄ (now a negative operation rather than a positive operation in the octahedron O_h or T11') and the x₁⁴x₂, x₁²x₂², and x₂x₄ operations generated from it. Thus the generation of the polarized hexagon from the larger groups T₉' and P₃[P₂]" is completely analogous to the generation of the trigonal prism D_{3h}[3C₂(L₂)] from the larger groups T₉' and O_h (T11').

Some of these observations on transitive signed permutation groups on six sites are summarized in Table 4, particularly the transitive signed permutation

groups (Table 2) having a given chiral ligand partition as a maximum symmetry chiral ligand partition.

8. Summary

This paper unifies the following ideas from diverse sources for the study of chirality polynomials in transitive skeletons:

- (1) The concept of chirality is generalized to permutation groups not corresponding to three-dimensional symmetry point groups by introducing the ideas of signed permutation groups, signed cycle indices, and signed subgroups (Section 2).
- (2) The Frobenius reciprocity theorem [29] provides a basis for determining the total dimension of the set of chiral ligand partitions (chiral dimensionality) thereby indicating the complexity of a qualitatively complete chirality polynomial (Section 3).
- (3) Signed permutation groups of which a given ligand partition is the maximum symmetry chiral ligand partition by the Ruch-Schönhofer partial ordering [4,5] allow the determination of corresponding chirality polynomials which are required to depend only upon differences between ligand parameters by Meinköhn's algebraic invariant theory [31,32] (Section 4). Such signed permutation groups need not correspond to actual three-dimensional point groups thereby making necessary the generalization of chirality noted above. These signed permutation groups relate to the permutations within non-racemic isomer mixtures for which qualitatively incomplete chirality polynomials of insufficient degree can vanish thereby indicating a higher effective symmetry of such systems. In all cases the actual symmetry point group of the skeleton must be a signed subgroup of any larger permutation groups indicating such higher effective symmetries.

These ideas are illustrated for the limited number of possible transitive

skeletons having four (tetrahedron, polarized square, allene, polarized rectangle), five (polarized pentagon), and six (octahedron, cyclopropane, polarized hexagon) sites. All skeletons having seven or more sites have such high chiral dimensionalities that study of their chiral ligand partitions by these methods appears intractably complicated.

Acknowledgment: I am indebted to the U.S. Office of Naval Research for partial support of this research. I am also indebted to two referees of an earlier version of this paper who made several extremely helpful suggestions which led to major improvements in the final version of this paper.

9. References

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TABLE 1

TRANSITIVE PERMUTATION GROUPS HAVING THREE TO SIX SITES
CORRESPONDING TO THREE-DIMENSIONAL POINT AND FRAMEWORK GROUPS

<u>Skeleton</u>	<u>Sites</u>	<u>Symmetry Operations</u>	<u>Point Group</u>	<u>Framework Group</u>	<u>Presentation</u>
Polarized triangle	3	6	C _{3v}	C _{3v} [3σ _v (L)]	a ² =b ² =(ab) ³ =1; a, b neg
Tetrahedron	4	24	T _d	T _d [4C ₃ (L)]	a ² =b ³ =(ab) ⁴ =1; a neg; b pos
Polarized square	4	8	C _{4v}	C _{4v} [2σ _v (L ₂)]	a ² =b ² (ab) ⁴ =1; a, b neg
Allene	4	8	D _{2d}	D _{2d} [2σ _d (L ₂)]	a ² =b ² =(ab) ⁴ =1; a neg; b pos
Polarized rectangle	4	4	C _{2v}	C _{2v} [X(L ₄)]	a ² =b ² =(ab) ² =1; a, b neg
Polarized pentagon	5	10	C _{5v}	C _{5v} [5σ _v (L)]	a ² =b ² =(ab) ⁵ =1; a, b neg
Octahedron	6	48	O _h	O _h [3C ₄ (L ₂)]	a ² =b ⁶ =(ab ²) ⁴ =(ab ³) ² =1; a pos; b neg
Polarized hexagon	6	12	C _{6v}	C _{6v} [3σ _v (L ₂)]	a ² =b ² =(ab) ⁶ =1; a, b neg
Cyclopropane	6	12	D _{3h}	D _{3h} [3C ₂ (L ₂)]	a ² =b ² =(ab) ⁶ =1; a neg; b pos

TABLE 2

TRANSITIVE SIGNED PERMUTATION GROUPS ON FIVE AND SIX SITES^a

Butler/ McKay Label ^b	Other Designation ^c	Order	Number of Classes	Signed Cycle Index	Chiral Ligand Partitions
<u>A) Five Sites</u>					
T5	P5	120	7	$x_1^5 + 15x_1x_2^2 + 20x_1^2x_3 + 24x_5$ $- 10x_1^3x_2 - 20x_2x_3 - 30x_1x_4$	(15)
T3	M5	20	5	$x_1^5 + 5x_1x_2^2 + 4x_5 - 10x_1x_4$	(15)+(32)
T3'	M5'	20	5	$x_1^5 + 4x_5 + 5x_1x_4 - 5x_1x_2^2 - 5x_1x_4$	(312)
T2	D5	10	4	$x_1^5 + 4x_5 - 5x_1x_2^2$	2(312)
<u>B) Six Sites</u>					
T16	P6	720	11	$x_1^6 + 45x_1^2x_2^2 + 40x_1^3x_3 + 40x_3^2 + 90x_2x_4^2 + 144x_1x_5$ $- 15x_1^4x_2 - 15x_2^3 - 120x_1x_2x_3 - 90x_1^2x_4 - 120x_6$	(16)
T14	PGL(2,5)	120	7	$x_1^6 + 15x_1^2x_2^2 + 20x_3^2 + 24x_1x_5$ $- 10x_2^3 - 30x_1^2x_4 - 20x_6$	(16)+(32)
T13	P ₂ [P ₃]	72	9	$x_1^6 + 4x_1^3x_3 + 4x_3^2 + 9x_1^2x_2^2 + 18x_2x_4$ $- 6x_1^4x_2 - 12x_1x_2x_3 - 6x_2^3 - 12x_6$	(16)+(212)
T13'	P ₂ [P ₃ ']	72	9	$x_1^6 + 4x_1^3x_3 + 4x_3^2 + 9x_1^2x_2^2 + 6x_2^3 + 12x_6$ $- 6x_1^4x_2 - 12x_1x_2x_3 - 18x_2x_4$	(214)+(23)
T13''	P ₂ [P ₃ ']	72	9	$x_1^6 + 4x_1^3x_3 + 4x_3^2 + 9x_1^2x_2^2 + 6x_1^4x_2 + 12x_1x_2x_3$ $- 6x_2^3 - 12x_6 - 18x_2x_4$	(32)+(51)
T11	P ₃ [P ₂]	48	10	$x_1^6 + 9x_1^2x_2^2 + 8x_3^2 + 6x_2x_4$ $- 3x_1^4x_2 - 7x_2^3 - 6x_1^2x_4 - 8x_6$	(16)+(2212)+(32)
T11'	Oh	48	10	$x_1^6 + 8x_3^2 + 3x_1^2x_2^2 + 6x_1^4x_4 + 6x_2^3$ $x_2^3 - 8x_6 - 3x_1^4x_2 - 6x_2x_4 - 6x_1^2x_2^2$	(313)+(23)

TABLE 2 (Continued)

Butler/ MacKay Label ^b	Other Designation ^c	Order	Number of Classes	Signed Cycle Index	Chiral Ligand Partitions
T11"	P ₃ [P ₂]"	48	10	$x_1^6 + 3x_1^4 x_2^2 + 3x_1^2 x_2^2 + 8x_6^2 + 8x_3^2 + x_2^3 - 6x_2^3 - 6x_1^2 x_2^2 - 6x_1^2 x_4^2 - 6x_2^2 x_4$	(41 ²) + (32)
T10'	P ₂ [P ₃]/C ₂	36	9	$x_1^6 + 9x_1^2 x_2^2 + 4x_1^3 x_3 + 4x_3^2 - 18x_2^2 x_4$	(21 ⁴) + (2 ³) + (32) + (51)
T9	P ₂ [P ₃]/C ₂	36	9	$x_1^6 + 9x_1^2 x_2^2 + 4x_1^3 x_3 + 4x_3^2 - 6x_2^3 - 12x_6$	(16) + (221 ²) + (32) + (51)
T9'	D _{3h} xC ₃	36	9	$x_1^6 + 4x_1 x_3 + 4x_3^2 + 3x_2^3 + 6x_6 - 9x_1^2 x_2^2 - 3x_2^3 - 6x_6$	(31 ³) + (41 ²)
T8	P ₃ [P ₂]/C ₂	24	5	$x_1^6 + 3x_1^2 x_2^2 + 8x_3^2 - 6x_2^3 - 6x_1^2 x_4$	(16) + (221 ²) + (41 ²) + 2(32)
T7'	O _h /C ₂	24	5	$x_1^6 + 3x_1^2 x_2^2 + 8x_3^2 - 6x_1^2 x_2^2 - 6x_2^2 x_4$	(31 ³) + (2 ³) + (41 ²) + (32)
T6		24	8	$x_1^6 + 3x_1^2 x_2^2 + 8x_3^2 - x_2^3 - 3x_1^4 x_2 - 8x_6$	(16) + (221 ²) + (31 ³) + (2 ³) + (32)
T5		18	9	$x_1^6 + 4x_1^3 x_3 + 4x_3^2 - 3x_2^3 - 6x_6$	(16) + (221 ²) + (31 ³) + (41 ²) + (32) + (51)
T3		12	6	$x_1^6 + 3x_1^2 x_2^2 + 2x_3^2 - 4x_2^3 - 2x_6$	(16) + 2(221 ²) + (321) + 2(32) + (41 ²) + (51)
T3'	D _{3h} or D _{3d}	12	6	$x_1^6 + 2x_3^3 + 3x_2^3 - x_2^3 - 3x_1^2 x_2^2 - 2x_6$	(2 ³) + 2(31 ³) + (321) + (41 ²) + (42)
T3"	C _{6v}	12	6	$x_1^6 + 2x_3^2 + x_2^3 + 2x_6 - 3x_1^2 x_2^2 - 3x_2^3$	(221 ²) + (31 ³) + (321) + (32) + 2(41 ²)

(a) This includes groups not corresponding to three-dimensional point or framework groups.

(b) See Butler, G., MacKay, J.: Comm. Algebra **11**, 863 (1983).

(c) P_n = symmetric group, M_n = metacyclic group, P_n[P_m] = wreath product group; different extraordinary signed permutation groups with the same unsigned cycle index are differentiated by primes.

TABLE 3

LIGAND PARTITIONS FOR FIVE SITES

<u>Ligand Partition</u>	<u>Chirality Polynomial Degree</u>	<u>Irreducible Representation Dimension</u>	<u>Signed Transitive Permutation Groups Having the Ligand Partition as a Max. Symmetry Chiral Ligand Partition^a</u>
(1 ⁵)	10	1	P ₅
(21 ³)	6	4	(tetrahedron + center)
(2 ² 1)	4	5	(trigonal bipyramid)
(31 ²)	3	6	M ₅ ', D ₅
(32)	2	5	M ₅
(41)	1	4	
(5)	0	1	

- (a) There is no transitive signed permutation group for which the (21³) or (2²1) ligand partition is the maximum symmetry chiral ligand partition corresponding to a minimum degree chirality polynomial. In both of these cases an intransitive permutation group is indicated in parentheses having these ligand partitions as the maximum symmetry chiral ligand partition.

TABLE 4

LIGAND PARTITIONS FOR SIX SITES

<u>Ligand Partition</u>	<u>Chirality Polynomial Degree</u>	<u>Irreducible Representation Dimension</u>	<u>Transitive Signed Permutation Groups Having the Ligand Partition as a Maximum Symmetry Chiral Ligand Partition</u>
(1 ⁶)	15	1	P ₆
(21 ⁴)	10	5	
(2 ² 1 ²)	7	9	P ₂ [P ₃]
(31 ³)	6	10	O _h
(2 ³)	6	5	O _h , P ₂ [P ₃] [']
(321)	4	16	
(41 ²)	3	10	P ₃ [P ₂] ^{''} , D _{3h} ×C ₃ , O _h /C ₂ , C _{6v}
(3 ²)	3	5	P ₃ [P ₂] ^{''} , O _h /C ₂ , C _{6v}
(42)	2	9	D _{3h} , D _{3d}
(51)	1	5	
(6)	0	1	

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