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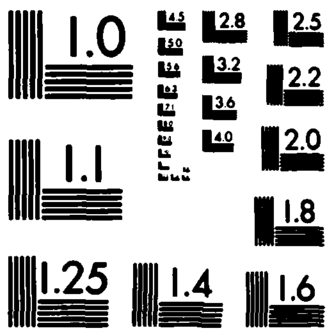
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Abstract

The asymptotic stress and deformation fields of a crack propagating steadily and quasi-statically into an elastic-plastic material, characterised by sub 2 J_2-flow theory with linear strain-hardening, were first determined by Amazigo and Hutchinson (1977) for the cases of mode III and mode I (plane strain and plane stress). Their solutions were approximate in that they neglected the possibility of plastic reloading on the crack faces. This effect was taken into account by ~~Ponte Castañeda (1985c),~~ the author who formulated the (eigenvalue) problem in terms of a system of first order O.D.E.'s in the angular variations of the stress and velocity components. The strength of the singularity, serving as the eigenvalue, and the angular variations of the fields were determined as functions of the hardening parameter.

The above analysis does not determine the amplitude factor of the near-tip asymptotic fields, or *plastic* stress intensity factor. In this work, a simple, approximate technique based on a variational statement of compatibility is developed under the assumption of small-scale yielding. A trial function for the stress function of the problem, that makes use of the asymptotic information in the near-tip and far-field limits, is postulated. Such a trial function depends on arbitrary parameters that measure the intensity of the near-tip fields and other global properties of the solution. Application of the variational statement then yields optimal values for these parameters, and in particular determines the plastic stress intensity factor, thus completing the knowledge of the near-tip asymptotic fields. The results obtained by this novel method are compared to available finite element results.

1. Introduction

In the propagation of cracks in ductile materials, a stage of stable crack growth, where the efects of inertia are unimportant, is often observed under continuously increasing external load, and immediately following the onset of crack growth. This phenomenon is intrinsic to the plastic deformation at the tip of a moving crack: as the crack propagates in an elastic-plastic material, it leaves behind a wake of residual plastic deformation, which typically accounts for all of the energy available from the external source, and thus reduces the tendency of the crack to grow.

Under small-scale yielding, which assumes that the size of the active plastic zone surrounding the crack tip is small compared to the crack size and the overall dimensions of the specimen in consideration, an appropriate measure of the external loading is the stress intensity factor of the enclosing elastic singularity fields. In accord with the discussion in the previous parragraph, it has been postulated that the graph of such elastic stress intensity factor versus the amount of crack growth, called the resistance curve, is a material property for small-scale yielding.

Central to these notions, and to the continued development of fracture mechanics in general, is the knowledge of the stress and deformation fields near the tip of the advancing crack, and important advances have been made in recent years.

Most of the work, however, has been in the context of the ideal elastic-plastic solid. McClintock (1958) and McClintock and Irwin (1965) were the first to demonstrate that the key to stable crack growth lies in the nature of the plastic deformation, and found that the strains in the active plastic zone near the tip of the crack depend logarithmically on the radial distance from the crack tip, r. Later, Chitaley and McClintock (1971) gave the correct asymptotic solution to the problem of a steadily and quasi-statically growing crack in an ideal elastic-plastic material with Mises yield criterion. Rice (1974) demonstrated that the same logarithmic dependence of the strains on r would be expected for the cases of plane stress and plane strain. Slepyan (1974) obtained the first correct asymptotic solution of the plane strain problem in modes I and II using a Tresca yield criterion. Rice *et al.* (1980), following Rice and Sorensen (1978), and Gao and Hwang (1980), independently, solved the plane strain problem in mode I using a Mises yield criterion, but the validity of their solution was restricted to the case of $\nu = 1/2$. Somewhat later, Drugan *et al.* (1982) generalised the results to the case of $\nu \neq 1/2$. The plane-stress mode I problem has remained most elusive, but recently, Ponte Castañeda (1985a) produced a solution to the corresponding mode II problem. In addition, a general discussion of the subject can be found in Rice (1982).

The corresponding work for the hardening elastic-plastic solid is at a less developed stage, and most of this work relates to the linear-hardening case. Slepyan (1973), using a modified J_2-deformation theory, produced the first asymptotic solution to the problem of a mode III crack propagating steadily in a linear-hardening material. Much later, Ponte Castañeda (1985b) extended Slepyan's work to the mode I case. Amazigo and Hutchinson (1977), using J_2-flow theory, produced steady-state asymptotic solutions for the moving crack in anti-plane strain, as well as in plane strain and plane stress mode I. For a linear-hardening material, with Young's modulus, E, and tangent modulus, E_t, they found a variable-separable solution with stress and strain fields having a power singularity of strength, s, which is a function of the hardening ratio $\alpha = E_t/E$, and vanishes in the perfectly-plastic limit. These solutions, however, neglected the possibility of plastic reloading on the crack flanks which is known to occur in the elastic-perfectly plastic solutions mentioned above. In a recent publication, Zhang *et al.* (1983) included reloading (as well as a Bauschinger effect) in their solution of the plane-strain mode I problem. More recently, but independently, Ponte Castañeda (1985c) formulated the problem efficiently in terms of a system of first order O.D.E.'s in the angular variations of the components of the stress and velocity fields, and included as well the effect of reloading in the solution of the anti-plane strain, plane strain and plane stress (modes I and II) problems.

These asymptotic solutions leave undetermined certain parameters in the expressions for the stress and deformation fields near the tip of the moving crack. In the perfectly-plastic case, the parameter in question is the size of the plastic zone ahead of the crack, but this parameter does not appear in the lowest order term of the asymptotic expansion of the fields as $r \rightarrow 0$, and its determination is not critical. On the other hand, in the linear-hardening case, by analogy with the elastic problem, the amplitude of the lowest order term of the asymptotic expansion of the stress and deformation fields, as measured by the *plastic* stress intensity factor, is undetermined from the asymptotic analysis, and its determination introduces an important problem.

In this paper we use the near-tip information known from the work of Ponte Castañeda (1985c) to complete the knowledge of the asymptotic fields of a steadily propagating crack in a linear-hardening material under small-scale yielding conditions by determining the plastic stress intensity factor of these near-tip fields as a function of the hardening parameter. Other relevant information, such as the size and shape of the active plastic zones, will result as by-products of our calculations.

We develop a simple, approximate technique based on a direct use of a variational statement of compatibility. Thus we propose a trial function for the stress function of the problem, that makes use of the known asymptotic information in the near-tip and far-field limits, and depends on arbitrary parameters that measure the intensity of the near-tip fields and other global properties of the solution. Application of the variational statement then yields optimal values for these parameters, and hence determines the plastic stress intensity factor of the near-tip fields. The results obtained by this novel method will be compared to the finite element method results of Dean and Hutchinson (1980), Dean (1983) and Mataga (1986) for the linear-hardening material.

2. Anti-plane strain (mode III)

2.1 *The plastic stress intensity factor in small-scale yielding*

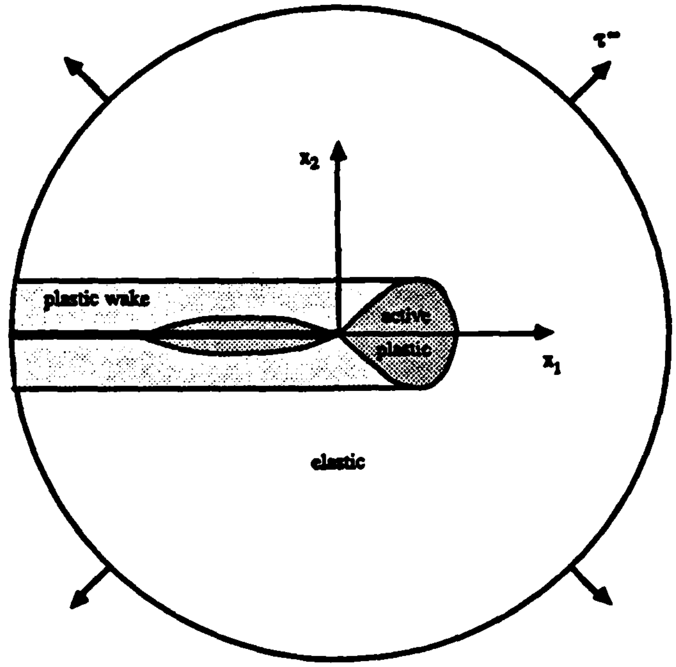

Figure 1—Steadily-propagating crack in small-scale yielding.

Consider the steadily propagating semi-infinite crack depicted in Figure 1. Under the small-scale yielding assumption, the elastic singularity field is imposed in the remote limit so that

$$\begin{aligned} \tau(r,\theta) &\sim K^{el}\ \bar{\tau}^{\infty}(\theta)\,(2\pi r)^{-1/2} \equiv \tau^{\infty} \\ & \qquad\qquad\qquad\qquad \text{as } r \to \infty,\ |\theta| < \pi \qquad (2.1) \\ v(r,\theta) &\sim -2\,(V/G)\,K^{el}\ \bar{v}^{\infty}(\theta)\,(2\pi r)^{-1/2} \equiv v^{\infty} \end{aligned}$$

where τ^{∞}, v^{∞} refer to the well-known elastic singularity solution, K^{el} is the *elastic* stress intensity factor, G is the modulus of elasticity in shear, V is the propagating speed of the crack, and $\bar{\tau}^{\infty}(\theta)$, $\bar{v}^{\infty}(\theta)$ are the angular variations of the elastic fields in mode III, normalized such that $\bar{\tau}_2^{\infty}(0) = 1$ for consistency with the definition of K^{el}.

On the other hand, we require that in the near-tip limit

$$\begin{aligned} \tau(r,\theta) &\sim K^{pl}\ \bar{\tau}^{o}(\theta)\,(2\pi r)^{s} \equiv \tau^{o} \\ & \qquad\qquad\qquad\qquad \text{as } r \to 0 \qquad (2.2) \\ v(r,\theta) &\sim K^{pl}\,(V/G)\ \bar{v}^{o}(\theta)\,(2\pi r)^{s}/s \equiv v^{o} \end{aligned}$$

where τ^o, v^o refer to the near-tip solutions of Ponte Castañeda (1985c), K^{pl} is defined to be the *plastic* stress intensity factor in small-scale yielding, $s = s(\alpha)$ is the strength of the plastic singularity, and we have normalized the angular variations of these near-tip fields, which are also functions of α, such that $\bar{\tau}_2^o(0) = 1$.

It is convenient to work with dimensionless variables, and in anti-plane strain we select the following:

$$\begin{aligned} \underline{r} &= (\tau_0 / K^{el})^2\, r \\ \underline{\tau}_\alpha &= \tau_\alpha / \tau_0 \qquad\qquad (2.3) \\ \underline{v} &= (G/\tau_0)\,(v/V) \end{aligned}$$

where τ_0 is the yield stress in shear.

In terms of these new variables the remote fields can be rewritten as

$$\begin{aligned} \underline{\tau}^{\infty}(r,\theta) &= \bar{\tau}^{\infty}(\theta)\,(2\pi \underline{r})^{-1/2} \\ & \qquad\qquad\qquad\qquad (2.4) \\ \underline{v}^{\infty}(r,\theta) &= -2\ \bar{v}^{\infty}(\theta)\,(2\pi \underline{r})^{-1/2} \end{aligned}$$

and the near-tip fields as

$$\begin{aligned} \underline{\tau}^{o}(r,\theta) &= \kappa\ \bar{\tau}^{o}(\theta)\,(2\pi \underline{r})^{s} \\ & \qquad\qquad\qquad\qquad (2.5) \\ \underline{v}^{o}(r,\theta) &= \kappa\ \bar{v}^{o}(\theta)\,(2\pi \underline{r})^{s}/s \end{aligned}$$

where κ is the nondimensional plastic stress intensity factor, and is a function of α only.

It follows from this nondimensionalization that the plastic stress intensity factor may be expressed as

$$K^{pl} \equiv \lim_{r \to 0} (2\pi \underline{r})^{-s}\, \tau_2(r,0) = \kappa(\alpha)\,(\tau_0)^{1+2s}\,(K^{el})^{-2s} \qquad (2.6)$$

and it now remains to determine the functional dependence of κ on α.

Note however that in the elastic limit we must have $K^{pl} = K^{el}$, which necessarily implies that

$$\kappa(1) = 1$$

Alternatively, in the perfectly-plastic limit we expect $K^{pl} = \tau_o$, and therefore that

$$\kappa(0) = 1$$

2.2 *The variational statement of the problem*

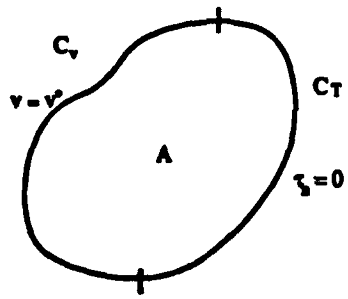

Figure 2—Domain for the general boundary value problem.

We wish to develop a simple, approximate technique to determine the functional dependence of κ on α. In this connection we propose to make use of the following variational statement of compatibility:

> Let A be the domain of the boundary value problem, with boundaries C_T and C_v, shown in Figure 2.
>
> If the stress field $\boldsymbol{\tau}$ satisfies equilibrium
>
> $$\nabla \cdot \boldsymbol{\tau} = 0 \quad \text{in} \quad A \tag{2.7a}$$
>
> and the boundary condition
>
> $$\tau_n = \boldsymbol{\tau} \cdot \mathbf{n} = 0 \quad \text{on} \quad C_T \tag{2.7b}$$
>
> and if the functional relation
>
> $$\int_A G \cdot \delta\tau \, dA = \int_{C_v} v^* \, \delta\tau_n \, ds \tag{2.7c}$$

between the engineering strain-rate G, and the velocity field, prescribed on the boundary C_v, v^*, holds for arbitrary variations of the stress field $\delta\tau$, that satisfy conditions (2.7a) and (2.7b).

Then the strain-rate field G is compatible

$$G = \nabla v \tag{2.7d}$$

and the associated velocity field v meets the boundary condition

$$v = v^* \text{ on } C_v \tag{2.7e}$$

The constitutive relation to be used in conjunction with this statement is

$$\underline{G} = \underline{T} + \underline{G}^{pl} \tag{2.8}$$

where $\underline{G}$ is the dimensionless engineering strain-rate vector, $\underline{T}$ is the dimensionless stress-rate vector, and

$$\underline{G}^{pl} = \begin{cases} 0 & \text{in } A_e \\ (\alpha^{-1}-1)\,(\underline{\tau}_e^{\cdot}/\,\underline{\tau}_e)\,\underline{\tau} & \text{in } A_p \end{cases} \tag{2.9}$$

is the dimensionless plastic strain-rate vector, which depends on whether the material particle is in an *active* plastic zone A_p, or in a zone of current *elastic* behavior A_e.

The advantage of this particular choice for a variational statement (compatibility over equilibrium) is that with this choice it is easier to compute all the quantities involved in the variational statement. In fact, with the introduction of a stress function f, that ensures satisfaction of (2.7a) and (2.7b), we can easily compute all the quantities involved in the variational statement via simple differentiation

$$\begin{aligned} \underline{\tau} &= \underline{\nabla} \times (f e_3) \\ \underline{T} &= -\underline{\tau}_{\perp} \end{aligned} \tag{2.10}$$

Our approximate technique then consists in postulating the following *trial* function for f

$$f(\underline{r},\theta) = p(\underline{r})\, f^{o}(\underline{r},\theta) + q(\underline{r})\, f^{\infty}(\underline{r},\theta) \tag{2.11}$$

where

$$f^o(r,\theta) = \kappa(\alpha)\, g^o(\theta)\ r\,(2\pi r)^s, \quad g^o(\theta) = -\ \bar{\tau}_\theta{}^o(\theta)\,/\,(1+s)$$

is the stress function associated with the near-tip fields,

$$f^\infty(r,\theta) = g^\infty(\theta)\ r\,(2\pi r)^{-1/2}, \qquad g^\infty(\theta) = -2\cos(\theta/2)$$

is the stress function associated with the remote fields, and where $p(r)$ and $q(r)$ are chosen so that the resulting stress and deformation fields conform with the given asymptotic conditions, i.e. equations (2.1) and (2.2). Hence, we must require

$$p(0) = 1 \qquad p(\infty) = 0$$

$$q(0) = 0 \qquad q(\infty) = 1$$

Also, since we must have that $f = f^o = f^\infty$ for $\alpha = 1$, we require that $p(r) + q(r) = 1$ for $\alpha = 1$. For simplicity, we select our trial functions $p(r)$ and $q(r)$ such that $p(r) + q(r) = 1$ for all values of α. Thus, we take

$$p(r) = \gamma^2\,/\,(\gamma^2 + r^2) \qquad \text{and} \qquad q(r) = r^2\,/\,(\gamma^2 + r^2) \tag{2.12}$$

where γ is arbitrary.

Having specified f, we evaluate τ and T, by computing the appropriate derivatives as indicated in (2.10)

$$\tau_1 = f_{,2} = \sin\theta\, f_{,r} + r^{-1}\cos\theta\, f_{,\theta}$$

$$\tau_2 = -f_{,1} = -\cos\theta\, f_{,r} + r^{-1}\sin\theta\, f_{,\theta}$$

$$T_1 = -f_{,21} = -(1/2)\sin 2\theta\,(f_{,rr} - r^{-1} f_{,r} - r^{-2} f_{,\theta\theta}) - \cos 2\theta\ (r^{-1} f_{,r\theta} - r^{-2} f_{,\theta})$$

$$T_2 = f_{,11} = \cos^2\theta\, f_{,rr} + \sin^2\theta\,(r^{-2} f_{,\theta\theta} + r^{-1} f_{,r}) - \sin 2\theta\ (r^{-1} f_{,r\theta} - r^{-2} f_{,\theta})$$

Here the computation of the partial derivatives with respect to r is straightforward, but yields lengthy expressions which we will not include here, and the computation of the partials with respect to θ involve the following derivatives of $g^o(\theta)$

$$(g^o)'(\theta) = \bar{\tau}_r{}^o(\theta) \qquad (g^o)''(\theta) = (\bar{\tau}_r{}^o)'(\theta)$$

which are expressed in terms of quantities readily available from the numerical integration of system (2.5) in Ponte Castañeda (1985c).

We then evaluate the variation of $\boldsymbol{\Sigma}$ by assuming that κ and γ are independent parameters. We simply have that

$$\delta\boldsymbol{\Sigma} = (\partial\boldsymbol{\Sigma}/\partial\kappa)\,\delta\kappa + (\partial\boldsymbol{\Sigma}/\partial\gamma)\,\delta\gamma$$

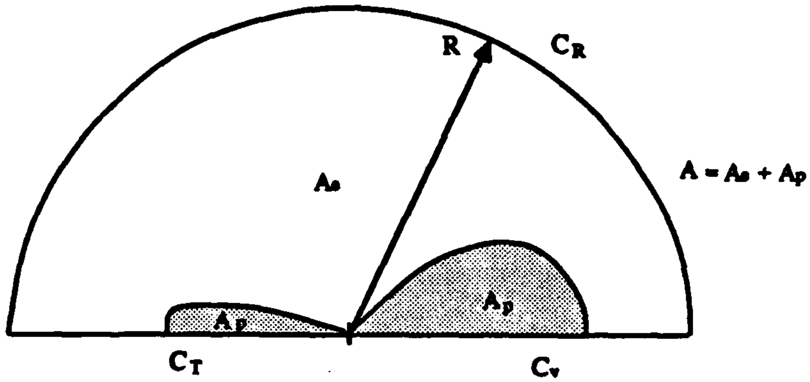

Figure 3—Domain for the variational statement of the steadily-propagating crack in SSY.

Next we apply our variational statement to an appropriate domain A as shown in Figure 3. A has been chosen for convenience to be a semi-circle of large radius R, and is composed of two parts: the active plastic zones A_p and the zone of current elastic behavior A_e. Note that on the crack face C_T, $\tau_n = 0$ as required by (2.7a), and that on the line ahead of the crack C_v

$$v^* = 0 \quad \Rightarrow \quad \int_{C_v} v^*\,\delta\tau_n\,ds = 0$$

On the rest of the boundary C_R, we assume that R is large enough that we can take $v^* = v^\infty$. Then

$$v^* = O(r^{-1/2}) \text{ and } \delta\tau_n = O(r^{s-2}) \quad \Rightarrow \quad \int_{C_R} v^*\,\delta\tau_n\,ds \to 0 \quad \text{as } R \to \infty$$

Thus the variational statement reduces to

$$\int_A \boldsymbol{\Sigma}\cdot(\partial\boldsymbol{\Sigma}/\partial\kappa)\,dA \;+\; (\alpha^{-1}-1)\int_{A_p}(\Sigma_e'/\Sigma_e)\,\boldsymbol{\Sigma}\cdot(\partial\boldsymbol{\Sigma}/\partial\kappa)\,dA \;=\; 0$$

$$\int_A \boldsymbol{\Sigma}\cdot(\partial\boldsymbol{\Sigma}/\partial\gamma)\,dA \;+\; (\alpha^{-1}-1)\int_{A_p}(\Sigma_e'/\Sigma_e)\,\boldsymbol{\Sigma}\cdot(\partial\boldsymbol{\Sigma}/\partial\gamma)\,dA \;=\; 0 \qquad (2.13)$$

where A is now the upper-half plane $x_2 \geq 0$, and A_p refers only to the active plastic zones. Note, therefore, that the plastic wake does not enter directly in the calculation of $\kappa(\alpha)$, and also that no integrations are needed beyond the quadrature of the variational integrals. Note further that although the fields (and in fact also some of the integrands in (2.13)) are singular and the domain of integration is unbounded, the integrals themselves are bounded.

Equations (2.13) amount to a system of two algebraic equations in κ and γ to be solved numerically using a secant method available as a package from I.M.S.L. The area integrals were evaluated as iterated integrals using successive Gaussian quadratures. The difficulty in the known singularity in r as $r \to 0$ was treated by subtracting the singularity from the integrand and evaluating its contribution analytically. Alternatively, the difficulty in the known singularity in θ as $\theta \to \pi$, was treated by a stretching technique.

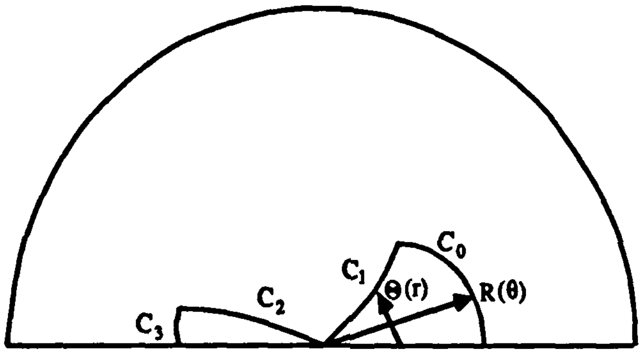

Figure 4—The elastic-plastic boundaries for the steadily-propagating crack in SSY.

Finally, let us state for completeness the conditions that determine the elastic-plastic boundaries shown in Figure 4. The plastic loading boundary C_0 is determined by

$$\Sigma_e^2 = \Sigma_1^2 + \Sigma_2^2 = 1$$

which is interpreted as an algebraic equation for the radial coordinate of the boundary $\mathbf{R}$ as a function of θ.

The elastic unloading boundary C_1 is determined by

$$\Sigma_e \Sigma_e^{\cdot} = \Sigma_1 T_1 + \Sigma_2 T_2 = 0$$

which in turn is interpreted as an algebraic equation for the angular coordinate of the boundary Θ as a function of $\mathbf{r}$.

In anti-plane strain, we will not consider values of the hardening parameter for which reloading takes place, and therefore we will not discuss reloading here, but it is clear that a similar treatment of the plastic reloading boundary C_2 and the secondary elastic unloading boundary C_3 (if necessary) is appropriate.

2.3 *Results*

The results of the numerical calculations are given in Table 1, and are depicted in Figures 5 and 6. It is seen that for mode III, $\kappa(\alpha)$ remains roughly constant, and γ varies dramatically, especially near $\alpha = 1$. As we have already remarked, $\kappa(1)$ is expected to be unity, but on the other hand, $\gamma(1)$ is undetermined by our choice of $p(\underline{r})$ and $q(\underline{r})$ (since $p + q = 1$, independently of the value of γ), and its steep behavior near $\alpha = 1$ is not too disturbing. In the limit of small α, the behavior of κ seems to be consistent with $\kappa \rightarrow 1$, as expected, but our results are not conclusive since we do not trust our trial function for f for very small α. This is due to the fact that the limit as $\alpha \rightarrow 0$ of the near-tip fields is not uniform, and the range of dominance of these fields vanishes in the limit as $\alpha \rightarrow 0$ (see Ponte Castañeda, 1985c).

The active plastic zones corresponding to several values of α are shown in Figure 7 using nondimensional coordinates in the axes. It is seen that the plastic zone gets shifted forward as α decreases. Also remark that there is a sharp kink at the intersection of the plastic loading and elastic unloading boundaries, and that the unloading boundary naturally intersects the plastic loading boundary at the place where its slope vanishes.

Next we will compare our results to the finite element method results of Dean and Hutchinson (1980) for linear-hardening in mode III, as improved by Mataga (1986). Figures 8 are plots of the logarithm of $\underline{\tau}_2$ in the line ahead of the crack versus the logarithm of $\underline{r}$. Assuming that the power singularity solution holds close to the crack tip we have

$$\underline{\tau}_2(r,0) = \kappa^* \underline{r}^{s^*}$$

so that

$$\ln \underline{\tau}_2 = \ln \kappa^* + s^* \ln \underline{r}$$

We fit a straight line through those points in Figures 8 that are close to the tip (far from the elastic-plastic boundaries which show as kinks in the graphs), and we compute the values of s^* and κ^*. Table 2 shows the results, and how they compare with our results for s and κ. It is found that s^* is not in very good *quantitative* agreement with the theoretically correct s, particularly for the smaller values of α. Given this, we can only conclude that κ^* is in adequate *qualitative* agreement with κ.

The active plastic zones corresponding to the finite element results (crosses) are shown in Figures 9, together with our plastic zones (solid lines) for comparison. It is found that our plastic zones have the same general shape as the finite element plastic zones. However, it is not clear which plastic zones are more accurate. The finite element plastic zones probably suffer from inaccuracy due to the relative coarseness of the grid, particularly at the unloading boundary far away from the tip, where the results are very sensitive due to the flat variation of τ_e near unloading. Alternatively, our results are probably not very good for the smaller values of α due to the expected decrease in the range of dominance of the near-tip fields.

3. Plane strain (mode I)

3.1 *The plastic stress intensity factor in small-scale yielding*

In plane strain the appropriate dimensionless variables are

$$\begin{aligned} \underline{r} &= (\sigma_0 / K^{el})^2 r \\ \underline{\sigma}_{\alpha\beta} &= \sigma_{\alpha\beta} / \sigma_0 \qquad \underline{\sigma}_{33} = \sigma_{33} / \sigma_0 \\ \underline{v}_\alpha &= (E / \sigma_0)(v_\alpha / V) \end{aligned} \qquad (3.1)$$

where σ_0 is the yield stress in tension.

In small-scale yielding the remote fields are

$$
\left.\begin{aligned}
\underline{\boldsymbol{\sigma}}^{\infty}(r,\theta) &= \bar{\boldsymbol{\sigma}}^{\infty}(\theta)\,(2\pi \underline{r})^{-1/2} \\
\underline{\mathbf{v}}^{\infty}(r,\theta) &= -2\,\bar{\mathbf{v}}^{\infty}(\theta)\,(2\pi \underline{r})^{-1/2}
\end{aligned}\right\} \quad |\theta| < \pi \qquad (3.2)
$$

where $\bar{\sigma}^{\infty}(\theta)$ and $\bar{v}^{\infty}(\theta)$ are well-known, and normalised such that $\bar{\sigma}_{22}^{\infty}(0) = 1$.

The near-tip fields are

$$
\begin{aligned}
\underline{\boldsymbol{\sigma}}^{o}(r,\theta) &= \kappa\,\bar{\boldsymbol{\sigma}}^{o}(\theta)\,(2\pi \underline{r})^{s} \\
\underline{\mathbf{v}}^{o}(r,\theta) &= \kappa\,\bar{\mathbf{v}}^{o}(\theta)\,(2\pi \underline{r})^{s} / s
\end{aligned} \qquad (3.3)
$$

where $s = s(\alpha,\nu)$, and the angular variations have been normalized such that $\bar{\sigma}_{22}^{o}(0) = 1$. It is then clear that for small-scale yielding

$$\kappa = \kappa(\alpha,\nu)$$

and the dimensional plastic stress intensity factor is given by

$$K^{pl} = \lim_{r \to 0} (2\pi \underline{r})^{-s}\,\sigma_{22}(r,0) = \kappa(\alpha,\nu)\,(\sigma_0)^{1+2s}\,(K^{el})^{-2s} \qquad (3.4)$$

Note that in the elastic limit we must have

$$\kappa(1,\nu) = 1$$

and that due to the high level of triaxiality in the line ahead of the propagating crack in the perfectly-plastic material (see Rice, 1980), we expect to have

$$\kappa(0,\nu) \cong 3$$

depending on the exact value of ν.

3.2 *The variational statement of the problem*

In plane strain the variational statement of compatibility is:

If

$$\sigma_{\alpha\beta,\beta} = 0 \quad \text{in} \quad A \qquad (3.5a)$$

$$T_{\alpha} = \sigma_{\alpha\beta}\,n_{\beta} = 0 \quad \text{on} \quad C_T \qquad (3.5b)$$

and if

$$\int_A D_{\alpha\beta}\,\delta\sigma_{\alpha\beta}\,dA = \int_{C_v} v_{\alpha}^{*}\,\delta T_{\alpha}\,ds \qquad (3.5c)$$

for arbitrary $\delta\sigma_{\alpha\beta}$ satisfying (3.5a) and (3.5b).

Then

$$D_{\alpha\beta} = (1/2)\,(v_{\alpha,\beta} + v_{\beta,\alpha}) \tag{3.5d}$$

and

$$v_\alpha = v_\alpha^* \text{ on } C_v \tag{3.5e}$$

where A, C_T and C_v have the same meaning as before.

The constitutive relation in dimensionless form states

$$\underline{D}_{\alpha\beta} = (1+\nu)\,\underline{\Sigma}_{\alpha\beta} - \nu\,(\underline{\Sigma}_{\alpha\alpha} + \underline{\Sigma}_{33}) + \underline{D}_{\alpha\beta}{}^{pl} \tag{3.6}$$

where $\underline{D}$ is the strain-rate tensor, $\underline{\Sigma}$ is the stress-rate tensor, and

$$\underline{D}_{\alpha\beta}{}^{pl} = \begin{cases} 0 & \text{in } A_e \\ (3/2)\,(\alpha^{-1}-1)\,(\underline{\sigma}_e{}^{\cdot}/\,\underline{\sigma}_e)\,\underline{S}_{\alpha\beta} & \text{in } A_p \end{cases} \tag{3.7}$$

refers to the plastic strain-rate tensor. Here

$$\underline{\sigma}_e{}^2 = \underline{\sigma}_{11}{}^2 + \underline{\sigma}_{22}{}^2 + \underline{\sigma}_{33}{}^2 + 3\,\underline{\sigma}_{12}{}^2 - \underline{\sigma}_{11}\,\underline{\sigma}_{22} - \underline{\sigma}_{22}\,\underline{\sigma}_{33} - \underline{\sigma}_{33}\,\underline{\sigma}_{11}$$

We can automatically satisfy (3.5a) by means of a stress function f such that

$$\underline{\sigma}_{11} = f_{,\underline{22}} \qquad \underline{\sigma}_{22} = f_{,\underline{11}} \qquad \underline{\sigma}_{12} = -f_{,\underline{12}} \tag{3.8}$$

In addition, we require $\underline{\sigma}_{33}$, which is determined by the plane-strain condition

$$\underline{D}_{33} = 0 = \begin{cases} \underline{\Sigma}_{33} - \nu\,\underline{\Sigma}_{\alpha\alpha} & \text{in } A_e \\ \underline{\Sigma}_{33} - \nu\,\underline{\Sigma}_{\alpha\alpha} + (3/2)\,(\alpha^{-1}-1)\,(\underline{\sigma}_e{}^{\cdot}/\,\underline{\sigma}_e)\,\underline{S}_{33} & \text{in } A_p \end{cases} \tag{3.9}$$

but this in general involves solving a complicated differential equation in A_p. The difficulty is avoided by restricting our consideration to the case of $\nu = 1/2$, when

$$\underline{\sigma}_{33} = (1/2)\,\underline{\sigma}_{\alpha\alpha} \qquad \text{in } A \tag{3.10}$$

For this reason we will consider here only the case of $\nu = 1/2$. The procedure for determining κ is then analogous to that of the mode III case. We take

$$f(\underline{r},\theta) = p(\underline{r})\,f^{o}(\underline{r},\theta) + q(\underline{r})\,f^{\infty}(\underline{r},\theta) \tag{3.11}$$

where

$$f^{o}(\underline{r},\theta) = \kappa\, g^{o}(\theta)\; \underline{r}^{2}\,(2\pi \underline{r})^{s}, \qquad g^{o}(\theta) = \bar{\sigma}_{\theta\theta}{}^{o}(\theta)\,/\,[(1+s)(2+s)]$$

$$f^{\infty}(\underline{r},\theta) = g^{\infty}(\theta)\; \underline{r}^{2}\,(2\pi \underline{r})^{-1/2}, \qquad g^{\infty}(\theta) = (1/3)\,[3\cos(\theta/2) + \cos(3\theta/2)]$$

and where $p(\underline{r})$ and $q(\underline{r})$ are again given by (2.12).

Given f we use (3.8) to compute the stress components. We also compute the stress-rate components from

$$\Sigma_{11} = -f_{,\underline{22}} \qquad \Sigma_{22} = -f_{,\underline{11}} \qquad \Sigma_{12} = f_{,\underline{12}} \qquad \Sigma_{33} = (1/2)\,\Sigma_{\alpha\alpha}$$

In these calculations the following derivatives of $g^{o}(\theta)$ are needed

$$(g^{o})'(\theta) = -\,\bar{\sigma}_{r\theta}{}^{o}(\theta)\,/\,(1+s)$$

$$(g^{o})''(\theta) = \bar{\sigma}_{rr}{}^{o}(\theta) - (2+s)\,g^{o}(\theta)$$

$$(g^{o})'''(\theta) = (\bar{\sigma}_{rr}{}^{o})'(\theta) - (2+s)\,(g^{o})'(\theta)$$

all of which are expressed in terms of quantities readily available from the numerical integration of system (3.5) in Ponte Castañeda (1985c).

The variation of the stresses is obtained by assuming κ and γ to be independent

$$\delta\underline{\sigma}_{\alpha\beta} = (\partial\underline{\sigma}_{\alpha\beta}/\partial\kappa)\,\delta\kappa + (\partial\underline{\sigma}_{\alpha\beta}/\partial\gamma)\,\delta\gamma$$

and again it is found that if we take A to be the upper-half plane

$$\int_{C_v} v_{\alpha}{}^{*}\,\delta T_{\alpha}\,ds = 0$$

Thus the variational statement reduces to

$$\int_{A} [(1+\nu)\,\Sigma_{\alpha\beta}\,\delta\underline{\sigma}_{\alpha\beta} - \nu\,(\Sigma_{\alpha\alpha} + \Sigma_{33})\,\delta\underline{\sigma}_{\alpha\alpha}]\,dA \;+ \ldots$$

$$\ldots + (\alpha^{-1}-1)\int_{A_p} (\underline{\tau}_e{}^{\cdot}/\,\underline{\tau}_e)\,[(3/2)\,\underline{\sigma}_{\alpha\beta}\,\delta\underline{\sigma}_{\alpha\beta} - (1/2)\,(\underline{\sigma}_{\alpha\alpha} + \underline{\sigma}_{33})\,\delta\underline{\sigma}_{\alpha\alpha}]\;dA = 0 \qquad (3.12)$$

where A_p refers to the active plastic zones (both plastic loading and reloading).

Again, for the case of $\nu = 1/2$, no integrations are needed beyond those explicitly shown in the variational statement, which provides a system of two nonlinear algebraic equations κ and γ to be solved for given α.

3.3 *Results*

The results of the numerical calculations for $\nu = 1/2$ are presented in Table 3, and are shown graphically in Figures 5 and 6. It is observed that κ increases from unity as α decreases, whereas γ remains roughly constant for all α. More specifically, it is found that even though the resulting values of κ for $\alpha = 0.05$ and 0.1 seem to be inconsistent with the rest of the values of κ, the general trend of κ as α approaches zero, outlined in the figure, is consistent with the perfectly-plastic result of $\kappa = 2.9476$ (Slepyan, 1974). The anomalous behavior of the results for κ for small α is seen to be a consequence of the deficiency in the values of s for small α (see Ponte Castañeda, 1985c).

The active plastic zones corresponding to several values of α are shown in Figure 10, where the axis coordinates have been appropriately nondimensionalised. In spite of our comments in the previous paragraph, we have included in this figure the plastic zones corresponding to values of α equal to 0.05 and 0.1, for the purpose of explicitly showing some plastic reloading zones, which are an important feature of the plane-strain mode I problem. Another interesting feature is the fact that the extent of the plastic zones in the line ahead of the crack, r_p, is small, and in fact vanishes for $\alpha = 1$ and again for $\alpha \sim 0.1$. The fact that r_p vanishes for $\alpha = 1$ is intrinsic to the $\nu = 1/2$ case, but the fact that it vanishes again for $\alpha \sim 0.1$, depending on the specific value of ν, is more interesting, and its source can be traced to a change in sign of the quantity $\underline{\sigma}_{rr}^{o}(r,0) - \underline{\sigma}_{\theta\theta}^{o}(r,0)$, which forces $\bar{\underline{\sigma}}_{e}^{o}(r,0)$ to vanish when α takes on the critical value.

Unfortunately, no finite element results are available for the linear-hardening plane-strain mode I problem to compare with our results. Dean and Hutchinson (1980) only considered power hardening in mode I, and their results are quite different from ours.

4. Plane-stress (mode I)

4.1 *The plastic stress intensity factor*

In plane stress the appropriate nondimensional variables are the same as in plane strain as given by (3.1) with the exception of $\underline{\sigma}_{33}$, which is set equal to zero. The far fields are as given by (3.2) where the $\bar{\sigma}_{\alpha\beta}^{\infty}(\theta)$ are the same as in plane strain, but the $\bar{v}_{\alpha}^{\infty}(\theta)$ are different. The near-tip fields have the same form as (3.3), but the strength of the singularity, and the angular variation of the fields are as given in Section 4 of Ponte Castañeda (1985c). The plastic stress intensity factor has then the same general form as (3.4), where again we expect $\kappa = \kappa(\alpha, \nu)$.

4.2 *The variational statement of the problem*

The same development as for plane strain applies for plane stress, provided that we set $\underline{\sigma}_{33}$ equal to zero, and we evaluate the pertinent quantities that go into the variational statement (3.12) using the appropriate near-tip and remote fields. Note that because $\underline{\sigma}_{33}$ is equal to zero, independently of the value of ν, the variational statement applies equally well for any value of ν. For definiteness, however, we will perform the computations here for $\nu = 1/2$ only.

Note that unlike s, which was found to be independent of ν, κ appears to be a functions of ν as well as α. This is analogous to what happens in the stationary plastic crack problem. Nevertheless, we expect κ to become independent of ν in the limit as α approaches zero, since it can be shown (Rice, 1982) that the stress field of the perfectly-plastic problem in plane stress is independent of ν.

4.3 *Results*

The results of the calculations corresponding to the value of $\nu = 1/2$ are presented in Table 4, and in Figures 5 and 6. It is seen that for plane-stress, κ increases slightly from unity as α decreases, and γ remains roughly constant as α changes. Note that in the elastic limit κ approaches unity, but more interestingly, in the perfectly-plastic limit κ seems to be consistent with the result that one would expect from the solution of the perfectly-plastic problem with a centered-fan sector centered about the crack line (see Rice, 1982 and Dean, 1983) $\kappa = 2/\sqrt{3} \cong 1.1547$.

The active plastic zones obtained are shown in Figure 11. They have a shape similar to the corresponding anti-plane strain zones, but they are taller and skinnier, at least for the given range of α.

Dean (1983) produced the corresponding finite element method results for this problem. His results show again that the finite element method is not very good at predicting the strength of the plastic singularity for small values of α. However, we find that our plastic zones are similar in shape and size to his.

5. Discussion

It is apparent that this new approximate method to determine the plastic stress intensity factor of a steadily-moving crack in a linear-hardening material is at least more efficient than the corresponding finite element method. It also seems reasonable that it should produce more accurate results for the near-tip fields as long as the range of validity of the imposed near-tip singularity field is not too small. This method, however, suffers from some difficultties that we will address next.

The first of these difficulties, addressed here for convenience in the context of mode III, has to do with the fact that Σ_1, as calculated from our guess for f, becomes unbounded as $\theta \to \pi$, whereas it is known (see Slepyan, 1973 for mode III, or Ponte Castañeda, 1985b for the analogous result in mode I) that Σ_1 in fact remains bounded as $\theta \to \pi$. Indeed, we find by making use of the results for the analytical near-tip fields in the elastic sector found in Ponte Castañeda (1985c), and the given expression for f that our approximate $\mathcal{T}_1$ has the following behavior

$$\mathcal{T}_1 \sim [C\,\mathfrak{r}\,p'(\mathfrak{r})\,(\pi-\theta)^s \ + D + \ldots]\ \mathfrak{r}^{\,s-1} \qquad \text{as} \ \ \theta \to \pi$$

where

$$\mathfrak{r}\,p'(\mathfrak{r}) = -2\,\mathfrak{r}^2 / (\mathfrak{r}^2 + \gamma^2)^2$$

On the other hand, we have that the exact f is given by

$$f(r,\theta) \ = f^h(r,\theta) + f^{pa}(x_2)$$

where f^h yields a bounded contribution to $\mathcal{T}_1$ as $\theta \to \pi$, but f^{pa} yields no contribution at all, so that the exact $\mathcal{T}_1$ remains bounded as $\theta \to \pi$.

Nonetheless, this difficulty should not affect the values of κ and γ significantly. Recall that the stress $\mathfrak{t}_1$ itself becomes unbounded as $\theta \to \pi$, and therefore the fact that $\mathcal{T}_1 = -\mathfrak{t}_{1,\perp}$ should stay bounded as $\theta \to \pi$ is a very special circumstance, that does not carry much weight in the evaluation of the integrals in (2.13). The reason for this is that even though the approximate $\mathcal{T}_1$ becomes unbounded like $(\pi-\theta)^s$ as $\theta \to \pi$, the coefficient of this term vanishes faster both as $\mathfrak{r}$ approaches zero and infinity than the next order term in θ, which behaves like D as $\theta \to \pi$, making the artificial singularity in θ unimportant in relative terms.

A way to eliminate the difficulty, however, would be to start with a guess for the stress-rate function

$$\mathfrak{S}(r,\theta) = p(\mathfrak{r})\,\mathfrak{S}^o(r,\theta) + q(\mathfrak{r})\,\mathfrak{S}^\infty(r,\theta)$$

but this would require integrating $\mathfrak{S}$ with respect to x_1 to obtain $\mathfrak{t}_1$. It is conceivable, nonetheless, that one may be able to chose $p(\mathfrak{r})$ and $q(\mathfrak{r})$ such that the integration is easy, but we will not pursue this idea here any further.

Another important difficulty is the fact that the range of dominance of the near-tip fields, which are a key element of this method, is expected to shrink to zero as $\alpha \to 0$. The difficulty is compounded for the plane-strain mode I case where, as we have seen, we trust our small-α results even less. Finally, we should add that improved results would also be expected if we included the correct first order perturbation of the remote fields.

6. Concluding remarks

In this work, we have completed the knowledge of the near-tip stress and deformation fields of a crack propagating steadily and quasi-statically in a linear-hardening material under small-scale yielding conditions, by determining the plastic stress intensity factor associated with these fields. The form of these fields, and the associated strength of the singularity has been given by Ponte Castañeda (1985c).

The procedure, which so far has only been exploited in its crudest form, is fairly successful, provided that the strain-hardening is not too small, but this is not seen to be a major drawback because we expect that the correct results for small strain-hardening would not be too different from the well-known perfectly-plastic results. Nevertheless, the method should be subject to overall improvement by use of a more sophisticated choice for the stress function.

Finally, it should be pointed out that although the present work does not directly address the theoretical prediction of resistance curves, it seems reasonable that it could be extended to do so.

Acknowledgements

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Tables

α	κ	γ
0.9	1.030	2.895
0.8	1.044	1.880
0.7	1.054	1.524
0.6	1.058	1.318
0.5	1.055	1.163
0.4	1.044	1.006
0.3	1.005	0.685
0.2	0.976	0.513
0.1	0.943	0.509

Table 1—The values of κ and γ as functions of α for anti-plane strain.

α	s	s^*	κ	κ^*
0.9	-0.483	-0.475	1.030	1.01
0.5	-0.394	-0.375	1.055	1.02
0.3	-0.325	-0.297	1.005	1.01
0.2	-0.276	-0.247	0.976	0.99
0.1	-0.207	-0.169	0.943	0.98

(*) F.E.M. results.

Table 2—A comparison of the values of s and κ for anti-plane strain.

α	κ	γ
0.9	1.085	1.001
0.8	1.170	1.001
0.7	1.268	0.999
0.6	1.392	0.995
0.5	1.565	0.989
0.4	1.817	0.980
0.3	2.184	0.968
0.2	2.568	0.960
0.1	2.640	0.960

Table 3—The values of κ and γ for plane strain mode I ($\nu = 1/2$).

α	κ	γ
0.9	1.031	1.000
0.8	1.063	1.000
0.7	1.093	1.001
0.6	1.122	1.001
0.5	1.148	1.000
0.4	1.169	1.000
0.3	1.179	1.000
0.2	1.173	0.999
0.1	1.137	0.997

Table 4—The values of κ and γ for plane stress mode I (ν =1/2).

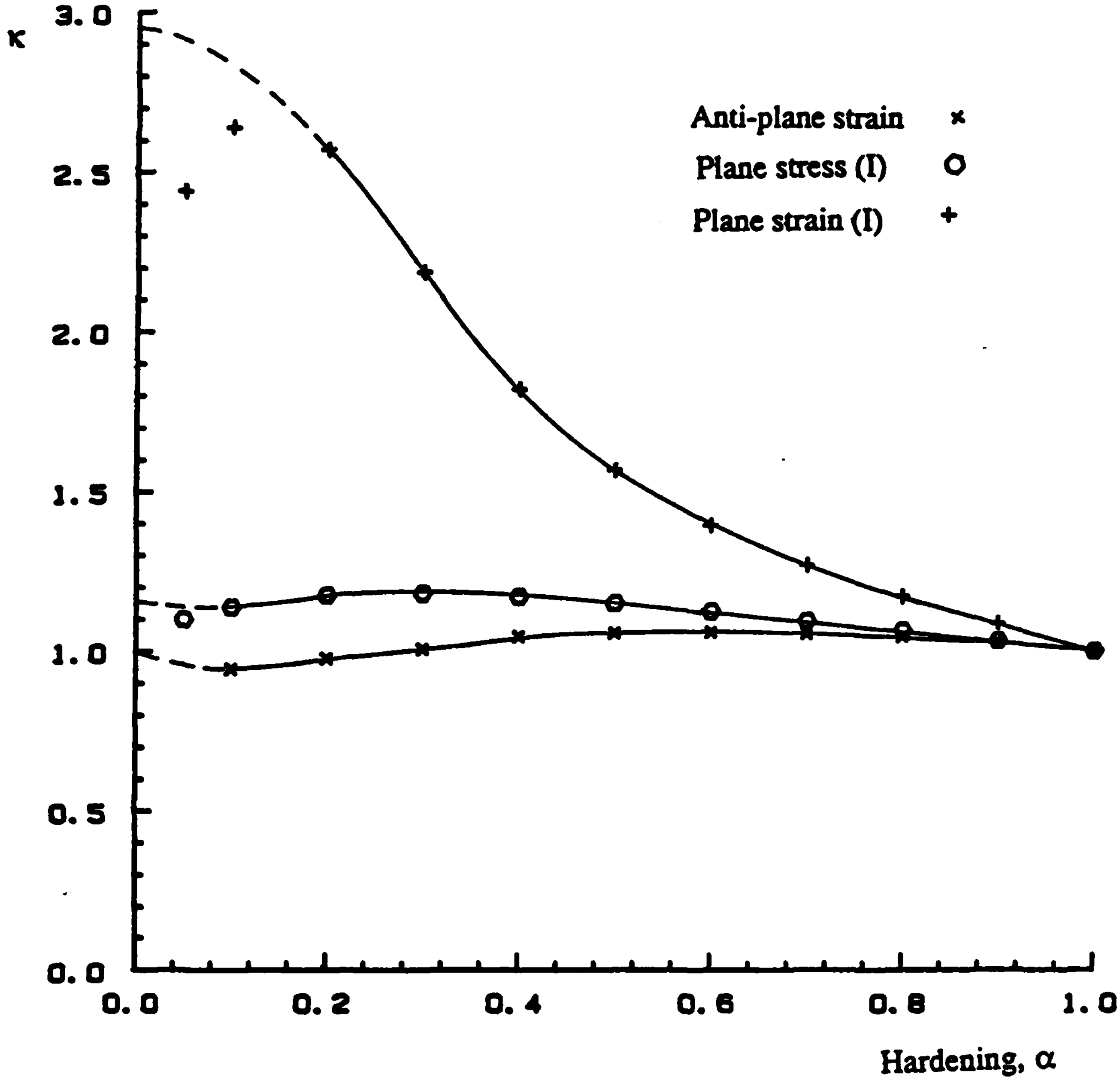

Figure 5—Plots of κ versus α for anti-plane strain, plane strain (I) and plane stress (I).

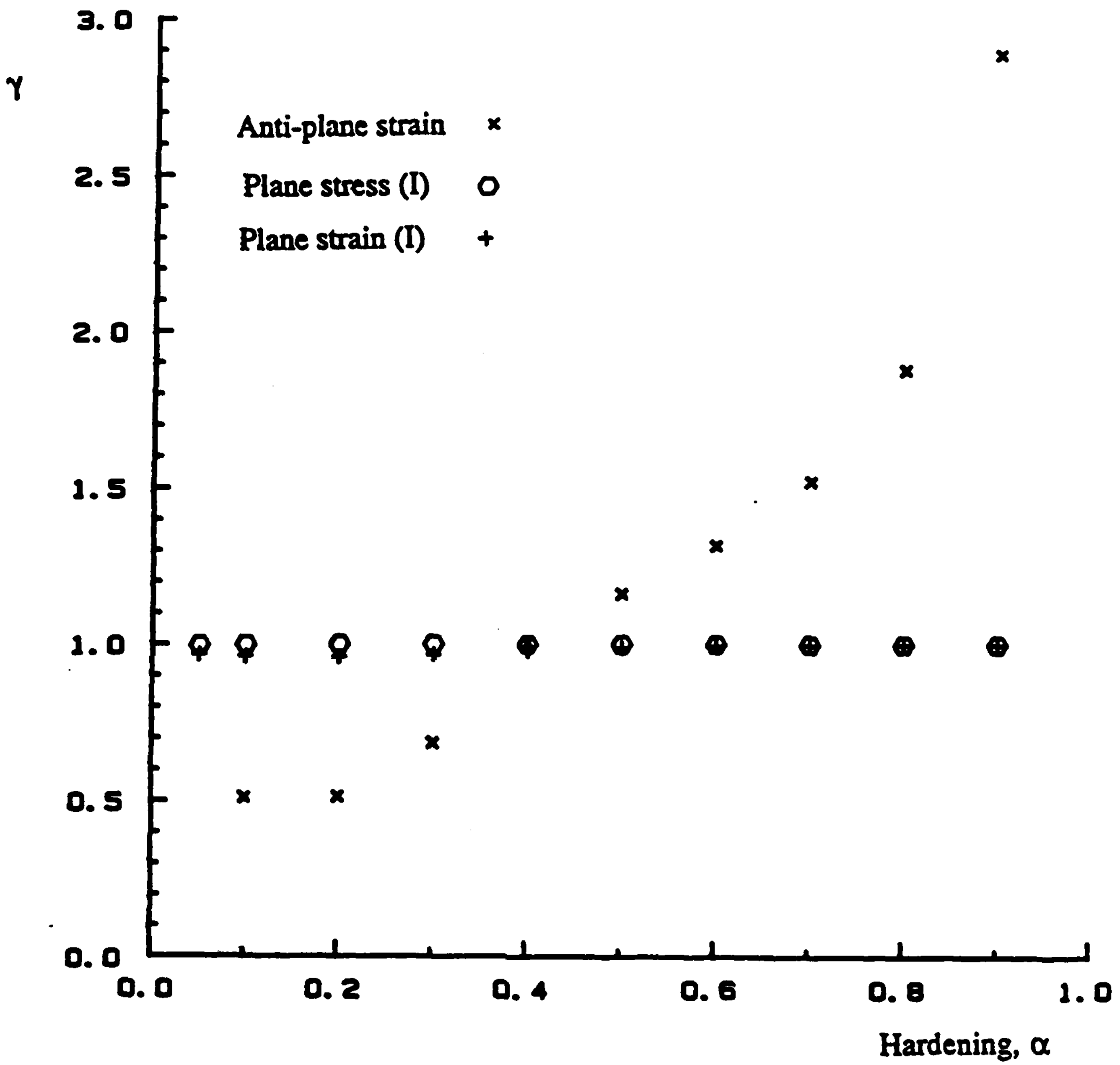

Figure 6—Plots of γ versus α for anti-plane strain, plane strain (I) and plane stress (I).

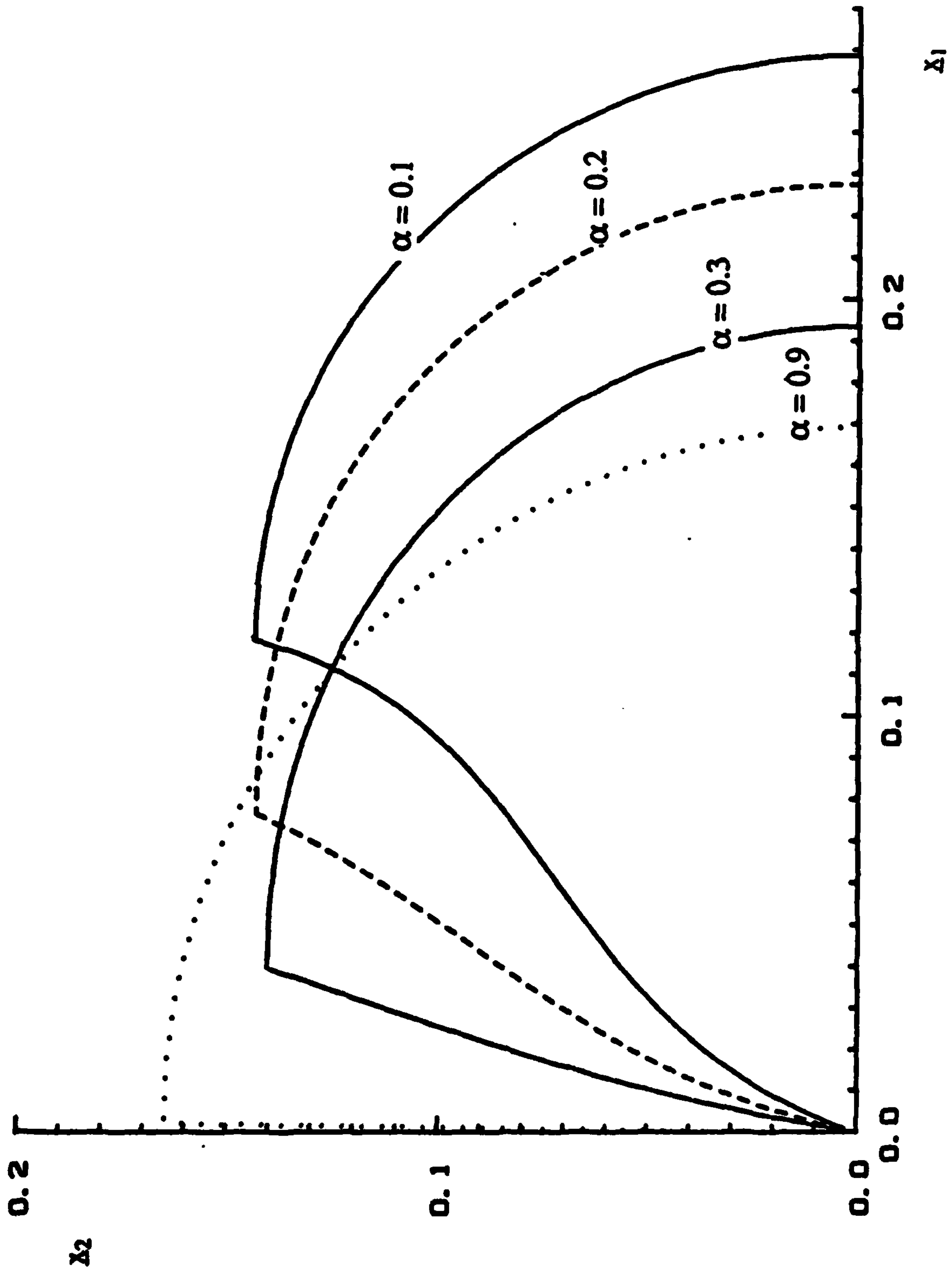

Figure 7—Active plastic zones in anti-plane strain.

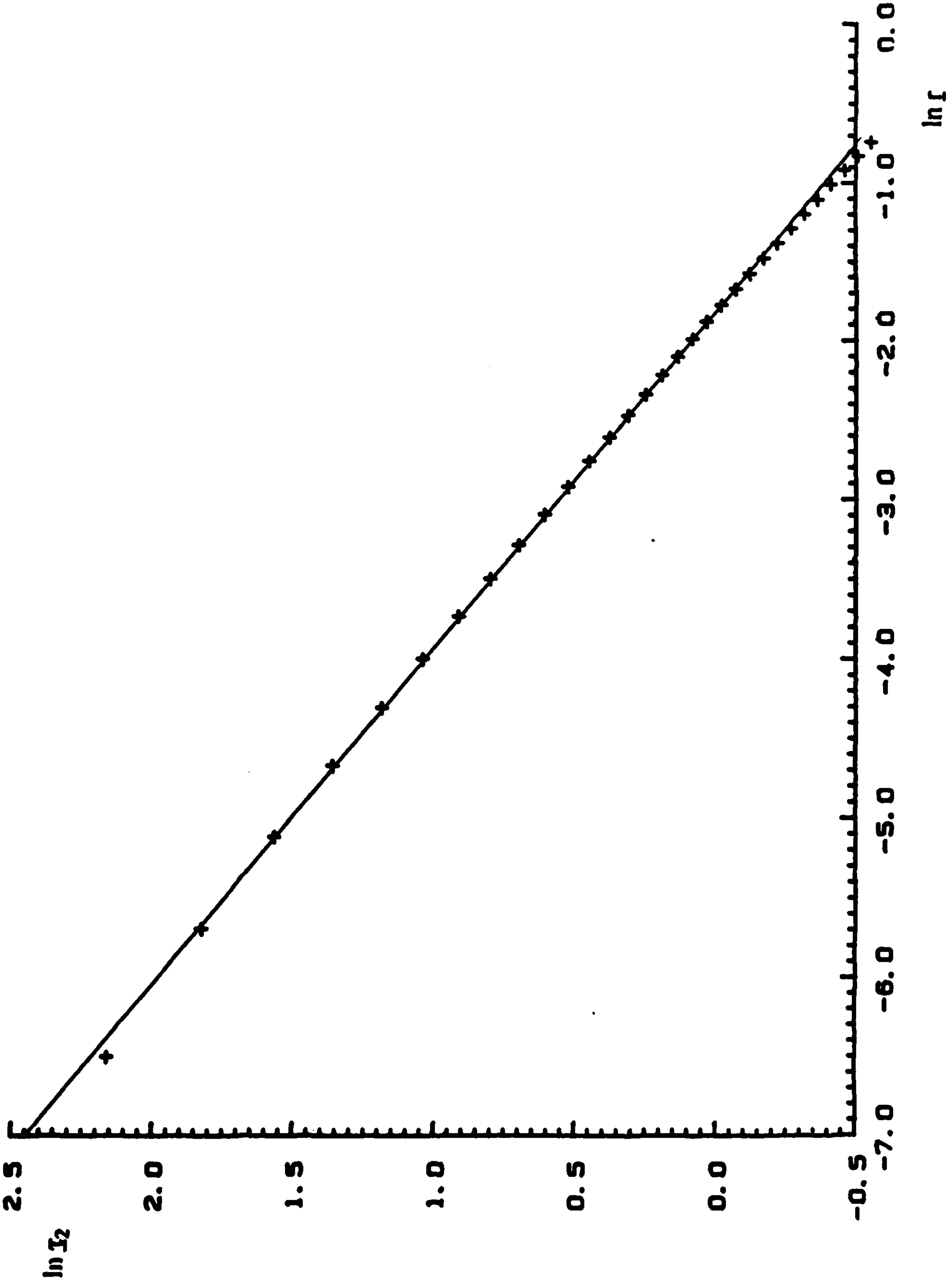

Figure 8a—F.E.M. results for the stress ahead of the crack ($\alpha = 0.9$)

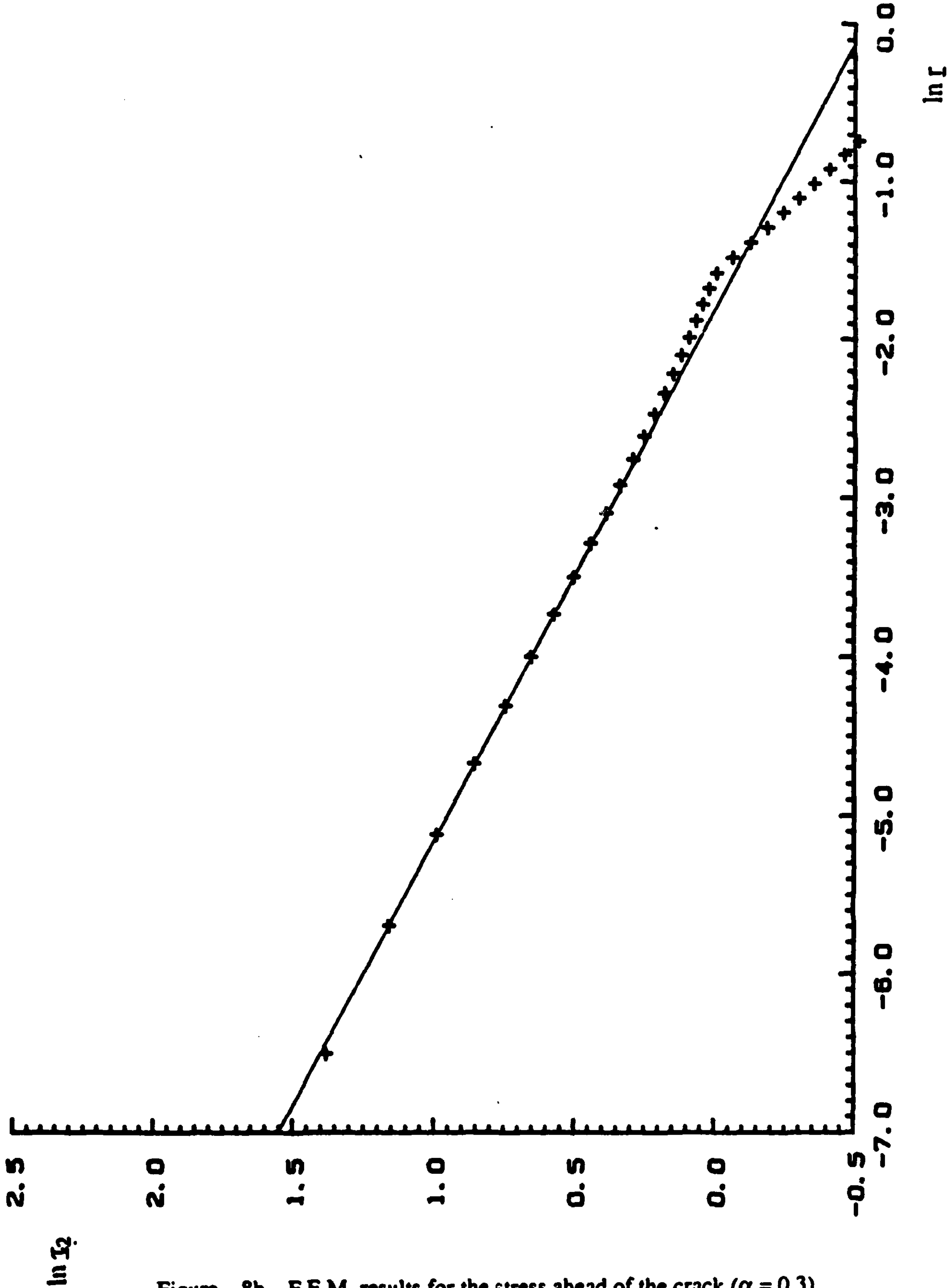

Figure 8b—F.E.M. results for the stress ahead of the crack ($\alpha = 0.3$)

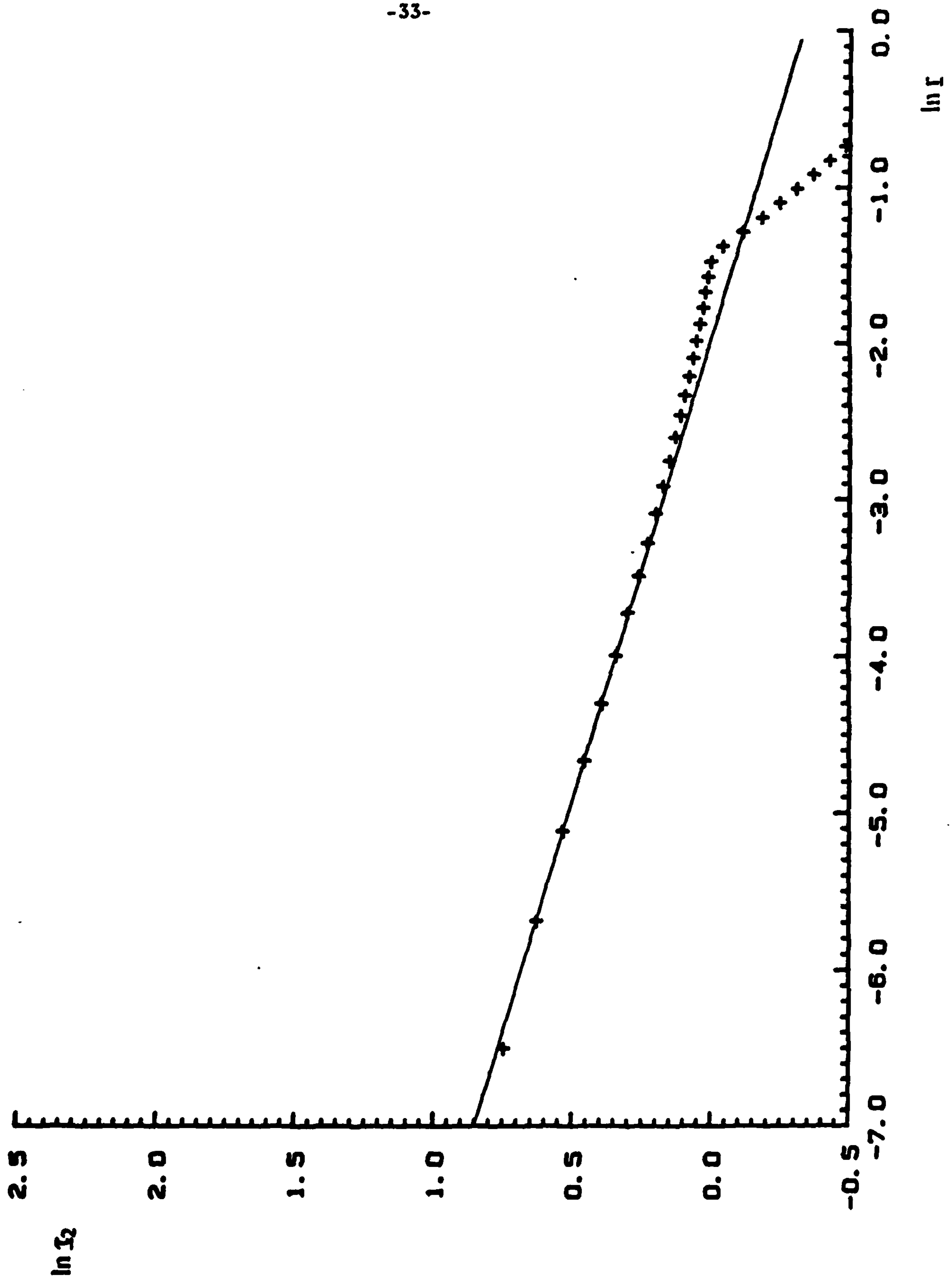

Figure 8c—F.E.M. results for the stress ahead of the crack ($\alpha = 0.1$)

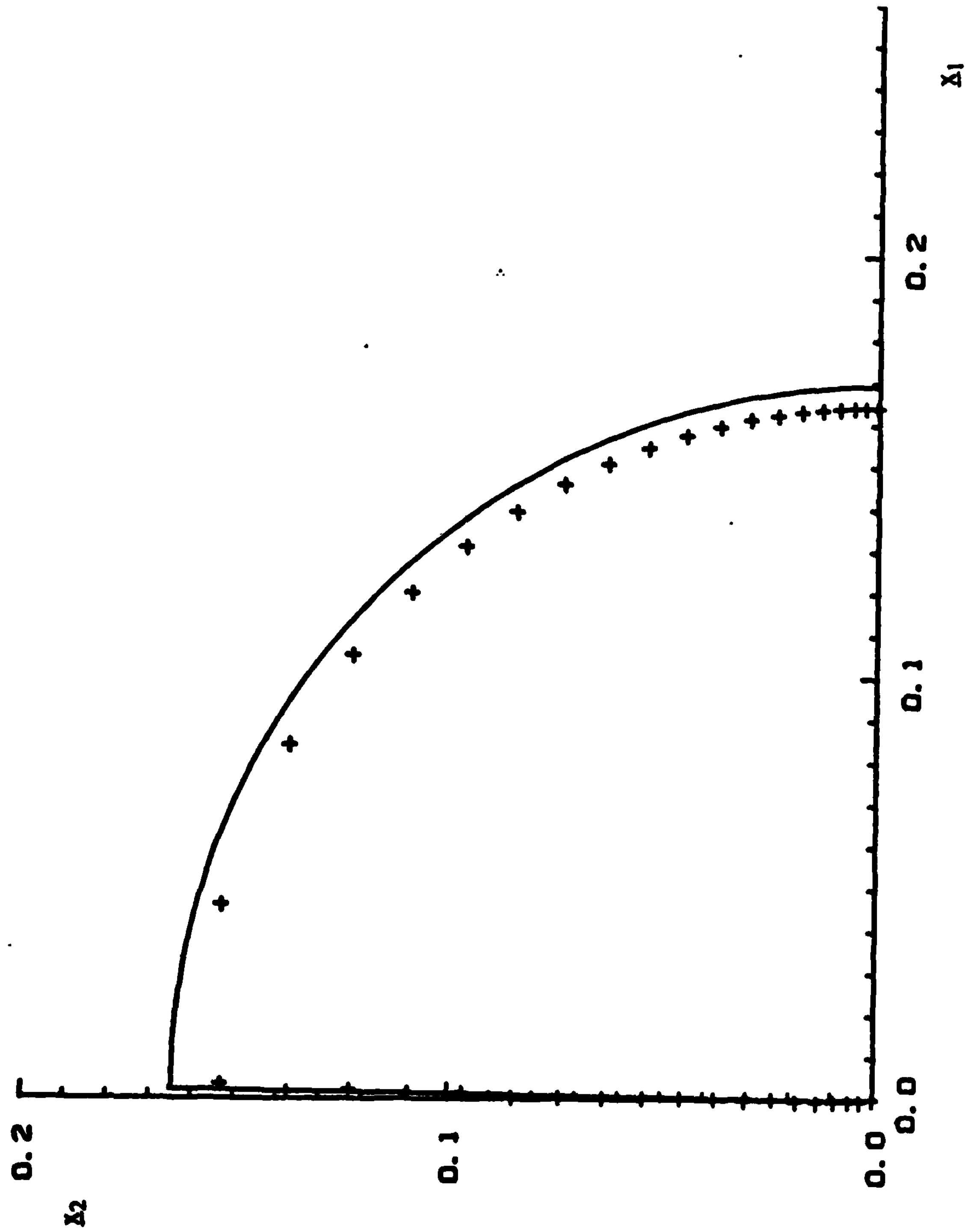

Figure 9a—Comparison of the F.E.M. active plastic zones in anti-plane strain ($\alpha = 0.9$)

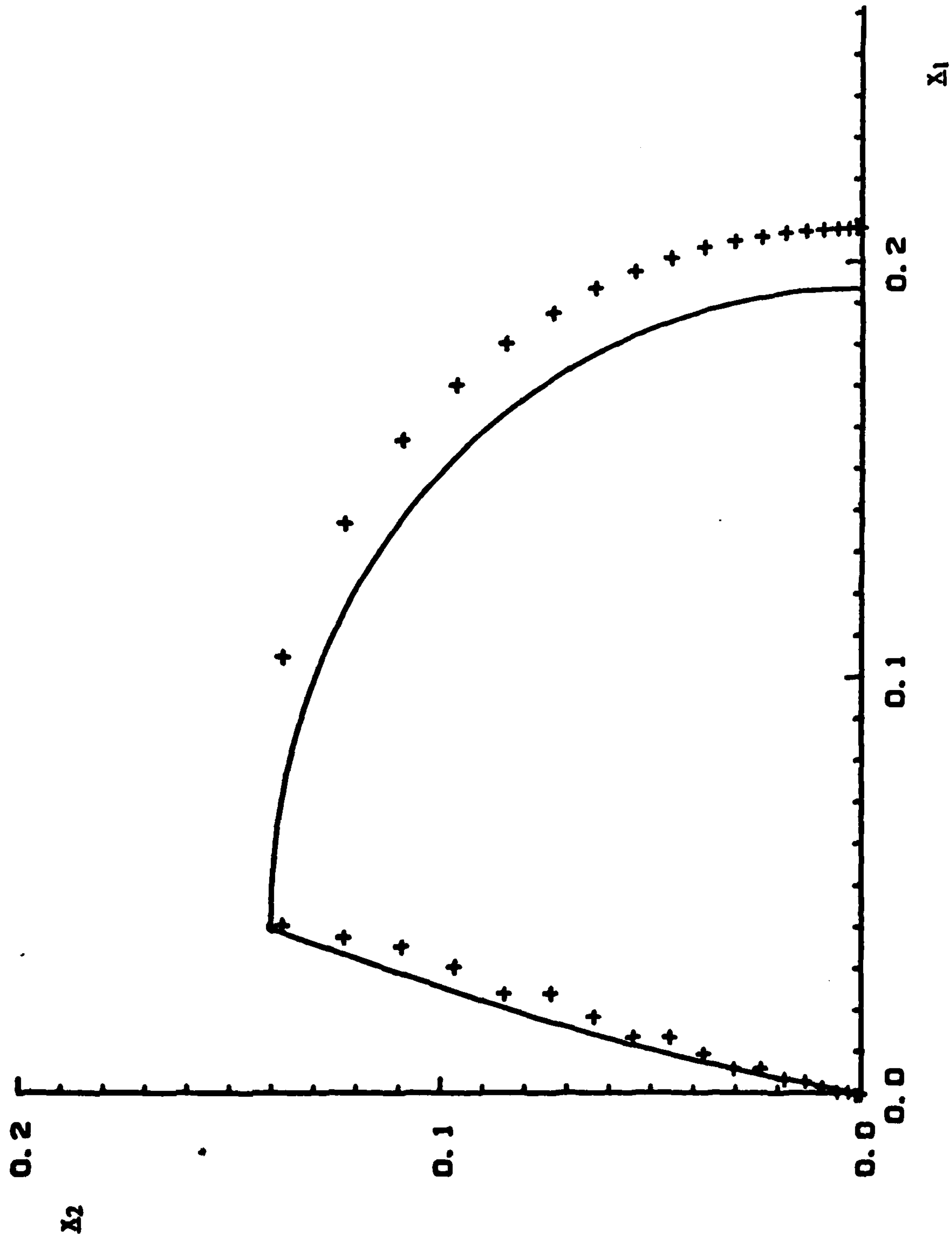

Figure 9b—Comparison of the F.E.M. active plastic zones in anti-plane strain ($\alpha = 0.3$)

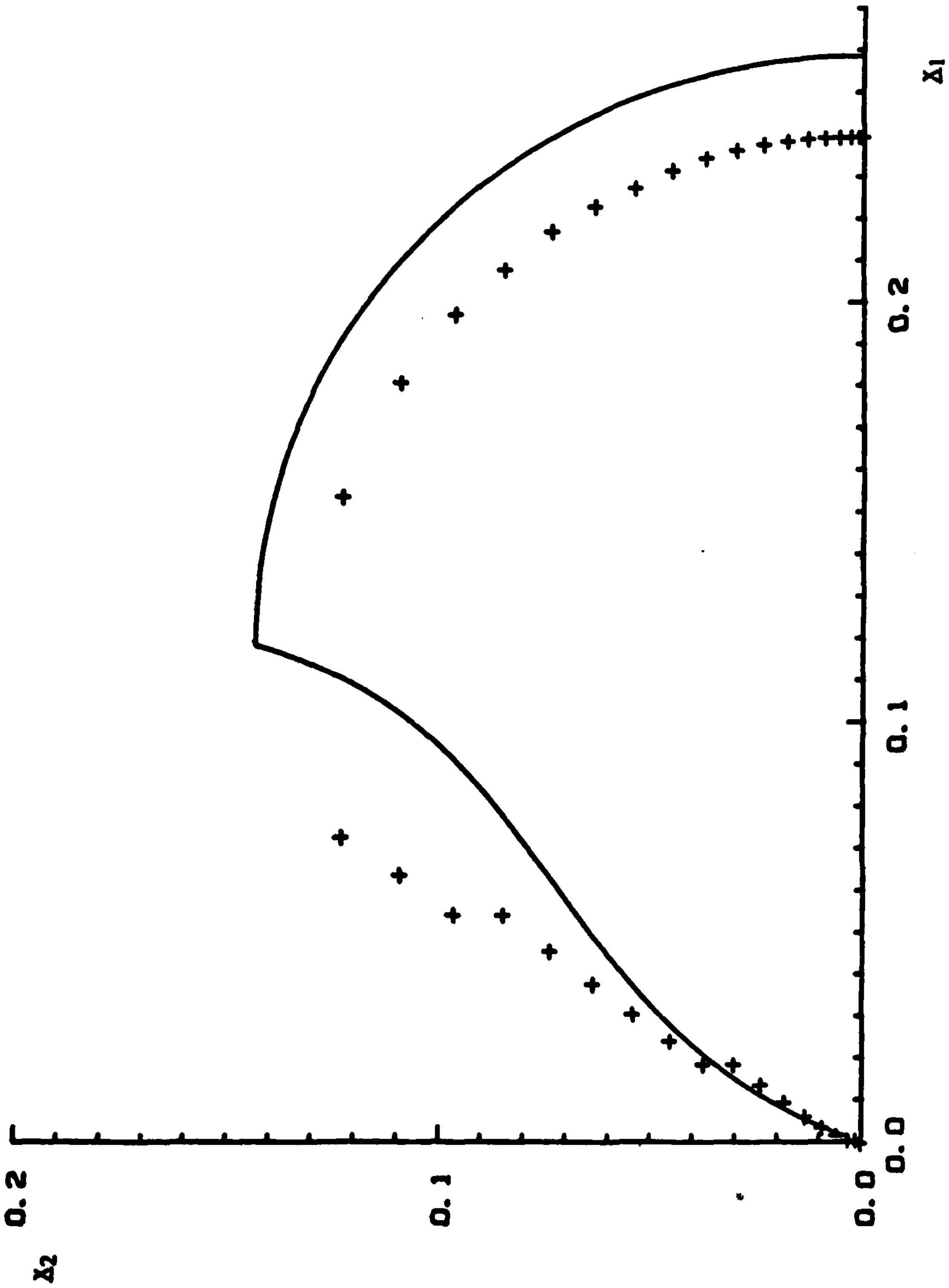

Figure 9c—Comparison of the F.E.M. active plastic zones in anti-plane strain ($\alpha = 0.1$)

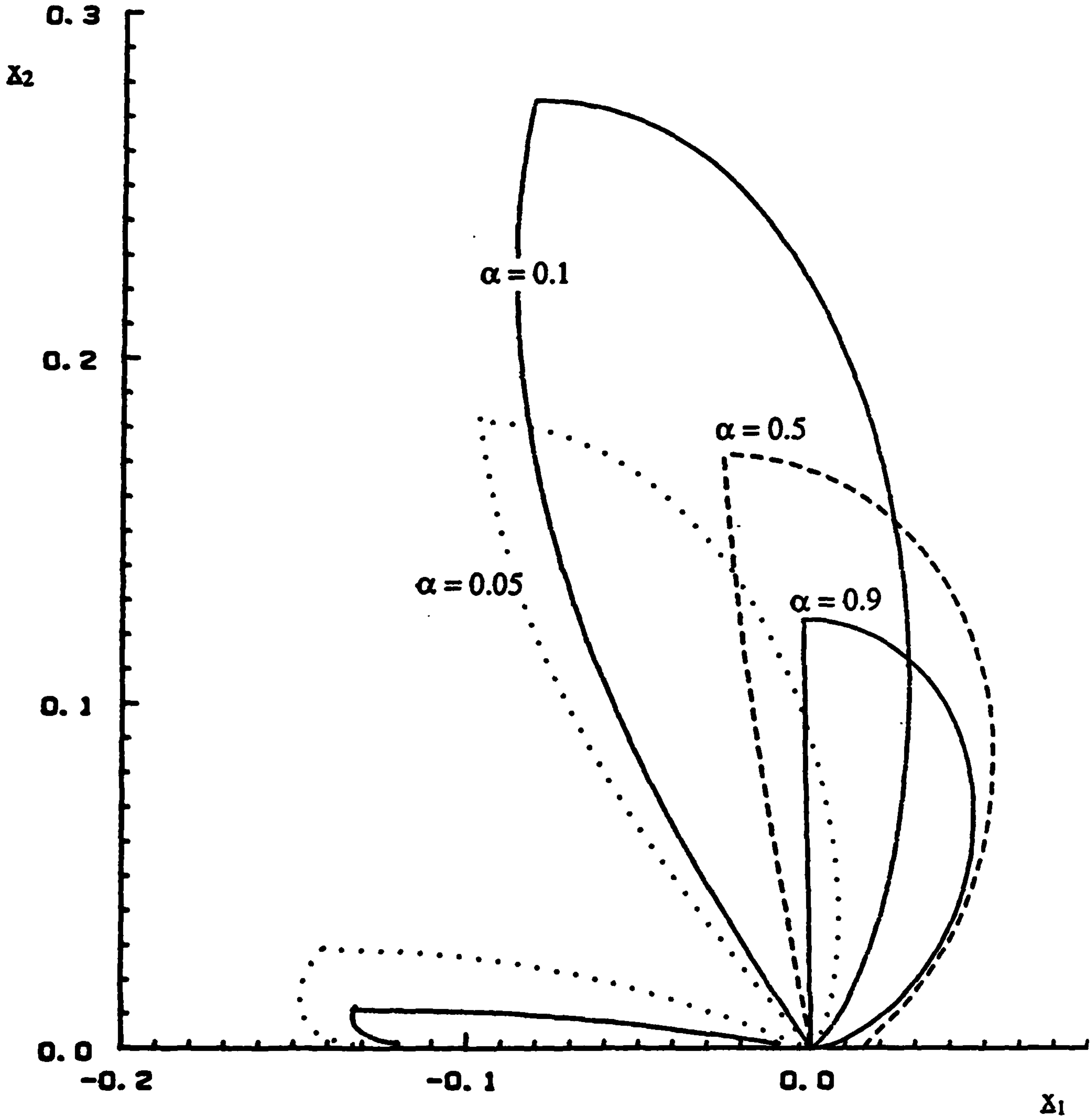

Figure 10—Active plastic zones in plane-strain mode I.

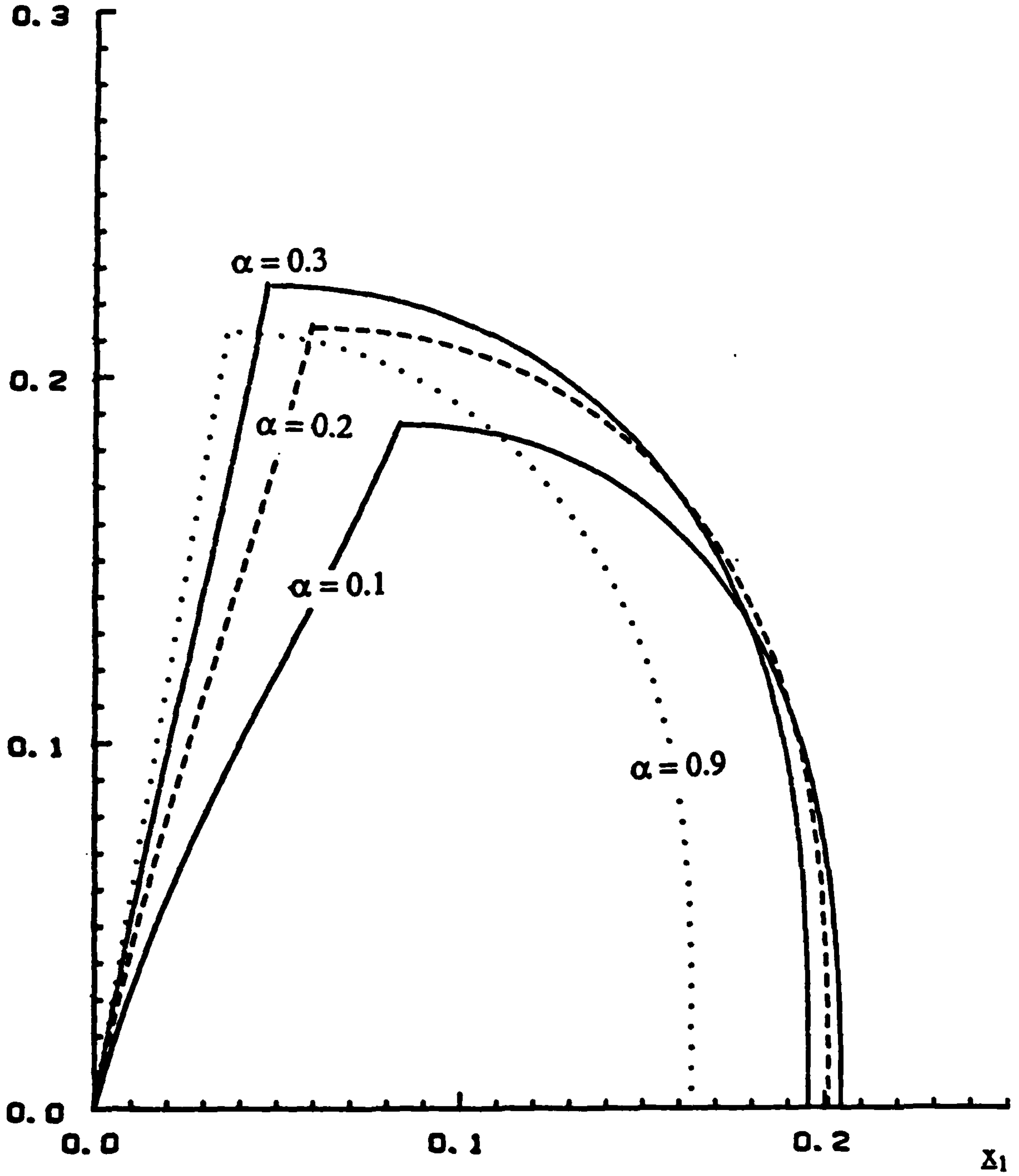

Figure 11—Active plastic zones in plane-stress mode I.

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