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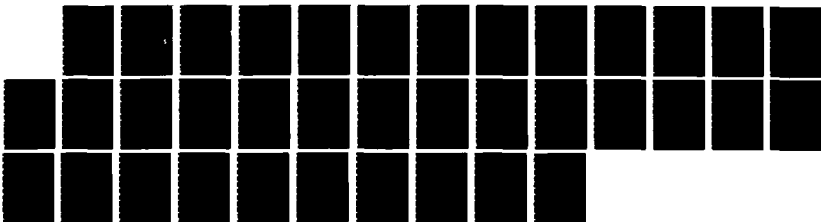
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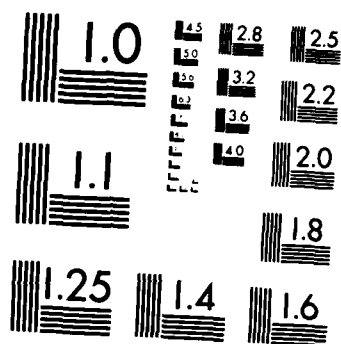
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CONSTRUCTING MARKOV PROCESSES WITH RANDOM TIMES OF BIRTH AND DEATH

by

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0. Introduction

Kuznetsov [11] (see also [12]) introduced a Kolmogorov-type construction in which he constructs a stationary measure Q_m from a transition semigroup $P_t(x, dy)$ and an excessive measure m . In fact, his theorem has other interesting consequences outside of the Markovian framework, but we do not discuss these here. While Kuznetsov's proof is "elementary", it is rather involved. The purpose of this paper is to give an alternate construction of Q_m in the case of right processes. We consider both the time homogeneous and time inhomogeneous cases. Our construction does not extend to cover the other interesting cases of Kuznetsov's theorem, but our approach may yield some insight into the measures Q_m and may aid the

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reader interested in recent articles [5,10] in which the measure Q_m has played an important role. Mitro [13] has obtained a result similar to ours under duality hypotheses on the underlying processes, but her construction is quite different from ours.

We must confront squarely the complexities of the subject soon, but first we try to introduce Q_m gently to the reader by discussing the example which motivated our investigation. Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ be a right process on a Lusin state space $(E, \mathcal{E})$ with semigroup P_t and resolvent U^q . Let m be an excessive measure for X , and assume that m is in fact a measure potential. That is, $m = \mu U$ for some positive measure μ . We can "easily" construct Q_m once we introduce the measurable space on which Q_m must sit. To do this, adjoin a "birth" point a and a "death" point b to E to obtain E_b^a . Let W be the set of all maps w from $\mathbb{R}$ to E_b^a so that there is a non-empty open interval $]\alpha(w), \beta(w)[$ on which w is E -valued and right continuous, $w(t) = a$ for $t < \alpha(w)$, and $w(t) = b$ for $t > \beta(w)$. Let $Y_t(w) = w(t)$, and let $\mathcal{X}^0 = \sigma\{Y_t: t \in \mathbb{R}\}$. For each $t \in \mathbb{R}$, define a map $\rho_t: \{\zeta > 0\} \rightarrow W$ by

$$[\rho_t(w)](s) = X_{s-t}(w) \text{ if } t < s \text{ and } s - t < \zeta(w)$$

$$= a \quad \text{if } t > s$$

$$= b \quad \text{if } s - t > \zeta(w).$$

Let Q^t be the image of the measure P^μ under the map ρ_t : note that Q^t is a measure on $(W, \mathcal{X}^0)$. Then $Q_m = \int_{\mathbb{R}} Q^t dt$.

It is simple to check that

$$(0.1) \quad Q_m(x_t \in dx; \alpha < t < \beta) = m(dx) \text{ for every } t \text{ in } R; \text{ and}$$

$$(0.2) \quad \text{if } t_1 < t_2 < \dots < t_n, \text{ then}$$

$$\begin{aligned} Q_m(\alpha < t_1, Y_{t_1} \in dx_1, \dots, Y_{t_n} \in dx_n, t_n < \beta) \\ = m(dx_1)P_{t_2-t_1}(x_1, dx_2) \dots P_{t_n-t_{n-1}}(x_{n-1}, dx_n). \end{aligned}$$

The key to this construction is the fact that m is a measure potential μU . In general, excessive measures are not measure potentials, but they can be decomposed into the sum of an invariant part m^i and a potential part m^p . The potential part can be represented as an integral of an entrance law (v_t) . This proves to be enough to imitate the steps above. The representation $m^p = \int_0^\infty v_t dt$ is well-known, but we do not know where a direct proof of it can be found in the generality we need. In [4], Dynkin derives it as a corollary to the representation of excessive measures in terms of minimal elements. Fitzsimmons and Maisonneuve [5] have a very nice proof using the existence of Q_m . It is proved for finite m in [9]. In section 1, we give a direct proof of the representation of m^p (1.4). The decomposition of m is summarized in Theorem (1.10). Section 2 contains a generalization of this representation for entrance rules for a time inhomogeneous transition operator P_t^S . The main result is Theorem (2.11). Section 3 contains the construction of Q_m . In fact, we proceed more generally and construct the measure corresponding to an entrance rule and

a time-inhomogeneous transition operator P_t^s .

We make the following suggestion to the reader interested only in the case of an excessive measure m and a (temporally homogeneous) right process with semigroup P_t . After reading section one, read the interpretation of the representation (1.10) given in the paragraph just below the statement of Theorem 2.33; in particular, the form (2.34) of (1.10). Then read section three with $P_t^s = P_{t-s}$ for $s < t$ and use (1.10) - that is, (2.34) - in place of (2.33) in the proof of Theorem 3.8.

We use what is essentially standard notation. Here are a few examples. Let E be a set and $\mathcal{X}$ a class of numerical functions on E . Then $b\mathcal{X}$ and $p\mathcal{X}$ denote the classes of bounded and positive functions in $\mathcal{X}$, respectively. If $(E, \mathcal{E})$ is a measurable space, then $\mathcal{E}$ is used to denote both the underlying σ -algebra and the class of all $\mathcal{E}$ -measurable numerical functions on E . Thus, for example, $bp\mathcal{E} = pb\mathcal{E}$ is the class of bounded, positive, measurable functions on E . Also $\mathcal{E}^*$ denotes the σ -algebra of universally measurable sets over $(E, \mathcal{E})$. If μ is a measure on $(E, \mathcal{E})$ and $h \in p\mathcal{E}$, then $h\mu$ or $h \cdot \mu$ denotes the measure $h(x)\mu(dx)$. If $(F, \mathcal{F})$ is another measurable space and ϕ is a measurable mapping from $(E, \mathcal{E})$ to $(F, \mathcal{F})$, then $\phi(\mu)$ is the image of μ on $(F, \mathcal{F})$; that is, $\phi(\mu)(A) = \mu[\phi^{-1}(A)]$ for $A \in \mathcal{F}$. As usual, $\mathbb{R}$ denotes the reals and $\mathcal{B}(\mathbb{R})$ is the σ -algebra of Borel subsets of $\mathbb{R}$. Similarly, $\mathbb{R}^+$ denotes the positive (i.e. non-negative) reals and $\mathcal{B}(\mathbb{R}^+)$ the Borel σ -algebra of $\mathbb{R}^+$.



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1. Excessive Measures of Right Processes.

Fix a U-space $(E, \mathcal{E})$ (i.e. E is homeomorphic to a universally measurable subset of a compact metric space), and let $X = (0, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ be a right process on E as described in [6]. Let (P_t) and (U^q) denote the semigroup and resolvent of X , respectively.

(1.1) DEFINITION. A σ -finite measure m on $(E, \mathcal{E})$ is said to be excessive for X (or P_t or U^q) if $mP_t < m$ for every $t > 0$. (Here, mP_t is the measure defined by $mP_t(f) = m(P_t f)$).

It is well known that an excessive measure m also has the property that $mP_t(f)$ increases to $m(f)$ as t decreases to zero for every $f \in p\mathcal{E}$; e.g. see ([8], (1.4)).

(1.2) DEFINITION. An entrance law for X (or P_t or U^q) is a family of σ -finite measures $(v_t)_{t>0}$ on $(E, \mathcal{E})$ so that $v_t P_s = v_{t+s}$ for every $t > 0$ and $s > 0$.

Note that $t \mapsto v_t(f)$ is $\mathcal{B}(R^{++})$ -measurable if $f \in p\mathcal{E}$.

The main result of this section is the theorem below connecting an excessive measure m with an entrance law v_t . But first, we introduce the following useful convention.

(1.3) NOTATION. Let $(m_t)_{t \in R}$ be a collection of σ -finite measures on $(E, \mathcal{E})$ with $m_t > m_{t+s}$ for every $s > 0$ and for every $t \in R$. Then there exists a unique σ -finite measure π on $(E, \mathcal{E})$ so that whenever $f \in p\mathcal{E}$ with $m_s(f) < \infty$ for some

$s \in \mathbb{R}$, one has $\pi(f) = \lim_{t \rightarrow \infty} m_t(f)$. We write $\pi = \lim_{t \rightarrow \infty} m_t$. Also, if μ and ν are σ -finite measures on $(E, \mathcal{E})$ with $\mu < \nu$, then there exists a unique σ -finite measure λ with $\mu + \lambda = \nu$. We write $\lambda = \nu - \mu$.

(1.4) THEOREM. Let m be an excessive measure for X so that $\lim_{t \rightarrow \infty} m P_t = 0$. Then there is a unique entrance law $(v_t)_{t>0}$ so that $m = \int_0^\infty v_t dt$.

PROOF. Choose $f \in b\mathcal{E}$ with $f > 0$ and $m(f) < \infty$. Since m is excessive, $m(U^q f) < \infty$ for every $q > 0$. Let A_t be the increasing function on $]0, \infty[$ defined by setting

$$A_t = -m P_t U^q f = -e^{qt} \int_t^\infty e^{-qu} m P_u(f) du.$$

Since A_t is the product of two locally bounded absolutely continuous functions, A_t is absolutely continuous on bounded intervals. Therefore, dA_t is a positive measure on $]0, \infty[$ which is absolutely continuous with respect to Lebesgue measure. If $g \in p\mathcal{E}$, define the increasing function $A_t(g)$ on $]0, \infty[$ by setting $A_t(g) = -m P_t(g)$. Note that if $m(g) < \infty$, then

$$\int_0^\infty dA_t(g) = \lim_{r \rightarrow 0} m P_r(g) - \lim_{r \rightarrow \infty} m P_r(g) = m(g).$$

In what follows, $0 < c < \infty$, and c may change from line to line. If $0 < g < c U^q f$, then $c A_t = A_t(g) + A_t(c U^q - g)$. Since both functions on the right side of this last equality are increasing, it follows that $dA_t(g) \ll dt$. Consequently if $g \in \mathcal{E}$ and $|g| < c U^q f$,

$dA_t(g)$ defines a signed measure on $]0, \infty[$ that is absolutely continuous with respect to Lebesgue measure, where $A_t(g) = -mP_t(g)$ for such g . Let $dA_t(g) = H_t(g)dt$ with g as above and with $H_t(g)$ being a finite Borel measurable version of the density. Set $E_0 = \{U^q f > 1\}$ and $E_n = \{(n+1)^{-1} < U^q f < n^{-1}\}$ for $n > 1$. Since $f > 0$, $E = \bigcup_{n>0} E_n$. If $g \in b\mathcal{E}$, then $|gl_{E_n}| < cU^q f$ and so $dA_t(gl_{E_n}) = H_t(gl_{E_n})dt$. Let $(g_k) \subset p\mathcal{E}$ be a sequence increasing to $g \in b\mathcal{E}$. Then $H_t[(g - g_k)l_{E_n}] < H_t(gl_{E_n})$ and $\int_0^\infty H_t(gl_{E_n})dt = m(gl_{E_n}) < m(cU^q f) < \infty$. Hence we may apply the dominated convergence theorem to conclude

$$\begin{aligned} \lim_k \int_0^\infty H_t[(g - g_k)l_{E_n}]dt &= \lim_k \int_0^\infty H_t[(g - g_k)l_{E_n}]dt \\ &= \lim_k m[(g - g_k)l_{E_n}] = 0. \end{aligned}$$

This shows that $H_t(g_k l_{E_n})$ increases to $H_t(gl_{E_n})$ a.e. (dt) as k approaches infinity. By a standard result on regularizing pseudo-kernels ([7], (4.5)) or ([2], IX-11 and 13) there exists a bounded kernel $\mu_t^n(\cdot)$ from $(R^+, \mathcal{B}(R^+))$ to $(E, \mathcal{E})$ that is carried by E_n so that $H_t(gl_{E_n}) = \mu_t^n(g)$ a.e. (dt) for $g \in b\mathcal{E}$. Set

$$\mu_t = \sum_{n>0} \mu_t^n.$$

Then μ_t is a kernel from $(R^+, \mathcal{B}(R^+))$ to $(E, \mathcal{E})$. Let $g \in pb\mathcal{E}$ with $m(g) < \infty$ and let $g_n = \sum_{k=0}^n gl_{E_k}$. Then

$$\int_0^\infty dA_t(g - g_n) = m(g - g_n) \rightarrow 0$$

as $n \rightarrow \infty$, and so the positive measures $dA_t(g_n)$ increase to $dA_t(g)$. Consequently

$$dA_t(g) = \mu_t(g)dt$$

for $g \in pb\mathcal{E}$ with $m(g) < \infty$ and then with one more passage to a limit for $g \in pL^1(m)$. Moreover $m = \int_0^\infty \mu_t dt$. Since on $]0, \infty[$, $\mu_{t+s}(g)dt = -d_t m(P_{t+s}g) = dA_t(P_s g) = \mu_t(P_s g)dt$ and $\mu_t(g) < \infty$ a.e. (dt) for $g \in pL^1(m)$, we have $\mu_{t+s}(g) = \mu_t(P_s g)$ a.e. (dt). But $\mathcal{E}$ is countably generated since $(E, \mathcal{E})$ is a U-space, and so for each $s > 0$

$$(1.5) \quad \mu_{t+s} = \mu_t P_s \text{ a.e. (dt).}$$

Thus (μ_t) is a "crude" version of the desired entrance law which we shall obtain by "regularizing" (μ_t) as follows. Let L be Lebesgue measure on $\mathbb{R}^+$. Applying Fubini's theorem to (1.5), we obtain

$$\iint 1_{\{\mu_{t+s} \neq \mu_t P_s\}} ds dt = 0.$$

That is, there is a set $\Gamma \subset \mathbb{R}^+$ with $L(\Gamma^c) = 0$ so that for each t in Γ , there is another set $\Lambda(t) = \Lambda_t \subset \mathbb{R}^+$ with $L(\Lambda_t^c) = 0$ and $\mu_{t+s} = \mu_t P_s$ for every s in Λ_t . Choose a sequence $(t_n) \subset \Gamma$ decreasing to zero so that $\mu_{t(n)}(U^q f) < \infty$ for each n . (This can be done since $\mu_t(U^q f) < \infty$ a.e.). For $s > 0$, define

$$(1.6) \quad \nu_{t_n+s}^n = \mu_{t_n} P_s.$$

Note that

$$(1.7) \quad v_{t_n+s}^n = \mu_{t_n+s} \text{ for every } s \in \Lambda(t_n),$$

and

$$(1.8) \quad v_{t_{n+1}+(t_n-t_{n+1})+s}^{n+1} = \mu_{t_{n+1}}^P P_{t_n-t_{n+1}+s}.$$

But whenever $t_n - t_{n+1} + s$ is in $\Lambda(t_{n+1})$,

$$\mu_{t_{n+1}}^P P_{t_n-t_{n+1}+s} = \mu_{t_n+s}.$$

Since $L(\Lambda(t_n)^c \cup \Lambda(t_{n+1})^c) = 0$, we have

$$\mu_{t_n}^P s = \mu_{t_{n+1}}^P P_{t_n-t_{n+1}+s} \text{ a.e. } (ds).$$

In particular, whenever $0 < g < f$,

$$\mu_{t_n}^P s U^g = \mu_{t_{n+1}}^P P_{t_n-t_{n+1}+s} U^g \text{ a.e. } (ds).$$

Since each side is finite and right continuous in s , they agree for all s ; that is,

$$\mu_{t_n}^P s U^g = \mu_{t_{n+1}}^P P_{t_n-t_{n+1}+s} U^g.$$

By the uniqueness theorem for potentials ([8], (1.1)), this implies

$$\mu_{t_n}^P s = \mu_{t_{n+1}}^P P_{t_n-t_{n+1}+s}.$$

Thus, for every $s > 0$, using (1.6) and (1.8),

$$v_{t_n+s}^n = v_{t_{n+1}+(t_n-t_{n+1})+s}^{n+1}$$

If $t > t_n$,

$$(1.9) \quad v_{t_n+(t-t_n)}^n = v_{t_{n+1}+(t_n-t_{n+1})+(t-t_n)}^{n+1} = v_{t_{n+1}+(t-t_{n+1})}^{n+1}$$

Define

$$v_t = \lim_{n \rightarrow \infty} v_{t_n+(t-t_n)}^n$$

If $t > t_n$ and if $s > 0$, then (using (1.6)) $v_t^n P_s = v_{t+s}^n$. It follows that (v_t) is an entrance law. By (1.7) and (1.9), $v_t = \mu_t$ a.e., so $m = \int_0^\infty v_t dt$.

To prove v_t is unique, let γ_t be another entrance law with $m = \int \gamma_t dt$. Then

$$mP_s = \int_0^\infty \gamma_t P_s dt = \int_0^\infty \gamma_{t+s} dt = \int_0^\infty \gamma_s P_t dt = \gamma_s U.$$

Similarly, $mP_s = v_s U$. Since $mP_s < m$, and m is σ -finite, $\gamma_s = v_s$ by ([8], (1.1)). Q.E.D.

We can now give the representation of an excessive measure which was mentioned in the introduction.

(1.10) THEOREM. Let m be an excessive measure for X . There is a unique invariant measure m^i (i.e. $m^i P_t = m^i$ for every $t > 0$) and a unique entrance law v_t so that

$$m = m^i + \int_0^\infty v_t dt.$$

This is an immediate consequence of Theorem 1.4 since it is well-known and easy to check that $m^P = \lim_{t \rightarrow \infty} m^P_t$ defines an excessive measure satisfying the hypothesis of (1.4) and that $m^i = m - m^P$ defines an invariant measure. Also see the discussion following the statement of Theorem 2.33.

2. Representing an entrance rule.

Fix a U-space $(E, \mathcal{E})$. For each s and t in R with $s < t$ and for each x in E , let $P(s, x; t, dy)$ be a sub-probability measure on $(E, \mathcal{E})$. For each $s < t$, define

$$P_t^s f(x) = \int P(s, x; t, dy) f(y)$$

whenever $f \in b\mathcal{E}$; P_t^s is called a transition operator if (2.1), (2.2), and (2.3) are satisfied:

$$(2.1) \quad (s, t, x) \mapsto P_t^s f(x) \text{ is } \mathcal{B}(R) \times \mathcal{B}(R) \times \mathcal{E}$$

measurable for each $f \in b\mathcal{E}$.

$$(2.2) \quad P_t^s (P_u^t f) = P_u^s f \text{ whenever } s < t < u \text{ and } f \in b\mathcal{E}.$$

$$(2.3) \quad \text{For each } s \in R, P_t^s 1 \text{ increases to } 1 \\ \text{as } t \text{ decreases to } s.$$

We also need a type of "right" hypothesis.

(2.4) DEFINITION. A function $h_t(x)$ is called an exit rule if $h_t \in \mathcal{P}^*$ for every t and if, for each s , $P_t^s h_t$ increases to h_s as t decreases to s .

REMARKS. (i) The argument in Dynkin ([3], Lemma (5.1)) shows that a finite exit rule must be $\mathcal{B}(R) \times \mathcal{E}^*$ measurable. Standard arguments show that any entrance rule is an increasing limit of bounded entrance rules, so every entrance rule must be $\mathcal{B}(R) \times \mathcal{E}^*$ -measurable.

(ii) Note that h_t is an exit rule provided the function $H(t, x) = h_t(x)$ is excessive for the homogenous space-time semigroup

$$T_t((r, x); ds \times dy) = \varepsilon_{r+t}(ds)P(r, x; t + r, dy).$$

The last condition we assume for P_t^s is the following.

(2.5) For every bounded continuous function f on E and every bounded exit rule h_t , $\lim_{t \downarrow s} P_t^s(fh_t) = fh_s$.

If T_t is a right semigroup of a right process [6], then (2.5) is satisfied. In particular, if $P_t^s = P_{t-s}$ is a time-homogeneous right semigroup of a right process, then (2.5) holds.

(2.6) NOTATION. If ν is a σ -finite measure on $(E, \mathcal{E})$, then νP_t^s denotes the measure defined by $\nu P_t^s(f) = \nu(P_t^s f)$.

(2.7) DEFINITION. (i) An entrance rule for (P_t^s) is a family of σ -finite measures $(\nu_t)_{t \in R}$ on $(E, \mathcal{E})$ so that for

each t in $\mathbb{R}$, $v_s P_t^s$ increases to v_t as s increases to t .

(ii) Let $-\infty < s < \infty$. An entrance law at s is an entrance rule (v_t) so that $v_t = 0$ if $t < s$ and $v_t P_u^t = v_u$ whenever $s < t < u$.

(2.8) REMARK. One may apply an argument similar to that used in Dynkin ([3], Lemma (5.1)) to show that whenever (v_t) is an entrance rule $t \rightarrow v_t(f)$ is $\mathcal{B}(\mathbb{R})$ -measurable for $f \in p\mathcal{E}$. This result may also be obtained as a corollary of Theorem (2.33) below.

(2.9) LEMMA. Let (v_t) be an entrance rule for (P_t^s) . For each $t \in \mathbb{R}$, there is a function $f_t(x)$ on E so that $0 < f_t(x) < 1$, $v_t(f_t) < 1$ and $(t, x) \rightarrow f_t(x)$ is $\mathcal{B}(\mathbb{R}) \times \mathcal{E}$ measurable.

PROOF. For each t , v_t is σ -finite, and we may choose an $\mathcal{E}$ -measurable function k_t with $0 < k_t < 1$ and $v_t(k_t) < 1$. For each rational number r , choose $a_r > 0$ so that $\sum_{r \in \mathbb{Q}} a_r < 1$, and define

$$(2.10) \quad f_t(x) = \sum_{r > t, r \in \mathbb{Q}} a_r P_r^t k_r(x)$$

By (2.1), $(t, x) \rightarrow f_t(x)$ is $\mathcal{B}(\mathbb{R}) \times \mathcal{E}$ -measurable, and $f_t(x) < \sum_{r \in \mathbb{Q}} a_r < 1$. Moreover,

$$v_t(f_t) = \sum_{r > t} a_r v_t P_r^t k_r < \sum_{r > t} a_r v_r(k_r) < 1$$

since $v_r(k_r) < 1$. Recalling (2.3) and the fact that $k_r > 0$, we see that $P_r^t k_r(x) > 0$ for some rational $r > t$.

Therefore $f_t(x) > 0$.

Q.E.D.

(2.11) THEOREM. Let (v_t) be an entrance rule for (P_t^s) so that $\lim_{s \rightarrow -\infty} v_s P_t^s = 0$ for each $t \in R$. Then there is a finite measure ϕ on $(R, \mathcal{B}(R))$ and a collection of measures $(v_t^s)_{s, t \in R}$ so that

(2.12) for each $s \in R$, $v^s \equiv (v_t^s)_{t \in R}$ is an entrance law at s for (P_t^r) ;

(2.13) for each $f \in p\mathcal{E}$, $(s, t) \rightarrow v_t^s(f)$ is

$\mathcal{B}(R) \times \mathcal{B}(R)$ - measurable;

(2.14) for each $t \in R$, $v_t = \int_R v_t^s \phi(ds)$.

In addition, there is a strictly positive function $g_t(x)$ in $\mathcal{B}(R) \times \mathcal{E}$ so that $v_t^s(g_t) < \infty$ for every t and s .

PROOF. Step 1: Reducing the Problem.

For $s < t$, define $Q_t^s = e^{-(t-s)} P_t^s$, and set $\mu_t = e^{-t} v_t$. One can easily check that Q_t^s is a transition operator and (μ_t) is an entrance rule for (Q_t^s) with $\lim_{s \rightarrow -\infty} \mu_s Q_t^s = 0$. Thus, (μ_s) and (Q_t^s) satisfy the hypotheses of the theorem and have the following extra property:

$$(2.15) \quad \int_s^\infty Q_t^s 1 \, dt < \int_s^\infty e^{-(t-s)} \, dt = 1.$$

We now observe that it suffices to prove the theorem for (μ_s) and (Q_t^s) . For suppose we can produce a family of

entrance laws $(\mu^s) \equiv (\mu_t^s)$ for (Q_t^s) and a measure ϕ so that $\mu_t^s = \int \mu_t^s \phi(ds)$. Set $v_t^s = e^t \mu_t^s$. Then $v_t = e^t \mu_t = \int v_t^s \phi(ds)$. If $s < t < u$, then $v_t^s P_u^t = e^t \mu_t^s e^{u-t} Q_u^t = e^u \mu_u^s = v_u^s$. We now devote our attention to proving the theorem for (μ_s) and (Q_t^s) .

Let (f_t) be the functions described in (2.9) relative to (P_t^s) , and define

$$(2.16) \quad g_s = \int_s^\infty Q_t^s f_t dt.$$

Then $g_s > 0$, $(s, x) \rightarrow g_s(x)$ is $\mathcal{B}(R) \times \mathcal{E}$ -measurable, and, since $f_t < 1$,

$$g_s < \int_s^\infty Q_t^s 1 dt < 1.$$

Also, since $v_t(f_t) < 1$,

$$(2.17) \quad \mu_s(g_s) = \int_s^\infty \mu_s Q_t^s f_t dt < \int_s^\infty e^{-t} v_t(f_t) dt < e^{-s} < \infty.$$

$$(2.18) \quad \mu_t(f_t) < e^{-t}.$$

If $s < t$, then

$$(2.19) \quad Q_t^s g_t = Q_t^s \int_t^\infty Q_u^t f_u du = \int_t^\infty Q_u^s f_u du < g_s,$$

and as t decreases to s , $Q_t^s g_t$ increases to g_s .

Consequently, g_t is an exit rule for (Q_t^s) . If one defines $\tilde{Q}_t^s(x, dy) = g_s(x)^{-1} Q_t^s(x, dy) g_t(y)$ and $\tilde{\mu}_t = g_t \mu_t$, then $\tilde{Q}_t^s$ is a transition operator and $\tilde{\mu} = (\tilde{\mu}_t)$ is an entrance rule for $(\tilde{Q}_t^s)$ which satisfies $\tilde{\mu}_t(1) < \infty$ for all t . This additional

reduction does not seem to be particularly useful in our construction, and so we shall not use it.

Step 2: Constructing ϕ .

For each t in $\mathbb{R}$ and $f \in \mathcal{P}$, define an increasing function on $]-\infty, t[$ by setting

$$(2.20) \quad A(t, f; s) = \mu_s Q_t^s(f)$$

Note that $\lim_{s \uparrow t} A(t, f; s) = \mu_t(f)$ and $\lim_{s \rightarrow -\infty} A(t, f; s) = 0$, provided $\mu_t(f) < \infty$. If $\mu_t(f) < \infty$, then the increasing function $A(t, f; s)$ is the distribution function of the measure $\mathcal{A}(t, f)(ds) = d_s A(t, f; s)$ on $]-\infty, t[$. Note that $\mathcal{A}(t, f)(\mathbb{R}) = \mu_t(f)$. If $0 < f < g_t$, then $\mathcal{A}(t, g_t) = \mathcal{A}(t, f) + \mathcal{A}(t, g_t - f)$, so $\mathcal{A}(t, f) \ll \mathcal{A}(t, g_t)$. We let $\phi_t = \mathcal{A}(t, g_t)$, and we observe that ϕ_t is a finite measure on $]-\infty, t[$ since $\mu_t(g_t) < e^{-t}$.

Note that $\mathcal{A}(u, f_u)(\mathbb{R}) = \mu_u(f_u) < e^{-u}$ by (2.18) and that $u \mapsto \mathcal{A}(u, f_u)(h)$ is $\mathcal{B}(\mathbb{R})$ -measurable whenever $h \in \mathcal{P}$. Set

$$(2.21) \quad \phi = \int_{-\infty}^{\infty} \mathcal{A}(u, f_u) du.$$

Since $\mathcal{A}(u, f_u)$ is carried by $]-\infty, u[$,

$$\begin{aligned} \phi([t, \infty[) &= \int_t^{\infty} \mathcal{A}(u, f_u)([t, \infty[) du \\ &< \int_t^{\infty} \mathcal{A}(u, f_u)(\mathbb{R}) du < e^{-t}. \end{aligned}$$

Therefore, ϕ is a Radon measure. (We shall observe at the end of the proof that ϕ can be replaced with a finite measure as promised.) From the definitions of g_t and ϕ_t , we have

$$\phi_t = (\int_t^\infty \mathcal{A}(u, f_u) du) 1_{]-\infty, t[} \ll \phi.$$

Therefore, there is a function $\rho_t(s)$ so that $\phi_t(ds) = \rho_t(s)\phi(ds)$ with $0 < \rho_t(s) < 1$ and $\rho_t(s) = 0$ if $t < s$. Since $t \mapsto \mu_s^Q g_t^s$ is Borel measurable on $]s, \infty[$, ρ_t may be chosen jointly measurable in (s, t) ([2], V.T.58).

Step 3: Disintegrating $\mathcal{A}(t, f)$.

From Step 2 and ([2], V-T.58), we know there is a density $\alpha(t, f; s)$ which is jointly measurable in (s, t) , so that

$$\mathcal{A}(t, f)(ds) = \alpha(t, f; s)\phi_t(ds)$$

whenever $0 < f < cg_t$. (Here c is any positive constant.)

If $|f| < cg_t$, then $\mathcal{A}(t, f)(ds) = d_{\mu_s^Q} f$ defines a signed measure which is absolutely continuous with respect to ϕ_t .

Hence $\mathcal{A}(t, f)(ds) = \alpha(t, f; s)\phi_t(ds)$ whenever $|f| < cg_t$.

(The rest of this paragraph is a kernel construction analogous to the one in the first paragraph of (1.4).) Let $E_0 = \{g_t > 1\}$ and $E_n = \{(n+1)^{-1} < g_t < n^{-1}\}$. If $f \in \mathcal{P}$, then

$$\mathcal{A}(t, f 1_{E_n})(ds) = \alpha(t, f 1_{E_n}; s)\phi_t(ds)$$

since $|f| < (n+1)g_t$ on E_n . Suppose $(h_k) \subset b\mathcal{E}$ is a sequence of functions decreasing to zero. Then if $D \in \mathcal{B}(R)$,

$$\mathcal{A}(t, h_k 1_{E_n})(D) < \mathcal{A}(t, h_k 1_{E_n})(R) = \mu_t(h_k 1_{E_n}),$$

and this last term goes to zero as k increases to infinity since $h_k < |h_1|_\infty$ and $\mu_t(E_n) < (n+1)\mu_t(g_t) < \infty$. Consequently, if $(h_k) \subset b\mathcal{E}$ is a sequence of positive functions increasing to h , then $\alpha(t, h_k 1_{E(n)}; \cdot)$ increases to $\alpha(t, h 1_{E(n)}; \cdot)$ a.e. (ϕ_t) . By a standard result on kernels ([7], (4.5)), there exists a bounded kernel $K_t^n(s, dx)$ from $(]-\infty, t[, \mathcal{B}(]-\infty, t[))$ to $(E, \mathcal{E})$ which is carried by E_n so that for each $f \in b\mathcal{E}$, (i) $(s, t) \mapsto K_t^n(s, f)$ is $\mathcal{B}(R) \times \mathcal{B}(R)$ -measurable, and (ii) $\alpha(t, f 1_{E(n)}; \cdot) = K_t^n(\cdot, f)$ a.e. (ϕ_t) . Set

$$K_t(s, f) = \sum_{n \geq 0} K_t^n(s, f)$$

Let $f \in b\mathcal{E}$ with $\mu_t(f) < \infty$, and let

$$h_n = \sum_{k=0}^n 1_{E_k} f.$$

Then $\mathcal{A}(t, f - h_n)(R) < \mu_t(f) < \infty$, and it follows that $\alpha(t, h_n; \cdot)$ increases to $\alpha(t, f; \cdot)$ a.e. (ϕ_t) . Hence

$$\mathcal{A}(t, f)(ds) = K_t(s, f) \phi_t(ds)$$

whenever $\mu_t(f) < \infty$.

If $s < t < u$,

$$\Lambda(u, f; s) = \mu_s Q_u^s f = \mu_s Q_t^s Q_u^t f = \Lambda(t, Q_u^t f; s),$$

so $\mathcal{A}(u, f) = \mathcal{A}(t, Q_u^t f)$ on $] -\infty, t[$. This implies

$$K_u(s, f) \phi_u(ds) = K_t(s, Q_u^t f) \phi_t(ds) \text{ on }] -\infty, t[\text{ or}$$

$$\rho_u(s) K_u(s, f) \phi(ds) = \rho_t(s) K_t(s, Q_u^t f) \phi(ds) \text{ on }] -\infty, t[.$$

Since ϕ is a Radon measure, we conclude

$$(2.22) \quad \rho_u(s) K_u(s, f) = \rho_t(s) K_t(s, Q_u^t f) \text{ a.e. } (\phi) \text{ in } s \\ \text{on }] -\infty, t[.$$

Set.

$$(2.23) \quad \lambda_t^s(f) = \rho_t(s) K_t(s, f)$$

Then $(s, t) \rightarrow \lambda_t^s(f)$ is $\mathcal{B}(R) \times \mathcal{B}(R)$ -measurable for every $f \in p\mathcal{E}$, and $\lambda_t^s = 0$ if $t < s$. Now (2.22) implies that for $t < u$, $\lambda_u^s(f) = \lambda_t^s(Q_u^t f)$ a.e. (ϕ) in s on $] -\infty, t[$. Since λ_u^s and λ_t^s are σ -finite measures, we have that if $t < u$,

$$(2.24) \quad \lambda_u^s = \lambda_t^s Q_u^t \text{ a.e. } (\phi) \text{ in } s \text{ on }] -\infty, t[.$$

Also,

$$\begin{aligned} \int \lambda_t^s \phi(ds) &= \int] -\infty, t[\rho_t(s) K_t(s, \cdot) \phi(ds) \\ &= \int] -\infty, t[K_t(s, \cdot) \phi_t(ds) = \mathcal{A}(t, \cdot)(R) = \mu_t(\cdot). \end{aligned}$$

So (λ_t^S) is a "crude" version of the desired family (μ_t^S) .

Step 4: Regularizing (λ_t^S) .

By the results above and (2.17),

$$\int \lambda_t^S(g_t) \phi(ds) = \mu_t(g_t) < e^{-t}.$$

Thus, for each t , $\lambda_t^S(g_t) < \infty$ a.e. (ϕ) . In particular, if L denotes Lebesgue measure on $\mathbb{R}$, and if

$$\Pi = \{(r, u) : r < u, \lambda_u^r(g_u) < \infty, \lambda_t^r = \lambda_{u-r}^r\}$$

a.e. L in t on (u, ∞) ,

then by (2.24),

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Pi^C(r, u) 1_{\{u > r\}} \phi(dr) L(du) = 0.$$

By Fubini's theorem and the change of variables $u \rightarrow u + r$, we may rewrite this as

$$\int_{-\infty}^{\infty} \int_0^{\infty} \Pi^C(r, u + r) L(du) \phi(dr) = 0.$$

Thus there is a set $\Gamma \subset \mathbb{R}$ with $\phi(\Gamma^C) = 0$ and so that if $r \in \Gamma$, then

$$(2.25) \quad \lambda_{u+r}^r(g_{u+r}) < \infty \text{ a.e. } (L) \text{ on } \{u > 0\}$$

$$(2.26) \quad \lambda_t^r = \lambda_{u+r}^r Q_t^{u+r} \text{ a.e. } (L \times L) \text{ on}$$

$$\{(t, u) : u > 0, t > u + r\}.$$

From (2.25) and (2.26), we know there is a sequence (u_n) decreasing to zero so that

$$(2.27) \quad \lambda_t^r = \lambda_{u_n+r}^r Q_t^{u_n+r} \text{ a.e. } (L) \text{ on } \{t : t > u_n + r\}$$

$$(2.28) \quad \lambda_{u_n+r}^r(g_{u_n+r}) < \infty.$$

Set $s_n = s(n) = u_n + r$, and note that s_n depends measurably on r . For each n , (2.27) implies there is a set $\Lambda_n \subset]s_n, \infty[$ of full Lebesgue measure so that

$$(2.29) \quad \lambda_t^r = \lambda_{s_n}^r Q_t^{s_n} \text{ for all } t \in \Lambda_n.$$

Define

$$(2.30) \quad \gamma_t^n = \lambda_{s_n}^r Q_t^{s_n} \text{ for } t > s_n.$$

Note that $\gamma_t^n = \lambda_t^r$ for every t in Λ_n . If $t > s_n > s_{n+1}$

$$\gamma_t^{n+1} = \lambda_{s_{n+1}}^r Q_t^{s_{n+1}}.$$

But if $t \in \Lambda_n \cap \Lambda_{n+1}$, then

$$(2.31) \quad \gamma_t^n = \lambda_{s_n}^r Q_t^{s_n} = \lambda_t^r = \lambda_{s_{n+1}}^r Q_t^{s_{n+1}}.$$

Since g_t is an exit rule for Q_t^S , it follows from (2.5) that $Q_t^{S_n}(fg_t)$ and $Q_t^{S_{n+1}}(fg_t)$ are right continuous in t on $]s_n, \infty[$ whenever f is bounded and continuous. Since $L((\Lambda_n^C \cup \Lambda_{n+1}^C) \cap]s_n, \infty[) = 0$, we conclude that

$$\lambda_{s_n}^r Q_t^{S_n} = \lambda_{s_{n+1}}^r Q_t^{S_{n+1}} \text{ for every } t > s_n > s_{n+1}.$$

Thus for each $t > r \in \Gamma$, the limit

$$(2.32) \quad \mu_t^r = \lim_{n \rightarrow \infty} \lambda_{s_n}^r Q_t^{S_n}$$

exists, and for each r , (μ_t^r) is a (Q_t^S) -entrance law at r . If $r \notin \Gamma$, set $\mu_t^r = 0$ for all t . By (2.31), for every $r \in \Gamma$ and $t \in \bigcup_k \bigcap_{n>k} \Lambda_n$, $\mu_t^r = \lambda_t^r$. So $\mu_t^r = \lambda_t^r$ a.e. (L) in t on $]r, \infty[$. Let $\Phi = \{(r, t) : \mu_t^r \neq \lambda_t^r\}$. Then $\int \int \Phi(r, t) L(dt) \Phi(dr) = 0$. Applying Fubini's theorem, we see there is a set $G \subset \mathbb{R}$ with $L(G^C) = 0$ so that for every t in G , $\mu_t^r = \lambda_t^r$ a.e. (Φ) . Since $\int \lambda_t^r \Phi(dr) = \mu_t$, we have $\int \mu_t^r \Phi(dr) = \mu_t$ for every t in G . Fix $t \in \mathbb{R}$ and choose a sequence $(t_n) \subset G$ increasing to t . Then

$$\mu_{t_n} = \int \mu_{t_n}^r \Phi(dr) = \int_{]-\infty, t_n[} \mu_{t_n}^r \Phi(dr),$$

so

$$\mu_{t_n} Q_t^{t_n} = \int_{]-\infty, t_n[} \mu_{t_n}^r Q_t^{t_n} \Phi(dr) = \int_{]-\infty, t_n[} \mu_t^r \Phi(dr).$$

Since (μ_t) is an entrance rule, $\mu_{t(n)} Q_t^{t(n)}$ increases to μ_t as t_n increases to t . The integral increases to

$$\int_{]-\infty, t[} \mu_t^r \phi(dr) = \int \mu_t^r \phi(dr),$$

and we have shown that $\mu_t = \int \mu_t^r \phi(dr)$. Now we show that $\mu_t^r(g_t) < \infty$ for every t and r . If $r \notin \Gamma$, this is clear since $\mu_t^r = 0$. Let $r \in \Gamma$ and recall that $\mu_t^r = \lambda_{t-}^r$ a.e. (L) in t on $]r, \infty[$. By (2.25), $\mu_t^r(g_t) < \infty$ a.e. (L). In particular, there is a sequence $(t(n))$ decreasing to r with $\mu_{t(n)}^r(g_{t(n)}) < \infty$. But if $u > t(n)$, $\mu_u^r(g_u) = \mu_{t(n)}^r(Q_u^{t(n)} g_u) < \mu_{t(n)}^r(g_{t(n)}) < \infty$ since g_t is a (Q_t^S) -exit rule.

This essentially finishes the proof: we have produced the desired (μ_t^S) . All that remains is to observe that we can replace ϕ with a finite measure if desired. To do this, choose a strictly positive function z on R so that $\phi(z) < \infty$. Set $\bar{\mu}_t^S = z(s)^{-1} \mu_t^S$ and

$$\bar{\phi}(ds) = z(s)\phi(ds)$$

Q.E.D.

REMARK. Theorem (1.4) can be obtained by carefully checking through the proof of (2.11). It does not seem to be an immediate corollary of the statement of (2.11).

We can now give the representation of entrance rules.

(2.33) THEOREM. Let $v = (v_t)_{t \in R}$ be an entrance rule for (P_t^S) . Then for each $s, -\infty < s < \infty$, there exists an entrance law at s , $v^s = (v_t^s)$ and a finite measure ϕ on R so that

- (i) $(s, t) \mapsto v_t^s(f)$ is $\mathcal{B}(R) \times \mathcal{B}(R)$ -measurable; and
- (ii) $v_t = v_t^{-\infty} + \int_R v_t^s \phi(ds)$ for every $t \in R$.

Before we give the proof, let us re-interpret the time homogeneous situation (1.10) in this context. If we set $v_t = m$ for every t , and $P_t^s = P_{t-s}$, then v_t is an entrance rule for P_t^s . Set $v_t^{-\infty} = m^1$ for every t and for each $s \in \mathbb{R}$, set $v_t^s = \mu_{t-s}$ if $t > s$, $v_t^s = 0$ if $t < s$. In this case, we may take $\phi(dt) = dt$ to obtain

$$(2.34) \quad m = v_t^{-\infty} + \int_{\mathbb{R}} v_t^s \phi(ds).$$

PROOF of (2.33). By definition, the measures $v_s P_t^s$ decrease as s decreases to $-\infty$. For each t , define

$\mu_t = \lim_{s \rightarrow -\infty} v_s P_t^s$. Then μ_t is a σ -finite measure with $\mu_t < v_t$. Let $f \in \mathcal{P}$ with $v_t(f) < \infty$; then for $s < t$,

$$\mu_s P_t^s f = \lim_{r \rightarrow -\infty} v_r P_s^r P_t^s f = \lim_{r \rightarrow -\infty} v_r P_t^r f = \mu_t(f),$$

so $\mu = (\mu_t)_{t \in \mathbb{R}}$ is an entrance law at $-\infty$. For each $t \in \mathbb{R}$, $\lambda_t = v_t - \mu_t$ is a σ -finite measure and $\lambda = (\lambda_t)$ is an entrance rule such that $\lim_{s \rightarrow -\infty} \lambda_s P_t^s = 0$. Apply Theorem (2.11) to λ and set $v^{-\infty} = \mu$ to obtain

$$v_t = \mu_t + \lambda_t = v_t^{-\infty} + \int_{\mathbb{R}} v_t^s \phi(ds).$$

3. Constructing the measures.

In this section, E denotes a Lusin topological space with Borel field $\mathcal{E}$ (i.e. E is homeomorphic to a Borel subset of a compact metric space). In what follows, it would suffice to assume that E is a cosouslin metrizable space, but we leave such an extension to the interested reader. Fix a transition operator (P_t^S) on $(E, \mathcal{E})$ satisfying (2.1), (2.2), (2.3) and (2.5). In order to state our last assumption on (P_t^S) , we need to introduce some notation.

Let b be a point not in E , and set $E_b = E \cup \{b\}$. Topologize E_b so that E has its original topology and b is isolated in E_b . Then E_b is a Lusin topological space and the trace of its Borel field $\mathcal{E}_b$ on E is $\mathcal{E}$. We adopt the usual convention that a numerical function f on E is extended to E_b by setting $f(b) = 0$. For $-\infty < r < \infty$, let W_r denote the set of all right continuous maps from $]r, \infty[$ to E_b with b as cemetery. If $t > r$ and $w \in W_r$, let $Y_t(w) = w(t)$. Set $\mathcal{G}_r = \sigma\{Y_t : t > r\}$, and set $\beta(w) = \inf\{t : w(t) = b\}$. We now state our last assumption on (P_t^S) .

(3.1) ASSUMPTION. For each $x \in E$ and $r \in \mathbb{R}$, there exists a probability $P_{x,r}$ on $(W_r, \mathcal{G}_r)$ so that if $r < t_1 < t_2 < \dots < t_n$, then

$$\begin{aligned} P_{x,r}(Y_{t_1} \in dy_1, \dots, Y_{t_n} \in dy_n, t_n < \beta) \\ = P_{t_1}^r(x, dy_1) P_{t_2}^{t_1}(y_1, dy_2) \dots P_{t_n}^{t_{n-1}}(y_{n-1}, dy_n) \end{aligned}$$

(3.2) REMARKS (i) By (2.3), P_t^s decreases to 1 as t decreases to s , so $\lim_{t \downarrow r} P_{x,r}^s(t < \beta) = 1$. Thus $P_{x,r}$ is carried by $\{r < \beta\}$. It also follows from (3.1) that $x \mapsto P_{x,r}(F)$ is $\mathcal{G}$ -measurable whenever $F \in \mathcal{G}_r$.

(ii) If the space-time semigroup defined in Sec. 2 is the semigroup of a right process on $R \times E$, then (3.1) holds. In particular, if $P_t^s = P_{t-s}$, where (P_t) is the semigroup of a right process on E , then (3.1) holds.

(iii) Let $r \in R$, and set $W_r^+ = \{w \in W_r : \lim_{s \downarrow r} w(s) \text{ exists in } E_b\}$. In the usual set up for right continuous strong Markov processes, one obtains the measure $P_{x,r}$ concentrated on W_r^+ . We do not need this stronger assumption here: (3.1) will suffice.

(iv) Since E_b is a Lusin space, it follows easily from IV-19 of [1] that $(W_r, \mathcal{G}_r)$ is a U-space. We need this fact below.

The usual result on constructing measures via inverse limits is stated for probabilities. Here we need a version which will work for σ -finite measures. We state it here; its proof is given at the end of this section. First, recall the definition.

(3.3) DEFINITION. Let $(F_n, \mathcal{F}_n)_{n \geq 1}$ be U-spaces and let $P_n : F_{n+1} \rightarrow F_n$ be $\mathcal{F}_{n+1}/\mathcal{F}_n$ -measurable. The inverse limit $(F, \mathcal{F})$ of $(F_n, \mathcal{F}_n, P_n)$ is the subset of $\prod_{k \geq 1} F_k$ consisting of those $x = (x_k)$ with $P_k(x_{k+1}) = x_k$ for each $k \geq 1$ and $\mathcal{F} = \sigma(q_k : k \geq 1)$, where q_k is the natural projection $q_k : (x_n) \mapsto x_k$.

(3.4) THEOREM. Let $(F_n, \mathcal{F}_n)_{n \geq 1}$ be U-spaces and let $P_n : F_{n+1} \rightarrow F_n$ be $\mathcal{F}_{n+1}/\mathcal{F}_n$ -measurable. For each n , let μ_n be a measure on $(F_n, \mathcal{F}_n)$ so that $P_n(\mu_{n+1}) = \mu_n$. Assume μ_1 is σ -finite. Then there exists a unique σ -finite measure μ on $(F, \mathcal{F})$ so that $q_n(\mu) = \mu_n$ for each $n \geq 1$.

The next result is the basic step in our construction.

(3.5) PROPOSITION. Let $v = (v_t)$ be an entrance law at r , $-\infty < r < \infty$. Then there exists a σ -finite measure Q on $(W_r, \mathcal{G}_r)$ so that $Q(\beta = r) = 0$ and if $r < t_1 < \dots < t_n$, then

$$(3.6) \quad Q(Y_{t_1} \in dy_1, \dots, Y_{t_n} \in dy_n, t_n < \beta) \\ = v_{t_1}(dy_1) P_{t_2}^{t_1}(y_1, dy_2) \dots P_{t_n}^{t_{n-1}}(y_{n-1}, dy_n).$$

Note that $\beta > r$ on W_r . The uniqueness of Q will follow from the main Theorem (3.8) of this section.

PROOF. Let (s_n) be a sequence of numbers which strictly decreases to r . For the moment, fix $k > 1$, and for $n > k$, set $k_W^n = W_{s(n)} \cap \{s_k < \beta\}$. Since a Borel subspace of a U-space is a U-space, it follows that k_W^n is a U-space, and its Borel σ -algebra $k\mathcal{G}^n$ is the trace of $\mathcal{G}_{s(n)}$ on k_W^n . For $n > k$, let $P_n : k_W^{n+1} \rightarrow k_W^n$ by restriction; that is, $P_n w(t) = w(t)$ for $t > s_n$. Note that the image of k_W^{n+1} , $P_n(k_W^{n+1})$ is not all of k_W^n . In fact, it is the set $W_{s(n)}^+ \cap \{s_k < \beta\}$ defined in (3.2iii). But P_n is $k\mathcal{G}^{n+1}/k\mathcal{G}^n$ -measurable, and it is clear that $W_r \cap \{s_k < \beta\}$ may be identified with the inverse limit of

$(k_W^n, k_{\mathcal{G}}^n, p_n)_{n \geq k}$. In fact, $q_n(W_r \cap \{s_k < \beta\}) = p_n(k_W^{n+1})$, where q_n is the map from W_r to $W_{s(n)}$ defined by restriction. For each $n > k$, define k_Q^n on $(k_W^n, k_{\mathcal{G}}^n)$ by setting

$$(3.7) \quad k_Q^n(F) = \int v_{s(n)}(dx) P_{x, s(n)}(F; s_k < \beta).$$

One may check that $p_n(k_Q^{n+1}) = k_Q^n$ since v is an entrance law. Let $f_k > 0$ with $v_{s(k)}(f_k) < \infty$. Then $f_k(Y_{s(k)}) > 0$ on k_W^{k+1} , and $k_Q^{k+1}(f(Y_{s(k)})) < v_{s(k)}(f_k) < \infty$. Therefore, k_Q^{k+1} is σ -finite. Let $k_{\mathcal{G}_r}$ be the trace of $\mathcal{G}_r$ on $W_r \cap \{s_k < \beta\}$. By (3.4), there exists a σ -finite measure k_Q on $(W_r \cap \{s_k < \beta\}, k_{\mathcal{G}_r})$ so that $q_n(k_Q) = k_Q^n$ for $n > k$. We now regard k_Q as a measure on W_r carried by $\{s_k < \beta\}$. Set $s_0 = \infty$, and for $k \geq 1$, let $k_Q = 1_{\{s(k) < \beta < s(k-1)\}} \cdot k_Q$. Then set $Q = \sum_{k \geq 1} k_Q$. Then Q is a σ -finite measure since each term in the sum is σ -finite and they are carried by disjoint sets. If $r < t_1 < \dots < t_n$, then

$$\begin{aligned} & Q(Y_{t_1} \in dy_1, \dots, Y_{t_n} \in dy_n, t_n < \beta) \\ &= \sum_{k \geq 1} k_Q(Y_{t_1} \in dy_1, \dots, Y_{t_n} \in dy_n, t_n < \beta < s_{k-1}; \beta > s_k). \end{aligned}$$

Since the s_k decrease to r , there is an integer N so that $s_N < t_1$. Since the event $k_A^N = \{Y_{t_1} \in dy_1, \dots, Y_{t_n} \in dy_n, t_n < \beta < s_{k-1}; \beta > s_k\} \in k_{\mathcal{G}}^N$, we may rewrite the sum as

$$\sum_{k \geq 1} k Q^N(k, \lambda^n)$$

$$= \sum_{k \geq 1} \int v_{s(N)}(dx) P_{x, s(N)}(Y_{t_1} \in dy_1, \dots, Y_{t_n} \in dy_n,$$

$$t_n < \beta < s_{k-1}; \beta > s_k)$$

$$= \int v_{s(N)}(dx) P_{x, s(N)}(Y_{t_1} \in dy_1, \dots, Y_{t_n} \in dy_n; t_n < \beta)$$

$$= v_{t_1}(dy_1) P_{t_2}^{t_1}(y_1, dy_2) \dots P_{t_n}^{t_{n-1}}(y_{n-1}, dy_n)$$

since v_t is an entrance law. Finally, observe that

$Q(\beta = r) = 0$ because kQ is carried by $\{s_k < \beta\}$.

Now we come to the main result. Let a be another point not in E_b . Adjoin a as an isolated point to E_b to obtain the Lusin space E_b^a with Borel field $\mathcal{E}_b^a$. Let W be the set of all maps w from $\mathbb{R}$ to E_b^a so that there is a non-void open interval $]\alpha(w), \beta(w)[$ on which w is E -valued and right continuous, $w(t) = a$ for $t < \alpha(w)$ and $w(t) = b$ for $t > \beta(w)$. (Note that for each r , $-\infty < r < \infty$, $W_r \cap \{\beta > r\}$ may be identified with $W \cap \{\alpha = r\}$). If $w \in W$, let $Y_t(w) = w(t)$, $\mathcal{X}^0 = \sigma\{Y_t : t \in \mathbb{R}\}$.

(3.8) THEOREM. Let $v = (v_t)_{t \in \mathbb{R}}$ be an entrance rule.

Then there exists a unique measure Q_v on $(W, \mathcal{X}^0)$ so that if $t_1 < \dots < t_n$,

$$(3.9) \quad Q_v(\alpha < t_1, Y_{t_1} \in dy_1, \dots, Y_{t_n} \in dy_n, t_n < \beta)$$

$$= v_{t_1}(dy_1) P_{t_2}^{t_1}(y_1, dy_2) \dots P_{t_n}^{t_{n-1}}(y_{n-1}, dy_n).$$

Moreover, Q_v is σ -finite.

REMARK. Note that if Q is any measure satisfying (3.9), then $Q = Q_v$ and Q is σ -finite. It is not necessary to verify that Q is σ -finite a priori.

PROOF. For each rational r , we may choose a decomposition of E , $(G_{rk})_{k \geq 1} \subset \mathcal{E}$ so that $v_r(G_{rk}) < \infty$, since v_r is σ -finite. Let v^s , $-\infty < s < \infty$ be the entrance laws in the representation (2.33). Recall from the statement of Theorem (2.11) and (2.9) that there is a function $h_t(x) > 0$ in $\mathcal{B}(R) \times \mathcal{E}$ so that $v_t^s(h_t) < \infty$ for all s and t . Order the collection of sets $\{G_{rk} \cap \{l < h_r < l+1\} : l > 0, k > 1\}$ into a sequence $(E_{rk})_{k \geq 1}$ so that $\bigcup_{k=1}^{\infty} E_{rk} = E$, $v_r(E_{rk}) < \infty$ and $v_r^s(E_{rk}) < \infty$ for every k and for every $s \in [-\infty, \infty[$. Let $W_{rk} = \{w \in W : Y_r(w) \in E_{rk}\}$. Since $E_{rk} \cap E_{rj} = \emptyset$ if $k \neq j$, $W_{rk} \cap W_{rj} = \emptyset$ if $k \neq j$. Because $] \alpha(w), \beta(w)[$ is non-void for each $w \in W$, one has $\bigcup_{r \in Q, k \geq 1} W_{r,k} = W$.

STEP 1. Uniqueness.

Let Q and P be two measures on $(W, \mathcal{X}^0)$ for which (3.9) holds. Then $Q(W_{rk}) = v_r(E_{rk}) = P(W_{rk})$. Let Q_{rk} and P_{rk} be the restrictions of Q and P to W_{rk} . Then Q_{rk} and P_{rk} are finite measures on W_{rk} . Moreover, we have from (3.9) that $Q_{rk}(F) = P_{rk}(F)$ whenever F is of the form

$$(3.10) \quad F = \prod_{j=1}^n f_j \circ Y_{t_j},$$

for $t_1 \dots < t_n$, $(f_j) \subset \mathcal{B}$, $n > 1$. Such functions constitute a multiplication - stable vector space whose restriction to W_{rk} generates the trace of $\mathcal{X}^0$ on W_{rk} , because $\{Y_t = b\} \cap W_{rk}$ is empty if $t < r$ and $\{Y_t = a\} \cap W_{rk}$ is empty if $t > r$. Thus we have $Q_{rk} = P_{rk}$. It follows that $Q = P$.

STEP 2. Existence.

Let Q^s be the σ -finite measure on $(W_s, \mathcal{G}_s)$ carried by $\{s < \beta\}$ constructed from v^s in Proposition (3.5). For each s , $-\infty < s < \infty$, define maps $q_s: W_s \cap \{s < \beta\} \rightarrow W$ by setting

$$q_s w(t) = w(t) \text{ if } t > s$$

$$= a \quad \text{if } t < s.$$

Note that $\alpha \circ q_s = s$ and $\beta \circ q_s = \beta$. Let ${}^s Q = q_s(Q^s)$. Then ${}^s Q$ is a measure on $(W, \mathcal{X}^0)$ carried by $\{\alpha = s\}$. If $t_1 < \dots < t_n$, then from (3.6) we have

$$(3.11) \quad {}^s Q(Y_{t_1} \in dy_1, \dots, Y_{t_n} \in dy_n, t_n < \beta)$$

$$= v_{t_1}^s(dy_1) P_{t_2}^{t_1}(y_1, dy_2) \dots P_{t_n}^{t_{n-1}}(y_{n-1}, dy_n),$$

and note that this is zero if $t_1 < s$. In particular,

$${}^s Q(W_{rk}) = v_r^s(E_{rk}) < \infty. \text{ Thus each } {}^s Q \text{ is } \sigma\text{-finite.}$$

Next, we claim that $s \mapsto {}^s Q(F)$ is Borel measurable for each $F \in \mathcal{X}^0$. In view of the above and by "disjointing" the W_{rk} , it suffices to prove this for each $s \mapsto {}^s Q(F \cap W_{rk})$

with F of the form (3.10). But for such F ,

$$(3.12) \quad {}^sQ(F) = \int f_1(x_1) P_{t_2}^{t_1} f_2 P_{t_3}^{t_2} f_3 \dots P_{t_n}^{t_{n-1}} f_n(x_1) v_{t_1}^s(dx_1).$$

Note that this expression is zero if $s > t_1$ as it should be since $\alpha = s$ almost surely sQ , and that it is Borel measurable in s . Consequently, so is $s \mapsto {}^sQ(F)$ for all $F \in \mathcal{X}^0$. Finally, define Q_v on $\mathcal{X}^0$ by setting

$$(3.13) \quad Q_v(F) = {}^{-\infty}Q(F) + \int_{\mathbb{R}} {}^sQ(F) \phi(ds)$$

for $F \in \mathcal{X}^0$. Using (3.12) and (2.33), it is immediate that (3.9) holds for Q_v . Moreover, $Q_v(W_{rk}) = v_r(E_{rk}) < \infty$, so Q_v is σ -finite. This establishes the existence of Q_v . Q.E.D.

PROOF OF THEOREM (3.4).

Let h_1 be finite and strictly positive on F_1 with $\mu_1(h_1) = 1$. Define inductively $h_{n+1} = h_n \circ p_n$ for $n > 1$. Then $h_n \in \mathcal{F}_n$, $h_n > 0$ on F_n , and $\mu_n(h_n) = \mu_1(h_1) = 1$ for each n . Let $\nu_n = h_n \mu_n$. Then ν_n is a probability on $(F_n, \mathcal{F}_n)$, and one easily checks that $p_n(\nu_{n+1}) = \nu_n$. Hence by (III-53, [1]), there exists a unique probability ν on $(F, \mathcal{F})$ with $\nu_n = q_n(\nu)$. Define h on F by $h(x) = h_n(x_n) = h \circ q_n(x)$. (Here, $x = (x_k)$). Since $h_{n+1}(x_{n+1}) = h_n \circ p_n(x_{n+1}) = h_n(x_n)$, h is well-defined, $h \in \mathcal{F}$, and $h > 0$. Let $\mu = h^{-1} \nu$. Then μ is σ -finite, and one readily checks that $q_n(\mu) = \mu_n$. Finally, the uniqueness of μ follows from the uniqueness of $\nu = h\mu$. Q.E.D.

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