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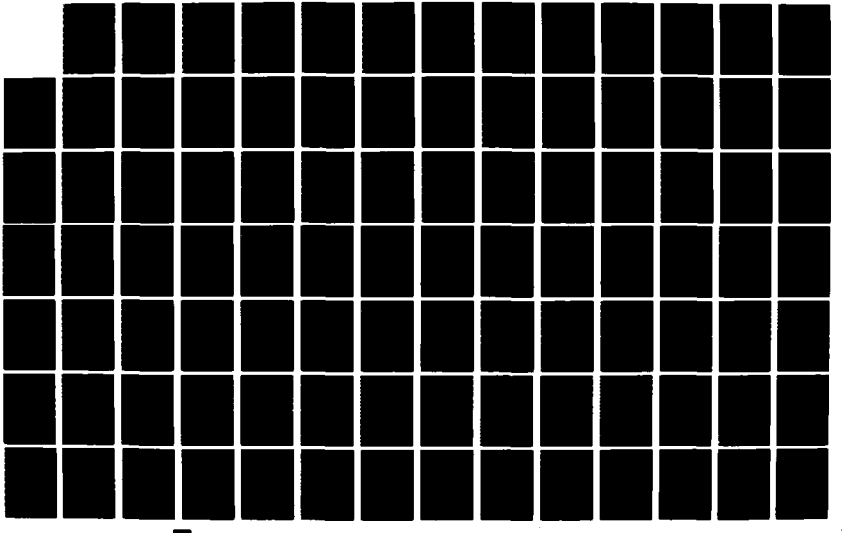

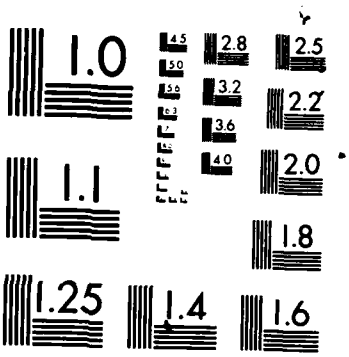

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A SURVEY OF PROBABILISTIC METHODS FOR DYNAMICAL SYSTEMS WITH UNCERTAIN PARAMETERS

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The purpose of this report is to present a survey of techniques that are available for studying the effect of random parameters on the response characteristics of linear dynamical systems. The report is directed towards the properties of the solutions of ordinary linear differential equations with random parameters. Both time independent (constant) random parameters, and time varying parameter systems are discussed. The motivation is the study of structures with random parameters.

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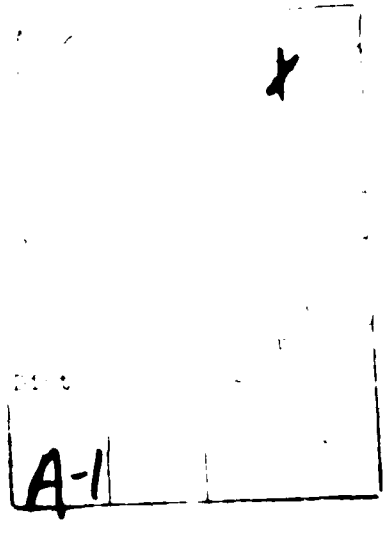

I. General Introduction

The purpose of this report is to describe the results of a survey on the techniques that are available for studying the effect of random parameters on the response characteristics for linear dynamical systems. We are, therefore, concerned with the properties of the solutions of ordinary linear differential equations with random parameters.

There are two classes of random parameters that we distinguish between in this report. They are time independent (constant) random parameters, and time varying random parameters. In both cases, structural applications are of interest.

Before, we outline the content of this report, we briefly consider the history of the general problem of parameter uncertainties and parameter fluctuations to see why they cannot be overlooked.

Very early in engineering design, factors of safety were introduced to account for our lack of precise knowledge of the structure and its loads, factors of safety quantified the fact that loads, material behavior, structural element properties, etc., could not be accurately estimated. Indeed Roebling and his engineers in 1880's, by means of very careful computations, estimated the factor of safety of the then new Brooklyn Bridge to be above five. Later estimates in 1944 placed this figure at four. These factors essentially tell us that column misalignment, residual stress due to manufacturing errors, reduction of working area due to corrosion, member weight, joint behavior, etc., etc., would have to be large indeed before the integrity of the structure could be put into question. In other words, parameter variability would have to be beyond all reasonable bounds before the bridge could fail. Roebling and his engineers were correct in their estimation of the effect of parameter variability on the safety of their bridge. However, in the first Quebec Railway Bridge (1907) errors in estimation of member weight, in the effect of column misalignment and in the behavior of new types of columns, lead to failure because the effect of parameter variability exceeded the bounds that the factors of safety could absorb.

Along the same lines, consider the buckling under axial loading of thin cylindrical and conical shells. The buckling load of such shells can be calculated. However, in 1950's, 1960's extensive tests of such shells revealed that the actual buckling loads were significantly lower than calculated. This discrepancy between calculated and actual was traced to the fact that there were random deviations from the regular geometrical shape assumed. These are reflected in the fact that the PDE's whose solution produces the buckling load contains random variations, in its parameters.

Randomly time varying parameters occur to a great extent as a result of environmental fluctuations that effect the system. The vibrations in aerospace vehicles due to atmospheric turbulence is one prime example. This is reflected in random load variations on the structure which are of particular significance, for example, on rotating lifting surfaces such as helicopter blades. There has been a great deal of attention to this problem due to the critical nature of the stability and safety of helicopters.

Liquid sloshing in the tanks which are undergoing vertical excitations, also is a problem that was actively studied for the stability of the initial atmospheric stages of the lifting of large rockets. In both of these problems, randomly fluctuating parameters are present in the analytical equations that model the response characteristics of these components. In general, inverted beams, pendulums as well as aerodynamic panels subjected to random end loads, will be described by models with randomly varying parameters.

Moreover, for the control of such systems with uncertain parameters, it is necessary to be able to characterize the response of the controlled system, in order to determine the accuracy required to achieve prescribed control accuracies.

Surveying the techniques and results in this overall class of problems, it appears that there exists a natural distinction between the case of random time dependent (fluctuating) parameters and the case of the random time independent (constant) parameters. We shall, therefore, distinguish between these two cases.

We shall carry on this distinction in the next sections of the Introduction, as well as the general survey of results.

I.1 Introduction - Time Varying Structural Equations

Time varying models of engineering systems occur for a number of geometrical, environmental as well as analytical reasons. They occur for geometrical reasons as a result of the location and directions that external excitations impact upon a system, while the system's physical parameters (damping, stiffness, etc.) are assumed to remain fixed. They occur as a result of environmental properties due to chemical effects, thermal effects, and radiation effects that are reflected in varying physical parameters for the components of the system. These may be of a periodic random nature or of a monotonic random nature due to ageing of the components in general. Finally, they occur in studies of non-linear systems. In particular, if one wishes to study the linearized equations of small oscillations about some specific (non-equilibrium) system response, time varying coefficients will be present.

In the geometrical case, those systems which are subjected to base excitations, such as pendulums and missiles; to end loadings, such as beams with various supports, and finally to boundary edge loadings, such as plates and shells, will be described by differential equations with time varying coefficients.

These cases are illustrated in the following figure

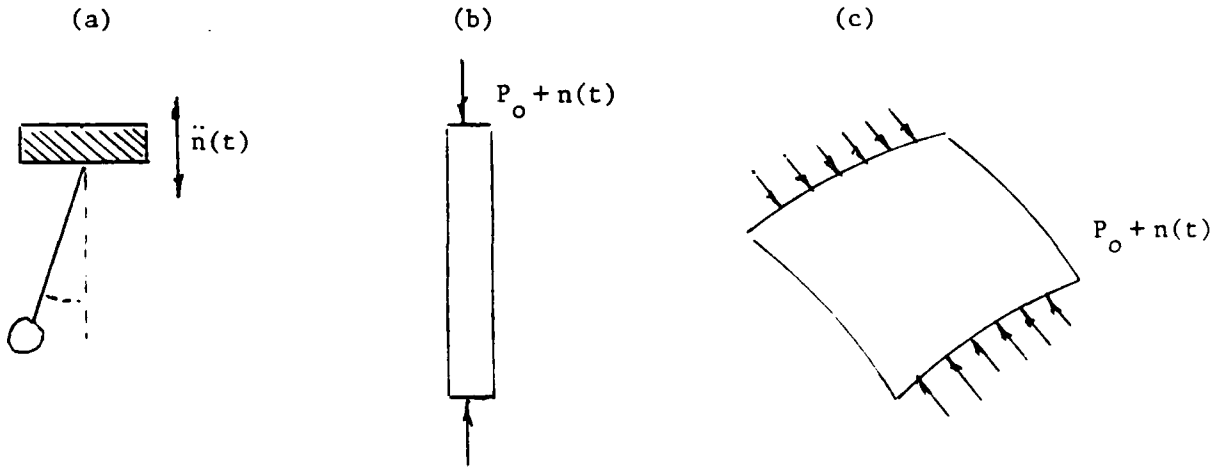

Figure 1

These cases lead to differential equations of the following form, [1.1, 1.2, 1.3]*

$$\underline{\text{Pendulum}}$$

$$\text{(a)}\quad \ddot{\theta} + c\dot{\theta} + (k + \ddot{n}(t))\theta = 0, \qquad (k < 0 \text{ for inverted pendulum})$$

$$\underline{\text{Single Supported Beam}} \tag{1.1}$$

$$\text{(b)}\quad w_{tt} + 2\beta w_t + (p_o + n(t))w_{xx} + w_{xxxx} = 0$$

$$w(o,t) = w_{xx}(o,t) = 0,\ w(1,t) = w_{xx}(1,t) = 0$$

$$\underline{\text{Infinite panel of unit width in supersonic flow}}$$

$$\text{(c)}\quad w_{tt} + 2\beta w_t + M w_x + (p_o + h(t))w_{xx} + w_{xxxx} = 0$$

(boundary conditions same as above)

Using modal expansions, (b) and (c) lead to equations of the form (a).

For the general linear structure that we shall be concerned with in this development, the model will be assumed to be of the form

$$M\ddot{\vec{x}} + C(t)\dot{\vec{x}} + K(t)\vec{x} = \vec{f}(t) \quad . \tag{1.2}$$

The response vector $\vec{x}$, will be n-dimensional for the n-mass structure, the nxn mass matrix M, will always assumed to be known and most often fixed (although this is not necessary). The nxn damping matrix C(t) and stiffness matrix K(t) will, in general, contain randomly varying elements. The n-vector $\vec{f}$ represents external excitations that may be random. The external excitation vector $\vec{f}$ does not pose any analytical difficulties and may be treated as the non-homogeneous part of any linear differential equation. It is the randomly fluctuating coefficients in the matrices C(t), K(t) that generate the difficulties. Time varying systems are difficult to analyze quantitatively even in the deterministic setting. Thus, the random setting will be at least as difficult. Naturally, all will depend upon the assumed properties of these random coefficients.

* References in Sections I-V are given after Section V.

If these random coefficients are due to fluctuating external loadings, they will have frequency spectra (power spectral densities in the second order stationary case) that will cover a band containing low frequency as well as high frequency components. If the bandwidth is large and relatively constant, then in many cases it makes sense to model the coefficient fluctuations as Gaussian white noise. If, on the other hand, there are definite peaks in the frequency spectrum and the larger frequency components are less pronounced (essentially band-limited) then Gaussian white noise is not a suitable model. In this case, so-called, physical noise is the proper model for the coefficient fluctuations. Finally, when the coefficient fluctuations are due to environmental effects (thermal, chemical, radiation, or ageing) it is natural to model the fluctuations as slowly varying (i.e., narrow band, low frequency). Small parameters can be applied via approximation schemes. In a related situation if there are small random fluctuations about a nominal value, again small parameters can be applied via approximation schemes. However, in this last case, we must be careful. For example for the simple undamped oscillator [1.4]

$$\ddot{x} + (\omega^2 + \varepsilon n(t))x = 0, \tag{1.3}$$

if $n(t)$ is the gaussian white noise then no matter how small $\varepsilon>0$, the second moments, as well as the sample solutions, will become unbounded. Thus, even though random fluctuations are small in their variances the system can still become unstable. This important point cannot be overstressed in analyzing systems with random coefficients. Therefore careful attention must be paid always to meaningful approximations, especially when small parameters are present.

The importance of the white noise assumption for the coefficient fluctuations is that the solution process is Markov allowing us to use the many tools available. In particular, the Fokker-Planck equation, the generator of the associated diffusion process and finally, the related Ito differential formula can be applied.

The statistical moments such as means, variances and covariances can be obtained explicitly for these linear models. In general, however, probability densities cannot be obtained explicitly.

For physical noise coefficient processes, the story is quite different. Here, we cannot even obtain the moments explicitly, unless very specific simplifying assumptions are made. For example, if the system coefficient matrices generate a Lie algebra [1.5] then the solution moments can be obtained explicitly. If the physical noise is almost a white noise, an associated system can be studied with white noise coefficients, having statistical properties that are similar to the original system [1.6], [1.7], [1.8]. Under assumptions of small parameters, an associated Markov process can be obtained, which also will yield valid approximations to the statistical properties of the original system. [1.9], [1.10], [1.11].

Finally, in lieu of all of these, successive approximations must be applied [1.12-1.18] simply based upon the assumed physical noise statistics.

The essential reason that the physical noise coefficient case cannot yield the exact statistical properties of the response, is simply that the coefficient process at any given time is correlated to the response at that time. To illustrate this fact, let us consider the simple first order scalar equation

$$\dot{x} + (a+n(t))x = 0, \tag{1.4}$$

Upon taking expectations, we find

$$\frac{d}{dt} E\{x(t)\} + a E\{x(t)\} + E\{n(t)x(t)\} = 0. \tag{1.5}$$

It is the term $E\{n(t)\,x(t)\}$, that is of concern to us. Since $n(t)$ is a physical noise, with dependence upon the past, we cannot simplify this term any further. On the other hand, if $n(t)$ is the gaussian white noise, then $n(t)$, $x(t)$ are independent random variables for any t, which allows us to write

$$E\{n(t)x(t)\} = E\{n(t)\}E\{x(t)\} \quad . \tag{1.6}$$

In view of (1.6), we can then write (1.5) as

$$a\,\dot{m}(t) + (a + E\{n(t)\})m(t) = 0, \tag{1.7}$$

which is a simple linear scalar equation for the mean response, $m(t) \equiv E\{x(t)\}$.

Naturally, this extends to higher order structural system equations as well. It is interesting to note at this time, that the many recursive schemes for obtaining the approximations to the moments, or probability densities, for the physical noise case base the initial approximation on the assumption of independence of the coefficient process with the response process. At this time there does not appear to be any general technique available to obtain even the exact moments for the linear system with physical noise coefficients.

There are other interesting properties that must be taken into account when studying the statistical properties of the dynamics of systems with uncertain or randomly fluctuating coefficients. It is possible that the average, statistical, properties of the response process may be quite distinct in character from the actual sample solutions themselves. In particular, the asymptotic behavior of the mean motion may be divergent or even be undefined (so-called finite explosion time) yet the actual, sample, behavior of the response will remain quite regular. This anomalie occurs for systems with random coefficients since we are dealing not only with the solutions of differential equations, but with their ensemble averages as well.

We illustrate this non-intuitive behavior with two simple first order scalar equations.

Example 1.1 - Finite explostion time for moments.

Let ,b, be a zero mean gaussian variable (constant in time), with unit variance. Thus, $p(b) = \frac{1}{\sqrt{2\pi}} e^{-\frac{b^2}{2}}$, the probability density for b.

We study the mean behavior of the solution to the scalar equation

$$\dot{x}(t) = b^2 x(t), \quad x(0) = 1 \quad . \tag{1.8}$$

We can easily find the sample solution to (1.8) to be

$$x(t) = e^{b^2 t}, \quad t \geq 0 \tag{1.9}$$

Clearly, the samples are exponentially increasing as $t \uparrow \infty$, with a rate depending upon the specific value $b^2 \epsilon [0, \infty)$.

Now, what can be said about the behavior of the moments, $m_k(t) = E\{x^k(t)\}$?

We merely have to evaluate the expectation

$$\left\{ \begin{aligned} E\{x^k(t)\} = E\{e^{kb^2 t}\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{b^2}{2}} e^{kb^2 t} db \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{b^2 (kt - \frac{1}{2})} db. \end{aligned} \right. \tag{1.10}$$

We see from (1.10), quite obviously, if $t > \frac{1}{2k}$, then the k^{th} moment does not exist! That is, it becomes infinite. Thus, for $t > \frac{1}{2}$ all moments are infinite, even though the sample solution behavior, from (1.9) is quite regular. Hence, the mean motion will <u>not exist</u> after finite time, even though the system response is well behaved as exponentially increasing curves.

We now look at a somewhat more complicated case, with randomly varying, physical noise, coefficients.

Example 1.2 - Increasing Moments - Decreasing Samples

For this example, we assume the fluctuating coefficient, scalar first order equation,

$$\dot{x}(t) + (a + n(t)\ x(t) = 0, \tag{1.11}$$

where $a > 0$ is a known constant and the stochastic process $\{n(t),\ t \geq 0\}$, is gaussian with zero mean and covariance function

$$\gamma_n(t,s) = E\{n(t)n(s)\} = \sigma^2 e^{-|t-s|} \ . \tag{1.12}$$

The n-process is a stationary gaussian process, whose spectral density $\frac{2\sigma^2}{1+\omega^2}$, is absolutely continuous. Therefore, it is known to be [1.19] ergodic.

Thus, we can equate time averages and ensemble averages, with probability one.

In particular, $\lim_{t \uparrow \infty} \frac{1}{t} \int_o^t n(s)ds = E\{n(\cdot)\} = 0$ with probability one.

The n-process is, in fact, the so-called Ornstein-Uhlenbeck process [1.20] whose sample functions are known to be continuous functions with probability one. (This is in contrast to the random telegraphic signal, a non-gaussian process whose covariance is given by (1.12), but whose samples are piecewise constant - (see [1.21] for a discussion)) .

Thus, we can integrate (1.11) directly, to yield the sample solutions for $x(0) = 1$,

$$x(t) = e^{-a\,t - \int_o^t n(s)ds} \ . \tag{1.13}$$

In order to study the asymptotic behavior of the response, $x(t)$, we form the limit,

$$\lim_{t \uparrow \infty} x(t) = \lim_{t \uparrow \infty} e^{-a\,t - \int_o^t n(s)ds} = \lim_{t \uparrow \infty} e^{\left(-a - \frac{t}{t}\int_o^t n(s)ds\right)\ t} \tag{1.14}$$

However, by the ergodic properties of the n-process, we have

$$\lim_{t\uparrow\infty} \left(-a - \frac{1}{t}\int_o^t n(s)ds\right) = -a - \lim_{t\uparrow\infty} \frac{1}{t}\int_o^t n(s)ds \qquad (1.15)$$

$$= -a, \quad \text{with probability one.}$$

Finally, since $e^{-at}\downarrow 0$ as $t\uparrow\infty$, it follows that $\lim_{t\uparrow\infty} x(t) = 0$ with probability one. This property is independent of the magnitude of σ^2! Therefore, what we have established, is that all sample solutions (i.e. with probability one) (1.13) of the equation (1.11) will approach zero asymptotically. In the general literature on stochastic linear systems, this is usually referred to as almost sure asymptotic stability (See e.g. [1.2] or [1.22]).

Now we study the statistical properties of the solution process (1.13) in order to determine their asymptotic behavior. We concentrate on the moments. To illustrate the basic results here, it is sufficient to study the mean motion, first moment, $E\{x(t)\}$.

Since the solution is known explicitly by (1.13), then the mean can be evaluated as,

$$E\{x(t)\} = E\{e^{-at - \int_o^t n(s)}\} = e^{-at}\, E\{e^{-\int_o^t n(s)ds}\}. \qquad (1.16)$$

The n-process is gaussian, therefore, since linear operators on gaussian processes, yield gaussian processes (e.g. see [1.20]), it follows that $\int_o^t n(s)ds = N(t)$ is a gaussian process.

We need merely calculate the mean and variance to obtain the associated gaussian density function for N(t), that is

$$p(N,t) = \frac{1}{\sqrt{2\pi}\,\sigma_N(t)}\, e^{-\frac{(N-m_N(t))^2}{2\sigma_N^2(t)}} \qquad (1.17)$$

We have, by direct calculation

$$m_N(t) = E\{N(t)\} = E\{\int_o^t n(s)ds\} = \int_o^t E\{n(s)\}ds = 0$$

$$\sigma_N^2(t) = E\{N^2(t)\} = E\{\left(\int_o^t n(s)ds\right)^2\} = E\{\int_o^t d\tau \int_o^t ds\ n(\tau)n(s)\} \tag{1.18}$$

$$= \int_o^t d\tau \int_o^t ds\ E\{n(\tau)n(s)\} = \int_o^t d\tau \int_o^t ds\ \sigma^2\ e^{-|\tau-s|}$$

$$= 2\sigma^2[t + e^{-t}-1].$$

Since it is easily seen by direct calculation, that

$$E\{e^{-N(t)}\} = \exp\ [\frac{\sigma^2_N(t)}{2}],$$

then upon substituting the result of (1.18) yields

$$E\{x(t)\} = e^{(\sigma^2-a)t}\ e^{\sigma^2(e^{-t}-1)}. \tag{1.19}$$

As t approaches infinity it is obvious that the term $e^{(\sigma^2-a)t}$ determines the asymptotic behavior of (1.19).

Therefore, for $\sigma^2>a$, the mean motion satisfies,

$$\lim_{t\uparrow\infty} E\{x(t)\} = \infty \tag{1.20}$$

with exponential rate of increase. Yet for any σ^2 we have seen that the samples approach zero asymptotically with probability one!

Again, this illustrates completely opposite behavior of the sample responses and the mean response for systems with random coefficients.

Such non-intuitive features of systems with uncertain or randomly fluctuating coefficients make it imperative that when assuming statistics of the coefficient processes the induced statistical properties of the response be carefully investigated when actual engineering design considerations are to be made for structures and their controllers.

Clearly, the statistical measures may be quite misleading as to the nature of the true response characteristics. (See [1.23] for an early discussion of this problem, and [1.24] for the exact statement relation asymptotic moment behavior and asymptotic sample behavior).

It follows that we must further question what the statistics can tell us about the structural response fluctuations.

We will deal with these questions along with the development of the statistical characteristics below.

We will first look at the white noise coefficient case, the physical noise coefficient case, and various approximation schemes in that order. We will concentrate on what statistical measures (moments, probability densities) can be obtained exactly, or approximately.

I.2 Introduction - Time Invariant Structural Equations

This introduction is concerned with response prediction in a time-invariant system when it is not possible to specify the system exactly. Two questions immediately come to mind. First, do situations exist where it is not possible to specify a system exactly as is required for exact response prediction? Second, does it make any great difference to the engineer (in response prediction) if he does not have accurate system specification? It is instructive to develop reasons why the answer to each of these questions is sometimes yes. Two preliminary remarks are in order before proceeding.

We know that in classical mechanics, applied mechanics, structures, small vibration, in text books, and in most design procedures, the system is regarded as known exactly. That is, masses, stiffnesses, and dissipation are regarded as known exactly. Response prediction in these "ideal" situations is thus exact, of course. The possibility that the system specification is in many practical engineering situations uncertain is never brought up for consideration.

For any rational discussion of the influence of system uncertainty on accuracy of response prediction, it is important to clearly specify the type of system under consideration so that the scope of the problem is kept within reasonable bounds. We assume the system is holonomic with generalized coordinates $q_1, \ldots, q_n$, where n is finite; thus, the system has n-degrees of freedom. We next assume the configuration $q_1 = \ldots = q_n = 0$ is one of stable equilibrium. We further assume small motion about this configuration of equilibrium, and take the dissipation to be viscous. To obtain the equations of motion by Lagrange's method, we require the kinetic energy T, the dissipation function F, the

potential energy V, and the virtual work δW of the external forces. We know [ ] that we may write

$$T = \frac{1}{2} m_{jk}\dot{q}_j\dot{q}_k \quad , \quad V = \frac{1}{2} k_{jk}q_jq_k$$
$$F = \frac{1}{2} c_{jk}\dot{q}_j\dot{q}_k \quad , \quad \delta W = f_j(t)\delta q_j \tag{1.21}$$

where summation convention is employed and the $f_j(t)$ are the external force components. T is a positive definite quadratic form with constant $m_{jk} = m_{kj}$. We assume that F is positive definite to ensure only dissipation and take constant $c_{jk} = c_{kj}$. Since the equilibrium configuration is stable, V is also positive definite and we take $c_{jk} = c_{kj}$ (constant). Lagrange's equations

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q_j} + \frac{\partial F}{\partial \dot{q}_j} + \frac{\partial V}{\partial q_j} = f_j(t) \quad , \quad j = 1, \ldots, n \tag{1.22}$$

thus provide the equations of motion from which response prediction are obtained. We can now discuss how uncertainty enters system specification in a significant manner.

First, observe that in a deformable structure (system) we require an infinity of coordinates to specify the configuration. Our choice of coordinates $q_1, \ldots, q_n$, where n is finite, immediately points up the fact that we do not have enough coordinates to specify the configuration of the system exactly even though we know the organization of the system in terms of members, joints, masses, etc. However, it is reasonable to assume we can select a set of $q_1, \ldots, q_n$ that will serve for our specific purpose, as is done in finite element modeling. Thus, we shall disregard uncertainty in system specification due to coordinate choice in what follows.

Next, consider V. The k_{jk} (stiffness coefficients) will be calcu-

lated from geometry and material behavior employing one of the standard methodology. Even if considerable effort (say, by finite element techniques) is expended in computation of these k_{jk} within the frame work of the selected $q_1, \ldots, q_n$, the influence of joint behavior, for example, on the value of the k_{jk} can never be estimated exactly. Further, changes in joint friction due to corrosion will cause joint behavior to slowly change with time. Since joint behavior, which determines end-conditions for the relevant members, frequently has a profound influence on member stiffness, uncertainty in joint behavior will produce uncertain k_{jk}. Further, structural members may rupture due to aging or may be partially inoperative due to assembly and/or manufacturing errors and remain undetected. No matter how hard we try to accurately estimate, at least some of the k_{jk} will actually have values different from what we estimate.

The entire area of passive dissipation mechanics is at best on an insecure foundation. Mathematical convenience has dictated the form of F in (1.21) [see Ray]; to be specific, the form assumed produces linear dissipative terms in the equations of motion. Mathematical convenience is an important point; however, experimental evidence is required (and we are not aware of it) to demonstrate that viscous damping does produce physically accurate response over the entire frequency range.

Given the form if F in (1.21), there are, in general, no reliable techniques for calculating the c_{jk} in a rational manner. In the absence of large concentrated dissipators (dampers), it is usual practice either to assume the c_{jk} values are roughly proportional to the k_{jk} and/or the m_{jk}, where the proportionality constant is adjusted to produce dissipation in the first mode of motion equal to that observed in similar

structures, or to assume the c_{jk} are more less constant in value over the structure with a multiplicative constant adjusted as in the previous case. The c_{jk} arising from concentrated dampers can be estimated over a frequency interval from test results; however, accuracy in the estimates is usually not greater than ±50%. In all events, we will rarely have accuracy in the c_{jk} values even comparable to that obtained for the k_{jk}.

We can calculate or weigh the elements with as great an accuracy as required. Thus, the physical elements that enter into the computation of T have accurately estimated masses. Given our choice of the $q_1, \ldots, q_n$, there are a number of methods for calculating the m_{jk} ["constant mass matrices"]. Accuracy in the values of the m_{jk} is obviously not going to be a problem for a time invariant system or for a system in which the total mass changes slowly in a known manner. Although we recognize that the live load in a building or bridge may not be known precisely, we shall assume on occasion in this report that the m_{jk} are accurately known.

We conclude from the discussion up to this point that uncertainties will exist in the c_{jk} and k_{jk} values, with the former being larger than in the latter. Return to the first question -

> "do situations exist where it is not possible to specify a system parameters exactly ...?"

The answer is obviously "yes" and we have indicated a few of the possibilities.

We now come to the second question -

> "does it make any difference to the engineer (in response prediction) if he does not have accurate system specification?"

A few preliminary analytical details will provide a framework with which to motivate the answer.

Let us first write out (1.22) employing (1.21) assuming the $q_1, \ldots, q_n$

have been selected so that the known (assume the m_{jk} are not uncertain) kinetic energy has the form

$$T = \frac{1}{2} \sum_{1}^{n} \dot{q}_j^2 \tag{1.23}$$

$$\ddot{q}_j + c_{jk}\dot{q}_k + k_{jk}q_k = f_j(t) , \quad j = 1, \ldots, n . \tag{1.24}$$

Next, introduce the (nxn) matrices

$$C = \{c_{jk}\} , \quad K = \{k_{jk}\} , \quad I = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}, \tag{1.25}$$

and the (nx1) column vector

$$q = \begin{pmatrix} q_1 \\ \cdot \\ \cdot \\ \cdot \\ q_n \end{pmatrix} , \quad f = \begin{pmatrix} f_1 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{pmatrix} . \tag{1.26}$$

Then we can write (1.24) as

$$I\ddot{q} + K\dot{q} + Kq = f_1 , \tag{1.27}$$

where I is the (nxn) unit matrix.

Let us now put the second order system in the first order form:

$$\dot{x} = Ax + f \tag{1.28}$$

by employing the substitutions

$$A = \begin{pmatrix} 0 & I \\ -K & -C \end{pmatrix}, x = \begin{pmatrix} q_1 \\ \cdot \\ \cdot \\ \cdot \\ q_n \\ \cdot \\ \dot{q}_1 \\ \cdot \\ \cdot \\ \cdot \\ \dot{q}_n \end{pmatrix} , \quad f = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ f_1 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{pmatrix} , \tag{1.29}$$

where A is a (2nx2n) matrix and x and f are (2nx1) column vectors. We note that (1.28), when written out, consists of 2n first order equations whereas (1.27) consists of n second order equations. The nice feature of (1.28) is that we can immediately write down its solution; if at $t = 0$, $x = x_o$, then,

$$x = e^{A(t-o)}x_o + \int_o^t e^{A(t-\tau)}f(\tau)d\tau \ , \quad t > 0 \tag{1.30}$$

We also can take the Laplace transform of (1.28) obtaining

$$X(s) = \{Is - A\}^{-1}F(s) \tag{1.31}$$

where I is the (2nx2n) unit matrix

$$X(s) = L[x(t)] \ , \quad F(s) = L[f(t)]$$

and we must add the initial condition x_o at $t = 0$. We can now indicate, employing (1.30) and (1.31), how uncertainties in the values of c_{jk} and k_{jk}, i.e. in C and K, can change the response significantly.

We conclude from (1.31) that the eigenvalues of A will determine the stability of the system, since

$$\{Is - A\}^{-1} = \frac{\text{cof } \{Is-A\}^T}{|Is-A|} \tag{1.32}$$

(where "T" denotes transpose), which shows that the eigenvalues of A, as determined by

$$|I_2 s - A| = 0 \ , \tag{1.33}$$

provide poles of the Laplace transform of x(t). If we do not know the elements of the (2nx2n) matrix A precisely, we cannot unequivocally state the eigenvalues of A are in the left-half complex s-plane. Even if we know the eigenvalues of A are in the left-half s-plane, but we do not know their precise location, then if we add a control system for the

purposes of controlling our original system orientation we have the possibility that the complete system might at the worst be unstable or have very poor control characteristics, or at best have excellent control characteristics. The uncertainty of how the system would behave cannot be tolerated in space where adjustment and/or repair would be difficult.

Suppose the eigenvalues of A are in the left-half s-plane so we know the system is at least stable and suppose the external forces f_l contain periodic elements. Then, the possibility of resonance phenomenon comes up. Notice, first, uncertainties in the elements k_{jk} of K means we are not certain where the undamped natural frequencies of the system lie; thus, we cannot be certain that some of the simple harmonic components in the periodic disturbances will have frequencies well separated from the undamped natural frequencies of the system. Second, uncertainties in the elements c_{jk} of the matrix C means that we cannot be certain that the forced amplitudes of the system response will be small if the frequencies of the simple harmonic components in the external forces (disturbances) happen to be close to one or more of the undamped natural frequencies of the system. If either of these situations occur, some of the components in x of (1.30) will be large leading ultimately to fatigue failures. Examples of past cases where these possibilities have occurred and caused problems readily come to mind.

We also have given specific examples in the General Introduction where uncertainties in parameter values make a difference between what is predicted to occur, ignoring these uncertainties, and what actually occurs. Thus, the answer to the second question is yes!

In the sections devoted to time invariant parameters, our concern is to describe how uncertainty in the parameters influences response and what techniques are currently available to quantitatively assess this influence. We shall divide our discussion into subsections that reflect different interests and methodologies. The two main areas of interest are derived from problems of physics and problems of engineering. We shall start with engineering.

The most important topic for engineering systems is how uncertain parameter values influence the accuracy of system response prediction. It often suffices to know how these uncertainties influence the accuracy in estimating the values of the natural frequencies and their corresponding normal modes of motion in a conservative system ($C = 0$). Since linear systems response prediction depends upon frequency response or impulsive admittance (i.e. Green's function, impulse function), our interest centers on:

natural frequencies
normal modes
frequency response
impulse response .

In some situations we will be directly interested in the response q.

It is important to note that there are two ways to quantitatively characterize uncertainty in the parameters. We may simply have bounds on the parameter values. Alternatively, we may consider the parameters as random variables, described by their joint probability distributions, or, at least, by their first two moments. We shall treat both of these characterizations, since each can arise in application. The broad classes of techniques available to pursue these subjects are:

perturbation methods
Liouville's equation
mean-square approximate systems.
bound determination.

We shall discuss each of these techniques separately, realizing that there will be overlap.

II. Moments - White Noise Coefficient Case

In this section we shall be concerned with the case that the coefficients of linear differential equations, contain white noise components. In particular, we are concerned with Gaussian white noise coefficients. The white noise $\{W_t,\ t\in[0,\infty]\}$, is characterized by the fact that its power spectral density is constant over the entire frequency domain $(-\infty,\infty)$. Therefore, by the Fourier transform relation between power spectral densities, $f(\omega)$, and covariance functions, $\gamma(\tau)$, it follows that the covariance for the white noise is an impulse function. Thus for the white noise, $f_w(\omega) \equiv S_o$, and its Fourier transform (the covariance) is $\gamma(\tau) = 2\pi S_o \delta(\tau)$. It immediately follows from $\gamma_w(\tau)$, that $W(t), W(t+\tau)$ are uncorrelated random variables, no matter how small τ. Furthermore since formally, we must have $E\{w^2(t)\} = 2\pi S_o \delta(0)$ which is undefined, we see that the white noise would possess infinite power, making it a mathematical concept rather than the model of a process that is observed in nature. Moreover, if we assume that $W(t)$ is a Gaussian random variable, then it follows that $W(t)$, $W(t+\tau)$ are not only uncorrelated, they are, indeed, independent random variables.

The question that comes up is, how do we interpret the meaning of differential equation models of real systems, with Gaussian white noise coefficients? It was the classic work of K. Ito [2.1], in the 1940's that answered this question, long before it was of importance to modern optimization and control applications.

Although a thorough development of these ideas is beyond the scope of this report, we shall illustrate a few of the basic ideas. The interested reader should consult texts such as [2.2], [2.3] for the fundamental development. The key point is that the Gaussian white noise, $W(t)$, possesses a representation via the Brownian motion process $\{B(t),\ t\in[0,\infty]\}$.

An especially readable account of the interpretation can be found in [2.4] Vol. II , as well as the recent book on parametric excitation by Ibrahim [2.5]. The Brownian Motion is discussed in almost all elementary tests on stochastic processes.

For $0<t_1<t_2\ldots<t_n$, for any $\{t_i\}$ and n, the joint density is

$$p(B_1,t_1;\ldots;B_n,t_n) = \frac{1}{(2\pi)^{n/2}\sigma^n\sqrt{t_1(t_2-t_1)\ldots(t_n-t_{n-1})}}\, e^{-\frac{1}{2\sigma^2}\left(\frac{B_1^2}{t_1}+\frac{(B_2-B_1)^2}{t_2-t_1}+\frac{(B_n-B_n^2)}{t_n-t_{n-1}}\right)} \tag{2.1}$$

The Brownian Motion process, also referred to as the Wiener process after Norbert Wiener who first made a thorough study of the properties in 1923 [2.6], and 1930 [2.7], is defined by the class of Gaussian joint probability densities (2.1). The Brownian Motion process is a Gaussian process with stationary, independent increments satisfying,

$$P\{B(0)=0\}=1,\ E\{B(t)\}=0,\ E\{B(s)B(t)\}=\gamma_B(s,t)=\sigma^2\min(s,t) \tag{2.2}$$

Among Wiener's fundamental contributions to the development of this process, was to determine that the sample functions of the process are continuous functions on any finite interval, and that the sample functions are nowhere differentiable.

Now from (2.2), we see from the covariance, that the Brownian Motion is non-stationary. Furthermore, from elementary properties of derivatives of processes, it follows that the covariance of $\dot{B}(t)$, is given as

$$\frac{\partial^2\gamma_B(s,t)}{\partial s\partial t}\ .$$

This result is immediately obtained formally, from

$$\frac{\partial^2\gamma_B(s,t)}{\partial s\partial t} = \frac{\partial^2}{\partial x\partial t}\ E\{B(s)B(t)\}$$

$$= E\{\dot{B}(s)\dot{B}(t)\}\ .$$

Upon taking these derivatives of $\gamma_B(s,t)$ given in (2.2), we also obtain the formal relation,

$$\frac{\partial^2 \gamma_B(s,t)}{\partial s \partial t} = \sigma^2 \delta(t-s) = \gamma_w(t-s) \ . \tag{2.3}$$

Thus, we obtain the representation for the Gaussian white noise as,

$$\dot{B}(t) \simeq W(t). \tag{2.4}$$

Therefore, for systems with Gaussian white noise coefficients, we can replace the white noise terms with the formally differentiated Browmian motions.

We write the general linear differential equation with white noise coefficients as,

$$\dot{x}(t) = Gx(t) + \sum_{i=1}^{K} H^i W_i(t)\, x(t) \ . \tag{2.5}$$

where the $n \times n$ matrices $G, \{H_i\}$ are known, and the $\{W_i(t)\}_{i=1}^{K}$ are independent white noise coefficients, and x is an n-vector. However, since we can represent $W_i(t) = \dot{B}_i(t)$, then (2.5) may be rewritten as,

$$\dot{x}(t) = Gx(t) + \sum_{i=1}^{K} H^i \dot{B}_i(t)\, x(t) \ . \tag{2.6}$$

As we stated above, $\dot{B}_i(t)$ is only a formal derivative, since as a result of Wiener's investigations, the Browmian motion does not possess derivatives. The approach of K. Ito, was to interpret (2.6) as an equation in differentials,

$$dx(t) = Gx(t)dt + \sum_{i=1}^{K} H^i dB_i(t)\, x(t) \ . \tag{2.7}$$

The meaning of this equation is via the integral equation,

$$x(t) - x(a) = \int_a^t Gx(s)ds + \sum_{i=1}^{K} \int_a^t H^i x(s) dB_i(s), \tag{2.8}$$

where the Integral in (2.8) is the so-called Ito stochastic Integral and the equation (2.7) is generally referred to in the literature as the Ito differential equation.

Thus, upon starting with (2.5) as the formal differential equation describing a system with white noise excitations, we are lead to the mathematically meaningful form (2.7).

The importance of this form lies in the many properties of the Brownian motion differential coefficients dB(t). In particular, since the Brownian motion process is a process with independent increments, then for each i, $dB_i(s)$, $dB_i(t)$ are independent random variables for $s \neq t$. (We define the differential, $dB(t) \equiv B(t+dt)-B(t)$ for positive time differentials $dt>0$). Also, as a result of this interpretation of the differential, it also follows, that $dB_i(t)$ is independent of x(t) for every i. We can interpret this formally as follows. Through the differential equality (2.7), x(t) is a functional of $\{B_i(s), s\leq t\}$ for all i. However, since the Brownian motion is an independent increment process, it follows that $dB_i(t) = B_i(t+dt)-B_i(t)$ is independent of all combinations (or functionals) of $\{B_i(s)\ s\leq t\}$. This independence is of fundamental significance. We can see the immediate effect of this property, when we investigate the mean value $E\{x(t)\}$, for the solution response to (2.7).

By taking expectations directly on (2.7) we obtain,

$$dE\{x(t)\} = G\ E\{x(t)\}dt + \sum_{i=1}^{K} H^i E\{dB_i(t)x(t)\}. \tag{2.9}$$

Due to the independence of x(t) and the $dB_i(t)$, we can write using the fact that $E\{dB(t)\} = 0$.

$$E\{dB_i(t)x(t)\} = E\{dB_i(t)\}\ E\{x(t)\} = 0\ . \tag{2.10}$$

Therefore, we immediately see that the mean equation from (2.9), (2.10) is

$$\frac{d}{dt}\ E\{x(t)\} = G\ E\{x(t)\}, \tag{2.11}$$

which is immediately solvable as a linear vector equation with constant coefficients.

Because of these properties, all moment statistics of the solution process to (2.7) may be obtained. Perhaps, the most important result of the properties of the Brownian motion, and the form of the Ito differential equation, is that the solution process $\{x(t),\ t\in[t_o,\infty)\}$, is a vector Markov process. See [2.3] and [2.12] for discussion of this deeply significant fact.

Moreover, it was the motivation to represent a Markov process explicitly that lead K. Ito to study the differential equations that bear his name.

The important tool by which we can set up the various moment equations is the so-called generator or backward operator for the Markov solution process of (2.7). We will obtain this generator via the characteristic functional, an approach that appeared in an important early article by Moyal [2.8] and is attributed to the statistician M.S. Bartlett. This approach will be used to obtain the Liouville equation in our review of the random constant coefficient case, following the approach in [2.9]. These ideas were also applied in a fundamental paper on statistical turbulence theory by E. Hopf [2.10] and later by the physicist S.F. Edwards 2.1 .

To illustrate these ideas, we write the development for the scalar case. The n-th order operators can then be written down immediately. For the simple, scalar Ito equation (general linear or non-linear) we write,

$$dx_t = g(x_t)\ dt + h(x_t)\ dB_t. \tag{2.12}$$

We are interested in the characterstic functional

$$\phi_x(t,u) = E\{e^{iux(t)}\} \tag{2.13}$$

Upon taking the differential with respect to t of (2.13), we will obtain, upon a formal interchange of expectation and differential operators -

$$\begin{aligned} d_t\phi_x(t,u) &= E\{d_t e^{iux(t)}\} \\ &= E\{[iud x(t) + \frac{1}{2}\ (iud x(t))^2 + o(dt)]\ e^{iux(t)}\} \\ &= iuE\{dx(t)\ e^{iux(t)}] + \frac{1}{2}\ (i\dot{u})^2\ E\{(dx(t))^2\ e^{iux(t)}\} \\ &\qquad + o(dt). \end{aligned} \tag{2.14}$$

Here, we have taken the first two terms in the expansion since, as we shall see from (2.13) there will be a contribution from the second order term. Higher order terms $(dx(t))^k$, $k>2$ will all be $o(\Delta t)$, and thus will not yield a contribution.

From (2.13), we have

$$\begin{aligned} d_t\phi_x(t,u) &= iu\, E\{[g(x(t)dt + h(x(t))dB(t)]\, e^{iux(t)}\} \\ &\quad + \frac{1}{2}(iu)^2\, E\{[(g(x(t))dt)^2 + 2g(x(t))h(x(t)dtdB(t) \\ &\qquad + h^2(x(t))\,(dB(t))^2]\, e^{iux(t)}\} \\ &\quad + o(dt) \end{aligned} \tag{2.15}$$

Now, since $dB(t)$ is independent of any function of $x(t)$, as discussed above, we have from (2.2), for any $f(x(t))$,

$$\begin{cases} E\{dB(t)\, f(x(t))\} = E\{dB(t)\}\, E\{f(x(t)\} = 0 \\ E\{(dB(t))^2\, f(x(t)\} = E\{(dB(t))^2\}\, E\{f(x(t)\} = \sigma^2 dt E\{f(x(t)\} \end{cases} \tag{2.16}$$

Therefore, we can write (2.15) as,

$$\begin{aligned} d_t\phi_x(t,u) &= iu\, E\{g(x(t))\, e^{iux(t)}\}\, dt \\ &\quad + \frac{\sigma^2}{2}(iu)^2\, E\{h^2(x(t))\, e^{iux(t)}\}\, dt \\ &\quad + o(dt) \end{aligned} \tag{2.17}$$

Finally, we obtain upon dividing by dt, and letting $dt \to 0$,

$$\frac{\partial \phi_x(t,u)}{\partial t} = E\{[iug(x(t)) + \frac{\sigma^2}{2}(iu)^2\, h^2(x(t))]\, e^{iux(t)}\} \tag{2.18}$$

We must now recall the elementary fact that

$$\phi_x(t,u) = \int_{-\infty}^{\infty} e^{iu} p(x,t)\, dx,$$

that is the characteristic function, and the corresponding probability density function are Fourier transform pairs.

Upon taking inverse transforms of the equation (2.18), using the operator identity $(iu)^k \rightarrow \frac{\partial^k}{\partial x^k}$, (2.18) yields the partial differential equation for $p(x,t)$,

$$\frac{\partial p(x,t)}{\partial t} = - \frac{\partial [g(x)p(x,t)]}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 [h^2(x)p(x,t)]}{\partial x^2} . \tag{2.19}$$

We recognize this equation as the Fokker-Planck equation that has been so important in the study of Markov diffusion processes.

The operator

$$\mathscr{L}_x^- \equiv \frac{-\partial [g(x) \cdot]}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 [h^2(x) \cdot]}{\partial x^2} \tag{2.20}$$

is referred to as the Forward operator in the literature. See [2.12] as well as other sources.

The adjoint operator

$$\mathscr{L}_x \equiv g(x) \frac{\partial}{\partial x} + \frac{\sigma^2}{2} h^2(x) \frac{\partial^2}{\partial x^2} , \tag{2.21}$$

is referred to as the *generator of the process* defined by the Ito differential equation (2.12).

The generator is the most important operator for diffusion processes since it always exists. It is known that the forward operator may not exist for certain Markov processes. This is an advanced concept which is usually discussed in fundamental studies of diffusion processes [See e.g. [2.13]].

It is the generator that is of interest to us. It is easily seen that for any f(x), from (2.19), since we may write,

$$\frac{d}{dt} E\{f(x(t))\} = \frac{d}{dt} \int_{-\infty}^{\infty} f(x)p(x,t)\, dx = \int_{-\infty}^{\infty} f(x) \frac{\partial p(x,t)}{\partial t}\, dx,$$

we obtain the equation for the mean value of f(x(t),

$$\frac{d}{dt} E\{f(x(t))\} = E\{\mathscr{L}_x f(x(t))\} . \tag{2.22}$$

The equality (2.22) is a simplified form of what is referred to in the literature as Dynkin's theorem, [see [2.13], [2.14]]. The apparent first application of these ideas to structural systems was presented in [2.15], where stability of the second moments of the linear oscillator with white noise coefficients was studied.

Considering only the moments $m_k(t) = E\{x^k(t)\}$, we have from (2.21), (2.22),

$$\frac{d}{dt} m_k(t) = kE\{g(x(t)x^{k-1}(t)\} + \frac{k(k-1)\sigma^2}{2} E\{h^2(x(t))x^{k-2}(t)\} \tag{2.23}$$

It is clear that this equation cannot be solved for arbitrary functions g(x),h(x). But, to our good fortune the linear case is completely determined by (2.23). Thus, for the linear form of (2.12) where g(x) = ax, h(x) = bx, the moment equation (2.23) becomes

$$\frac{d}{dt} m_k(t) = kE\{ax(t)x^{(k-1)}(t)\} + \frac{k(k-1)}{2}^2 E\{b^2x^2(t)x^{(k-2)}(t)\}$$

which the reader can easily put into the form,

$$\frac{d}{dt} m_k(t) = [ka + \frac{k(k-1)}{2} \sigma^2 b^2]\, m_k(t), \tag{2.24}$$

which yields,

$$E^k(t) = m_k(t) = m_k(0)e^{[ka + \frac{k(k-1)}{2} \sigma^2 b^2]t}$$

for the k^{th} moments.

In fact for any n^{th} order linear Ito equation we obtain a consistent set of equations for the k^{th} moments that can be solved exactly. We see this by first presenting the generator (backward operator), for the general linear n^{th} order Ito equation (2.7). It is obtained directly as

$$\mathscr{L}_x = \sum_{i=1}^{n} (\sum g_{ij} x_j) \frac{\partial}{\partial x_j} + \frac{1}{2} \sum_{i,j=1}^{n} b_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} \tag{2.25}$$

where $b_{ij}(x) = \sum_{\ell=1}^{K} \sigma_\ell^2 \sum_{r,s=1}^{n} h_{ir}^{\ell} h_{js}^{\ell} x_r x_s$,

$\sigma_\ell^2 \, dt = E\{(dB_\ell(t))^2\}$, and $H^\ell = (h_{ij}^\ell)$.

We note that, exactly as for the first order, scalar, example above, the coefficient of the $\frac{\partial}{\partial x_i}$ term is linear in x, and the coefficient of the $\frac{\partial^2}{\partial x_i \partial x_j}$ term is quadratic in x.

Hence, there will always be a closed set of equations to solve for the k^{th} moments.

In fact, we immediately have for the expectation of the general function $f(x_1(t),\ldots,x_n(t))$,

$$\frac{d}{dt} E\{f(x_1(t),\ldots,x_n(t))\} = E\{\mathscr{L} f(x_1(t),\ldots,x_n(t))\} , \tag{2.26}$$

similarly to (2.22).

We look at the classic example, apparently, first studied in [2.15]. This is the second order linear oscillator, with a white noise coefficient,

$$\ddot{x}(t) + 2\gamma\omega\dot{x}(t) + (\omega^2 + W(t))x(t) = 0 \tag{2.27}$$

Putting (2.27) into a linear Ito form, we set $x_1 = x$, $x_2 = \dot{x}$ and noting that $dB(t) = W(t)dt$, as in (2.7), the second order oscillator equation becomes

$$\begin{cases} dx_1 = x_2 dt \\ dx_2 = -(2\gamma\omega x_2 + \omega^2 x_1)dt - x_1 \, dB(t) . \end{cases} \tag{2.28}$$

The generator of the (x_1,x_2) diffusion process can immediately be written as

$$\mathscr{L} = x_2 \frac{\partial}{\partial x_1} - (2\zeta\omega x_2 + \omega^2 x_1) \frac{\partial}{\partial x_2} + \frac{\sigma^2}{2} x_1^2 \frac{\partial^2}{\partial x_2} . \tag{2.29}$$

For the three moments $m_{21}(t) = E\{x_1^2(t)\}$, $m_{12}(t) = E\{x_1(t)x_2(t)\}$, $m_{22}(t) = E\{x_2^2(t)\}$, the relation (2.26), for $\mathscr{L}$ given by (2.29) yields the equations,

$$\frac{d}{dt}\begin{pmatrix} m_{11}(t) \\ m_{12}(t) \\ m_{22}(t) \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 \\ -\omega^2 & -2\zeta\omega & 1 \\ \sigma^2 & -2\omega^2 & -4\eta\omega \end{pmatrix} \begin{pmatrix} m_{11}(t) \\ m_{12}(t) \\ m_{22}(t) \end{pmatrix} , \tag{2.30}$$

which can be solved explicitly.

For the general n^{th} order system (2.7), one can easily show that the second moments $E\{x_u(t)x_v(t)\} = m_{uv}(t)$, can be expressed as the solutions of the differential equations

$$\dot{m}_{uv}(t) = \sum_{j=1}^{n} [g_{uj}m_{jv}(t) + g_{vj}m_{ju}(t)] + \sum_{\ell=1}^{K} \sigma_\ell^2 \sum_{r,s,=1}^{n} h_{ur}^{\ell} h_{vs}^{\ell} m_{rs}(t) \tag{2.31}$$

where $u,v = 1,2,\ldots,n$.

Here, we have used (2.25) directly.

Finally, we shall mention a generalization of the 2^{nd} moment formula (2.31) that holds for all p^{th} order moments, [2.16], [2.17] for the linear Ito equation (2.7).

Motivated by the algebraic theory of linear differential equations, one can define for a given n-vector x and a given positive integer p, the associated vector $x^{[p]}$ whose components are

$$\sqrt{\begin{bmatrix} p \\ p_1 \end{bmatrix} \begin{bmatrix} p-p_1 \\ p_2 \end{bmatrix} \cdots \begin{bmatrix} p-p_1-p_2- \cdots -p_{p-1} \\ p_p \end{bmatrix}}\; x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n} \tag{2.32}$$

for

$$\sum_{i=1}^{n} p_i = p,\ p_i \geq 0 \quad .$$

The components of $x^{[p]}$ are ordered lexicographically. For example, for $n = p = 3$, the components of $x^{[3]}$, for

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{are, in order,}$$

$$x^{[3]} = \begin{bmatrix} x_1^3 \\ \sqrt{3}\, x_1^2 x_2^2 \\ \sqrt{3}\, x_1^2 x_3 \\ \sqrt{6}\, x_1 x_2 x_3 \\ \sqrt{3}\, x_1 x_3^2 \\ x_2^3 \\ \sqrt{3}\, x_2^2 x_3 \\ \sqrt{3}\, x_2 x_3^2 \\ x_3^3 \end{bmatrix} \tag{2.33}$$

The vector $x^{[p]}$ satisfies $\| x^{[p]} \| = \| x \|^p$, where $\|x\|^2 = (x,x)$ and more generally, $(x,y)^p = (x^{[p]}, y^{[p]})$. Furthermore, this concept extends to matrices through the definition $y = Ax$. One defines $A^{[p]}$, as that matrix that yields

$$y^{[p]} = A^{[p]} x^{[p]} \quad . \tag{2.34}$$

For linear systems

$$\dot{x}(t) = A(t)x(t) \tag{2.35}$$

the differential equality

$$x(t+h) = (I + hA(t))x(t) + 0(h^2)$$

holds, yielding from the definition (2.34)

$$x^{[p]}(t+h) = (I + hA(t))^{[p]} x^{[p]}(t) + 0(h^2) \quad . \tag{2.36}$$

Upon defining the limit

$$\lim_{h \to o} \frac{1}{h} \left((I + hA(t))^{[p]} - I^{[p]} \right) = A^{(t)}_{[p]} \quad ,$$

we now determine the associated differential equation for $x^{[p]}$,

$$\dot{x}^{[p]}(t) = \underset{[p]}{A(t)}\ x^{[p]}(t) \quad . \tag{2.37}$$

In order to relate these ideas back to the original Ito equation (2.7), one can apply the usual Ito calculus [2.12] to obtain the Ito differential equation for $x^{[p]}$ as

$$d\,x^{[p]}(t) = \left[\left(G - \frac{1}{2}\sum_{r=1}^{K} H_r^2\right)_{[p]} + \sum_{r=1}^{K} H_r^2\,{}_{[p]}\right] x^{[p]}(t)dt$$

$$+ \sum_{r=1}^{K} H_{r[p]}\, x^{[p]}(t)\, dB_r(t) \quad . \tag{2.38}$$

Applying expectations to the Equation (2.38) immediately yields the linear differential equation for the pth order moments as

$$\frac{d}{dt} E\{x^{[p]}(t)\} = \left[\left(G - \frac{1}{2}\sum_{r=1}^{K} H_r^2\right)_{[p]} + \sum_{r=1}^{K} H_{r[p]}^2\right] E\{x^{[p]}(t)\}$$

$$= A_p\, E\{x^{[p]}(t)\} \; . \tag{2.39}$$

The result (2.39), can also be obtained directly from application of (2.20) for $\mathscr{L}$ given by (2.25).

Although, in principle, the equations for the pth moments for linear homogeneous Ito differential equations have been known for many years, the general form (2.39) can be quite useful.

These ideas extend quite simply to the non-homogeneous case as well.

We mention that the results (2.32)-(2.39), have been derived from algebraic considerations and more particularly by consideration of the Lie algebras [see [2.16]] generated by the matrices $(G, H_1, H_2, \ldots H_K)$.

Finally, it should be noted that there is no conceptual or analytic difficulty to obtain the moments for the linear system with external forcing functions. The general linear Ito equation (2.7) would become,

$$dx(t) = Gx(t)dt + \sum_{i=1}^{K} H^i dB_i(t)x(t) + FdV(t),$$

where F is a constant nxm matrix, the vector V is an m-vector of Brownian motions that are independent of the $\{B_i(t)\}$, and are independent among their components.

For this case, the generator for (2.40) would be $\mathscr{L}_x + \mathscr{L}_F$, where $\mathscr{L}_x$ is given by (2.25) and

$$\mathscr{L}_F = \frac{1}{2} \sum_{i,j=1}^{n} \sigma_{ij} \frac{\partial^2}{\partial x \partial x_j} ,$$

where $\sigma_{ij} = \sum_{\ell=1}^{m} f_{i\ell} f_{j\ell}$,

and $F_{nxm} = (f_{ij})$.

The equation (2.40) is a non-homogeneous equation. The associated moment equations that are derived from the addition of $\mathscr{L}_F$ given by (2.41) to $\mathscr{L}_x$ given by (2.25) would also generate a set of non-homogeneous linear equations. Finally, we wish to comment on the probability densities for (2.7). We know that the probability densities for the solutions to Ito differential equations are the solutions to the Fokker-Planck equations, given by

$$\frac{\partial p}{\partial t} = \mathscr{L}^* p, \text{ where} \tag{2.42}$$

$\mathscr{L}^*$ is the adjoint of the generator .

Thus, for example, for the general linear equation, the adjoint of $\mathscr{L}_x$ given by (2.25) is

$$\mathscr{L}^*_x = - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} [\sum g_{iu} x_j \cdot] + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 [b_{ij}(x) \cdot]}{\partial x \partial x_j} . \tag{2.43}$$

For the simple scalar equation

$$dx(t) = \beta x(t)dt + x dB(t), \quad E\{(dB(t))^2\} = \sigma^2 dt, \tag{2.44}$$

whose Fokker-Planck equation is

$$\frac{\partial p(x,t)}{\partial t} = - \beta \frac{\partial [xp(x,t)]}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 [x^2 p(x,t)]}{\partial x^2} ,$$

we can obtain the probability density explicitly as,

$$p(x,t) = \frac{1}{\sqrt{2\pi t}\ \sigma x} e^{- [\log x - (\beta - \frac{1}{2}\sigma^2)t]^2 / 2\sigma^2 t} \tag{2.45}$$

Unfortunately, for higher order linear systems, one cannot, in general, solve for the probability densities. In certain cases, however, the stationary density as the solution of

$$0 = \mathscr{L}_x^* p(x) \tag{2.46}$$

can be obtained.

III. Approximately White Noise Coefficients

As we discussed in II, a white noise is distinguished by the fact that its spectral density is constant over the entire frequency domain $(-\infty, \infty)$.

Although the white noise itself is a mathematical abstraction, not present in nature, there are real or "physical" noise processes that are wide band. These processes may possess power spectral densities that are essentially flat over a broad frequency range and then exhibit a rapid drop-off to neglible frequency content.

It has been a common procedure throughout the development of stochastic methods to deal with problems of random excitations to replace such wide band processes with White noise.

In the case that the wide band gaussian physical noise, n(t), is an external excitation, such as in the simple oscillator,

$$\ddot{x}(t) + 2\zeta\omega\dot{x}(t) + \omega^2 x(t) = n(t), \tag{3.1}$$

the typical procedure, over the years, has been to replace n(t) by W(t) (gaussian white noise) and proceed with the analysis to obtain the solution process, moments, probability densities, etc. (In this linear case, $(x, \dot{x})$ will be gaussian random variables for gaussian excitations).

For external excitations this procedure can be justified. Difficulties occured when researchers in random vibrations attempted to make the same type of replacements for wide band random coefficients. In the random coefficient case, the replacement cannot be simply made. A deeper analysis is required. For the early discussions and ultimate clarification of these questions, the reader is referred to [3.1], [3.2], [3.3], [3.4], [3.5].

The basic problem can be illustrated via the first order differential equation.

$$\dot{x}(t) + n(t)\, x(t) = 0. \tag{3.2}$$

For any "physical" noise n(t), whose sample functions are well behaved in the sense that they are at the very least piecewise continuous, the solution may be represented as,

$$x(t) = x_o e^{-\int_o^t n(s)ds}, \tag{3.3}$$

where $x(0) = x_o$.

The natural question that occurs is as follows:

"If n(t) is a wide band gaussian process and we replace n(t) by the gaussian white noise W(t) in (3.2), will the solution be given by (3.3) where n(s) is replaced by W(s) in the integral?"

This is exactly what was done in the early 1960's in order to study oscillators with white noise coefficients. This would allow us to write (3.3) as

$$x(t) = x_o e^{-\int_o^t W(s)ds} = x_o e^{-\int_o^t dB(s)} = x_o e^{-B(t)}, \tag{3.4}$$

from the representation (2.4) of the gaussian white noise in terms of the Browmian motion. We would have to verify that the sample solution (3.4) satisfies the original equation

$$\dot{x}(t) + W(t)\, x(t) = 0,$$

or in the proper Ito differential form

$$dx(t) + x(t)dB(t) = 0 \tag{3.5}$$

(again identifying $W(t)dt \simeq dB(t)$).

This is simply obtained by taking the differential of the solution in (3.4), $x_o e^{-B(t)}$.

But, differentials of functions of Browmian motion possess a somewhat different form than differentials of the ordinary calculus.

This was, in fact, a major point in the development of Ito's stochastic differential equations [see eq. [3.6], [3.7]]. Essentially, since $E\{(B(t+\Delta t) - B(t)^2\} = \sigma^2 \Delta t$, for any $\Delta t > o$, it is known, on a sample property basis that $(B(t+\Delta t) - B(t))^2 \simeq \sigma^2 \Delta t$ with probability one. Therefore, upon taking a Taylor expansion

$$dF(B(t)) = F'(B(t)dB(t) + F''(B(t)) \frac{(dB(t))^2}{2!}$$

$$+ \ldots + F^{(n)}(B(t)) \frac{(dB(t))^n}{n!} + \ldots \tag{3.6}$$

we must keep the $(dB(t))^2 \simeq \sigma^2 dt$ term, all higher order terms $(dB(t))^k \simeq o(dt)$ for $k>2$.

This leads to the differential formula,

$$dF(B(t)) = F'(B(t))dB(t) + \frac{\sigma^2}{2} F''(B(t))dt, \tag{3.7}$$

which is referred to in the literature as the Ito differential formula. This is the proper relation that must be used to study the calculus of stochastic differential equations with gaussian white noise coefficients. For more general functions $F(B(t),t)$, one can show that

$$dF(B(t),t) = F_t dt + F_B dB + \frac{\sigma^2}{2} F_{BB} dt \ . \tag{3.7a}$$

Upon application of the formula (3.7) to the function (3.4), we find that

$$d(x_o e^{-B(t)}) = -x_o e^{-B(t)} dB(t) + \frac{\sigma^2}{2} x_o e^{-B(t)} dt,$$

or, equivalently,

$$dx(t) + x(t)dB(t) - \frac{\sigma^2}{2} x(t)dt = 0 \tag{3.8}$$

But (3.8) differs from (3.5), with the addition of the term $-\frac{\sigma^2}{2} x(t)dt$.

Thus, simply by replacing n(s) with W(s) in (3.3) does not yield the proper solution to (3.2) with n(t) replaced by W(t).

Thus, $x_o e^{-\int_o^t n(s)ds}$ is the solution to an equation that differs from the equation for which $x_o e^{-\int_o^t W(s)ds}$ is a solution. Yet, for the ordinary deterministic calculus, the equation would be the same. Thus, we must look further at how to study the wide band coefficient case by replacement with white noise coefficients. It appears from our simple example that the equation would have to be modified to yield the same analytical results.

We can show, simply, how the equation must be modified. Consider the scalar equation,

$$dx(t) = f(x(t))dt + g(x(t))\,dy(t), \tag{3.9}$$

where $\frac{dy(t)}{dt}$ exists as a physical noise process.

We assume that we can write the solution as

$$x(t) = F(y(t),t) \tag{3.10}$$

Thus, we would find

$$dx(t) = F_t(y(t),t)dt + F_y(y(t),t)dy(t) \tag{3.11}$$

Therefore, we must have by identifying terms in (3.9), (3.11),

$$F_t(y(t),t) = f(F(y(t),t))$$

$$F_y(y(t),t) = g(F(y(t),t)). \tag{3.12}$$

Now suppose we replace y(t) by B(t) in (3.12), we set a new x(t) equal to,

$$x(t) = F(B(t),t)\ . \tag{3.13}$$

Upon applying the Ito differential formula, (3.7a) we will have

$$dx(t) = F_t(B(t),t)dt + F_B(B(t),t)dB + \frac{\sigma^2}{2} F_{BB}(B(t),t)dt \tag{3.14}$$

now $F_B(B(t),t) = g(F(B(t),t))$,from (3.12). Therefore,

$$F_{BB}(B(t),t) = g'(F(B(t),t))\ F_B(B(t),t)$$

$$= g'(F(B(t),t))\ g(F(B(t),t)), \tag{3.15}$$

again from (3.12).

Upon realizing that (3.13) holds, then the substitutions of (3.12), (3.15) into (3.14) will give us the equation

$$dx(t) = f(x(t)dt + g(x(t))dB(t) + \frac{\sigma^2}{2} g'(x(t))g(x(t)dt \tag{3.16}$$

We see, therefore, the change in the equations (3.9), (3.16) through the term

$$\frac{\sigma^2}{2} g'(x(t))\ g(x(t))dt. \tag{3.17}$$

Therefore, it is just this term that must be added to the equation (3.9) when replacing the coefficient $\frac{dy(t)}{dt}$ by the Gaussian white noise $W(t)\left(\simeq\frac{dB(t)}{dt}\right)$ The term (3.17) is a correction term and is usually referred to in the literature as the Wong-Zakai correction term.

For the general n-dimensional vector differential equations, we will find that

$$dx(t) = f(x(t),t)\ dt + G(x(t),t)dy(t), \tag{3.18}$$

where x,f are n-vectors, y is an m-vector and G is an nxm matrix, will be replaced by the equations

$$dx_i(t) = \left[f_i(x,(t),t) + \frac{\sigma^2}{2} \sum_{k=1}^{n} \sum_{\ell=1}^{m} g_{k\ell}(x(t),t) \frac{\partial g_{i\ell}(x(t),t)}{\partial x_k} \right] dt$$

$$+ \sum_{\ell=1}^{m} g_{i\ell}(\vec{x}(t),t) dB_\ell(t) \quad , \quad i=1,\ldots,n \tag{3.19}$$

The immediate question for us, is how will this change the linear equations with random coefficients that are "almost" white noise. We have already seen how the simple first order equation (3.2) is modified to obtain the added term present in (3.8). Our interest, of course, is for higher order systems.

Thus, we can for example consider the case of the second order oscillator

$$\begin{cases} dx_1 = x_2 dt \\ dx_2 = -(2\zeta\omega x_2 + \omega^2 x_1)dt - x_1 dy \end{cases} \tag{3.20}$$

which corresponds to

$$\ddot{x} + 2\zeta\omega\dot{x} + (\omega^2 + \dot{y})x = 0,$$

where $\dot{y}$ is a physical noise coefficient.

Identifying terms in (3.18), (3.19) with the system (3.20), we see that

$$G(x) = \begin{pmatrix} 0 \\ -x_1 \end{pmatrix} = \begin{pmatrix} g_{11} \\ g_{21} \end{pmatrix} \tag{3.21}$$

Hence, the correction terms in (3.19) become

$$\sum_{k=1}^{2} g_{k1} \frac{\partial g_{i1}}{\partial x_k} = \begin{cases} 0 \ , & i = 1 \\ 0 \ , & i = 2 \ . \end{cases} \tag{3.22}$$

Thus, there is no correction required. The associated Ito equation is

$$\ddot{x} + 2\zeta\omega\dot{x} + (\omega^2 + B)x = 0$$

Indeed for the second order oscillator, a correction term is required only when there is a randomly fluctuating parameter in the damping term.

Indeed, for the n^{th} order general linear system, there will be a correction term required only when there is a random parameter appearing in the $(n-1)^{th}$ derivative terms. No corrections will be required to the linear system equations when the randomly fluctuating coefficients appear in terms lower than the $(n-1)^{th}$ derivative. The following example illustrates this property.

In exactly the same fashion as above, we can easily see that for the system

$$\begin{cases} dx_1 = x_2 dt \\ dx_2 = -(2\zeta\omega x_2 + \omega^2 x_1)dt - x_2 dy \end{cases} \tag{3.23}$$

corresponding to

$$\ddot{x} + (2\zeta\omega + \dot{y})\dot{x} + \omega^2 x = 0,$$

the G matrix is given as

$$G(x) = \begin{pmatrix} 0 \\ -x_2 \end{pmatrix} = \begin{pmatrix} g_{11} \\ g_{21} \end{pmatrix}. \tag{3.24}$$

Hence, the correction terms become,

$$\sum_{k=1}^{2} g_{kl} \frac{\partial g_{il}}{\partial x_k} = \begin{cases} 0 , & i=1 \\ x_2 , & i=2. \end{cases} \tag{3.25}$$

Thus, the associated Ito equation becomes

$$\begin{cases} dx_1 = x_2 dt \\ dx_2 = [-(2\zeta\omega x_2 + \omega^2 x_1) + \frac{2}{2} x_2]dt - x_1 dB, \end{cases} \tag{3.26}$$

which corresponds to

$$\ddot{x} + [2\zeta\omega - \frac{\sigma^2}{2} + \dot{B}]\dot{x} + \omega^2 x = 0. \tag{3.27}$$

Notice, that for the associated Ito equation the $-\frac{\sigma^2}{2}\dot{x}$ has a destabilizing effect on the overall system.

Naturally, we easily see how these correction terms will appear for the linear system. The important application is that of determining the approximate moments for the specific system.

For the physical noise coefficient system (3.20), we cannot determine the various moments.

However, for the associated Ito system, with no correction terms, we have the generator

$$\mathscr{L} = x_2 \frac{\partial}{\partial x_1} - (2\zeta\omega x_2 + \omega^2 x_1) \frac{\partial}{\partial x_2} + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x_2^2}, \tag{3.28}$$

from which we can obtain all moments, as in Section II.

Again for the physical noise coefficient system (3.23), we cannot obtain the moments directly. However, the associated Ito system (3.26) possesses the generator

$$\mathscr{L} = x_2 \frac{\partial}{\partial x_1} - [(2\zeta\omega - \frac{\sigma^2}{2})x_2 + \omega^2 x_1] \frac{\partial}{\partial x_2} + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x_2^2} \tag{3.29}$$

from which all moments may be obtained.

Finally, it is immediately seen from (3.19) that the backward operator for the system (3.19), which contains the correction terms is

$$\begin{aligned}\mathscr{L} = &\sum_{i=1}^{n} [f_i(x,t) + \frac{\sigma^2}{2} \sum_{k=1}^{n} \sum_{\ell=1}^{m} g_{k\ell}(x,t) \frac{\partial g_{i\ell}(x,t)}{\partial x_k}] \frac{\partial}{\partial x_i} \\ &+ \frac{1}{2} \sum_{i,j=1}^{n} \left[\sum_{r,s=1}^{m} g_{ir}(x,t) g_{js}(x,t) \sigma_{rs}\right] \frac{\partial^2}{\partial x_i \partial x_j}\end{aligned} \tag{3.30}$$

For a comprehensive study of the sample behavior of such physical noise coefficient linear systems and their associated Ito systems see [3.8].

In closing, we mention that physical noise coefficient systems with small parameters can also be related to associated Ito systems and their Markov process solutions. These techniques are based upon so-called averaging methods, which will be discussed in detail in Section V. We shall now turn to a more detailed discussion of the physical noise coefficient case.

IV. Physical Noise Coefficient Systems

For the case of linear systems that cannot be in some way approximated by differential equations with white noise coefficients or approximated by other methods, there are very few specific results available.

This, of course, is due to the fact that differential equations with time varying coefficients cannot, in general, be solved exactly. Indeed, unless we can solve the equations exactly, we cannot expect to obtain moments, nor probability densities. This is the opposite of the white noise coefficient case, where moments can be obtained exactly even though the differential equations cannot be solved.

The question that we must first consider is, what types of ordinary differential equations with time varying coefficients can we solve exactly.

The general equation of interest is

$$\dot{x}(t) = A(t)\, x(t), \tag{4.1}$$

where x is an n-vector, and A(t) is an n x n matrix which contains elements that are randomly time varying.

For the first order case

$$\dot{x}(t) = a(t)\, x(t), \tag{4.2}$$

we can write the solution as

$$x(t) = x_o e^{\int_o^t a(s)ds} \tag{4.3}$$

whose moment properties may be obtained under certain conditions.

This depends upon the assumptions we place upon the coefficient process $\{a(t),\ t \in [0,\infty)\}$.

It is a rather important point that in general, given the statistical properties of the a-process, (e.g. the joint probability densities) the probability densities of the integrated process $\int_o^t a(s)ds$ cannot be obtained.

Thus, even for the first order physical noise coefficient differential equation, we may not be able to obtain even its moments, exactly.

The reason here is that we must evaluate

$$E\{x^n(t)\} = E\{x_o^n\, e^{n\int_o^t a(s)ds}\} = x_o^n\, E\{e^{n\int_o^t a(s)ds}\}. \tag{4.4}$$

Without knowledge of the probability density for the integrated a-process, we cannot explicitly evaluate these expectations.

Fortunately, there is a class of coefficient processes for which we can determine the joint probabilities for the integrated process. This class is, of course, the class of Gaussian processes. For the Gaussian processes we know that any linear operation on the process will again yield a Gaussian process. This is the only general class for which we can make such a statement. Other processes simply do not allow us to make such a complete statement. Hence, this is another case in which the Gaussian assumption allows us to make a complete statement about the solution process.

If the a-process is zero mean Gaussian with covariance $\gamma_a(t_1,t_2)$, it immediately follows that the integral, $\int_o^t a(s)ds$ is also a zero mean Gaussian process with covariance given as

$$E\left\{\int_o^{t_1} a(s)ds \int_o^{t_2} a(s)ds\right\} = E\left\{\int_o^{t_1} ds_1 \int_o^{t_2} ds_2\, a(s_1)a(s_2)\right\}$$

$$= \int_o^{t_1} ds_1 \int_o^{t_2} ds_2\, E\{a(s_1)a(s_2)\}$$

$$= \int_o^{t_1} ds_1 \int_o^{t_2} ds_2\, \gamma_a(s_1,s_2)\ . \tag{4.5}$$

Thus, it follows that the integrated a-process is zero mean Gaussian with variance

$$\sigma^2(t) = \int_o^t ds_1 \int_o^t ds_2\, \gamma_a(s_1,s_2) \tag{4.6}$$

and the desired expectations (4.4) can be obtained explicitly as

$$E\{x^2(t)\} = x_o^n \, E\{e^{n\int_o^t a(s)ds}\}$$

$$= x_o^n \int_{-\infty}^{\infty} e^{ny} \frac{1}{\sqrt{2\pi\sigma(t)}} e^{-\frac{y^2}{2\sigma^2(t)}} dy$$

$$= x_o^n \; e^{\frac{n}{2}\sigma^2(t)}, \tag{4.7}$$

where σ^2 is given by (4.6).

For the non-Gaussian case one might consider an expansion of the exponential, leading us to the result

$$E\{e^{n\int_o^t a(s)ds}\} = \sum_{k=0}^{\infty} \frac{n^k \, E\{(\int_o^t a(s)ds)^k\}}{k!} . \tag{4.8}$$

Since the moments of the integrated a-process can be written as

$$E\{(\int_o^t a(s)ds)^k\} = \int_o^t ds_1 \; \ldots \; \int_o^t ds_k \;\; E\{a(s_1)\ldots a(s_k)\}, \tag{4.9}$$

then if the joint moments of the a-process are known, we can in principle write (4.8) in a series form. However, summing the series would be a problem of higher order of magnitude difficulty.

Thus, we see that only for the Gaussian assumption can we obtain explicit results for even the simplest first order physical noise coefficient system.

What can be said for higher order systems? The underlying difficulty here is simply that the general solution to the n^{th} order time varying system (4.1) cannot be written in a closed form. One is tempted to write the solution as the matrix exponential

$$x(t) = e^{\int_o^t A(s)ds} \, x_o, \tag{4.10}$$

However, as is well known (4.10) can be the solution of (4.1) only if a condition such as the commutativity of the matrices A(t) and $\int_o^t A(s)ds$, holds.

Otherwise, we cannot represent the solution as (4.10). Of course this does allow us to study systems satisfying the equality

$$A(t) = a(t)\, A_o, \tag{4.11}$$

where a(t) is a scalar random process, and A_o is a constant matrix. In this case A(t) and $\int_o^t A(s)ds \equiv \left(A_o \int_o^t a(s)ds\right)$ do commute, so that the solution of (4.1) with (4.11) is given by

$$x(t) = e^{A_o \int_o^t a(s)ds}\, x_o\,. \tag{4.12}$$

This is of course no different than in solving for the solution of the constant coefficient matrix case. The main distinction being that the characteristic values of the A_o matrix will pick up factors of $\int_o^t a(s)ds$. Therefore, if A_o, for example can be diagonalized by the matrix X, then we know that

$$X^{-1}A_oX = \Lambda \tag{4.13}$$

where Λ is the diagonal matrix of the characteristic values of A_o.

Hence (4.13) yields

$$X^{-1}A_o \int_o^t a(s)ds\, X = \Lambda \int_o^t a(s)ds = \begin{pmatrix} \lambda_1 \int_o^t a(s)ds & & \bigcirc \\ & \ddots & \\ \bigcirc & & \lambda_n \int_o^t a(s)ds \end{pmatrix} \tag{4.14}$$

which gives us the explicit solution

$$x(t) = X^{-1} e^{\Lambda \int_o^t a(s)ds} X x_o$$

$$= X^{-1} \begin{pmatrix} e^{\lambda_1 \int_o^t a(s)ds} & & \bigcirc \\ & \ddots & \\ \bigcirc & & e^{\lambda_n \int_o^t a(s)ds} \end{pmatrix} X x_o \qquad (4.15)$$

It follows that if the a-process is Gaussian, then as in the first order example, we can obtain all of the desired statistical properties of the solution vector process x(t).

Can we say anything specific for more general noise coefficient linear systems?

There is a class of linear random coefficient systems for which we can find explicit solutions. In order to describe this class, we first appeal to our usual first order case.

Let us consider the first order equation that generalizes (4.2), which we may write as,

$$\frac{dx(t)}{dt} = c\,x(t) + \left(\sum_{i=1}^{m} d_i a_i(t)\right)x(t) \ , \qquad (4.16)$$

where c, $\{d_i\}$ are given constants and $\{a_i(t)\}$ are m-coefficient processes which may or may not be correlated, may or may not be Gaussian, etc.

We can write the solution, with $x(0) = x_o$, as

$$x(t) = x_o e^{c\,t + \sum_{i=1}^{m} d_i \int_o^t a_i(s)ds} \ . \qquad (4.17)$$

Having this explicit solution, we can therefore obtain statistical properties as desired depending upon whether the joint statistics of the integrals $\int_o^t a_i(s)ds$ can be obtained (such as in the Gaussian case).

It immediately follows that if the matrices C, $D_1,\ldots,D_m$ are n x n diagonal matrices,

then, the vector equation

$$\frac{dx(t)}{dt} = C\,x(t) + [\sum_{i=1}^{m} a_i(t)D_i]\,x(t), \tag{4.18}$$

may be written as the n first-order equations

$$\frac{dx_j(t)}{dt} = c_j x_j(t) + [\sum_{i=1}^{m} d_{ij}a_i(t)]x_j(t),\ j = 1,2,\ldots,n. \tag{4.19}$$

The solutions to (4.19) may be written as in the form of (4.17).

The next level of complexity will occur when the C, $\{D_i\}$ matrices are all upper triangular. In this case, the solutions may be obtained sequentially.

Thus for

$$C = \begin{pmatrix} c_1 & & & X \\ & c_2 & & \\ & & \ddots & \\ \bigcirc & & & c_n \end{pmatrix}, \qquad D_i = \begin{pmatrix} d_{i1} & & & X \\ & d_{i2} & & \\ & & \ddots & \\ \bigcirc & & & d_{in} \end{pmatrix} \tag{4.20}$$

we would have

$$\frac{dx_n(t)}{dt} = c_n x_n(t) + [\sum_{i=1}^{m} d_{in} a_i(t)] x_n(t)$$

$$\frac{dx_{n-1}(t)}{dt} = c_{n-1} x_{n-1}(t) + c^n_{n-1} x_n(t) + [\sum_{i=1}^{m} d_{i(n-1)} a_i(t)]\, x_{n-1}(t) +$$

$$[\sum_{i=1}^{m} d^n_{in-1} a_i(t)]\, x_n(t)$$

etc. (4.21)

We see that there is a sequence of first order equations, all solvable. The first equation is first order homogeneous, all others are first order non-homogeneous. Again, if all the $[a_i(t)]$ are Gaussian processes, then the statistical properties of the solution vector process x are completely determined.

The next step in the generalization is to consider the case that the matrices $C,\{D_i\}$ can all be transformed into upper triangular matrices. This would allow us to reduce the problem to the form we have just considered. This problem can be expressed in a concise form by means of Lie algebra theory, [4.1], [4.2], [4.3], [4.4].

We briefly describe the fundamental ideas. A subspace L of n x n matrices is called a Lie algebra if for all C, D in L, the commutator product $[C,D] \equiv CD-DC$ also belongs to L. Let $L(C,D_1,\ldots,D_m)$ denote the smallest Lie algebra containing the matrices C, $D_1,\ldots D_m$. This is usually referred to as the Lie algebra generated by C, $D_1,\ldots D_m$. One defines the derived series of the algebra as follows.

$$
\begin{aligned}
L^0 &= L \\
L^1 &= [L, L] \\
&\vdots \\
L^{n+1} &= [L^n, L^n] \\
&\vdots
\end{aligned}
\qquad ,
$$

where $[L^k, L^k]$ denotes the set of all matrices formed by $[C^k, D^k] = C^k D^k - D^k C^k$, where C^k, $D^k \in L^k$.

The Lie algebra is said to be solvable if there is an integer N for which $L^N = \{0\}$. Clearly, an Abelian Lie algebra (where all matrices are pairwise commutative) is a special case of a solvable Lie algebra, since C, $D \in L$ satisfies $[C, D] = \{0\}$. Therefore, $[L, L] = \{0\}$.

How do these results impact upon our problem for random coefficient systems? The connection with our discussions above is contained in the following lemma. [4.5].

Lemma - A matrix Lie algebra, L, is solvable if,and only if, there exists a non-singular matrix P such that PMP^{-1} is upper triangular for any $M \in L$.

Thus, we see that if the n x n matrices C, $D_1, \ldots, D_m$ generate a solvable Lie algebra, L, then the linear system

$$
\frac{dx(t)}{dt} = C\,x(t) + \left[\sum_{i=1}^{m} a_i(t) D_i\right] x(t), \tag{4.22}
$$

can be transformed by setting $y = Px$ where P is guaranteed to exist by the lemma to yield

$$
\frac{dy(t)}{dt} = PCP^{-1}\, y(t) + \left[\sum_{i=1}^{m} a_i(t)\; PD_iP^{-1}\right] y(t), \tag{4.23}
$$

in which each matrix PCP^{-1}, PD_1P^{-1},...,PD_mP^{-1} is upper triangular. It follows that the solution vector y(t) can be obtained explicitly by sequential calculations. Finally the x-process is obtained by $x(t) = P^{-1}y(t)$.

Example [4.2]

Consider the second order system

$$\frac{dx(t)}{dt} = [C + D_1a_1(t) + D_2a_2(t)]x(t),$$

where

$$C = \alpha \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, D_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix},$$

and $a_1(t)$, $a_2(t)$ are Gaussian processes possibly correlated.

It is possible to show that the Lie algebra generated by (C, D_1, D_2) is solvable, with $P = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$.

We immediately find

$$PCP^{-1} = \alpha \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; PD_1P^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \; PD_2P^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

which gives us the transformed equation

$$\frac{dy(t)}{dt} = \begin{pmatrix} \alpha + a_1(t) & \alpha + a_1(t) \\ 0 & \alpha + a_2(t) \end{pmatrix} y(t). \qquad (4.24)$$

Clearly, (4.24) can be explicitly integrated, giving us the desired statistics for the vector solution process $x = p^{-1}y$.

There is one point that must be made here concerning the Lie algebra approach. Generally speaking, many of the linear equations that we study in mechanical vibrations do not generate solvable Lie algebras.

Thus, the classical oscillator equation yields

$$C = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\zeta\omega \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

for a randomly varying stiffness coefficient,

$$\ddot{x}(t) + 2\zeta\omega\dot{x}(t) + [\omega^2 + a(t)]x(t) = 0$$

Even in this simplest case therefore we cannot use the Lie algebra approach. For general linear systems, one would have to test whether they generate solvable algebras. This is not easy to accomplish, especially for structures with many components. For further details one may see [4.5], [4.6].

We shall finally consider what can be stated concerning bounds on the second moment statistics from a Lyapunov function point of view. We shall follow the derivation of Infante [4.7] [see [4.8], also]. For deterministic systems this basic idea was first studied by Wintner [4.9].

Towards this end, for the arbitrary n^{th} order linear system

$$\frac{dx(t)}{dt} = (A + F(t))x(t), \tag{4.25}$$

We define the quadratic form

$$V(x) = x^T P x, \tag{4.26}$$

where P is a positive definite symmetric matrix. Clearly if P is the identity, then

$$V(x) = \|x\|^2, \quad \|x\| = (\sum x_i^2)^{1/2}. \tag{4.27}$$

We evaluate the derivative of (4.26) along the trajectories of (4.25), thus obtaining,

$$\frac{d}{dt} V(x(t)) = x^T(t) \left[(A+F(t))^T P + P(A+F(t))\right] x(t) \tag{4.28}$$

Upon dividing by V(x(t)), we will obtain

$$\frac{1}{V(x(t))} \frac{dV(x(t))}{dt} = \frac{x^T(t)\left[(A+F(t))^T P + P(A+F(t))\right] x(t)}{x^T(t)\, Px(t)} \tag{4.29}$$

Clearly the quotient on the right hand side of (4.29) is quite complicated. However, by the properties of pencils of quadratic forms [4.10], as well as the min-max properties of eigenvalues, it is known that

$$\lambda_{min}(DP^{-1}) \leq \frac{x^T Dx}{x^T Px} \leq \lambda_{max}(DP^{-1}), \tag{4.30}$$

where λ_{min}, λ_{max} are, respectively, the minimum and maximum eigenvalues of the matrix DP^{-1}. Note since P is positive definite, then P^{-1} exists. Furthermore, the operator associated with the matrix DP^{-1} can be shown to be symmetric [4.9], hence the eigenvalues of DP^{-1} must be real.

Upon applying the inequalities (4.30) to the equality (4.29), we find that

$$\lambda_{min}(t) \leq \frac{1}{V(x(t))} \frac{dV(x(t))}{dt} \leq \lambda_{max}(t), \tag{4.31}$$

where

$$\lambda_{min,max}(t) = \lambda_{min,max}[(A+F(t))^T + P(A+F(t))P^{-1}].$$

Upon integrating (4.31) and taking exponentials of each term, we have

$$e^{\int_o^t \lambda_{min}(s)ds} \leq V(x(t)) \leq e^{\int_o^t \lambda_{max}(s)ds},$$

which leads to the expected values

$$E\{e^{\int_o^t \lambda_{min}(s)ds}\} \leq E\{V(x(t))\} \leq E\{e^{\int_o^t \lambda_{max}(s)ds}\}. \tag{4.32}$$

For the left inequality of (4.32), we may apply Jensen's theorem [4.11] which states that if $g(x)$ is convex (i.e., $g''(x) \geq 0$), then $g(E\{x\}) \leq E\{g(x)\}$.

Thus we have from (4.32), the lower bound

$$e^{E\{\int_o^t \lambda_{min}(s)ds\}} \leq E\{e^{\int_o^t \lambda_{min}(s)ds}\} \leq E\{V(x(t))\}. \tag{4.33}$$

Therefore, we must find

$$E\{\int_o^t \lambda_{min}(s)ds\} = \int_o^t E\{\lambda_{min}(s)\}ds.$$

For the oscillator $\ddot{x} + 2\zeta\omega\dot{x} + (\omega^2 + f(t))\,x = 0$, we have

$$\frac{dx(t)}{dt} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -2\zeta\omega \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 \\ -f(t) & 0 \end{pmatrix} x(t) \tag{4.34}$$

In the case that $P = I$, then we have $\lambda_{min,max}$ are obtained from, the eigenvalues of the matrix,

$$\begin{pmatrix} 0 & 1-(\omega^2 + f(t) \\ 1-(\omega^2 + f(t) & -2\zeta\omega \end{pmatrix} .$$

These are respectively

$$\zeta\omega \pm \sqrt{\zeta^2\omega^2 + (f(t) + \omega^2 - 1)^2} . \tag{4.35}$$

Hence, the lower bound on the $E\{\|x\|^2\}$, is obtained from (4.33) as

$$e^{\zeta\omega t - \int_0^t E\{\sqrt{\zeta^2\omega^2 + (f(s) + \omega^2 - 1)^2}\} ds} \leq E\{\|x\|^2\} \tag{4.36}$$

By the Schwartz inequality, we can write

$$E\{\|x\|\} \leq E\{\|x\|^2\}^{1/2} ,$$

Thus, it follows that

$$E\{\sqrt{\zeta^2\omega^2 + (f(s) + \omega^2 - 1)^2}\} \leq E\{\zeta^2\omega^2 + (f(s) + \omega^2 - 1)^2\}^{1/2}, \tag{4.37}$$

and (4.36) can finally be written as

$$e^{(\zeta\omega-\zeta^2\omega^2)t} - \int_o^t E\{(f(s)+\omega^2-1)^2\}ds \leq E\{\|x(t)\|^2\} . \qquad (4.38)$$

Thus, we have lower bounds that can be evaluated knowing only the second moments of the coefficient process.

Similar results can be obtained for upper bounds as well by the same concepts. For example, see [4.12].

This section contains essentially all general concepts that allow explicit solutions for the moments of linear systems with physical noise coefficients. Any other exact solutions are for specific cases. In general for the physical noise coefficient case, one must approximate. This is the subject of the next section.

V. Approximate Methods

Except for the various cases and examples that we have discussed in the previous sections for which exact results are available, all other linear systems with random coefficients must be looked at based upon some approximate method. These methods of approximation involve expansions in small parameters, asymptotic convergences in terms of small parameters in particular the various stochastic averaging techniques, and finally the so-called closure methods in which expansions in moments or cumulants are terminated based upon some assumed property. These are the main topics that we will attempt to describe in this section. Since exact results are few, one can be reasonably comprehensive in a survey. But approximate results are many, therefore, we cannot claim to be comprehensive. Instead, we try to present those results that are representative of the specific approach.

V.1 Expansion Techniques

As we stated in Sections II, IV, when the coefficient process is a physical (non-white) process, we cannot separate its statistics at a given time from the response statistics at a given time. Therefore, for the general physical noise we cannot, in general, even determine the first moment exactly. We can only approximate. One method of approximation is by expansion. The literature contains a great many examples of this approach.

Basically the idea is as follows.

For the linear system:

$$\frac{dx(t)}{dt} = (A + F(t))x(t), \tag{5.1}$$

as stated before, A is an n x n constant matrix and F(t) is an n x n matrix whose non-zero elements are stochastic processes. We can write (5.1) as the integral equation, with initial condition $x(t_o) = x_o$,

$$x(t) = \Phi(t,t_o)x_o + \int_{t_o}^{t} \Phi(t,s)F(s)x(s)ds, \tag{5.2}$$

where the transition matrix Φ, is given explicitly as the deterministic matrix exponential

$$\Phi(t,s) = e^{A(t-s)}, \tag{5.3}$$

where (t-s) is a scalar parameter.

A typical recursive approximation scheme represents the solution to (5.2) as,

$$\begin{cases} x_o(t) = \Phi(t,t_o)x_o \\ x_1(t) = \Phi(t,t_o)x_o + \int_{t_o}^{t} \Phi(t,s)F(s)x_o(s)ds \\ \quad \vdots \qquad \vdots \qquad \vdots \\ x_n(t) = \Phi(t,t_o)x_o + \int_{t_o}^{t} \Phi(t,s)F(s)x_{n-1}(s)ds \\ \quad \vdots \qquad \vdots \qquad \vdots \end{cases} \tag{5.4}$$

To a certain extent most approximations to the time varying linear system is of this form, or closely related to it. An associated procedure is to consider the expansion in ε of the system.

$$\frac{dx(t)}{dt} = (A + \varepsilon F(t))x(t), \tag{5.5}$$

Upon writing the associated integral equation equivalent to (5.2), we have

$$x(t) = \Phi(t,t_o)x_o + \varepsilon \int_{t_o}^{t} \Phi(t,s)F(s)x(s)ds. \tag{5.6}$$

Now expanding the solution $x(t)$ in powers of ε, we obtain the representation

$$x(t) = \sum_{k=0}^{\infty} \varepsilon^k x_k(t) \tag{5.7}$$

Substituting (5.7) into (5.6) and identifying the coefficients of the powers of ε, we obtain the sequence of terms $x_k(t)$, as

$$\begin{cases} x_o(t) = \Phi(t,t_o)x_o \\ x_1(t) = \int_{t_o}^{t} \Phi(t,s)F(s)x_o(s)ds \\ x_2(t) = \int_{t_o}^{t} \Phi(t,s)F(s)x_1(s)ds \\ \quad \vdots \\ x_n(t) = \int_{t_o}^{t} \Phi(t,s)F(s)x_{n-1}(s)ds \\ \quad \vdots \end{cases} \tag{5.8}$$

Both of these ideas are quite classical. For example, one can find the representation (5.7) and expansion (5.8) in the works of Kryloff and Bogoliubov [5.1]. For a discussion see [5.2]. Indeed, this procedure is attributed to Poisson and was studied further by Poincare in 1892. Naturally convergence is the problem of significance, especially if ε is not small. For $\varepsilon = 1$, this has been called Adomian's method in the stochastic literature. See for example [5.3] and [5.4]. Given such a classical approach, it appears somewhat questionable to attribute the procedure to anyone currently active.

Of course the objective of these expansions is to obtain the statistics of the solution process. Following the notation in [5.3] we can write (5.2) as

$$x(t) - \int_{t_o}^{t} N(t,s)x(s)ds = G(t,t_o), \tag{5.9}$$

where

$$N(t,s) = \Phi(t,s)F(s)$$
$$G(t,t_o) = \Phi(t,t_o)x_o \quad .$$

Defining the iterated kernels $N_k(t,s)$ as

$$N_1(t,s) = N(t,s)$$

$$N_k(t,s) = \int_{t_o}^{t} N(t,\tau)N_{k-1}(\tau,s)d\tau \tag{5.10}$$

$$k = 2,3,\ldots \quad ,$$

we define the resolvent kernel of $N(t,s)$ by the Neuman series

$$\Gamma(t,s) = \sum_{k=1}^{\infty} N_k(t,s) \tag{5.11}$$

This allows one to write the solution to (5.2) as

$$x(t) = \Phi(t,t_o)x_o + \int_{t_o}^{t} \Gamma(t,s)\Phi(s,t_o)x_o ds. \tag{5.12}$$

In this case, the mean vector $E\{x(t)\}$, may be obtained as

$$E\{x(t)\} = \Phi(t,t_o)E\{x_o\} + \int_{t_o}^{t} E\{\Gamma(t,s)\}\Phi(s,t_o)E\{x_o\}ds, \tag{5.13}$$

assuming that the statistics of F(t), and of x_o are independent of one another, and, furthermore, that the A matrix is deterministic. If the A matrix contains random constants, then the terms $E\{\Phi(t,t_o)\}$, $E\{\Phi(t,s)\}$ would appear in (5.13).

Although these procedures are straightforward applications of established solution expansion techniques, obtaining the desired statistics is not made truly simpler due to the complexities of the infinite number of terms to be considered. Expansions of moments using such ideas have been studied by many investigators. In the modern development of the topic, one can go back to the works of Samuels and Eringen [5.5], [5.6] who studied stability of the second moments of systems of the form (5.1) by such expansion techniques, (see also [5.3], [5.7].

For a general development of this topic, one might look at [5.8]. Notice that there is nothing fundamentally probabilistic about the expansion methods above. The so-called Neuman series for integral equations, as well as the perturbation expansions were developed for classical deterministic equations.

There are a number of papers which deal directly with approximations to the joint density functions via a sequence of integro-differential equations, that involve certain smoothing operations.

The first paper along these lines apparently is due to Kryloff and Bogoliubov in 1939 [5.9]. Subsequent studies appeared in [5.10] and in [5.11]. Also consult [5.12] for further referenced details.

V.2 Hierarchy Equations and Closure

Under certain assumptions, it is possible to obtain a set of equations that will yield the moments of the solutions of systems with randomly varying parameters. However, these equations may depend upon an infinite number of variables that render them impossible to solve by finite methods. This is typical of the type of problem one meets with non-linear systems in which lower order approximants may be a function of higher order approximants so that the equations are not closed. In that case, assumptions are put on the system (which may or may not be justified) which will reduce the infinite heirarchy of equations to a finite set that can be solved. After solution of the finite set, naturally the errors in the approximation must be studied. At this time, however, there is no general statement that can be made regarding the errors introduced.

We illustrate this problem for the case of the oscillator with a randomly varying spring coefficient. We assume that the random coefficient is generated by filtering a white noise. Thus, it is a Markov process, or more generally, it is the component of a Markov process. One of the earliest papers to treat the problem in this way was [5.13]. Thus, consider the oscillator,

$$\ddot{x}(t) + 2\zeta\omega\dot{x}(t) + [\omega^2 + y(t)]x(t) = 0, \tag{5.14}$$

where y(t) is generated by the Ito equation

$$dy(t) = -\beta y(t)dt + dB(t), \tag{5.15}$$

where the B-process is the usual Brownian motion. We recognize that the y-process is the Ornstein-Uhlenbeck process.

Now since y is generated by an Ito equation, then it follows that the vector $(x, \dot{x}, y)$ is a Markov process generated by the Ito system

$$\begin{cases} dx_1(t) = x_2(t)dt \\ dx_2(t) = -[2\zeta\omega x_2(t) + (\omega^2 + y(t))x_1(t)]dt \\ dt(t) = -\beta y(t)dt + dB(t). \end{cases} \tag{5.16}$$

Following along the concepts developed in Section II, we can write the generator of the system (5.16) as

$$\mathcal{L}_x = x_2 \frac{\partial}{\partial x_1} - [2\zeta\omega x_2 + (\omega^2 + y)x_1] \frac{\partial}{\partial x_2} - \beta y \frac{\partial}{\partial y} + \frac{\sigma^2}{2} \frac{\partial^2}{\partial y_2}, \tag{5.17}$$

and obtain the moment equations via the Dynkin formula, (2.22), as in Section II.

Upon denoting the n^{th} moment as

$$E\{x_1^j(t)x_2^k(t)y^{\ell}(t)\} = m_{jk\ell}(t), \text{ where } j+k+\ell = m \tag{5.18}$$

we shall find the first moment equations to be

$$\begin{cases} \dot{m}_{100} = m_{010} \\ \dot{m}_{010} = -\omega^2 m_{100} - 2\zeta\omega m_{010} - m_{101} \\ \dot{m}_{001} = -\beta m_{001} \end{cases} \tag{5.19}$$

Here we already see that the m_{010} equation contains a second moment,

$$m_{101} = E\{x_1(t)y(t)\} .$$

For the general case, we shall have

$$\dot{m}_{jk\ell} = j\, m_{j-1,k+1,\ell} - 2\zeta\omega k m_{jk\ell} - \omega^2 k m_{j+1,k-1,\ell} - k m_{j+1,k-1,\ell+1} \tag{5.20}$$

$$-\beta\ell m_{jk\ell} + \frac{\sigma^2}{2}\ell(\ell-1)m_{jk,\ell-2}$$

In (5.20), we see the term $m_{j+1,k-1,\ell+1}$ which is $j+k+\ell+1$ moment, one order higher than $m_{jk\ell}$.

Hence, the set of equations $m_{jk\ell}$ for $j+k+\ell \leq n$ is not a closed set. It is, instead an infinite hierarchy which we can express generically as

$$\dot{m}_n = G_n(m_1, m_2, \ldots, m_n, m_{n+1}), \tag{5.21}$$

always depending upon one order higher.

This n, of course, is not a surprise since the system (5.16) is non-linear in the variables (x_1, x_2, y) through the $x_1 y$ term in the second equation.

The objective of a closure procedure, would be to invoke a property or assumption that would allow us to write the approximate equations

$$\dot{\bar{m}}_n = \bar{G}_n(\bar{m}_1, \bar{m}_2, \ldots, \bar{m}_n). \tag{5.22}$$

There is no general theory that allows one to measure the errors made with such closure procedures. However, there have been many attempts to study this problem. In the study of turbulence, [5.14] contains investigations of closure. Closure ideas were applied to stochastic eigenvalue problems in [5.15]. Lie algebra ideas were applied in [5.16] to obtain closed first and second moment equations.

An investigation of error in the approximation, leads to results for one-dimensional systems in [5.17] and [5.18].

Two classes of closure concepts naturally arise. They are Gaussian closure methods and non-Gaussian closure methods.

The Gaussian closure idea is quite clear. Since all moments of a Gaussian process may be expressed in terms of the first and second moments, then the hierarchy is essentially closed by writing the equations for the first two moments. Any higher order moments that appear in the first two equations are then reduced to the appropriate expressions in the first and second moments. Thus, for example, for the equation (5.21) Gaussian closure reduces it to,

$$\dot{m}_n = H_n(m_1, m_2) \tag{5.23}$$

for some new function H_n, and therefore

$$\begin{cases} \dot{m}_1 = H_1(m_1, m_2) \\ \dot{m}_2 = H_2(m_1, m_2) \end{cases} \tag{5.24}$$

becomes a closed set that can be solved in all cases for linear systems.

Of course the problem that exists here is that systems with randomly varying coefficients do not in general lead to Gaussian solutions. The measure of error from the Gaussian assumption may be large. This in turn may generate large errors from the time moments. The reference [5.19] contains further discussion and references of these ideas as they relate to structural applications.

Non-Gaussian closure techniques assume that all moments of order $r > n$ can be expressed in terms of lower order moments $r \leq n$. One may also express the non-Gaussian closure assumptions in terms of cumulants. See [5.19] for a discussion, and [5.20] for a recent application. Cumulant approximations have been used in studies of turbulence [5.21], and many other stochastic continuing problems [5.22]. The basic idea comes from the characteristic function of the underlying probability density for the solution process.

To that end, let us recall that the characteristic function of the process $x(t)$, is given as

$$\phi_x(u,t) = E\{e^{iux(t)}\}, \tag{5.25}$$

which we can write in a Taylor expansion about the origin as

$$\phi_x(u,t) = \sum_{k=0}^{\infty} \frac{(iu)^k}{k!} m_k(t), \tag{5.26}$$

where $m_k(t) = E\{x^k(t)\}$, assuming all moments exist. (Otherwise (5.26) becomes a finite series with a remainder term).

An equally important expansion is that of $\log \phi_x(u,t)$, which we shall write as

$$\log \phi_x(u,t) = \sum_{k=0}^{\infty} \frac{(iu)^k}{k!} \lambda_k(t) . \tag{5.27}$$

The $\{\lambda_k(t)\}$ are referred to as the cumulants, or semi-invariants of the x-process. The connection between m_k and λ_k are obtained as follows [5.23],

$$\begin{cases} \lambda_1 = m_1 \\ \lambda_2 = m_2 - m_1^2 = \sigma^2 \\ \lambda_3 = m_3 - 3\, m_1 m_2 + 2m_1^3 \\ \lambda_4 = m_4 - 3\, m_2^2 - 4\, m_1 m_3 + 12\, m_1^2 m_2 - 6\, m_1^4 \\ \text{etc.} \end{cases} \tag{5.28}$$

Inversely, we can also write

$$\begin{cases} m_1 = \lambda_1 \\ m_2 = \lambda_2 + \lambda_1^2 \\ m_3 = \lambda_3 + 3\lambda_1\lambda_2 + \lambda_1^3 \\ m_4 = \lambda_4 + 3\lambda_2^2 + 4\lambda_1\lambda_3 + 6\lambda_1^2\lambda_2 + \lambda_1^4 \\ \text{etc.} \end{cases} \tag{5.29}$$

The cumulants are also a measure of the "distance" away from Gaussian. For example, if $m_1 = 0$, we have from (5.29),

$$\begin{cases} m_2 = \lambda_2 (= \sigma^2) \\ m_3 = \lambda_3 \\ m_4 = 3\lambda_2 + \lambda_4 \\ m_5 = 10\lambda_2\lambda_3 + \lambda_5 \\ \text{etc.} \end{cases} \tag{5.30}$$

But, for Gaussian processes one finds

$$\begin{cases} m_2 = \lambda_2 (= \sigma^2) \\ m_3 = 0 \\ m_4 = 3\lambda_2 \\ m_5 = 0 \\ \text{etc.} \end{cases} \tag{5.31}$$

The way in which the so-called cumulant neglect procedure is applied, is simply to assume that for some K, such that for $k > K$, we set $\lambda_k = 0$.

Since λ_k is a polynomial in $m_k, m_{k-1}, \ldots, m_1$, it follows that this will essentially close the moment equations. Thus, for our example (5.14), the second moment equations are

$$\begin{cases} \dot{m}_{200} = 2m_{110} \\ \dot{m}_{020} = -4\zeta\omega m_{020} - 2\omega^2 m_{110} - 2m_{111} \\ \dot{m}_{110} = m_{020} - 2\zeta\omega m_{110} - \omega^2 m_{100} - 2m_{201} \\ \dot{m}_{101} = m_{011} - \beta m_{101} \\ \dot{m}_{011} = -2\zeta\omega m_{011} - \omega^2 m_{101} - m_{102} - \beta m_{011} \\ \dot{m}_{002} = -2\beta m_{022} + \sigma^2 \end{cases} \tag{5.32}$$

One can use the formula for the characteristic function in terms of the cumulants

$$\phi(u_1,u_2,u_3,t) = E\{e^{i(u_1x_1(t) + u_2x_2(t) + u_3y(t))}\}$$

$$= \exp \sum_{p,r,s=0}^{\infty} \frac{\lambda_{prs}}{p!r!s!} i^{p+r+s} u_1^p u_2^r u_3^s \tag{5.33}$$

and the fact that

$$i^{j+k+\ell} m_{jk\ell} = \frac{\partial^j}{\partial u_1 j} \frac{\partial^k}{\partial u_2 k} \frac{\partial^\ell}{\partial u_3 \ell} \phi(u_1,u_2,u_3;t) \tag{5.34}$$

to obtain

$$\begin{cases} m_{100} = \lambda_{100} \\ \vdots \\ m_{200} = \lambda_{200} \\ \vdots \\ m_{111} = \lambda_{111} + \lambda_{011}\lambda_{100} + \lambda_{010}\lambda_{101} + \lambda_{001}\lambda_{110} + \lambda_{010}\lambda_{100} \\ \vdots \end{cases} . \tag{5.35}$$

Therefore, if one assumes that the third order cumulant λ_{111} is zero the relation (5.35) would yield

$$m_{111} = m_{011}m_{100} + m_{010}m_{101} + m_{001}m_{110} + m_{010}m_{100} \tag{5.36}$$

which relates the third moment m_{111} to first and second order moments.

In the same fashion we could find similar relations for m_{201} and m_{102}. Thus, (5.32) along with (5.19) would become a closed set of non-linear equations to solve for the moments. This, however, would require numerical methods for the solution. Explicit examples of these concepts can be found in [5.19]. For further discussion of these concepts one can consult [5.24].

V.3 Methods of Averaging

The method of averaging for classical deterministic equations has its origins in the works of Van der Pol [5.25], who studied the non-linear equation

$$\ddot{x}(t) + \omega^2 x(t) = \varepsilon f(x(t), \dot{x}(t)), \tag{5.37}$$

where $\varepsilon > 0$ is a small parameter.

He assumed a solution in the, now classical, form

$$x(t) = a(t) \cos(\omega t + \phi(t)), \tag{5.38}$$

where a, ϕ satisfy the equations

$$\begin{cases} \dot{a}(t) = \varepsilon f_1(a(t)) \\ \dot{\phi}(t) = \varepsilon f_2(a(t)) \end{cases} . \tag{5.39}$$

For small ε, they are, clearly, slowly varying and become constants as $\varepsilon \to 0$.

The idea was applied by a great many experts in non-linear oscillations such as Mandelstam, Papalexi, Krylov and Bogoliubov, Minorski as well as others. However, the method of averaging was not placed upon a firm mathematical foundation until the major studies of Bogoliubov [5.26].

There are many interpretations of the method of averaging especially, as a result of the concept playing an increasingly important role in applications to stochastic problems. For a general survey of the method of averaging for deterministic systems, see [5.27], for stochastic systems see [5.19].

The classic result of Bogoliubov is as follows: Let a system be written in "standard form", x, f are n-vectors

$$\dot{x}(t) = \varepsilon f(x(t),t), \quad x(0) = x_o, \tag{5.40}$$

where f satisfies boundedness and uniform Lipschitz conditions in x for some region $D \subset R_n$ (R_n is n-dimensional euclidean space). Furthermore, let the limit (time average)

$$\bar{f}(y) = \lim_{T\uparrow\infty} \frac{1}{T} \int_o^T f(y,t)dt \tag{5.41}$$

exist uniformly for $y \in D$. Then, given any $\eta > 0$, and an arbitrarily large T, there exists an ε_o, such that for $0 < \varepsilon < \varepsilon_o$, the solution y of the averaged equation,

$$\dot{y}(t) = \varepsilon \bar{f}(y), \; y(0) = x_o, \tag{5.42}$$

will satisfy,

$$\| y(t)-x(t) \| < \eta , \tag{5.43}$$

for $t \in [0, T/\varepsilon]$.

Of even more interest is the case that f(y,t) is almost periodic in t, uniformly for $y \in D$.

In that case it is a classical fact that the limit (5.41) exists. Furthermore, Bogoliubov established that the inequality (5.43) holds for $t \in (0,\infty)$. Therefore, the averaged solution y(t) becomes a uniformly close approximation to the true solution x(t) on the entire time domain $(0,\infty)$.

This is an especially important result, since it allows us to determine qualitative properties such as stability and boundedness of the solutions of the original system (5.40), in terms of the averaged constant coefficient system (5.42).

This is especially straightforward for linear systems of the form

$$\dot{x}(t) = \varepsilon A(t)x(t). \tag{5.44}$$

In order to apply these ideas, a linear system must be put in the standard form (5.44). We can see how this may be done through the following illustrative example.

Example

Consider the simple oscillator with periodic coefficient,

$$\frac{d^2x(t)}{dt^2} + (1 + \cos\omega t)x(t) = 0. \tag{5.45}$$

we may write this Mathieu type equation as

$$\begin{cases} \dfrac{dx_1(t)}{dt} = x_2(t) \\ \dfrac{dx_2(t)}{dt} = -(1 + \cos\omega t)x(t). \end{cases} \tag{5.46}$$

Let us assume that ω is very large as compared to unity. Thus, $\cos\omega t$ is a very rapidly oscillating coefficient term.

In particular let us assume that

$$\omega = \frac{\omega_o}{\varepsilon}, \text{ where } 0 < \varepsilon, \varepsilon << 1. \tag{5.47}$$

If we substitute into the cosine term

$$\cos\omega t = \cos \frac{\omega_o}{\varepsilon} t = \cos\omega_o \tau, \tag{5.48}$$

we can define the "fast" time $\tau \equiv \frac{t}{\varepsilon}$.

Upon making these transformations in (5.46) and noting that $\frac{d}{dt} = \frac{d}{\varepsilon d\tau}$ we obtain the standard form,

$$\begin{cases} \dfrac{dx_1(\tau)}{d\tau} = \varepsilon\, x_2(\tau) \\ \dfrac{dx_2(\tau)}{d\tau} = -\varepsilon\, (1 + \cos\omega_o \tau) x_1(\tau) \end{cases} \tag{5.49}$$

Thus, we have put equation (5.46) into the standard form (5.44), with

$$A(t) = -\begin{pmatrix} 0 & -1 \\ (1+\cos\omega_o t) & 0 \end{pmatrix} .$$

It is quite obvious that the time averaged matrix $\bar{A}$ is,

$$\bar{A} = \lim_{T\uparrow\infty} \frac{1}{T} \int_o^T A(t)dt = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{5.50}$$

Therefore, by Bogliubov's result the solution vector y(t) defined by

$$\begin{cases} \dfrac{dy_1(\tau)}{d\tau} = y_2(\tau) \\ \dfrac{dy_2(\tau)}{d\tau} = -y_1(\tau), \; y(0) = x(0), \end{cases} \tag{5.51}$$

will remain "close" to the solution of (5.46) for <u>all</u> time, $t \in [0,\infty)$.

The question now arises, is there a version of the averaging method that is possessed by systems

$$\dot{x}(t) = f(x(t), n(t)), \tag{5.52}$$

where n(t) is a stochastic process with suitable properties.

The motivation for asking this question is quite clear. It is a fundamental fact that if the elements $a_{ij}(t)$ of the stochastic matrix A(t) are stationary stochastic processes, then the time average

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T A(t)dt \tag{5.53}$$

exists with probability one.

Furthermore, if the elements $a_{ij}(t)$ are ergodic processes, then the limit (5.53) is equal to $E\{A(t)\}$ with probability one. As a result of this fact, it follows that there should be an averaging result for stochastic systems.

We can illustrate this for the case of the inverted pendulum [see Fig. 1], with base excitation n(t). The linearized equation is

$$\ddot{\theta} + (\frac{2c}{m})\dot{\theta} - \frac{1}{\ell}(\ddot{n}(t) + g)\theta = 0 \tag{5.54}$$

where m, c, ℓ are mass, damping factor, length, and g is the gravitational constant.

In terms of canonical variables, θ, $\phi = \frac{2L}{2\dot{\theta}}$, where $L = T - V$ is the Lagrangian and ϕ is the generalized momentum, we obtain the equations

$$\begin{cases} \dot{\phi} = -m[m\ell)^{-1}\phi + \dot{n}(t)\theta][\dot{n}(t) + 2\ \ell\ \frac{c}{m}] + mg\ell\ \theta \\ \dot{\theta} = (m\ell^2)^{-1}\ \phi + \ell^{-1}\ \dot{n}(t)\theta \end{cases} \tag{5.55}$$

as a phase space form of (5.54).

Now let us assume that the random base excitation function n(t) is of the form,

$$n(t) = \varepsilon w(\varepsilon^{-1}t), \tag{5.56}$$

where $\varepsilon > 0$ is a small parameter.

We see from (5.56) that the ε coefficient implies a small variance for n(t) if the variance of w is fixed. Furthermore, the time shift $t \to \varepsilon^{-1}t$ in w has the effect of shifting the average power to higher frequency ranges.

Upon setting $\varepsilon^{-1}t \equiv \tau$ in (5.56), (5.55), we will obtain, with "$'$" denoting $\frac{d}{d\tau}$,

$$\begin{cases} \dot{\phi} = -\varepsilon m[(m\ell)^{-1}\phi + w'(\tau)\theta][w'(\tau) + 2\frac{\ell c}{m}] + \varepsilon mg\ell\,\theta \\ \dot{\theta} = \varepsilon\ell^{-1}[(m\ell)^{-1}\phi + w'(\tau)\theta], \end{cases} \tag{5.57}$$

which is in the standard form to apply the theorem of Bogoliubov.

We assume that $w'(\tau)$ is stationary ergodic with

$$E\{w'(\tau)\} = 0,\ E\{(w'(\tau))^2\} = \sigma^2. \tag{5.58}$$

Upon writing the system (5.57) in the form (5.44), we find that the A-matrix is simply

$$A(t) = \begin{pmatrix} -\frac{1}{\ell}[w'(\tau) + 2\frac{\ell c}{m}] & -m[w'(\tau)^2 + w'(\tau)\,2\frac{\ell c}{m} - g\ell] \\ \frac{1}{m\ell^2} & \frac{1}{\ell}w'(\tau) \end{pmatrix} \tag{5.59}$$

whose time average, by the assumed ergodic assumption on the $w'(\tau)$ process, is obtained from (5.58) as,

$$\bar{A} = \begin{pmatrix} -2\frac{c}{m} & -m[\sigma^2 - g\ell] \\ \frac{1}{m\ell^2} & 0 \end{pmatrix}. \tag{5.60}$$

Therefore, Bogoliubov's theorem implies that the sample solutions of the original system (5.57) will be close to the solution of the constant coefficient deterministic system for time intervals of the order $0(1/\varepsilon)$. We note an immediate result of (5.60). If $\sigma^2 > g\ell$, then $\bar{A}$ is a stability matrix. Thus all sample solutions of (5.57) should decay exponentially, at least for time intervals of order $0(1/\varepsilon)$.

Simulation studies for the inverted pendulum with stochastic base motion [5.28] have suggested that the solutions might be close for all time if the solution process is bounded and the power is concentrated in the higher frequencies. A proof for a specific form of base excitation that is bounded was given by Bogdanoff in 1962 [5.29]. He considered excitations of the form

$$n(t) = \sum_{i=1}^{N} a_i \cos(\omega_i t + \nu_i), \tag{5.61}$$

where $\{a_i\}$, $\{\omega_i\}$ are given constants, all a_i are "small" in magnitude, ω_i are "large" and all $|\omega_i - \omega_j| > K$ for some constant K.

Finally, the $\{\nu_i\}$ are independent random variables all uniformly distributed on $[0, 2\pi]$. The functional form, with these assumptions does fit within the scope of Bogoliubov's theorem for almost periodic excitations. Bogdanoff's proof was based upon linearizations and comparing of small terms. Experimental results [5.30] corroborated the theory of [5.29] excellently, even to the loss of stability when two distinct frequencies ω_i, ω_j become close.

A generalization of the results in [5.29] was presented in [5.31] for linear systems in the standard form (5.44), where the ergodic matrix A(t) satisfies,

$$\begin{cases} \text{(a)} \quad \sup_t \| A(t) \| < M, \\ \text{(b)} \quad \sup_t \left\| \int_0^t [A(t)-P]dt \right\| < M_2, \end{cases} \tag{5.62}$$

where $P = E[A(t)]$.

It was established that if P is a stability matrix, then the solution to

$$\dot{y} = \varepsilon P y, \; y(0) = x(0), \tag{5.63}$$

satisfies

$$\lim_{\varepsilon \downarrow 0} \; \sup_{t \in [0,\infty)} \; \| x(t) - y(t) \| = 0 \, , \tag{5.64}$$

with probability one, where x(t) is a sample solution to the stochastic linear system (5.44).

Unfortunately, this result requires the rather strong assumption (5.62)(b). Although, this is known to hold for almost periodic matrices, A(t), we cannot state any general statistical properties that will guarantee this assumption to hold in the stochastic case. These assumptions were also used in [5.32]. For more recent results see [5.33].

We should mention at this point that even if the time average of the stochastic coefficient matrix A(t) exists, we cannot expect the solution process to be close to the solution of the ensemble averaged system, if there are no further assumptions such as ergodicity. This can be seen by the following simple example.

Example. Consider the first order scalar equation

$$\dot{x} = \varepsilon b x, \; x(0) = 1,$$

where b is a bounded constant random variable with $E\{b\} = p < 0$.

Thus, the ensemble averaged system is

$$\dot{y} = \varepsilon p y, \; y(0) = 1.$$

How do the solutions, x(t), y(t) compare? In particular, what can be said about

$$\lim_{\varepsilon \downarrow 0} |x(t) - y(t)| = \lim_{\varepsilon \downarrow 0} | e^{\varepsilon bt} - e^{-\varepsilon |p| t} |. \tag{5.65}$$

It can be shown that for $\alpha, \beta > 0$, $\alpha \neq \beta$, we have by simple calculus,

$$\sup_t |e^{-\alpha t} - e^{-\beta t}| = | (\frac{\beta}{\alpha})^{\frac{\alpha}{\alpha-\beta}} - (\frac{\beta}{\alpha})^{\frac{\alpha}{\alpha-\beta}} | > 0,$$

which occurs at $t = \frac{1}{\alpha-\beta} (\log\alpha - \log\beta) > 0$.

For our case (5.65), the ε cancels out, and we will obtain for $b < 0$,

$$\sup_t |e^{\varepsilon bt} - e^{\varepsilon pt}| = | (\frac{p}{b})^{\frac{b}{b-p}} - (\frac{p}{b})^{\frac{b}{b-p}} | > 0 \tag{5.67}$$

independent of ε!

Hence, the solutions cannot become close as $\varepsilon \downarrow 0$, assuming only that time averages exist!

For the case of the white noise excited system, we can write an averaging result that relates the stochastic response characteristics to the associated averaged deterministic system, [5.34].

In particular, for an Ito system of the form,

$$dx = \varepsilon(Axdt + F(x)dB), \; x(0) = x_o \tag{5.68}$$

where A is a constant or periodic matrix, F(x) is a matrix whose elements are linear functions of the vector x, and B denotes a vector of Browmian motions, it follows that the averaged system

$$\dot{y} = \varepsilon \bar{A} y, \quad y(0) = x_o \tag{5.69}$$

satisfies

$$\lim_{\varepsilon \downarrow 0} \text{Prob} \{ \sup_{t \in [0,\infty)} \| y(t) - x(t) \| \geq \delta \} = 0 \tag{5.70}$$

for any $\delta > 0$.

In this case, as before, $\bar{A}$ denotes the time averaged matrix defined by (5.53).

The effective importance of this result, is that if the structural excitations are wide band Gaussian then we can treat the system (5.68) for small ε, as the deterministic averaged system (5.69).

Furthermore, the response characteristics of the true system (5.68) will remain close to the solution of (5.69) for the entire time interval $[0,\infty)$.

The question of controlling the true system through the averaged model also becomes possible. The control problem is discussed in [5.34] as well.

Many averaging methods that have been developed for the study of stochastic systems have not taken the view that we have discussed above. The basic philosophy that we have discussed above is essentially to replace the true system with random excitations by a system that essentially averages out the random fluctuations leaving, in a sense, the mean system which is deterministic. If the true and the mean systems are "close" for all time, then we can restrict our study to the simpler mean system response, knowing that it will be very close to the true system dynamics. On the other hand, the philosophy that drives the work of Khazminski [5-35], [5-36], [5-37], Stratonovich [5-38] and Papanicolaou [5.39] is to replace the original random coefficient system by one that is simpler through an averaging procedure. Generally speaking their averaged systems are close to the true systems in the sense that the probability distributions generated by the averaged system are close to the probability distributions generated by the true system. This is referred to in probability theory as weak convergence.

It differs from the results we have described above, in which we are concerned with the actual response of the system being close to the response of the averaged model. This is called strong convergence, or almost sure sample convergence, which will imply that the probability distributions will be close. The important difference is that from the structural engineering point of view we are concerned primarily with the actual structural response of the systems. However, the weak convergence point of view does allow one to obtain probability distributions for averaged systems, when it may not be possible to determine the distributions for the true randomly excited system.

To a great extent this approach was motivated by [5-35], where an averaging result was established for partial differential equations. In particular, we consider

$$\frac{\partial u}{\partial t} = \varepsilon L(x,t)u, \tag{5.71}$$

where L is an elliptic or parabolic second order differential operat with sufficient regularity conditions relative to the (x,t) variables.

Let

$$L_o(x) = \lim_{T\uparrow\infty} \frac{1}{T} \int_0^T L(x,t)dt, \tag{5.72}$$

be the time averaged operator.

Then, if $v(x,t)$ is the solution to

$$\frac{\partial v}{\partial t} = \varepsilon L_o(x)v, \tag{5.73}$$

for the boundary condition

$$u(x,T/\varepsilon) = v(x,T/\varepsilon) = f(x). \tag{5.74}$$

it follows that,

$$\lim_{\varepsilon \downarrow 0} \sup_{(x,t) \in R_N [0,T/\varepsilon]} |u(x,t-v(x,t)| = 0 \tag{5.75}$$

The importance of this theorem for stochastically perturbed systems is the following; for the Ito system

$$dx = m(x,t)dt + \sigma(x,t)dB, \quad \sigma = (\sigma_{ij}) \tag{5.76}$$

where x,m are n-vectors, σ is an n x n matrix and B is an n-vector of Brownian motions, we know from Section II, that the generator $\mathscr{L}$ is given as

$$\mathscr{L} = \sum_{i=1}^{n} m_i(x,t) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} b_{ij}(x,t) \frac{\partial^2}{\partial x_i \partial x_j} \tag{5.76(A)}$$

where

$$b_{ij} = \sum_{k=1}^{n} \sigma_{ik}(x,t)\sigma_{jk}(x,t).$$

It immediately follows that the generator for

$$dx = \varepsilon m(x,t)dt + \sqrt{\varepsilon}\ \sigma(x,t)dB \tag{5.77}$$

becomes $\varepsilon\mathscr{L}$, where $\mathscr{L}$ is defined by (5.76(A)).

Hence, the theorem states that the probability distribution generated by

$$dy = \varepsilon \bar{m}(y)dt + \sqrt{\varepsilon}\ \bar{\sigma}\ (y)dB, \tag{5.78}$$

where

$$\begin{cases} \bar{m}(y) = \lim\limits_{T\uparrow\infty} \frac{1}{T} \int_0^T m(y,t)dt \\ \\ \bar{b}_{ij}(y) = \lim\limits_{T\uparrow\infty} \frac{1}{T} \int_0^T b_{ij}(y,t)dt, \end{cases} \tag{5.79}$$

($\bar{\sigma}(y)$ is obtained from $(\bar{b}_{ij}(y))$, is close to the probability distribution generated by (5.77) on the internal $[0,T/\varepsilon]$. Examples of this approach for non-linear systems may be found in [5.37].

Perhaps the most interesting form of averaging methods can be found in [5.36], [5.38], [5.39]. In a sense these constitute an extension of the central limit theorem, for diffusion processes. We shall mention the result in the form stated in [5.36] (which puts the results of [5.38] on a firm mathematical foundation).

Consider the standard form,

$$\dot{z}(t) = \varepsilon F(z(t), x(t), t), \quad z(0) = z_o \tag{5.80}$$

where z,F are n-vectors, x is a vector stochastic process, where for each component F_i of F, we assume that there is a constant C, such that

$$|F_i| < C, \ |\frac{\partial F_i}{\partial z_j}| < C, \ |\frac{\partial^2 F_i}{\partial z_j \partial z_k}| < C \tag{5.81}$$

uniformly in (z,x,t).

It is assumed that the x-process possesses the strong mixing property.

This essentially means that as time intervals $[0,t]$, $[t+s,\infty)$ become further separated (i.e. as $s \uparrow \infty$), events defined on the x-process, over these two intervals, will become independent.

Finally, it is assumed that $E\{F(z,x(t),t)\} = 0$, for fixed z, and the following limits exist uniformly in z,

$$\begin{cases} \lim_{T\uparrow\infty} \frac{1}{T} \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \; E\{ \sum_{j=1}^{n} \frac{\partial F_i(z,x(\tau),\tau_1)}{\partial z_j} F_j(z,x(\tau_2)\tau_2)\} = b_i(z) \\ \lim_{T\uparrow\infty} \frac{1}{T} \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \; E\{F_i(z,x(\tau_1),\tau_1)F_j(z,x(\tau_2),\tau_2)\} = a_{ij}(z). \end{cases} \tag{5.82}$$

Under these conditions, the process $z^{(\varepsilon)}(\tau)$ defined by

$$z^{(\varepsilon)}(\tau) = z(\tau/\varepsilon^2), \qquad \tau = \varepsilon^2 t, \tag{5.83}$$

converges weakly (i.e. in distribution) to a diffusion process whose generator, or backward operator, $\mathscr{L}$, is given as

$$\mathscr{L} = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(z) \frac{\partial^2}{\partial z_i \partial z_j} + \sum_{j=1}^{n} b_j(z) \frac{\partial}{\partial z_j} \tag{5.84}$$

Since we can always associate a white noise coefficient, Ito, differential equation with the generator (5.84), then it follows that the statistics of the solution process of (5.80), will converge to the statistics of the solution process of

$$dy = b(y)dt + G(y)dB, \tag{5.85}$$

where the vector $b(y)$ and matrix $G(y)$ are determined by the $(b_i(z)),(a_{ij}(z))$ of (5.82).

One important note, the convergence is weak convergence. Therefore, we cannot say that the actual sample response behavior of (5.80) will approach the response of (5.85), we can only say that their statistical properties as governed by their joint probability densities will become close as ε approaches zero. An extension and further development of these ideas are due to Papanicolaou, [5.39].

The following example of (5.39) is illustrative of the procedure. For further applications see [5.40] and [5.19].

Consider the undamped oscillator,

$$\ddot{x}(t) + (\omega^2 + \varepsilon n(t))x(t) = 0, \tag{5.86}$$

where the n-process is bounded, with covariance $\gamma(\tau)$. We can put (5.86) into the usual phase-space form as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 & 0 \\ -n(t) & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{5.87}$$

$$= A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \varepsilon N(t) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Upon applying the transformation

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = e^{-At} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} , \tag{5.88}$$

noting that the matrix exponential e^{-At} will generate trigonometric terms, the equation (5.87) may be written as

$$\begin{aligned} \dot{z}_1 &= \varepsilon n(t) \left[\frac{z_1}{2\omega} \sin^2\omega t + \frac{z_2}{\omega^2} \sin^2\omega t\right] \\ \dot{z}_2 &= \varepsilon n(t) \left[-z_1 \cos^2\omega t - \frac{z_2}{2\omega} \sin 2\omega t\right], \end{aligned} \tag{5.89}$$

One final transformation,

$$z_1 = e^r \cos\theta, \; z_2 = -\omega e^r \sin\theta, \tag{5.90}$$

transforms (5.89) into

$$\begin{cases} \dot{r} = \varepsilon n(t) G(\theta,t) \\ \dot{\theta} = \varepsilon n(t) H(\theta,t) \end{cases} \tag{5.91}$$

where G, H are trigonometric polynomials in (θ,t). Therefore, G,H are bounded and the averaging theorem of Khazminskii-Stratonovich-Papanicolaou can be applied to obtain the generator

$$\mathcal{L} = \frac{b}{2}\frac{\partial^2}{\partial r^2} + b\frac{\partial}{\partial r} + (a+\frac{b}{2})\frac{\partial^2}{\partial \theta^2} + c\frac{\partial}{\partial \theta}\ , \tag{5.92}$$

where from (5.82) one obtains

$$a = \omega\frac{S(0)}{4}\ ,\ b = \omega Re\left[\frac{S(2\omega)}{4}\right], c = \omega Im\left[\frac{S(2\omega)}{4}\right] \tag{5.93}$$

with

$$S(\omega) = \int_0^\infty \gamma(\tau)e^{i\omega\tau}d\tau.$$

Thus, the second moment approximation to $E\{x_1^2(t)\}$ on the internal $[0,T/\varepsilon^2]$ is

$$E\{x_1^2(t)\} \simeq \frac{1}{2}\exp\{\frac{\omega}{2}[ReS(2\omega) - 2S(0)]\varepsilon^2 t\}\cos(2\omega - \frac{\varepsilon^2\omega^2}{2}ImS(2\omega))t$$

$$+ \frac{1}{2}\exp[\omega ReS(2\omega)\varepsilon^2 t]. \tag{5.94}$$

We can further note that the generator $\mathcal{L}$ given by (5.92) implies that r, ϑ processes are independent. (Since $\mathcal{L} \equiv \mathcal{L}_r + \mathcal{L}_\theta$). Therefore, we can study the r-process through

$$\mathcal{L}_r = \frac{b}{2}\frac{\partial^2}{\partial r^2} + b\frac{\partial}{\partial r}\ , \tag{5.95}$$

whose associated Ito equation is

$$dr = bdt \pm \sqrt{b}\, dB. \tag{5.96}$$

Since the constant term b is positive, it follows that

$$\overline{\lim_{t\uparrow\infty}}\, r(t) = \infty \quad \text{with probability one.} \tag{5.97}$$

This will imply that (z_1,z_2) and therefore (x_1,x_2) are growing in an unstable manner on the internal $[0,T/\varepsilon^2]$.

This concludes our discussion of averaging methods.

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AD-A172 562 A SURVEY OF PROBABILISTIC METHODS FOR DYNAMICAL SYSTEMS WITH UNCERTAIN PA... (U) KOZIN BOGDANOFF AND ASSOCIATES INC WEST LAFAYETTE IN J L BOGDANOFF ET AL. MAY 86 2/2

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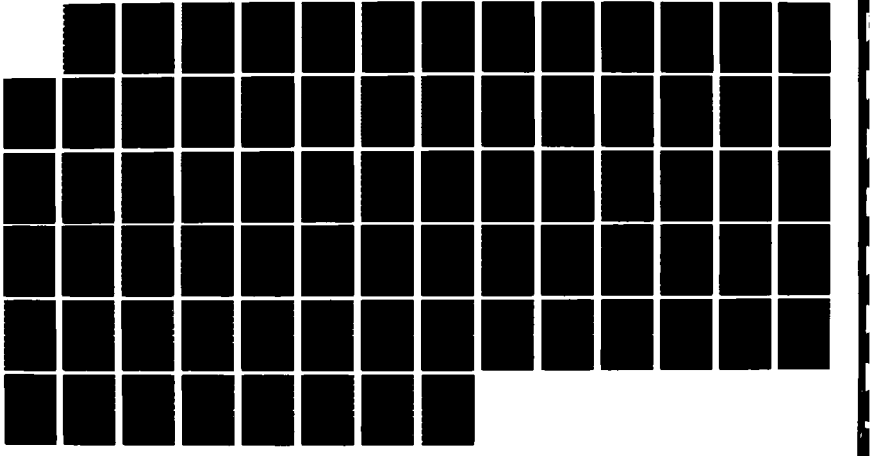

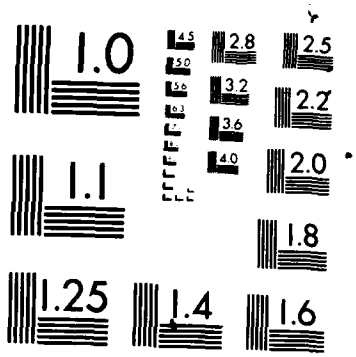

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VI. Perturbation methods*

Consider, for example, the rth natural frequency ω_r of a system. Assume the parameters depend upon random variables $X_1, \ldots, X_m$. Then, we seek an expression for ω_r, now regarded as a random variable, in the form

$$\omega_r = \underline{\omega}_r + \sum_{i=1}^{m} \lambda_i X_i + \sum_{i=1}^{m}\sum_{j=1}^{m} \lambda_{ij} X_i X_j + \ldots .$$

where the X_i are regarded as small perturbation terms, $\underline{\omega}_r$ represents the rth natural frequency of the mean system, and the λ_i, λ_{ij}, ... are to be determined. While some authors replace X_i with εY_i, ε being the perturbation parameter, we shall not usually do this. Once we know the λ_i, λ_{ij}, ..., we can obtain statistical properties of ω_r or any other quantity of interest. Let us consider a general formulation of this problem, considering natural frequencies and normal modes first.

Zarghame [27] sent one of the authors a method of this class which appears well suited to computation and which makes a useful suggestion on how to introduce random parameters that merits attention. This method has never appeared in print in so far as we know. Therefore, we shall write out the details in order to have it before us.

We are concerned in this subsection with the free motion of a conservative system. Thus, in (6), $C = 0$, $f_1 = 0$. Now the elements in the symmetric stiffness matrix K are determined by the bars, beams, columns, joints, etc. making up the structure. The uncertainties in

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* References in Sections VI-IX are given at the end of Section IX.

the structure reside in these elements. Let there be m structural elements in the structure, and let the stiffness matrix of the i^{th} structural element in terms of $q_1, \ldots, q_n$ be

$$K_i = (1 + X_i)\underline{K}_i \ , \quad i = 1, \ldots, m \ , \tag{6.1}$$

which produces the (nxn) random stiffness matrix

$$K = \sum_i K_i = \{K_{jk}\} \ . \tag{6.2}$$

The random variables $X_1, \ldots, X_m$ describe the uncertainty present in the structural elements and we assume

$$EX_i = 0 \ , \quad \text{Var } X_i = \sigma_i^2 \ , \tag{6.3}$$

$\underline{K}_i$ is the mean stiffness matrix of the i^{th} element, $K = K^T$, i.e. K is symmetric in the K_{jk}, K_{jk} is the random stiffness element corresponding to q_j and q_k, and we assume masses of the elements do not change. The advantage of (6.1) and (6.2) is that statistical dependence of the K_{jk} is brought in a straight-forward manner. We note also that we can write (6.2) as

$$K = \underline{K} + \sum X_i \underline{K}_i \ , \quad \underline{K} = \sum \underline{K}_i \tag{6.4}$$

which gives also

$$EK = \underline{K} \quad \text{and} \quad \frac{\partial K}{\partial X_i} = \underline{K}_i \ , \tag{6.5}$$

where $\underline{K}$ is the stiffness matrix of the structure with each number taking its mean stiffness.

We can now write (1.26) as

$$I\ddot{q} + Kq = 0 \ , \tag{6.6}$$

where q is the (nx1) column vector with transpose

$$q^T = \{q_1, \ldots, q_n\} . \tag{6.7}$$

Assume normal mode motion

$$q = \alpha \cos(wt + \phi) \tag{6.8}$$

with α the (nx1) column vector defined by

$$\alpha^T = \{\alpha_1, \ldots, \alpha_n\} . \tag{6.9}$$

Then, substituting (6.8) into (6.6), we obtain

$$(K - \omega^2 I)\alpha = 0, \tag{6.10}$$

where again I is the (nxn) unit matrix.

The squared natural frequencies ω_r^2 are determined by the n roots of the equation

$$\det|K - \omega^2 I| = 0 , \tag{6.11}$$

revealing that the ω_r^2 and ω_r are random variables since K contains random variables. Let the random mode corresponding to ω_r be the (nx1) column vector α_r. Then we can write

$$(K - \omega_r^2 I)\alpha_r = 0 , \tag{6.12}$$

with the usual orthogonality relations

$$\begin{aligned} \alpha_r^T I \alpha_s = 0 \quad , \quad \alpha_r^T K \alpha_s = 0 \quad \text{if } s \neq r \\ \alpha_r^T I \alpha_r = 1 \quad , \quad \alpha_r^T K \alpha_r = \omega_r^2 , \end{aligned} \tag{6.13}$$

where "super T" denotes transpose.

We are now interested in expressing the random variables ω_r and α_r in terms of a series in the random variables X_i. The expressions we

seek will be power series in the X_i. Notice that in this formulation w_r and α_r are regarded as functions of the uncertainty in stiffness in each individual structural member.

Differentiate (6.12) with respect to X_i:

$$(\underline{K}_i - 2\omega_r \frac{\partial \omega_r}{\partial X_i} I)\alpha_r + (K - \omega_r^2 I) \frac{\partial \alpha_r}{\partial X_i} = 0 \ . \tag{6.14}$$

Next premultiply (5.14) by α_r^T obtaining

$$\alpha_r^T(\underline{K}_i - 2\omega_r \frac{\partial \omega_r}{\partial X_i} I)\alpha_r = 0 \ , \tag{6.15}$$

since by the symmetry of K ($K = K^T$) and (5.12)

$$\alpha_r^T(K - \omega_r^2 I) = 0 \ .$$

Thus, with the last two of (5.13)

$$\frac{\partial \omega_r}{\partial X_i} = \frac{1}{2\omega_r} \alpha_r^T \underline{K}_i \alpha_r \ . \tag{6.16}$$

This is to be evaluated at $X_1 = \ldots = X_n = 0$ (i.e. $X = 0$); we obtain

$$(\frac{\partial \omega_r}{\partial X_i})_o = \frac{1}{2\underline{\omega}_r} \underline{\alpha}_r^T \underline{K}_i \underline{\alpha}_r \ , \tag{6.17}$$

where the underbarred quantities are to be evaluated for the system with mean stiffnesses. We note that (6.17) give the sensitivity coefficients [19,25] of ω_r with respect to the X_i.

The α_k, $k = 1, \ldots, n$ span the coordinate space; hence, we may write

$$\frac{\partial \alpha_r}{\partial X_i} = \sum_j \beta_{ri}^{(j)} \alpha_j \ . \tag{6.18}$$

We substitute (6.18) into (6.14):

$$(\underline{K}_i - 2\omega_r \frac{\partial \omega_r}{\partial X_i} I)\alpha_r + (K - \omega_r^2 I)\Sigma_j \beta_{ri}^{(j)} \alpha_j = 0 \ . \qquad (6.19)$$

Now premultiply (6.19) by α_k^T, obtaining

$$\alpha_k^T(\underline{K}_i - 2\omega_r \frac{\partial \omega_r}{\partial X_i} I)\alpha_r + \alpha_k^T(K - \omega_r^2 I)\Sigma_j \beta_{ri}^{(j)} \alpha_j = 0 \ ,$$

or, for $k \neq r$ and with the use of (6.13)

$$(\omega_k^2 - \omega_r^2)\beta_{ri}^{(k)} = -\alpha_{\underline{k}}^T \underline{K}_i \alpha_r \ .$$

Differentiating the next to last of (6.13) with respect to X_i gives

$$\alpha_r^T I \frac{\partial \alpha_r}{\partial X_i} = 0 \ ,$$

which on premultiplying (6.18) by $\alpha_r^T I$ then demonstrates that $\beta_{ri}^{(r)} = 0$. Thus,

$$\beta_{ri}^{(k)} = -\frac{\alpha_{\underline{k}}^T \underline{K}_i \alpha_r}{\omega_k^2 - \omega_r^2} \ , \quad k \neq r \ ,$$

$$\beta_{rj}^{(r)} = 0 \ , \qquad (6.20)$$

and hence from (6.18)

$$\frac{\partial \alpha_r}{\partial X_i} = \Sigma_j' \frac{\alpha_{\underline{k}}^T \underline{K}_i \alpha_r}{\omega_r^2 - \omega_k^2} \alpha_k \ , \qquad (6.21)$$

where the prime on Σ' means that j does not take the value r. When evaluated at X = 0, we have

$$(\frac{\partial \alpha_r}{\partial X_i})_o = \Sigma_j' \frac{\underline{\alpha}_j^T \underline{K}_i \underline{\alpha}_r}{\underline{\omega}_r^2 - \underline{\omega}_j^2} \ , \qquad (6.22)$$

where again the under barred quantities are evaluated when members take

on their mean stiffnesses. We note that (6.22) gives the sensitivity coefficients [28] of the mode shapes with respect to the X_i.

Next differentiate (6.14) with respect to X_j:

$$-2\left(\frac{\partial\omega_r}{\partial X_j}\frac{\partial\omega_r}{\partial X_i}+\omega_r\frac{\partial^2\omega_r}{\partial X_i\partial X_j}\right)I\alpha_r+\left(\underline{K}_i-2\omega_r\frac{\partial\omega_i}{\partial X_i}I\right)\frac{\partial\alpha_r}{\partial X_j} \tag{6.23}$$

$$+\left(\underline{K}_j-2\omega_r\frac{\partial\omega_r}{\partial X_j}I\right)\frac{\partial\alpha_r}{\partial X_i}+(K-\omega_r^2I)\frac{\partial^2\alpha_r}{\partial X_i\partial X_j}=0\ .$$

Next employ (34) for $\partial\alpha_r/\partial X_i$ and $\partial\alpha_r/\partial X_j$ in (36) and premultiply by α_r^T; these operations yield

$$-2\left(\frac{\partial\omega_r}{\partial X_i}\frac{\partial\omega_r}{\partial X_j}+\omega_r\frac{\partial^2\omega_r}{\partial X_i\partial X_j}\right)\alpha_r^TI\alpha_r+\Sigma'_k\beta_{ij}^{(k)}\alpha_r^T\underline{K}_i\alpha_k \tag{6.24}$$

$$-2\omega_r\frac{\partial\omega_r}{\partial X_i}\Sigma'_k\beta_{ij}^{(k)}\alpha_r^TI\alpha_k+\Sigma'_k\beta_{ri}^{(k)}\alpha_r^T\underline{K}_j\alpha_k$$

$$-2\omega_r\frac{\partial\omega_r}{\partial X_j}\Sigma'_k\beta_{li}^{(k)}\alpha_r^TI\alpha_k\ .$$

Since by (6.13) and (6.20),

$$\Sigma'_k\beta_{ri}^{(k)}\alpha_i^TI\alpha_k=\Sigma'_k\beta_{rj}^{(k)}\alpha_r^TI\alpha_k=0\ ,$$

and by (6.20)

$$u_r^T\underline{K}_j\alpha_k=\beta_{kj}^{(r)}(\omega_k^2-\omega_k^2),\quad \alpha_r^T\underline{K}_i\alpha_k=\beta_{ki}^{(r)}(\omega_k^2-\omega_r^2)\ ,$$

we find

$$\frac{\partial^2\omega_r}{\partial X_i\partial X_j}=\frac{1}{2}[\Sigma'_k(\beta_{ri}^{(k)}\beta_{kj}^{(r)}+\beta_{rj}^{(k)}\beta_{ki}^{(r)})(\omega_k^2-\omega_r^2) \tag{6.25}$$

$$-2\frac{\partial\omega_r}{\partial X_i}\frac{\partial\omega_r}{\partial X_j}]$$

or at $X=0$,

$$\left(\frac{\partial^2 \omega_r}{\partial X_i \partial X_j}\right)_o = \frac{1}{2}\Big[\sum_k \left(\beta_{ri}^{(k)}\beta_{kj}^{(r)} + \beta_{rj}^{(k)}\beta_{ki}^{(r)}\right)\left(\omega_k^2 - \omega_r^2\right) - 2\left(\frac{\partial \omega_i}{\partial X_i}\right)_o \left(\frac{\partial \omega_i}{\partial X_j}\right)_o\Big] \tag{6.26}$$

To go further, we must first evaluate $\partial^2 \alpha_r / \partial X_i \partial X_j$. Let

$$\frac{\partial^2 \alpha_r}{\partial X_i \partial X_j} = \sum_k \beta_{r,ij}^{(k)} \alpha_k \ , \tag{6.27}$$

since as with (6.18) the α_k span the coordinate space. The substitution of (6.27) into (6.23) yields

$$-2\left(\frac{\partial \omega_r}{\partial X_i}\frac{\partial \omega_r}{\partial X_j} + \omega_r \frac{\partial^2 \omega_r}{\partial X_i \partial X_j}\right) I \alpha_r + \left(\underline{K}_i - 2\omega_r \frac{\partial \omega_r}{\partial X_i} I\right)\frac{\partial \alpha_r}{\partial X_j} + \left(\underline{K}_j - 2\omega_r \frac{\partial \omega_r}{\partial X_j} I\right)\frac{\partial \alpha_r}{\partial X_i} + \left(K - \omega_r^2 I\right)\sum \beta_{r,ij}^{(k)} \alpha_k = 0 \ . \tag{6.28}$$

Premultiplication of this equation by $\alpha_{\underline{1}}^T$, $\underline{1} \neq r$, and the employment of similar relations as used to obtain (6.25) ultimately yields for $\underline{1} \neq r$.

$$\beta_{r,ij}^{(1)} = \frac{1}{\omega_r^2 - \omega_{\underline{1}}^2}\Big[\alpha_{\underline{1}}^T \left(\underline{K}_i - 2\omega_i \frac{\partial \omega_i}{\partial X_i}\right)\frac{\partial \alpha_r}{\partial X_j} + \alpha_{\underline{1}}^I \left(\underline{K}_j - 2\omega_r \frac{\partial \omega_r}{\partial X_j}\right)\frac{\partial \alpha_r}{\partial X_i}\Big] \ . \tag{6.29}$$

If we differentiate the equation before (6.20) with respect to X_j, we find

$$\alpha_r^T I \frac{\partial^2 \alpha_r}{\partial X_i \partial X_j} = -\frac{\partial \alpha_r^T}{\partial X_j} I \frac{\partial \alpha_r}{\partial X_i} \ .$$

Then on premultiplying (6.27) by $\alpha_r^T I$ and employing this relationship the result

$$\beta^{(r)}_{i,ij} = -\frac{\partial \alpha^T_r}{\partial X_j} I \frac{\partial \alpha_r}{\partial X_i} \tag{6.30}$$

follows. Equations (6.29) and (6.30) can now be evaluated at X = 0, and we obtain

$$(\frac{\partial^2 \alpha_r}{\partial X_i \partial X_j})_o = \Sigma_k \beta^{(k)}_{r,ij} \underline{\alpha}_k \ . \tag{6.31}$$

We could rewrite (6.29) and (6.30) employing previously determined expressions for the first partial derivatives contained therein but this is not particularly helpful. The procedure for going further is straight forward but we shall not write out the details in order to conserve space. Let us summarize our results up to this point.

We have for the random variable

$$\omega_r = \underline{\omega}_r + \Sigma_i (\frac{\partial \omega_r}{\partial X_i})_o X_i + \frac{1}{2}\Sigma\Sigma_{ij} (\frac{\partial^2 \omega_r}{\partial X_i \partial X_j})_o X_i X_j + \ \ldots \ , \tag{6.32}$$

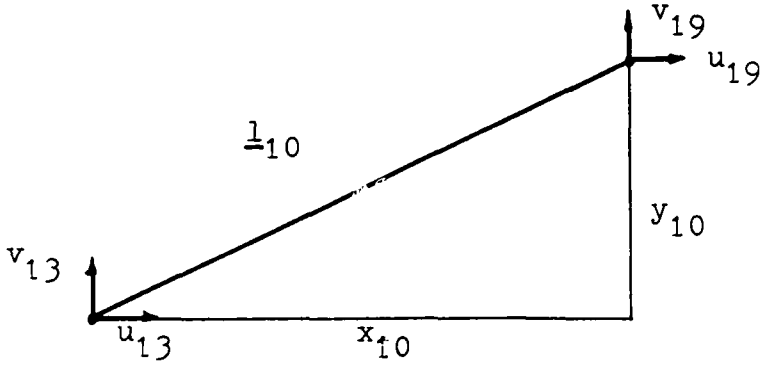
where the partial derivatives are supplied by (6.17) and (6.26). We also have for the random variable

$$\alpha_r = \underline{\alpha}_r + \Sigma_i (\frac{\partial \alpha_r}{\partial X_i})_o X_i + \frac{1}{2}\Sigma\Sigma_{ij} (\frac{\partial^2 \alpha_r}{\partial X_i \partial X_j})_o X_i X_j + \ \ldots \ , \tag{6.33}$$

where (6.22) supplies the first partial derivative, and (6.29), (6.30) supplies the derivatives in the double sum. Before looking at the statistics of ω_r and α_r, let us consider how the $\underline{K}_i$ and K are computed.

Consider, for simplicity, a plane frame consisting of pin-ended straight bars. Let the 10^{th} bar connect joints 13 and 19, for example. Let this bar have length l_{10}, mean area A_{10}, mean modulus E_{10}, and mean direction cosines λ_{10} and μ_{10}, as shown. Let the elastic displacements

at the joints be (u_{13}, v_{13}) and (u_{19}, v_{19}).



The mean elastic potential energy stored in the bar is

$$\underline{V} = \frac{1}{2} \frac{A_{10}E_{10}}{\underline{l}_{10}} \delta_{10}^2 , \tag{6.34}$$

where to first order in small displacements

$$\delta_{10} = (u_{19}-u_{13})\lambda_{10} + (v_{19}-v_{13})\mu_{10} . \tag{6.35}$$

The substitution of (6.35) into (6.34) gives, on arranging terms, $\underline{K}_{10}$

columns

rows	u_{13}	v_{13}	...	u_{19}	v_{19}
u_{13}	$\frac{AE\lambda^2}{\underline{l}}$	$\frac{AE\lambda\mu}{\underline{l}}$	...	$-\frac{AE\lambda^2}{\underline{l}}$	$-\frac{AE\lambda\mu}{\underline{l}}$
v_{13}	$\frac{AE\lambda\mu}{\underline{l}}$	$\frac{AE\mu^2}{\underline{l}}$	...	$-\frac{AE\lambda\mu}{\underline{l}}$	$-\frac{AE\mu^2}{\underline{l}}$
.	.	.	...	.	.
.	.	.	...	.	.
.	.	.	...	.	.
u_{19}	$-\frac{AE\lambda^2}{\underline{l}}$	$-\frac{AE\lambda\mu}{\underline{l}}$	...	$\frac{AE\lambda^2}{\underline{l}}$	$\frac{AE\lambda\mu}{\underline{l}}$
v_{19}	$-\frac{AE\lambda\mu}{\underline{l}}$	$-\frac{AE\mu^2}{\underline{l}}$	...	$\frac{AE\lambda\mu}{\underline{l}}$	$\frac{AE\mu^2}{\underline{l}}$

$= \underline{K}_{10}$, (6.36)

where all other entries in the $\underline{K}_{10}$ matrix are zero, and, hence

$$K_{10} = (1 + X_{10})\underline{K}_{10} \ . \tag{6.37}$$

Thus, according to (6.2)

$$K = \sum_{i=1}^{m} K_i \ . \tag{6.2}$$

Note that X_{10} represents the combination of all the random variables that may enter A_{10}, E_{10}, I_{10}, λ_{10}, and μ_{10}, assuming uncertainty comes only from the bar and not the joints. If the major share of the uncertainty in stiffness of bar 10 comes from the joint connections, then X_{10} must describe this fact; in this case, X_{10} may be dependent on the random variables associated with those bars sharing the joints with bar 10. Finally, it may be advantageous to introduce random variables associated with only joint behavior if its behavior has a substantial influence on stiffness of the structure. We conclude from this brief discussion that the $X_1, \ldots, X_m$ may be independent random variables if, for example, only bar stiffness need be considered and the bars do not interact with each other; however, it is possible the $X_1, \ldots, X_{10}$ may be dependent if the bars interact through joint behavior. Equation (6.36) demonstrates that by attaching random variable to physical element's stiffness coefficient, statistical dependence in K is easily included whether or not the $X_1, \ldots, X_m$ are dependent. Let us consider the statistics of ω_r, etc ...

Consider equation (6.36) first. We have, on taking expectation,

$$E\{\omega_r\} = \underline{\omega}_r + \frac{1}{2}\Sigma\Sigma\left(\frac{\partial^2 \omega_r}{\partial X_i \partial X_j}\right)_o E\{X_i X_j\} + \ldots \ . \tag{6.38}$$

Even if the X's are independent $E\{\omega_r\} \neq \underline{\omega}_r$, since the $EX_j^2 \neq 0$ terms are still present. Now square (6.32) and take expectation:

$$E\{\omega_r^2\} = \underline{\omega}_r^2 + 2\underline{\omega}_r \sum_{ij}\sum (\frac{\partial^2 \omega_r}{\partial X_i \partial X_j})_o E\{X_i X_j\} \tag{6.39}$$
$$+ \sum_{ij}\sum (\frac{\partial \omega_r}{\partial X_i})_o (\frac{\partial \omega_r}{\partial X_j})_o E\{X_i X_j\}$$
$$+ 2\sum_{ijk}\sum\sum (\frac{\partial \omega_r}{\partial X_i})_o (\frac{\partial^2 \omega_r}{\partial X_j \partial X_k})_o E\{X_i X_j X_k\}$$
$$+ \sum_{ijk\underline{l}}\sum\sum\sum (\frac{\partial^2 \omega_r}{\partial X_i \partial X_j})_o (\frac{\partial^2 \omega_r}{\partial X_k \partial X_{\underline{l}}})_o E\{X_i X_j X_k X_{\underline{l}}\} + \ldots .$$

We can now approximate Var ω_r; it is defined as

$$\text{Var } \omega_r^2 = E\{\omega_r^2\} - [E\{\omega_r\}]^2 . \tag{6.40}$$

Thus, it is a straight forward task to approximate the first two moments of ω_r.

If we extend (6.32) to cubic, quartic, ... terms in the X_i, then (6.38) and (6.39) would contain additional terms. How far we should continue this process will depend on relative size of the terms containing $E\{X_i X_j\}$, $E\{X_i X_j X_k\}$, etc. and what information we have that would enable us to evaluate these expectations. It is not usual that we can evaluate any more than $E\{X_i X_j\}$.

Let us consider the determination of the distribution of ω_r. To do this, we require the joint distribution of $X_1, \ldots, X_m$; denote the joint probability distribution by $f_m(x_1, \ldots, x_m)$. Then,

$$F_{\omega_r}(\omega) = P\{\omega_r \leq \omega\} \tag{6.41}$$
$$= \int \ldots \int f_m(x_1, \ldots, x_m) dx_1 \ldots dx_m$$

where the multiple integral is over all x_i such that $\omega_r \leq \omega$, and where from (6.32),

$$\omega_r(X_1, \ldots, X_m) = \underline{\omega}_r + \sum_i \left(\frac{\partial \omega_r}{\partial X_i}\right)_o X_i + \frac{1}{2}\sum\sum \left(\frac{\partial^2 \omega_r}{\partial X_i \partial X_j}\right)_o X_i X_j + \ldots \quad (6.42)$$

Evaluation of (6.41) is a formidable task even for m reasonably small. This paragraph is thus largely cultural in so far as practical application is concerned.

We have put in this detailed treatment of Zarghame's method since it is not in the literature and appears useful as mentioned earlier. In particular, we note that it gives sensitivity coefficients for natural frequencies and correspondence normal modes plus series expansions for these quantities in terms of the random variables $X_1, \ldots, X_m$ that define the uncertainty present in the stiffness matrix K. Moments of the quantities are easily obtained, but it is practically impossible to obtain distributions for the natural frequencies and corresponding normal modes. For confidence interval location and size for a natural frequency, for example, we must approximate using

$$E\{\omega_r\} \pm 3\sqrt{\text{Var } \underline{\omega}_r} \quad (6.43)$$

as a rough indication of a 99% confidence interval. This interval gives us some idea of the spread in a natural frequency and it could be employed to make reasonably sure that no steady excitation frequencies were contained therein for all ω_r. Alternatively, we might employ the signal to noise ratio

$$\frac{E\{\omega_r\}}{\sqrt{\text{Var } \omega_r}} \quad (6.44)$$

to obtain an idea of how important stiffness uncertainty is for natural frequency; if (6.44) is greater than 20 or 30 say, we would regard the

location of ω_r as deterministic; on the other hand, if (6.44) is less than 5-10, it might be unwise to ignore this level of variability in the location of ω_r, depending, of course, on the consequences of such uncertainty.

Next, let us consider frequency response. The Fourier transform of (1.26), with the unit mass matrix I replaced by M, is

$$(K - \omega^2 M + i\omega C)Q = F \ , \tag{6.45}$$

where

$$Q = \frac{1}{2\pi}\int_{-\infty}^{\infty} q e^{i\omega t} d\omega, \ F = \frac{1}{2\pi}\int_{-\infty}^{\infty} f e^{i\omega t} dt \ . \tag{6.46}$$

Let the (nxn) matrix

$$(K - \omega^2 M + i\omega C) = Z(\omega) = Z \ . \tag{6.47}$$

The inverse of Z, written $Z^{-1}(\omega) = Z^{-1}$, satisfies

$$ZZ^{-1} = Z^{-1}Z = I \ . \tag{6.48}$$

Writing (6.45) as

$$ZQ = F \ , \tag{6.49}$$

we find on using (6.48) that

$$Q = Z^{-1}F \ , \tag{6.50}$$

where the explicit dependence of Q, F, and Z^{-1} on ω is not shown.

The matrix Z^{-1} is called the frequency response matrix of the system. The physical meaning of Z^{-1} is as follows: write

$$Z^{-1}(\omega) = \{Z^{-1}_{jk}(\omega)\} \ ;$$

then the element Z^{-1}_{jk} is the complex response amplitude at q_j due to a

unit force (with proper dimensions)

$$e^{i\omega t}$$

acting at q_k; i.e.

$$Z^{-1}_{jk}(\omega)e^{i\omega t}$$

is the response at q_j (output) due to the above force at q_k (input). The symmetry of the matrices M, K, and C establishes the symmetry $Z^{-1}_{jk} = Z^{-1}_{kj}$ or $(Z^{-1})^T = Z^{-1}$. In a lightly damped structural system, when amplitude $|Z^{-1}_{jk}(\omega)|$ is plotted as ordinate against ω as absicssa, sharp peaks will appear in these curves in the neighborhood of natural frequencies of the undamped system. This means that resonance (large amplitudes) occurs in at least some of the q_j due to this force at q_k, when ω is near to one of the undamped natural frequencies of the system. Put another way, if we regard the system as a mechanical filter, only frequencies close to the undamped natural frequencies where there is a peak in $|Z^{-1}_{jk}(\omega)|$ will show up in the output q_j for the input q_k.

Knowing $Z^{-1}(\omega)$ we obtain the response vector q by taking the inverse Fourier transform of (6.50); thus,

$$q = \int_{-\infty}^{\infty} e^{i\omega t}\, Z^{-1}(\omega)F(\omega)d\omega \ . \qquad (6.51)$$

If the excitation vector f is a wide sense stationary random process, or if f is periodic, $Z^{-1}(\omega)$ is also the quantity needed to obtain the response q. Hence, our interest in Z^{-1}.

The inverse Fourier transform of $Z^{-1}(\omega)$ produces the impulse response matrix H(t), a (nxn) matrix:

$$H(t) = \int_{-\infty}^{\infty} e^{i\omega t} Z^{-1}(\omega) d\omega \ , \quad t \geq 0 \ , \tag{6.52}$$

$$= 0 \ , \qquad t < 0 \ .$$

The element $h_{jk}(t)$ is the response at q_j when a unit velocity is produced only at q_k, for $q_1 = \ldots = q_n = 0$ at $t = 0$. Again by symmetry, $h_{jk} = h_{kj}$. By the convolution theorem of Fourier transforms, we may replace (6.51) by

$$q = \int_{o}^{t} H(t-\tau) f(\tau) d\tau \ . \tag{6.53}$$

Equation (6.52) shows that the impulse response matrix H(t), which is in the time domain, is equivalent to the frequency response matrix $Z^{-1}(\omega)$, which is in the frequency domain.

We noticed, while writing out Zarghame's expansion method for natural frequencies and normal modes, that it is possible to extend it to include frequency response.

We assume Z is given by (6.47):

$$Z = K - \omega^2 M + i\omega C \ . \tag{6.47'}$$

Let $X_1, \ldots, X_m$ be the set of random variables that describes the distributions of member properties as contained in K, M, and C. Now m is the total number of distinct sources of variability in K, M, and C, which may be bigger than the number of members in this case. (For example, parallel acting but distinct sources of damping and stiffness in the same number would require two different X's.) We recall that before m equals at most the number of members according to Zarghame's formulation.

We seek an expression of the type

$$Z^{-1} = \underline{Z}^{-1} + \sum_j \left(\frac{\partial Z^{-1}}{\partial X_j}\right)_o X_j + \frac{1}{2!} \sum_{jk} \left(\frac{\partial^2 Z^{-1}}{\partial X_j \partial X_k}\right)_o X_j X_k + \ldots \quad . \tag{6.54}$$

We also seek a simple method for evaluation of the partial derivatives.

Consider

$$ZZ^{-1} = I \quad . \tag{6.48'}$$

The differentiation of this equation with respect to X_j gives

$$\frac{\partial Z}{\partial X_j} Z^{-1} + Z \frac{\partial Z^{-1}}{\partial X_j} = 0 \; ; \tag{6.55}$$

premultiplication by Z^{-1} produces

$$\frac{\partial Z^{-1}}{\partial X_j} = -Z^{-1} \frac{\partial Z}{\partial X_j} Z^{-1} \quad . \tag{6.56}$$

Since by (6.47'), noting that ω is a parameter,

$$\frac{\partial Z}{\partial X_j} = \underline{K}_j - \omega^2 \underline{M}_j + i\omega \underline{C}_j \; , \tag{6.57}$$

we now find

$$\left(\frac{\partial Z^{-1}}{\partial X_j}\right)_o = \underline{Z}^{-1} (\underline{K}_j - \omega^2 \underline{M}_j + i\omega \underline{C}_j) \underline{Z}^{-1} \; , \tag{6.58}$$

where under bar means evaluation is for the system with all member parameters taking their mean values. Equation (6.58) is the sensitivity coefficient of the matrix Z^{-1}.

Next, the differentiation of (6.55) with respect to X_k produces

$$\frac{\partial^2 Z}{\partial X_j \partial X_k} Z^{-1} + \frac{\partial Z}{\partial X_j} \frac{\partial Z^{-1}}{\partial X_k} + \frac{\partial Z}{\partial X_k} \frac{\partial Z^{-1}}{\partial X_j} + Z \frac{\partial^2 Z^{-1}}{\partial X_j \partial X_k} = 0 \quad . \tag{6.59}$$

Again, premultiply by Z^{-1} and rearrange to obtain

$$\frac{\partial^2 Z^{-1}}{\partial X_j \partial X_k} = -Z^{-1} \frac{\partial^2 Z}{\partial X_j \partial X_k} - Z^{-1} \frac{\partial Z}{\partial X_j} \frac{\partial Z^{-1}}{\partial X_k} - Z^{-1} \frac{\partial Z}{\partial X_k} \frac{\partial Z^{-1}}{\partial X_j} . \qquad (6.60)$$

We now use (6.56) and (6.57) to yield

$$\left(\frac{\partial^2 Z^{-1}}{\partial X_j \partial X_k}\right)_o = \underline{Z}^{-1}(\underline{K}_j - \omega^2 \underline{M}_j + i\omega \underline{C}_j)\underline{Z}^{-1}(\underline{K}_k - \omega^2 \underline{M}_k + i\omega \underline{C}_k)\underline{Z}^{-1} \qquad (6.61)$$
$$+ \underline{Z}^{-1}(\underline{K}_k - \omega^2 \underline{M}_k + i\omega \underline{C}_k)\underline{Z}^{-1}(\underline{K}_j - \omega^2 \underline{M}_j + i\omega \underline{C}_j)\underline{Z}^{-1}$$

since also by (6.57)

$$\frac{\partial^2 Z}{\partial X_j \partial X_k} = 0 . \qquad (6.62)$$

It is straight forward to obtain the higher derivatives of Z^{-1}. Thus, (6.54) can be carried as far as needed.

The first obvious advantages of this formulation is that it applies to any type of structure with variability in K, M and C. Second the needed information is contained in the system with mean elements only; once this information is in hand, everything else follows. Because of (6.62), the next differentiation of (6.59) with respect to X_1, say, will have no second or higher derivatives of Z, and hence, to proceed further is not difficult. It follows that this formulation appears attractive. Convergence requires attention, of course. Further, numerical computational ease must be established. case in [44] although the ideas there expressed are interesting. Normal coordinates?

Randomness in parameters can be introduced into the coefficients in K, M, and C, or through structural elements as suggested above, or by assuming that the natural frequencies are themselves random variables as in [35, 36].

The purpose of this short digression is to emphasize to the reader

that the choice of the equations of motion merits careful consideration so that what is important in a problem is included. Let us now return to the above references [1,9,11,21,24,27,28,30,31].

Lord Rayleigh [1] was not interested in our topic. However, he did, for a different reason, find formulas for the deterministic change in a natural frequency and corresponding normal mode when the coefficients in T and V (in the normal coordinate formulation) are subject to small deterministic changes. His formula for the changed natural frequency is (in our notation)

$$\omega_r^2 = \frac{k_r+\delta k_{rr}}{m_r+\delta m_{rr}} - \sum_{s=1}^{n}{}' \frac{(\delta k_{rs}-\omega_s^2\delta m_{rs})^2}{m_s m_r(\omega_s^2-\omega_r^2)} ,$$

where $s \neq r$ in Σ'. Randomness can be introduced by replacing δk_{rs} and δm_{rs} by small random variables. The first term represents the change in ω_r due to change in mass and stiffness without changing the mode shape; the second term is due to the change in mode shape. This formula is not used today, since changes in parameters in the normal coordinate formulation are not of direct concern. We bring it in because it shows that this master of small iteration was aware that small parameter changes may alter ω_r^2 by large amounts, which is of concern today.

Reference [9] is concerned with the sensitivity coefficients of the buckling load of plates with random thickness and [] discusses the vibration and buckling of a column with spring supports and axial loads treated as random variables and with material and geometric properties considered as correlated random functions. The key formula in [9], attributed by the author to Jacobi, is the same as (36.17), which is the

interesting point. Reference [31] employs a perturbation method (only linear term in (6.32)) for eigenvalue change; the interesting point is that a Monte Carlo simulation is employed to check the analysis. The computer simulation is briefly described. The results from analysis and simulation diverge as the variance in the parameters increase and this is displayed graphically.

References [21,24,28,31] employ either sensitivity coefficients or linear term perturbation expansions (first two terms on right of (6.32) and (6.33)) to examine influence of random parameters on natural frequencies and normal modes. Their techniques are in the same general form as given above following [27]. Except for the last, randomness resides in the terms in K, M, C; in the last, the component mode synthesis method [16] is used directly to derive the perturbation equations, which makes this paper [31] potentially interesting to those confronted with an actual problem, and hence randomness may reside in the structural elements. Numerical difficulties in carrying out the computations are discussed in [28] and [30]. Reference [31] mentions a computer code of NASA.

The linear chain geometry assumed for the physical system in [11] makes it possible to employ a different technique to derive the perturbation expansion than described above. This technique employs a transfer matrix method [68,69] and it is applicable whenever a

The references [1,9,11,21,24,27,28,30,31] address the eigenvalue (natural frequency) and eigenvector (normal mode) problem in structural systems by perturbation methods. Before discussing methods or techniques involved, it is important to understand at the outset that

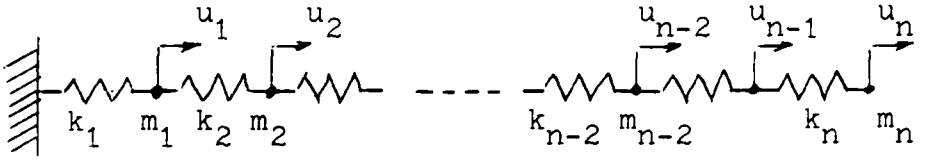
the geometry of the structure, how its equations of motion are assembled, the final mathematical form of the equations of motion, and how randomness in parameters is introduced have a profound influence on the nature of the results obtained.

A structure's geometry can be in the form of a linear array (chain) of elements that may, for example, consist of simple harmonic oscillators strung together in a line, beam segments continuously connected at a sequence of supports in a line, etc. The geometry is the simplest possible in such arrangements. Plate or shell type structures have a two dimensional grid-like geometry and are next in order of complexity. Finally, we have the general case in which one or two dimensional geometries are either missing or are interconnected in a complex manner.

The equations of motion depend on the coordinate choice, particularly when the fact that mass is always distributed is taken into account. Reference [26] discusses methods of making this choice and illustrates the substantial difference in response that can occur due to different choices. Reference [16] also discusses a component mode synthesis method for selecting coordinates and assembling the equations of motion. We cannot present any of this material here in spite of its importance.

A coordinate transformation of the equations of motion is sometimes employed as in [9, 44]. The altered form of the equations may be advantageous for our purpose as in [9] bu. this does not appear to be the structure consists of a chain of cells or units. Let us consider this method in some detail.

Consider the system shown in Figure which is linear chain of oscillators (See [11]) and where



we have, for the moment, ignored randomness in the parameters. The kinetic and potential energies are

$$2T = \sum_1^n m_j \dot{u}_j^2 \ , \tag{6.63}$$

$$2V = \sum_1^n k_j (u_j - u_{j-1})^2 \ , \quad u_o = 0 \ ,$$

which gives as equations of motion

$$m_j \ddot{u}_j + k_j(u_j - u_{j-1}) - k_{j+1}(u_{j+1} - u_j) = 0, \quad j = 1, \ldots, n-1 \tag{6.64}$$

$$m_n \ddot{u}_n + k_n u_n = f_o e^{i\omega t} \ .$$

Introduce a new coordinate

$$w_j = u_{j+1} - u_j \ ; \tag{6.65}$$

let

$$u_j = x_j e^{i\omega t} \ , \quad x_o = 0 \ , \tag{6.66}$$

$$w_j = y_j e^{i\omega t} \ ,$$

and let

$$d_j = \begin{pmatrix} x_j \\ y_j \end{pmatrix}. \tag{6.67}$$

We now write the equations of motion as

$$\begin{aligned} d_j &= \begin{pmatrix} 1 - \frac{m_{j+1}\omega^2}{k_{j+1}} & -\frac{k_{j+2}}{k_{j+1}} \\ \frac{m_{j+1}\omega^2}{k_{j+1}} & +\frac{k_{j+1}}{k_{j+1}} \end{pmatrix} d_{j+1} \\ &= (I + T_{j+1})d_{j+1}\ , \quad j = 0, \ldots, n-2\ , \\ d_{n-1} &= \begin{pmatrix} 1 - \frac{m_n\omega^2}{k_n} & -1 \\ \frac{m_n\omega^2}{k_n} & 1 \end{pmatrix} \begin{pmatrix} x_n \\ \frac{fo}{kn} \end{pmatrix} \\ &= (I + T_n)d_n\ , \end{aligned} \tag{6.68}$$

where I is the 2x2 unit matrix

$$T_j = \begin{pmatrix} -\frac{m_j\omega^2}{k_j} & -\frac{k_{j+1}}{k_j} \\ \frac{m_j\omega^2}{k_j} & \frac{k_{j+1}}{k_j} - 1 \end{pmatrix}, \quad j = 1, \ldots, n-1 \tag{6.69a}$$

$$T_n = \begin{pmatrix} -\frac{m_n\omega^2}{k_n} & -1 \\ \frac{m_n\omega^2}{k_n} & 0 \end{pmatrix},$$

$$d_o = \begin{pmatrix} 0 \\ y_o \end{pmatrix}, \quad d_n = \begin{pmatrix} x_n \\ \frac{fo}{kn} \end{pmatrix}. \tag{6.69b}$$

We can now relate the first displacement vector d_o to the last d_n by

$$d_o = \prod_1^n (I+T_j)d_n \ . \tag{6.70}$$

The matrix $I + T_j$ transfers d_{j+1} to d_j; hence, the name "transfer matrix". Any structure whose geometry is a linear array of units, such as in the above Figure, or of beams, etc, can be treated in this manner.

Let the elements in the (2x2) matrix $\prod^n(I+T_j)$ be $a_{jk}(\omega)$. Then (6.70) becomes

$$\begin{pmatrix} 0 \\ y_o \end{pmatrix} = \begin{pmatrix} a_{11}(\omega) & a_{12}(\omega) \\ a_{21}(\omega) & a_{22}(\omega) \end{pmatrix} \begin{pmatrix} x_n \\ \frac{fo}{kn} \end{pmatrix} . \tag{6.71}$$

The natural frequency equation of the chain is obtained by letting $f_o = 0$ and taking the first equation in (6.71); it is

$$a_{11}(\omega)x_n = 0 \tag{6.72}$$

Since x_n cannot be zero, the frequency equation is

$$a_{11}(\omega) = 0 \ . \tag{6.73}$$

In [11], details, which do not concern us here, are worked out for the natural frequencies when $k_j = k$, $m_j = m$.

Now assume the masses are random variables taking the form

$$m_j = (1 + X_j)m \ , \tag{6.74}$$

where X_j is a dimensionless random variable with mean zero. Introduce

$$T = \begin{pmatrix} -\frac{m\omega^2}{k} & -1 \\ +\frac{m\omega^2}{k} & 0 \end{pmatrix}, \quad E_j = \begin{pmatrix} -\frac{m\omega^2 X_j}{k} & 0 \\ +\frac{m\omega^2 X_j}{k} & 0 \end{pmatrix}; \tag{6.75}$$

then, with $k_j = k$,

$$I + T_j = I + T + E_j \,, \tag{6.76}$$

Now (6.70) takes the form

$$d_o = \prod_1^n (I+T+E_j)d_n \tag{91}$$

or

$$d_o = [(I+T)^n + \sum_{j=1}^{n} (I+T)^{j-1}E_j(I+T)^{n-j} \tag{6.78}$$
$$+ \sum_{j=1}^{n-1}\sum_{k=j+1}^{n} (I+T)^{j-1}E_j(I+T)^{k-j-1}E_k(I+T)^{n-k}$$
$$+ \ldots + \prod_1^n E_j]d_n \,.$$

A number of substitutions then make it possible to expand ω_r in a series in the X_j:

$$\omega_r = \underline{\omega}_r + \sum_{j=1}^{n} \omega_{1j}X_j + \sum_{j=1}^{n} \omega_{2j}X_j^2 + \sum_{j=1}^{n-1}\sum_{k=j+1}^{n} \omega_{1jk}X_jX_k + \ldots \,. \tag{6.79}$$

The main point to note is that normal modes of the mean system are <u>not</u> employed. Let us contrast this method with that of Zarghame's.

Return to (6.63), let $k_j = k$ and $m_j = m$ and employ (6.74) to obtain

$$M = \underline{M} + \Sigma \underline{M}_j X_j \ , \quad K = \underline{K} \ . \tag{6.80}$$

$$\underline{M} = \begin{pmatrix} m & 0 & 0 & \dots & 0 \\ 0 & m & 0 & \dots & 0 \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & m \end{pmatrix},$$

$$\underline{K} = \begin{pmatrix} 2k & -k & 0 & \dots & 0 & 0 \\ -k & 2k & 0 & \dots & 0 & 0 \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ 0 & 0 & 0 & \dots & 2k & -k \\ 0 & 0 & 0 & \dots & -k & k \end{pmatrix}, \tag{6.81}$$

$$\underline{M}_j = \begin{array}{c} \begin{matrix} 1 & 2 & \dots & j \dots & n \end{matrix} \\ \begin{pmatrix} 0 & 0 & \dots & 0 \dots & 0 \\ 0 & 0 & \dots & 0 \dots & 0 \\ . & . & \dots & \dots & . \\ . & . & \dots & \dots & . \\ . & . & \dots & \dots & . \\ 0 & 0 & \dots & m \dots & 0 \\ . & . & \dots & \dots & . \\ . & . & \dots & \dots & . \\ . & . & \dots & \dots & . \\ 0 & 0 & \dots & 0 \dots & 0 \end{pmatrix} \end{array} \begin{matrix} 1 \\ 2 \\ . \\ . \\ . \\ j \\ . \\ . \\ . \\ n \end{matrix}$$

Let $\underline{\omega}_r$ and $\underline{\alpha}_r$ be respectively the r^{th} natural frequency and normal mode of the system with mean elements (i.e. $M = \underline{M}$, $K = \underline{K}$). There, from (A7)

$$\left(\frac{\partial \omega_r}{\partial X_j}\right)_o = -\frac{\omega_r}{2}\,\underline{\alpha}_r^T \underline{M}_j \underline{\alpha}_r \,, \tag{6.82}$$

$$\beta_{rj}^{(k)} = -\frac{\omega_r^2 \underline{\alpha}_k^T \underline{M}_j \underline{\alpha}_r}{\omega_r^2 - \omega_k^2} \,, \quad k \neq r \,, \tag{6.83}$$

$$\beta_{rj}^{(r)} = -\frac{\underline{\alpha}_r^T \underline{M}_j \underline{\alpha}_r}{2} \,,$$

$$\begin{aligned}\left(\frac{\partial^2 \omega_r}{\partial X_j \partial X_k}\right)_o = \frac{1}{2\omega_r} \Big[& -2 \left(\frac{\partial \omega_r}{\partial X_j}\right)_o \left(\frac{\partial \omega_r}{\partial X_k}\right)_o \\ & + \sum_1{}' (\beta_{rj}^{(l)} \beta_{lk}^{(r)} - \beta_{lj}^{(r)} \beta_{rk}^{(l)})(\omega_l^2 - \omega_r^2) \\ & + 4\omega_r \left(\left(\frac{\partial \omega_r}{\partial X_j}\right)_o \beta_{rk}^{(r)} - \left(\frac{\partial \omega_r}{\partial X_k}\right)_o \beta_{rj}^{(r)}\right) \Big] \,.\end{aligned} \tag{6.84}$$

We observe from (6.82) that the j^{th} sensitivity coefficient of ω_r depends only upon $\underline{\omega}_r$, $\underline{\alpha}_r$ and $\underline{M}_j$. Let

$$\underline{\alpha}_r^T = (\underline{\alpha}_r^{(1)}, \underline{\alpha}_r^{(2)}, \ldots, \underline{\alpha}_r^{(n)}) \,.$$

Then (6.82) becomes

$$\left(\frac{\partial \omega_r}{\partial X_j}\right)_o = -\frac{m\omega_r}{2} (\underline{\alpha}_r^{(j)})^2 \,; \tag{6.85}$$

remembering from (A6) that the $\underline{\alpha}_r^{(j)}$ have the dimension $1/\sqrt{m}$, we see that the right of (6.85) has the correct dimension. To see that the sign is correct, observe from the first two terms on the right of (6.32) that

$$\omega_r = \underline{\omega}_r - \frac{m\omega_r}{2} \sum (\alpha_r^{(j)})^2 X_j \tag{6.86}$$

Positive X_j mean increase in mass. We know from Rayleigh's Principle [1, Sect.] that an increase in mass lowers or leaves unchanged every natural frequency; thus, the negative sign on the right of (6.85) is correct.

The importance of the sensitivity coefficients (6.85) and (7A) resides in the fact that they reveal by their magnitudes those ω_r that are either sensitive or insensitive to uncertainty in member values.

On evaluating the $\beta_{rj}^{(k)}$ from (6.83), we can write down the right of (6.84). The main point to observe is that now we need <u>all</u> $\underline{\alpha_\tau}$ and $\underline{\omega_\tau^2}$. Thus, we can obtain the partial derivative values appearing in (6.32). We refrain from these obvious details to conserve space.

Let us contrast Zarghame's procedure with that given in [11].

In the former, we need all $\underline{\alpha_\tau}$ and $\underline{\omega_\tau}$ to obtain (6.32); in the latter we need all $\underline{\omega_\tau}$ but not the normal modes since the intermediate coordinates were eliminated in obtaining (6.70). Thus, Zarghame's procedure does require more information than required in [11]. However, since all computer codes produce the $\underline{\alpha_\tau}$ along with the $\underline{\omega_\tau}$, the additional information required in the former is always available anyway. Hence, relative to effort the two procedures do not differ substantially.

The sensitivity coefficient $\omega_{\underline{1}j}$ in (6.79) is given by

$$\omega_{\underline{1}j} = 2\sqrt{\frac{\bar{k}}{m}}\,\beta_{1j}\cos\beta_o\,, \tag{6.87}$$

where

$$\beta_o(r,\,n) = \frac{(2r-1)\pi}{2(2n-1)}\,, \tag{6.88}$$

$$\beta_{1j} = -\frac{2}{2n+1}\sin^2 2j\,\beta_o\tan\beta_o$$

While these are easy to evaluate, the physical significance of changes in particular parameter values is not as easy to grasp in (6.87) as is the case with the corresponding formula in (7A). This difficulty increases with ω_{2j}, and ω_{1jn} in [11]. Hence, from the point of view of

physical understanding of what is of significance in a system relative to variability in natural frequencies, Zarghame's procedure appears superior to the procedure given in [11].

The main advantage of the method given in [9] is that by exploiting the system geometry explicit formulas can be written out for the quantities of interest. However, since all computations are now performed on a computer, this is no longer an advantage.

Two other points deserve comment. First, the formula for ω_r in [30] only employs the first two terms on the right of (6.32). This makes

$$E\{\omega_r\} = \underline{\omega}_r \ ,$$

whereas in (6.32)

$$E\{\omega_r\} = \underline{\omega}_r + \frac{1}{2} \Sigma\Sigma(\frac{\partial^2 \omega_r}{\partial X_j \partial X_k})E\{X_j X_k\} + \ldots \ .$$

Thus, in [30], the mean of ω_r is the mean system $\underline{\omega}_r$. However, from the formula above this is not correct not the case. Also, (6.32) gives a different formula for variance of ω_r than given in [30]. It follows that the method given in [30] is incomplete. Second, sensitivity coefficients also play an important role in other types of system behavior analysis such as automatic control [14,15,18,19,25]. Since large space structures will contain control systems, uncertainty in control system parameters coupled with uncertainties in the structural system parameters must be kept in mind.

References [5,8,11,20,29,35,36,37,39,40,42,43,48,49,50,52] discuss frequency response, impulse function (Green's function, impulsive

admittance, impedence) mainly by perturbation methods. Those that do not involve perturbation methods are [5,8,20]. Let us take up the latter group first.

The technique involved in [5,8] for assessing the variability of frequency response (or impedance) is what might be called the direct method. For a one-degree of freedom system, we have

$$Z^{-1}(\omega) = \frac{1}{K-\omega^2 M+i\omega C} , \tag{6.89}$$

where K, M, C are scalar random variables. There are cases where knowing the distributions of independent K, M, and C explicit results can be obtained. While interesting results can be obtained by this method, it is clear this technique is of limited practical use in complex systems.

Reference [20] starts by making assumptions about the statistical characteristics of the Green's function G of a system, and writes the response as

$$q(t) = \int_{-\infty}^{\infty} G(t,\tau)f(\tau)d\tau . \tag{6.90}$$

G is then related to the equation of motion

$$\dot{q} = -Aq \tag{6.91}$$

by assuming

$$A = \underline{A} + N(t) , \tag{6.92}$$

where $\underline{A}$ is a constant matrix and N(t) is a normal noise process. On letting

$$W(t,\tau) = \int_{\tau}^{t} N(\theta)d\theta \ , \tag{6.93}$$

we find

$$G(t,\tau) = [\underline{A}+N(\tau)]e^{-\underline{A}(t-\tau)+W(t,\tau)} \ . \tag{6.94}$$

Means and moments of q(t) can now be obtained in terms of the moments of $G(t,\tau)$. If we take N(t) to be independent of time, then this technique applies to our problem.

We observe from (6.94) that even the expectation of G is not easy to evaluate because of $N(\tau)$ in the coefficient and $W(t,\tau)$ in the exponent. By assuming $N(\tau)$ is generated by filtering a white noise process, it is shown that the moments of $G(t,\tau)$ can be evaluated. Further investigation of this interesting technique is necessary before we can determine if it can be cast in a form useful to us. Even if it can, additional work is necessary to establish that it can be applied when N(t) is constant or only slowly varying.

Advantage is taken of a specific structural shape from the outset in [11,39,50]. The first two assume a linear chain of similar elements differing in the random nature of element parameters; the structure in [50] is a circular chain. Linear one degree of freedom damped oscillator elements are assumed in the first reference; damped Bernoulli-Euler beam elements with random lengths are assumed in the second; and a continuous distribution of linear spring connected linar one degree of freedom oscillator elements, as in buckets or a turbine disc, is assumed in the third. When damping is assumed, it is taken as small so that undamped normal modes can be employed. The transfer matrix technique is employed in [11,39] since a chain-like structure is

assumed; this makes it possible to obtain analytical results that are complex indeed. Because of this complexity, it will require considerable effort to obtain numerical results, and it is by no means clear that what is proposed is as easy to use as (6.54). The periodic circular structure in [50] makes it possible to employ Fourier series; results on natural frequencies, normal modes, and response are obtained; on those rare occasions when a structure has a circular form, the technique employed could be considered but not otherwise. Excellent graphical results that assist in understanding in a qualitative sense the effect of disorder on response are presented in [11,39]. All point out that high variability in response will occur in lightly damped disordered systems. The techniques employed lack the generality of (6.54) and are not of direct interest.

Let us consider [35,36,37,42] next. In each of these references random parameters are introduced in special ways which renders the technqiues used and results obtained of limited use. [42] does introduce a new quantity of some interest; they consider the equation

$$m\ddot{x} + k(1+\varepsilon)x = f(t),$$

where ε is random with zero mean, and introduce

$$V[x^2(t)] = \left| \frac{E\{x^2(t)\} - x^2(t)_{t=0}}{\langle x^2(t) \rangle_t} \right| \tag{6.95}$$

as a measure of the time for the mean square response $E\{x^2(t)\}$ to deviate from the unperturbed response, i.e. $x^2(t)$ with $t = 0$. The normalization with respect to the time average $\langle x^2(t) \rangle_t$ is selected so that $V[x^2(t)] \to 1$ as $t \to \infty$. A simple expression is then produced for

the envelope of V, namely

$$V[x^2(t)]_{env.} = 1 - e^{-\frac{1}{2}\omega^2\sigma^2t^2}, \qquad (6.96)$$

where $\omega^2 = k/m$, $\sigma^2 = \text{Var } \varepsilon$. V is essentially the growth in uncertainty in the response as a function of σ^2 and t. This is a nice idea that merits development since it says that when

$$\frac{1}{2}\omega^2\sigma^2t^2 > 4$$

response location is lost.

Papers [48,49] merit attention not for techniques involved but rather for some qualitative results that may be of interest in connection with large space structures. Reference [48] discusses wave propagation in long beams with many supports, where there is random variation in length among the beam segments. The point of interest is that the random variation in length among the beam segments has a substantial influence on which type of flexural waves will propagate and which will attenuate. Reference [49] is concerned with the confinement of vibration to certain regions of a structure due to structural disorder. The reason for noticing them resides in the fact that in a large space structure it may be desirable to introduce structural irregularity in order to prevent wave motion from propagating throughout the structure and/or confining vibration to a favorable region of the structure. These references would then form a useful starting point. It should be noted that the "receptance method" [50], frequency response method, and the mobility method provide essentially the same approach; the advantage of the first and third of these methods lies in techniques

for obtaining the frequency response in a sequential manner. Referene [52] investigates failure probability in a structure with uncertain properties; it emphasizes the importance of considering these uncertainties when estimating such probabilities.

VII. Liouville Equation

The techniques discussed in Section 5 are based directly upon the equations of motion. For example, the perturbation expansion of the frequency response $Z^{-1}(\omega)$ employs the equations of motion to derive $Z^{-1}(\omega)$ and the relations it satisfies. There is another approach based upon the Liouville equation for the time evolution of the joint probability distribution function of the state space (2nx1) column vector x and the system parameters. We consider that approach in this Section.

The use of the Liouville equation in mechanics and statistical mechanics is of long standing [51,52,53, for example and going back to Maxwell]. These references do not assume system parameters are random variables, and average of quantities under equilibrium conditions is of main interest. While not of direct interest to us, it is possible to adapt these early methods to our needs. We derive the needed form of the Liouville equation following a procedure suggested by Kozin [5].

Consider the equations of motion in the form given by (1.27) with f = 0:

$$\dot{x} = Ax \ , \qquad (1.27')$$

where x is the (2nx1) column vector whose transpose x^T has the form $x^T = \{q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n\}$ and A is the (2nx2n) matrix given by the first of (1.28). The vector x is the state space form for representing the system response; the components of x will be denoted by $x_k(t)$, k = 1, ..., 2n. The random variables in A are, as before, denoted by $X_1, \ldots, X_m$. We will rewrite (1.27′) in component form as

$$\dot{x}_k = g_k(x_1, \ldots, x_{2n}; X_1, \ldots, X_m; t) , \quad k = 1, .., 2n . \qquad (7.1)$$

Let $p(x_1, \ldots, x_{2n}; X_1, \ldots, X_n; t)$ be the joint probability distribution of the random quantities $x_1, \ldots, x_{2n}; X_1, \ldots, X_n$. We define the characteristic function Φ as

$$\Phi = E[\exp i(\sum_{k=1}^{2n} \theta_k x_k(t) + \sum_{j=1}^{m} \phi_j X_j)], \quad i = \sqrt{-1} . \qquad (7.2)$$

The differentiation of (112) with respect to time gives

$$\frac{\partial \Phi}{\partial t} = E[i \sum_{k=1}^{2n} \theta_k \dot{x}_k(t) \exp i(\sum_{k=1}^{2n} \theta_k x_k(t) + \sum_{j=1}^{m} \phi_j X_j)] . \qquad (7.3)$$

The use of (6.1) in (6.3) yields

$$\frac{\partial \Phi}{\partial t} = i \sum_{k=1}^{2n} \theta_k E[g_k \exp i(\sum_{k=1}^{2n} \theta_k x_k(t) + \sum_{j=1}^{m} \phi_j X_j)] . \qquad (7.4)$$

Since (7.2) is essentially the Fourier transform of the joint density function $p(x_1, x_{2n}; X_1, \ldots, X_m; t)$, the inverse Fourier transform of (7.4) produces

$$\frac{\partial p}{\partial t} = -\sum_{1}^{n} \frac{\partial (g_j p)}{\partial x_j} . \qquad (7.5)$$

The solution of (7.5) for p is given by a suitable function of the independent integrals of the Lagrangian system

$$\frac{dt}{1} = \frac{-dp}{p(\frac{\partial f_1}{\partial X_1} + \ldots + \frac{\partial f_{2n}}{\partial X_{2n}})} = \frac{dx_1}{f_1} = \ldots = \frac{dx_{2y}}{f_{2n}} . \qquad (7.6)$$

Let $u_1, \ldots, u_{2n}$ be 2n-independent integrals of (7.6). Then we know that the general solution of (7.5) is

$$p(x_1, \ldots, x_{2n}; X_1, \ldots, X_m; t) = h(u_1, \ldots, u_{2u}; X_1, \ldots, X_m; t) \quad L$$

where h is an arbitrary function whose form is determined by the initial

conditions on x.

The 2n-integrals $u_1, \ldots, u_{2n}$ may be determined in one of two ways. First, the general solution of (7.6) is

$$x = e^{At}x_o;$$

thus

$$e^{-At}x = u, \tag{7.8}$$

where $u^T = \{u_1, \ldots, u_{2n}\}$. Further, for distinct eigenvalues of A,

$$e^{-At} = U^{-1}\begin{pmatrix} e^{-\lambda_1 t} & & & & \\ & e^{-\lambda_2 t} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & e^{-\lambda_{2n} t} \end{pmatrix} U \tag{7.9}$$

where $\lambda_1, \ldots, \lambda_{2n}$ are the eigenvalues of A, and U is the (2nx2n) matrix of corresponding eigenvectors of A. This is one way to determine the 2n independent integrals u required in (7.7).

The second method employs Laplace transforms and the equation

$$\dot{x} = Ax \quad \text{with} \quad x_o = u\ . \tag{7.10}$$

Then from the Laplace transform of this equation, we find

$$L^{-1}[(Is - A)^{-1}]x = u\ ; \tag{7.11}$$

it should be noted that the u's depend on the x's and the X's. To evaluate the inverse Laplace transform, we need the eigenvalues of -A or, equivalently, the roots of (Is-A) = 0.

Either employing (7.8) or (7.11), we have the required integrals. Hence, in principal, we find with (7.7) the exact expression for the joint density p, the arbitrary function h being determined by the specifics of the problem addressed.

Consider the case where the parameters $X_1, \ldots, X_m$ are independent of the integrals (i.e. the initial vector). Then (6.7) can be written as

$$p(x_1, \ldots, x_{2n}; X_1, \ldots, X_m; t) = h_1(u_1, \ldots, u_{2n}; t)h_2(X_1, \ldots, X_m) ,$$

where h_2 is the joint density function of the parameters. Suppose we are interested only in the impulsive response for q_1; this means at $t = 0$, $x_{n+1} = 1$, with all other $x_j = 0$. Then at $t = 0$,

$$p(x_1, \ldots, x_{2n}; X_1, \ldots, X_m; 0) \tag{7.12}$$
$$= \delta(x_1)\ldots\delta(x_n)\delta(x_{n+1}-1)\ldots = - \delta(x_{2n})h_2(X_1, \ldots, X_m) .$$

In view of the fact that at $t = 0$, $u = x_o$, we see that

$$h_1(u_1, \ldots, u_{2n}; t) = \delta(u_1)\ldots\delta(u_n)\delta(u_{n+1}-1)\ldots\delta(u_{2n}) ,$$

where $\delta(.)$ is the delta function. Finally,

$$p(x_1, \ldots, x_{2n}; X_1, \ldots, X_m; t) \tag{7.13}$$
$$= \delta(u_1)\ldots\delta(u_n)\delta(u_{n+1}-1)\ldots\delta(u_{2n})h_2(X_1, \ldots, X_m) .$$

Let us now illustrate this process with a simple example.

Consider again the undamped one degree of freedom linear oscillator. Let $K/M = \Omega^2$ be a random variable with sample value denoted by ω^2. The first order system representation of the equation of motion is

$$\dot{x}_1 = x_2 = f_1 \ , \tag{7.14}$$
$$\dot{x}_2 = -\Omega^2 x_1 = f_2 \ .$$

Equation (7.5) becomes

$$\frac{\partial p}{\partial t} + x_2 \frac{\partial p}{\partial x_1} - x_1 \Omega^2 \frac{\partial p}{\partial x_2} = 0 \ , \tag{7.15}$$

and (7.6) takes the form

$$\frac{dt}{1} = \frac{dx_1}{x_2} = -\frac{dx_2}{x_1 \Omega^2} \ . \tag{7.16}$$

We now find the integrals of (7.16); they are

$$u_1 = x_1 \cos \Omega t - \frac{x_2}{\Omega} \sin \Omega t \ , \tag{7.17}$$
$$u_2 = x_1 \Omega \sin \Omega t + x_2 \cos \Omega t \ .$$

Assume, for example, Ω^2 has a discrete distribution given by

$$h_2(\omega^2) = \sum_1^n p_i \delta(\omega^2 - \omega_i^2) \ . \tag{7.18}$$

If we are interested in the impulse function, at $t = 0$ we have $x_1 = 0$, $x_2 = 1$. There, (7.13) becomes

$$p(x_1, x_2, \omega^2; t) \tag{7.19}$$
$$= \delta(u_1)\delta(u_2 - 1) \sum_1^n p_i \delta(\omega^2 - \omega_i^2)$$
$$= \delta(x_1 \cos \omega t - \frac{x_2}{\omega} \sin \omega t)\delta(\omega x_1 \sin \omega t + x_2 \cos \omega t - 1) \sum_1^n p_i \delta(\omega^2 - \omega_i^2) \ .$$

Let us determine the mean of x_1 to illustrate a possible use for (7.19); we have

$$E\{x_1\} = \int dx_1 \int dx_2 \int d\omega^2 [x_1 p(x_1, x_2, \omega^2; t] \ . \tag{7.20}$$

Some manipulation (see [5]) yields

$$E\{x_1\} = \sum_1^n p_i \frac{\sin \omega_i t}{\omega_i} . \tag{7.21}$$

Other illustrations, including damping, are given in [5,6]. We are frequently concerned with the moments of x. Let us next consider how (7.5) can be employed to do this.

To keep the details simple, consider the linear damped one degree of freedom system with equation of motion

$$\begin{aligned} \dot{x}_1 &= x_2 = g_1 , \\ \dot{x}_2 &= -\frac{K}{M} x_1 - \frac{C}{M} x_2 = f_2 . \end{aligned} \tag{7.22}$$

Equation (7.5) now takes the form

$$\frac{\partial p}{\partial t} + \frac{\partial (f_1 p)}{\partial x_1} + \frac{\partial (f_2 p)}{\partial x_2} = 0 . \tag{7.23}$$

Assume the M, K, C are independent of the initial vector with $h_2(m,k,c)$ the pdf (prob. den. function) of these parameters, and write

$$p(x_1, x_2; m,k, c;t) = p_1(x_1, x_2; t)h_2(m,k,c) \tag{7.24}$$

Then $p_1(x_1, x_2; t)$ is the joint pdf of x_1 and x_2 <u>conditional</u> on M = m, K = k, C = c. Since h_2 is independent of x_1, x_2, and t, we can write (7.23) as

$$\frac{\partial p_1}{\partial t} + \frac{\partial (g_1 p_1)}{\partial x_1} + \frac{\partial (g_2 p_1)}{\partial x_2} = 0 . \tag{7.23'}$$

Let the expectations

$$E\{x_1|m,k,c\} = m_{1,0}(t), \; E_c\{x_2|m,k,c\} = m_{0,1}(t) \tag{7.25}$$

be conditional on M, K, C. Let us evaluate these expectations;

$$E_c E\{x_1|m,k,c\}\{x_1\} = m_{1,0}(t) = \iint x_1 p_1 dx_1 dx_2$$
$$E_c E\{x_2|m,k,c\}\{x_2\} = m_{0,1}(t) = \iint x_2 p_1 dx_1 dx_2$$

Differentiation (partial) of these equations with respect to time produces

$$\dot{m}_{1,0} = \iint x_1 \frac{\partial p_1}{\partial t} dx_1 dx_2 \tag{7.26}$$
$$\dot{m}_{0,1} = \iint x_2 \frac{\partial p}{\partial t} dx_1 dx_2$$

But, from (7.23′),

$$\frac{\partial p_1}{\partial t} = -\frac{\partial(g_1 p_1)}{\partial x_1} - \frac{\partial(g_2 p_2)}{\partial x_2} . \tag{7.27}$$

The substitution of (7.27) into (7.26), the employment of (7.22), and integration by parts of the resulting terms on the right of (7.26) finally yields

$$\dot{m}_{1,0} = m_{0,1} , \tag{7.28}$$
$$\dot{m}_{0,1} = -\frac{k}{m} m_{10} - \frac{c}{m} m_{0,1} .$$

The same procedure will produce the equations for the conditional moments $E_c\{x_1^2|m,k,c\}$, $E_c\{x_2^2|m,k,c\}$, etc. We note that for the first conditional moments we could have taken the conditional expectation of (7.22) to produce (7.28); however, this procedure only applies to the first moments.

We integrate the moment equations (7.28) to obtain the conditional moments as a function of time. On multiplying these moments by $h_2(m,k,c)$ and integrating over m, k, and c, we obtain the moments of x_1 and x_2.

It is clear from the above discussion that the Liouville equation will provide the exact solution for the joint probability density

function $p(x_1, \ldots, x_{2u}; X_1, \ldots, X_m; t)$ in the absence of external forces provided the integrals $u_1, \ldots, u_{2n}$ can be obtained. Further, it provides a straight forward method for determining the moments of x from which means and variances of x can be obtained.

The Liouville equation applies when there are no external forces. We are interested in the case when external forces are present, of course. Let us see what can be done along these lines.

The Fokker-Planck equation is the natural extension of the Liouville equation. This equation has already been derived above. (See Section 2 and also [8], [6b]). We confine attention to the case in which the external force vector f can be obtained by passing gaussian white noise through a stable linear damped system. We have as equations of motion, conditional on M = m, K = k, and C = c,

$$\begin{aligned} dx_1 &= x_2 dt \ , \\ dx_2 &= -\frac{k}{m} x dt - \frac{c}{m} x_2 dt + x_3 dt \ , \\ dx_3 &= -\beta x_3 dt + dB \ , \quad x_3 = 0 \text{ at } t = 0 \ . \end{aligned} \tag{7.29}$$

where we have employed the differential notation in this case, set $f = x_3$, and where dB is the Brownian motion increment (see Section 2) with

$$E\{dB\} = 0 \ , \quad E\{(dB)^2\} = \sigma^2 dt \ . \tag{7.30}$$

The last equation of (7.29) represents the fact that the excitation is obtained by passing a gaussian white noise through a linear first order stable filter. We notice that for the Ito system (7.29) $x^T = \{x_1, x_2, x_3\}$ is a vector Markoff process that generates a Fokker-Planck equation.

It can be shown that in this case the Fokker-Planck equation for the conditional probability density function p_1 is

$$\frac{\partial p_1}{\partial t} = -\frac{\partial}{\partial x_1}(x_2 p_1) - \frac{\partial}{\partial x_2}\{(-\frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{1}{m}x_3)p_1\} - \frac{\partial}{\partial x_3}(-\beta x_3 p_1) + \frac{\sigma^2}{2}\frac{\partial^2 p_1}{\partial x_3^2} . \quad (7.31)$$

We observe that all but the last term on the right are the same as would have occurred in the Liouville equation in the absence of f. Let the conditional moments be

$$m_{k_1,k_2,k_3} = E\{x_1^{k_1} x_2^{k_2} x_3^{k_3}\} \quad (7.32)$$
$$= \iiint x_1^{k_1} x_2^{k_1} x_3^{k_3} p_1(x_1, x_2, x_3)dx_1 dx_2 dx_3$$

Then, proceeding as in the development of (7.28), we find

$$\begin{aligned} \dot{m}_{1,0,0} &= m_{0,1,0} \\ \dot{m}_{0,1,0} &= -(\tfrac{k}{m})\, m_{1,0,0} - (\tfrac{c}{m})\, m_{0,1,0} + (\tfrac{1}{m})\, m_{0,0,1} \\ \dot{m}_{0,0,1} &= -\beta m_{0,0,1} \end{aligned} \quad (7.33)$$

On multiplying the solutions of (7.33) by $h_2(m,k,c)$ and integrating out the condition in these three conditional moments, we finally obtain the moments of x as a function of time. We obtain in analogous fashion the differential equations for the second conditional moments; we do not do this as the steps are of a mechanical nature and not of direct interest. The main point to notice is that differential equations for the conditional moments of x can be obtained when an external force is present in the equations of motion provided this force is produced by passing white noise through a suitable filter.

It is important to point out that for any gaussian external excitation the solution vector is gaussian conditioned on the random parameters. Therefore, all conditional moments can be obtained but not as easily as above [79].

The Liouville equation enabled us to obtain, in a straight forward manner, the exact expression for the conditional probability density function p_1. Reference to (7.31) suggests that it will be much more difficult to obtain p_1 from this equation and we shall not pursue this line of thought further.

VIII. Mean Square Approximate Systems

We consider, in this section, a technique for including disorder or parameter uncertainty that follows a different line than taken in Sections a) and b). Specifically, mean square systems are employed. We begin by introducing this type of system [13,17].

Let us begin with a very simple example in which there is no disorder and no damping. Let the coordinates be $q_1, \ldots, q_n$. Then, with

$$2T = m_{jk}\dot{q}_j\dot{q}_k, \quad 2V = k_{jk}q_jq_k, \quad \delta W = f_j(t)\delta_{qj} , \tag{8.1}$$

where summation convection. Then (1.24), with mass coefficients included, can be rewritten as

$$m_{jk}\ddot{q}_k + k_{jk}q_k = f_j(t) . \tag{8.2}$$

Let, with f_{oj} constant,

$$f_j(t) = f_{oj}\cos(\omega t + \phi) . \tag{8.3}$$

Then, the forced motion

$$q_k = u_k\cos(\omega t + \phi) \tag{8.4}$$

satisfies

$$(k_{jk} - \omega^2 m_{jk})u_k = f_{oj} . \tag{8.5}$$

These equations state that given the f_{oj} and ω, the u_k are determined by the solution of this linear system of equations. Further, if ω is the natural frequency ω_r and the u_k define the r^{th} mode shape α_{rk}, then the f_{oj} must vanish. Let us look at the natural frequency problem in an unorthodox manner.

Suppose we pick an ω and a set of u_k which may not be one of the natural frequencies and normal modes. Then the right of (8.5) will not be zero and we need force amplitudes ε_j to produce this motion:

$$(k_{jk} - \omega^2 m_{jk})u_k = \varepsilon_j \ . \tag{8.6}$$

The ε_j are the amplitudes required to maintain the assumed motion; we regard the ε_j as the amplitudes of the constraint forces required to produce the motion.

Consider next

$$I(n,\omega) = \sum_1^n \varepsilon_j^2 \geq 0 \ . \tag{8.7}$$

For a fixed ω, this is a positive definite quadratic function of the u's. We can use this equation to find the natural frequencies ω_r and corresponding normal modes α_{rk}. Assume the u's are normalized in some manner (for example, $u_n = 1$ or better $m_{jk}u_j u_k = 1$). For fixed ω, we find the minimum of $I(u, \omega) \geq 0$. Notice that if $\omega = \omega_r$ the u's that produce a minimum are the α_{rk} and $I(\alpha_{rk}, \omega_r^2) = 0$, since the $\varepsilon_j = 0$, $j = 1, \ldots, n$, in this case. It follows that if for a specified ω we find the minimum of $I(u, \omega^2)$ and this minimum equals zero, then this ω is a natural frequency and the u that produce this zero minimum are proportional to the corresponding normal mode. Let us consider another interesting aspect of this method.

Consider a frequency window $g(\omega)$ with the following properties:

$$g(\omega) \geq 0 \ , \quad \omega' \leq \omega \leq \omega'' \ , \tag{8.8}$$

$$\int_{\omega'}^{\omega''} g(\omega)d\omega = 1 \ , \quad \int_{\omega'}^{\omega''} \omega^2 g(\omega)d\omega < \infty \ .$$

Replace (8.7) with

$$I(u, g) = \int_{\omega'}^{\omega''} \Sigma\varepsilon_j^2 d\omega \ . \tag{8.9}$$

Find the u that makes (8.9) a minimum. The interesting feature of this method is that if there is a natural frequency of the system in the frequency interval (ω', ω''), the u in min I(u, g) determine the normal mode of this natural frequency. Let these u be in component form $\{u_1^{(r)}, \ldots, u_n^{(r)}\}$; then the corresponding natural frequency is determined by the Rayleigh quotient:

$$\omega_r^2 = \frac{k_{jk}u_j^{(r)}u_k^{(r)}}{m_{jk}u_j^{(r)}u_k^{(r)}} \ , \tag{8.10}$$

where we assume we have found the r^{th} normal mode and its natural frequency. It follows that if there is concern that an interval (ω', ω'') contains a natural frequency, we have method for determining if this is the case without determining <u>all</u> natural frequencies of the system. References [13,17] give details on this matter we cannot discuss in this Report.

The computational problem of finding the minimum of $I(u, \omega^2)$ is carried out using one of a number of computer codes based upon conjugate gradient techniques, and, hence, is not a problem.

So far, there has been no disorder in our system; i.e. the parameters m_{jk} and k_{jk} have been assumed to take definite values. Let

us assume at this point that mass and stiffness contain random variables. We define, in this case,

$$I(u,g) = E\{\int_{\omega}^{\omega''} g(\underline{\omega})\sum_{1}^{n}\varepsilon_j^2 d\underline{\omega}\} \ , \tag{8.11}$$

where, in vector-matrix form

$$\sum_{1}^{n}\varepsilon_j^2 = u^T(K - \underline{\omega}^2 M)^T(K - \underline{\omega}^2 M)u \ . \tag{8.12}$$

Since E only operates on $\sum_{1}^{n}\varepsilon_j^2$ in (7.11), we have

$$E\{\sum_{1}^{n}\varepsilon_j^2\} = E\{u^T(K - \underline{\omega}^2 M)^T(K - \underline{\omega}^2 M)u\} \ , \tag{8.13}$$

and ω is a fixed parameter in (8.13). We assume the u are parameters to be determined. Thus, (8.13) takes the form

$$E\{\sum_{1}^{n}\varepsilon_j^2\} = u^T E\{(K - \underline{\omega}^2 M)^T(K - \underline{\omega}^2 M)\}u \ . \tag{8.14}$$

We note that $(K - \underline{\omega}^2 M)^T = K - \underline{\omega}^2 M$ because of the symmetry assumed for K and M. In all events, means and second moments of K and M are all the information needed to determine the expectation in (8.14).

We then proceed as in the deterministic case, since I(u, g) has a deterministic form.

To relate (7.7) to (7.13), all we have to do is assume

$$g(\underline{\omega}) = \delta(\omega - \underline{\omega}) \ , \tag{8.15}$$

where $\delta(.)$ is the delta function. The substitution of (8.15) into (8.11) yields

$$I(u,\omega) = u^T E\{(K - \omega^2 M)^T (K - \omega^2 M)\}u \ . \tag{8.16}$$

This expression differs from (8.7) because of the assumed random parameters in K and M. If in (8.7), ω is a natural frequency of the deterministic system, $I(u,\omega) = 0$. The $I(u,\omega) > 0$ in (8.16) because of the random parameters. Use of this fact has been made in [32] to obtain an estimate of the variance of natural frequency ω_r; the formula is

$$\text{Var } \omega_r = \frac{I(u_r, \omega_r^2)}{4\omega_r^2} \ , \tag{8.17}$$

where ω_r is the r^{th} natural frequency and u_r is the corresponding normal mode for the system with mean parameter values. Monte Carlo simulation [32] reveals that (8.17) can be conservative and a correction is suggested. Equation (8.17) is easy to use since a minimum for I is not required. Further, (8.17) provides a much simpler method for estimating the variance of ω_r than given in Section V. However, mean square approximate systems do not provide any information on $E\{\omega_r\}$ or on variability in mode shape. Let us next consider how these systems apply to estimating frequency response with parameter uncertainty present.

We take the equations of motion in the form

$$M\ddot{q} + C\dot{q} + Kq = f \ . \tag{8.17}$$

The frequency response $Z^{-1}(\omega)$ and $Z(\omega)$ are defined by (6.47) and (6.48). For the external force,

$$f = \delta_{jr} e^{i\omega t} \ , \tag{8.18}$$

with r fixed and $\delta_{jr} = 0$ for $j \neq r$, $\delta_{rr} = 1$, the component form of (8.17) is

$$q_j = Z^{-1}_{jr}(\omega)e^{i\omega t} \tag{8.19}$$

which is exact.

Suppose we try to approximate (8.19) with

$$q_j = \beta_{jr}e^{i\omega t} , \tag{8.20}$$

where the β_{jr} are not known in advance. The equations of motion now are not satisfied and we must introduce constraint forces ε_j to bring about their satisfaction as in (6.1):

$$(K_{jk} - \omega^2 M_{jk} + i\omega C_{jk})\beta_{kr} - \delta_{jr} = \varepsilon_{jr} . \tag{8.21}$$

Form

$$I(\beta, \omega) = E\{\sum_1^n \varepsilon^*_{jr}\varepsilon_{jr}\} , \tag{8.22}$$

where asterisk denotes complex conjugate. This I is just like (8.11) except the β have replaced the u. We find the β that make (8.22) a minimum, denote this β by $\hat{\beta}$. Then, $\hat{\beta} = \{\hat{\beta}_1, \ldots, \hat{\beta}_n\}$ is the mean square approximate to the $Z^{-1}_{jr}(\omega)$. It can be shown that if the system is deterministic (i.e. contains no random parameters) the $\hat{\beta}$ are exactly the $Z^{-1}_{jr}(\omega)$.

The ε_{jr} are complex; hence, the right of (8.22), when written out, is

$$\sum_{j=1}^n E\{[(K_{jk}-\omega^2 M_{jk}-i\omega C_{jk})\beta^*_{k1}\delta_{jr}][(K_{j1}-\omega^2 M_{j1}+i\omega C_{j1})\beta_{1r}-\delta_{jr}]\} . \tag{8.23}$$

It follows that the minimum of (8.22) is for the real and imaginary parts of β_{kr}. This added complication poses no additional computational problem [39, 42].

The method also supplies an error criterion that makes it possible to judge the accuracy of the β_{kr}.

References [39,42] describe in some detail how the above technique can be applied to estimating the frequency response in a number of structures with specific attention being paid to numerical details of the computations. Reference [51] also describes how these techniques can be applied to the construction of a sequence of approximants for a complex system by starting from a highly constrained initial system and gradually relaxing the constraints. In these three reference, extensive use is made of the error criterion to determine when the estimated quantities (usually frequency response) are sufficiently accurate for the purpose in hand. A comment on what mean square approximate system provide is now in order.

We observe, for example, that these systems enable us to estimate frequency response $Z_{kr}^{-1}(\omega)$ by means of $\beta_{kr}(\omega)$. The $\beta_{kr}(\omega)$ are deterministic numbers that take into account the means and variances of the statistical parameters of the structure. Thus, the $\beta_{kr}(\omega)$ provide a deterministic estimate for $Z_{kr}^{-1}(\omega)$. In the form given above and in [39,41,42], it is not possible to obtain statistical information concerning the $Z^{-1}(\omega)$. However, it is possible to employ Monte Carlo methods to obtain estimates for the $\beta_{kr}(\omega)$ given the parameters are sample values to obtain sample values for the $\beta_{kr}(\omega)$ from which statistical information can be obtained.

The statistical energy approach (SEA) merits mention at this point since it also employs average energy concepts [55,56,57,59,80]. Basically, SEA estimates the average flow of energy from one part of a

structure to another. For example, if there is energy input into one part of a complex structure, this method provides an estimate of how this energy flows into another part of the structure. Thus, it is possible to estimate average vibrational energy present in any part of the structure. Information of this type is frequently the only type of information it is possible to obtain about the response in an extremely complex structure containing a large number of undamped natural frequencies in 1 hertz. In so far as we know, nothing has yet been done to include the influence of statistical parameters; however, the work given in [59] suggests it might be possible to do this.

IX. Bound Determination

Sections V and VI discuss techniques for response estimation in lumped linear systems whose parameters take time independent uncertain values in a probabilistic setting; thus, it is possible by these techniques to obtain statistical information about a natural frequency and its normal mode, a frequency response, and a response. The techniques discussed in Section VII provide a deterministic frequency response in which account has been taken of the statistical properties of the parameters. This section assumes that all we know is a bound on the parameter values; we then will be interested in what can be said about bounds on the quantities of interest.

Needless to say the stability and control of systems in which only bounds are known on parameter values and the disturbances continues to be of interest to those working in automatic control, economic analysis, and stability theory [34,58,60,61,62]. Maximum response in structural systems has also been and continues to be of interest to structural engineers [58,61,62]; however, it is usually assumed the parameters in these systems are known exactly. Let us consider our problem from the point of view of those in automatic control.

Let us consider the simplest possible linear system in order to fix ideas. Assume the system is described by the first order linear differential equation in the single variable x:

$$\dot{x} = -(a + \Delta a)x + f(t) \ , \tag{9.1}$$

where f(t) is the external disturbance, a is a known positive constant, Δa is an unknown constant with $-\overline{\Delta a} \leq \Delta a \leq \overline{\Delta a}$, $\overline{\Delta a}$ is a known positive

constant, and $a-\Delta a > 0$. Let the initial condition on x be

$$x(0) = 0 \ . \tag{9.2}$$

We are interested in bounds on x for $t > 0$.

We consider as nominal solution of (9.1) the case where $\Delta a = 0$; thus let $\bar{x}$ satisfy

$$\dot{\bar{x}} = -a\bar{x} + f(t) \tag{9.3}$$

with initial condition $\bar{x}(0) = 0$. The solution of (9.3) is

$$\bar{x} = \int_0^t e^{-a(t-\tau)} f(\tau) d\tau \ . \tag{9.4}$$

Introduce the error function

$$y = x - \bar{x} \ . \tag{9.5}$$

Then y satisfies

$$\dot{y} = -(a + \Delta a)y - \Delta a\bar{x}(t) \ , \quad y(0) = 0 \ . \tag{9.6}$$

Define a Liapunov function V by

$$V(t) = \frac{1}{2} y^2(t) \ . \tag{9.7}$$

Then,

$$\begin{aligned} \dot{V} &= \dot{y}y \qquad (9.8) \\ &= [-(a + \Delta a)y - \Delta a\bar{x}]y \\ &= -(a + \Delta a)y^2 - \Delta a\bar{x}y \ ; \end{aligned}$$

Let $\Delta a = -\overline{\Delta a}$. We shall see shortly that this yields an upper bound on V which is equivalent to bounding the errror given by (9.5). Then, on taking the absolute value of the second term on the right of (9.8), we obtain

$$\dot{V} = -|a-\overline{\Delta a}|y^2 + \overline{\Delta a}|\overline{x}(t)||y| \tag{9.9}$$

which from (9.7) gives

$$\dot{V} \leq -2(a - \overline{\Delta a})V + \overline{\Delta a}|\overline{x}(t)|(2V)^{1/2} .$$

Next, let

$$\eta = V^{1/2} \tag{9.10}$$

which gives

$$\dot{\eta} = \frac{1}{2}\frac{\dot{V}}{V^{1/2}}$$

We now can write (9.9) as

$$\dot{\eta} \leq -(a - \overline{\Delta a})\eta + \frac{\overline{\Delta a}}{\sqrt{2}}|\overline{x}(t)| \tag{9.11}$$

Let us solve (9.11) for the equality:

$$\dot{\eta} = -\underline{a}\eta + \frac{\overline{\Delta a}}{\sqrt{2}}|\overline{x}(t)| \ , \quad \eta(0) = 0 \tag{9.12}$$

where $\underline{a} = a - \overline{\Delta a}$, obtaining

$$\eta = \frac{\overline{\Delta a}}{\sqrt{2}}\int_0^t e^{-\underline{a}(t-\tau)}|\overline{x}(\tau)|d\tau \ , \tag{9.13}$$

or, by (9.10) and (9.11),

$$V^{1/2} \leq \frac{\overline{\Delta a}}{\sqrt{2}}\int_0^t e^{-\underline{a}(t-\tau)}|\overline{x}(\tau)|d\tau \tag{9.14}$$

Finally, by appealing to the Schwarz inequality for integrals, we can write (9.14) as

$$V^{1/2} \leq \frac{\overline{\Delta a}}{\sqrt{2}}\left|\int_0^t e^{-2\underline{a}(t-\tau)}d\tau\right|^{1/2}\left|\int_0^t \overline{x}^2(\tau)d\tau\right|^{1/2} \tag{9.15}$$

Let us examine this expression.

First, the first integral on the right can be evaluated as

$$\left|\frac{1}{2\underline{a}}\left(1 - e^{-2\underline{a}t}\right)\right|^{1/2} . \tag{9.16}$$

Second, the left equals $\frac{|y(t)|}{\sqrt{2}}$, by (9.7). Hence (9.15) reduces to

$$|y(t)| \leq \frac{\overline{\Delta a}}{\sqrt{2(a-\overline{\Delta a})}}\left[1 - e^{-2(a-\overline{\Delta a})t}\right]^{1/2}\left[\int_0^t \overline{x}^2(\tau)d\tau\right]^{1/2} \tag{9.17}$$

At $t = 0$, we get $|y(a)| = 0$, as we should. The term (9.16) increases to unity as time increases. The term

$$\left[\int_0^t \overline{x}^2(\tau)d\tau\right]^{1/2} \tag{9.18}$$

will increase with increasing time; if $\overline{x}^2(t)$ approaches 0 as t approaches ∞, then (7.18) will increase to a positive limit. The bound is proportional to

$$\frac{\overline{\Delta a}}{\sqrt{2(a-\overline{\Delta a}}} ,$$

indicating the larger $\overline{\Delta a}$, which determines the bounds on a, the larger $|y(t)|$. We note that the bound on $|y(t)|$ increases with time using (9.16); (9.14) will provide a smaller bound, of course. In all events, equation (9.5) makes it possible to assert that x(t) lies within the bounds

$$\overline{x}(t) \pm |y(t)| . \tag{9.19}$$

Thus, the technique described enables us to put bounds on the response given bounds on the system parameter.

The example employed above is for a first order linear ordinary differential equation. References [ ] suggest that the technique can be

extended to vector differential equations of this type. However, the details will have to be worked out to determine if this promising technique can be put to practical use, since it may turn out that the smallest possible bounds are too large to be of practical value.

X. Physics

The physicists have for a long time been concerned with the vibrational properties of disordered systems. In particular, they have been interested in crystals in which disorder is present (See [63-75]). One type of disorder is called "substitutional disorder"; here one or more atoms in a regular crystal are replaced with another atom or atoms different from those in the crystal (while the organization of the crystal is not changed). The other type of disorder is called "topological disorder", in this case the basic organization of the structure is changed. A moments reflection will indicate that our main concern is with the first type of disorder, where member properties may change but not the structural organization. For the second type to occur, structural members would have to be removed, added or rearranged differently. Much progress has occurred in dealing with substitutional disorder; much more modest gains have been made for topological disorder. It might be hoped that much of what the physicist has done could be adopted in toto in studying the response in substitutionally disordered structures. Based on a relatively short study, this does not appear to be the case because of different interests.

We are interested in natural frequencies, normal modes, frequency response, impulse function, etc. in systems with a relatively small number of degrees of freedom as a rule. By and large, the physicist is interested in estimating the number of natural frequencies in a specified frequency interval; to do this, a frequency spectrum must be estimated. Hence, they are interested in the effect of various degrees of disorder on this spectrum from which optical, thermodynamic,

electrical, etc. properties are obtained. Thus, our interests are very different from theirs.

The early work of Hori and Asaki which introduced the method of transfer matrices [68,69] did provide a technique that has proved useful when dealing with chain like structures. However, mechanical structures do not usually possess organizational regularity as in crystals. Hence, as noted above, transfer matrices have a limited range of application.

References [73,74] indicate how a Green's function (impulse function) can be employed to derive information about the frequency spectra. So far, we have not been able to determine how this technique might apply to structures. Nonetheless, detailed study may reveal there are possibilities that have been overlooked in this brief survey.

The early work of Born and Brillouin [3,6] on the effect of experimental errors on the motion of classical mechanics systems is of course classic and is worth reading just for cultural reasons.

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Appendix A

Zarghame's method with mass variability included.

Assume the mass matrix M can be written in the form

$$M = \underline{M} + \sum_{1}^{m_1} Y_j \underline{M}_j \tag{1A}$$

where $\underline{M}$ is the mass matrix when all structural numbers take on their mean masses, and $\underline{M}_j$ the mean stiffness matrix of the j^{th} element. Now some of the Y_j may correspond to some of the X_i in (14). Hence, to include this possibility, we set

$$\{W_1, \ldots, W_{m_2}\} = \{X_1, \ldots, X_m\} + \{Y_1, \ldots, Y_{m_1}\} \tag{2A}$$

with $m_2 \leq m + m_1$. We thus regard

$$\begin{aligned} K &= \underline{K}(W_1, \ldots, W_{m_2}) \\ M &= \underline{M}(W_1, \ldots, W_{m_2}) \end{aligned} \tag{3A}$$

Equation (19) is replaced by

$$\ddot{Mq} + Kq = 0 \tag{4A}$$

where q is the same (nx1) column vector as in (20). Equation (23) is replaced by

$$(K - \omega^2 M)\alpha = 0 \tag{5A}$$

and α is the same (nx1) column vector as in (22).

We apply the same procedure as employed before to obtain the partial derivatives of ω_r and α_r with respect to the W_i, replacing the orthogonality conditions (26) by

$$\alpha_r^T M \alpha_s = 0 \ , \quad \alpha_r^T K x_s = 0 \quad \text{if} \quad s \neq r$$

$$\alpha_r^T M \alpha_r = 1 \ , \quad \alpha_r^T K \alpha_r = \omega_r^2 \ . \tag{6A}$$

We summarize the results for only ω_r:

$$\frac{\partial \omega_r}{\partial W_i} = \frac{1}{2w_r} \left[\alpha_{r-1}^T K_i \alpha_r - \omega_r^2 \alpha_{r-1}^T M_i \alpha_r \right] \tag{7A}$$

$$\beta_{ri}^{(k)} = \frac{\alpha_{k-1}^T K_i \alpha_r - \omega_r^2 \alpha_{k-1}^T M_i \alpha_r}{\omega_r^2 - \omega_k^2} \ , \quad k = r$$

$$\beta_{rj}^{(r)} = - \frac{\alpha_{r-1}^T M_i \alpha_r}{2} = - \frac{\alpha_{r-1}^T K_i \alpha_r}{2\omega_r^2} + \frac{1}{\omega_r} \frac{\partial \omega_r}{\partial W_i} \tag{8A}$$

$$\frac{\partial^2 \omega_r}{\partial W_i \partial W_j} = \frac{1}{2\omega_r} \left[-2 \frac{\partial \omega_r}{\partial W_i} \frac{\partial \omega_r}{\partial W_j} + \Sigma' (\beta_{ri}^{(k)} \beta_{kj}^{(r)} - \beta_{ki}^{(r)} \beta_{rj}^{(k)}) (\omega_k^2 - \omega_r^2)^{-1} \right.$$

$$\left. + 4\omega_r \left(\frac{\partial \omega_r}{\partial W_i} \beta_{rj}^{(r)} + \frac{\partial \omega_r}{\partial \omega_j} \beta_{ri}^{(r)} \right) \right]$$

END

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