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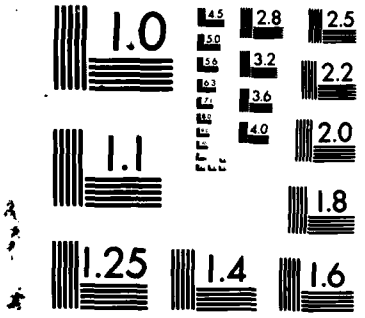

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by

H. D. Mittelmann

Technical Report No. 90
January 1986

Arizona State University
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DEPARTMENT OF MATHEMATICS

ARIZONA STATE UNIVERSITY

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The Investigator has employed multi-grid methods to solve an important class of boundary value problems in two space dimensions that arises in VLSI device simulation. Methods for the numerical solution of bifucation problems are also under development. Six papers have been produced so far under this effort.

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Multi-grid Continuation and Spurious Solutions for Nonlinear Boundary Value Problems

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Abstract Recently the author together with R. Bank has developed, implemented and successfully applied a continuation technique for the numerical solution of parameter-dependent nonlinear elliptic boundary value problems. The method was integrated into an existing multi-grid package based on an adaptive finite element discretization. We present the continuation method and proves an important theoretical result for the corrector iteration. For the Bratu problem $-\Delta u = \lambda e^{u}$ on the square with homogeneous Dirichlet conditions we show how spurious solutions may be encountered while computing relevant solutions, how the program handles those and how it allows to detect them.

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1. Introduction

In the following we consider the parameter-dependent nonlinear problem

$$G(u,\alpha) = 0, \tag{1.1}$$

where $G : X^m \times \mathbb{R}^p \to X^m$, X a suitable function space and α a vector of real parameters. (1.1) will typically represent a parameter-dependent nonlinear elliptic system of dimension m and order two. Important examples are the VLSI device simulation equations

$$\begin{aligned} g_1(u,v,w) &\equiv -\Delta u + e^{u-v} - e^{w-u} - k_1 = 0 , \\ g_2(u,v,w) &\equiv \nabla\cdot\mu_n e^{u-v} \nabla v + k_2 = 0 , \\ g_3(u,v,w) &\equiv -\nabla\cdot\mu_p e^{w-u} \nabla w + k_2 = 0 . \end{aligned} \tag{1.2}$$

In this case the parameters α enter the boundary conditions if, for example, current-voltage characteristics are to be determined (cf.[3]).

The solution manifold $G^{-1}(0)$ of (1.1) may have a very complex structure. In practical applications it is frequently desirable and sufficient to compute certain cross-sections of it, i.e. all parameters α are kept fixed at certain values and for simplicity suppressed except one which we will call λ. A simple graphical representation of this cross-section may be obtained if a functional of the solution is depicted versus λ. It should be noted, however, that usually in these applications such functionals are in fact important pieces of information, such as the currents for (1.2).

We assume that a vector

$$R(u(\lambda)), \qquad R : X^m \to \mathbb{R}^k \tag{1.3}$$

of differentiable functionals is given and that most of the important information about the problem can be obtained from the $k \times p$ diagrams

$$R_i = R_i(u(\alpha_j)),\ i = 1, \ldots, k,\ j = 1, \ldots, p \tag{1.4}$$

and the corresponding solutions.

The solutions $u(\lambda)$, $\lambda = \alpha_j$, in (1.4) and from them the curves $R_i(u(\lambda))$ usually have to be computed in a continuation fashion. Starting from a known solution $u(\lambda_0)$ another solution is computed by utilizing a predictor-corrector scheme. Frequently an initial solution to start this continuation procedure is not known and has itself to be determined from an approximate solution. Along the solution curves singular points have to be expected and our goal is to provide methods to locate these points, to overcome turning points and to switch branches at bifurcation points.

In the following section we will outline a continuation algorithm that has been successfully used for the scalar case ($m = 1$ in (1.1)). It provides an efficient and robust method and has been implemented for a general class of second order elliptic boundary value problems discretized by finite elements in [2]. The continuation method is applied on a coarse triangulation of the given domain and fine grid approximations at arbitrarily specified points are obtained by multi-grid.

2. The Continuation Method

In the following we assume that a choice of the parameter $\lambda = \alpha_j$ in (1.1), (1.3) and also of the functional R_i in (1.4) has been made and denote (1.1) by

$$G(u,\lambda) = 0, \qquad G : X^m \times \mathbb{R} \to X^m, \tag{2.1}$$

and R_i by r.

The parametrization of the solution u of (2.1) by λ and thus of r by λ need not be possible in general. Let s denote the arclength along a solution curve $(u,\lambda) = (u(s), \lambda(s))$ of (2.1). Then under sufficient smoothness and regularity assumptions this latter dependence is differentiable and thus ($G_u^0 = G_u(u_0, \lambda_0)$ etc.).

$$G_u^0 \dot{u}_0 + G_\lambda^0 \dot{\lambda}_0 = 0, \tag{2.2a}$$

$$\|\dot{u}_0\|^2 + \dot{\lambda}_0^2 = 1 \tag{2.2b}$$

in a solution $(u_0,\lambda_0) = (u(s_0), \lambda(s_0))$ of (2.1). Here, subscripts stand for partial derivatives and a dot for differentiation with respect to s.

In singular points (u_0,λ_0) of (2.1) the linear operator G_u^0 in (2.2a) is singular and, for example, a solution by Newton's method breaks down. This singularity may be removed or reduced by augmenting (2.1) by a normalizing equation, a well-known example of which is

$$N_\sigma(u,\lambda,\sigma) \equiv \Theta\, \dot{u}_0(u-u_0) + (2-\Theta)\dot{\lambda}_0(\lambda-\lambda_0) - \delta\sigma = 0 \tag{2.3}$$

which approximates (2.2b). σ is called the pseudo-arclength parameter. θ satisfying $0 \leq \theta \leq 2$ may be used as a weighting parameter.

Instead of (2.3) we propose to use

$$N_r(u,\lambda,\sigma) \equiv \theta\dot{r}_0(r-r_0) + (2-\theta)\dot{\lambda}_0(\lambda-\lambda_0) - \delta\sigma = 0, \tag{2.4}$$

where $r_0 = r(u_0)$, $\dot{r}_0 = \left(\frac{d}{d\sigma} r(u)\right)_{\sigma = \sigma_0}$ and r denotes the above functional. From (2.4) we see that the three values $\theta = 0,1,2$ are special in the sense that for $\theta = 0(2)$ the augmenting equation characterizes points with a fixed $\lambda(r)$ - value, while for $\theta = 1$ this point lies on a hyperplane orthogonal to the solution arc in (r_0,λ_0).

This suggests to use target values in the two (physically relevant) variables r and λ. Such a strategy seems appropriate as long as starting from a point on the solution curve another point corresponding to a suitably chosen target value can be computed in a small number of (corrector) iterations. This was achieved for the scalar case of (2.1) and the choice

$$r(u) = \|u\|. \tag{2.5}$$

The following class of nonlinear boundary value problems in the plane

$$\begin{aligned} -\nabla \cdot a(x,y,u,\nabla u,\lambda) + f(x,y,u,\nabla u,\lambda) &= 0 && \text{in } \Omega, \\ u &= g_1(x,y,\lambda) && \text{on } \partial\Omega_1, \\ a \cdot n &= g_2(x,y,u,\lambda) && \text{on } \partial\Omega_2 = \partial\Omega \setminus \partial\Omega_1, \end{aligned} \tag{2.6}$$

was considered. Here, Ω is a connected domain in R^2 with boundary $\partial\Omega$, n is the

unit normal vector on $\partial\Omega$, $a = (a_1,a_2)^T$ and a_1,a_2,f,g_1,g_2 are given scalar functions.

In [2] the program PLTMG, an implementation of an adaptive finite element multi-grid method combined with the continuation technique presented below is described in detail. In addition to some theoretical results on the continuation method [12, 13] contain numerical results for several problems from the applications including problems with turning-points, symmetry-breaking and other bifurcation points as well as parameter-switching for cases depending on several ($p > 1$ in (1.1)) parameters. [13] also contains a comparison with the classical pseudo-arclength continuation method. Other problems to which the program has been applied successfully are Steklov eigenvalue problems for which the Dirichlet condition in (2.6) is parameter-dependent.

The predictor step which is crucial for the performance of the method determines predicted values of the form

$$\begin{pmatrix} u_p \\ \lambda_p \end{pmatrix} = \begin{pmatrix} (1+\gamma)u_0 + \beta\dot{u}_0 \\ \lambda_0 + \alpha\dot{\lambda}_0 \end{pmatrix} \tag{2.7}$$

such that α,β,γ satisfy the three equations

$$\begin{aligned} N(u_p,\lambda_p,\sigma) &= 0, \\ (u_0,G(u_p,\lambda_p)) &= 0, \\ (\dot{u}_0,G(u_p,\lambda_p)) &= 0 \end{aligned} \tag{2.8}$$

in a least-squares sense. Here it is assumed that X is a Hilbert space and (. , .) denotes its inner product.

In the corrector iteration the augmented system

$$F(y) = \begin{pmatrix} G(u,\lambda) \\ N(u,\lambda,\sigma) \end{pmatrix} = 0, \quad y^T = (u^T,\lambda) \tag{2.9}$$

is solved by Newton's method starting in (u_p,λ_p). The restriction of the problem (2.1) to the (r,λ)-plane in case $N = N_r$ is used in (2.8) introduces an additional singularity in exceptional points.

If in a solution (u_0, λ_0) simultaneously $\dot{r}_0 = 0$ and $\dot{\lambda}_0 = 0$ then the normalizing condition (2.4) is not well-defined. This situation occurs in symmetry-breaking pitchfork bifurcation points, see [11]. We propose to switch temporarily to the augmenting equation (2.3) which, as explained in [12] may be viewed as asymptotically equivalent to a version of the generalized inverse iteration ([10]) exploiting symmetries in such bifurcation points. The precise choice of N is given by (cf. [13]).

$$\begin{aligned} &|\dot{r}_0| < \varepsilon \, \|\dot{u}_0\| : N = N_\sigma, \ \theta \neq 0, \\ &|\dot{r}_0| > (1-\varepsilon) \, \|\dot{u}_0\| : N = N_r, \ \theta \neq 0, \ \beta = 0, \\ &\varepsilon \, \|\dot{u}_0\| \leq \dot{r}_0 \leq (1-\varepsilon) \, \|\dot{u}_0\| : N = N_r, \ 0 \leq \theta \leq 2. \end{aligned} \tag{2.10}$$

3. The Corrector Iteration

A point (u_p, λ_p) computed by the predictor of the previous section is used as a starting guess for an iterative solution of the augmented system (2.9). This is done by Newton's method and thus the Jacobian

$$F_y = \begin{pmatrix} G_u & G_\lambda \\ N_u & N_\lambda \end{pmatrix} \tag{3.1}$$

has to be evaluated and inverted.

While (2.10) prescribes which N to choose it does not completely define the augmenting equation. The weighting parameter θ still has to be picked and we assume that r respectively λ is not used as parameter to continue past a turning point in r respectively λ. This yields $\theta \neq 0$ if $\dot{\lambda}_0 = 0$ and $\theta \neq 2$ if $\dot{r}_0 = 0$. Under these assumptions it was shown in Proposition 4.2 of [13] that F_y is regular except in bifurcation points.

When computing in a neighborhood of a bifurcation point the growth of $\|F_y^{-1}\|$ is crucial for the convergence of Newton or other iterative methods used as corrector, cf. [7]. We give here the detailed proof of Proposition 4.3 in [13]. Let locally near the simple bifurcation point $y_0 = (u_0,\lambda_0)$ solutions be parametrized by s such that without restriction of generality $y_0 = (u(0),\lambda(0))$. We assume in the following that $\dot{y}_0 = (\dot{u}(0),\dot{\lambda}(0))$ is a nondegenerate ray in the sense of [8].

Theorem 3.1 If the normalizing condition N is chosen according to (2.10) then on any of the branches passing through the simple bifurcation point $y_0 = (u(0),\lambda(0))$ of (2.1) for $\delta > 0$ sufficiently small

(3.2) $$\|F_y^{-1}(y(s))\| = O(|s|^{-1}),\ 0 < |s| < \delta.$$

Proof. As mentioned in [13] we have to generalize the proof of Theorem 1 in [8]. This proof covers the case $N = N_\sigma$ so that it remains to consider $N = N_r$. We note that the row-vector $a_0 = (N_u, N_\lambda)$ in (3.1) is not constant in this case, in fact

(3.3) $$a_0(s) = \left(\frac{\theta \dot{r}_0 u(s)}{\|u(s)\|},\ (2-\theta)\dot{\lambda}_0\right).$$

Let us further denote $H(y) = G(u,\lambda)$ and thus

$$H' = (G_u, G_\lambda) : \mathbf{R}^{n+1} \to \mathbf{R}^n, \qquad F_y = \begin{pmatrix} H' \\ a_0 \end{pmatrix}.$$

As in [8] we introduce the one-dimensional space $X_1 \subset \mathbf{R}^{n+1}$ spanned by $\dot{y}_0 = (\dot{u}_0, \dot{\lambda}_0)^T$ on one of the branches, the one-dimensional subspace $X_2 \subset \mathbf{R}^{n+1}$ such that the kernel $K(H'(y_0))$ satisfies $K = X_1 \oplus X_2$, and the $(n-1)$-dimensional subspace X_3 such that $\mathbf{R}^{n+1} = X_1 \oplus X_2 \oplus X_3$. Similarly let $Y_1 = \mathbf{R}$, $Y_3 = R(H'(y_0))$, where $R(A)$ denotes the range of the linear operator A, and let Y_2 be the one-dimensional subspace of $\mathbf{R}^n$ such that $\mathbf{R}^n = Y_2 \oplus Y_3$. We define the orthogonal projectors $Q : \mathbf{R}^n \to Y_3$ and $I - Q : \mathbf{R}^n \to Y_2$. For $0 < |s| < \delta$ we define the linear operator $B(s) : X_1 \times X_2 \times X_3 \to Y_1 \times Y_2 \times Y_3$ by

$$B(s)(x_1,x_2,x_3) = (a_0(s)(x_1+x_2+x_3),\ (I-Q)H'(y(s))(x_1+x_2+x_3),\ QH'(y(s))(x_1+x_2+x_3)).$$

$B(s)$ is equivalent to $F_y(y(s))$ in the sense that

$$F_y = C\,B\,D$$

where C and D are regular matrices independent of s. B(s) can be decomposed in a natural way into nine suboperators $B_{ij}(s) : X_i \to Y_j$, $1 \le i,j \le 3$. From the above definitions it follows that $B_{21}(0)$, $B_{22}(0)$, $B_{31}(0)$, $B_{32}(0)$ and $B_{23}(0)$ vanish identically. From (2.10) we conclude the crucial fact

$$a_0(0)\dot{y}_0 = \theta\dot{r}_0^2 + (2-\theta)\dot{\lambda}_0^2 \neq 0$$

This and the definition of X_3, Y_3 imply that $B_{11}(0)$ and $B_{33}(0)$ define bijections, while as in [8] it follows that $\dot{B}_{21}(0) = 0$ and that $\dot{B}_{22}(0)$ is a bijection between X_2 and Y_2. We can thus expand B(s) in a neighborhood of s = 0 as

$$B(s) = \begin{pmatrix} B_{11}(0) & B_{12}(0) & B_{13}(0) \\ s^2\tilde{B}_{21}(0) & s\dot{B}_{22}(0) & s\tilde{B}_{23}(0) \\ s\tilde{B}_{31}(0) & sB_{32}(0) & B_{33}(0) \end{pmatrix} + \text{h.o.t.}$$

where $B_{ij}(0)$, $\dot{B}_{ij}(0)$ and $\tilde{B}_{ij}(0)$ are linear operators from X_i into Y_j which are uniformly bounded with respect to s. Thus $\det B(s) = O(|s|)$ and hence $B^{-1}(s)$ can be written as

$$B^{-1}(s) = \begin{pmatrix} E_{11}(s) & s^{-1}E_{12}(s) & E_{13}(s) \\ sE_{21}(s) & s^{-1}E_{22}(s) & E_{23}(s) \\ sE_{31}(s) & E_{32}(s) & E_{33}(s) \end{pmatrix}$$

where similarly the E_{ij} are uniformly bounded and $E_{22}(0) = (\dot{B}_{22}(0))^{-1}$ is regular.

4. Spurious Solutions

It is well-known that discretized problems possess frequently many more solutions than the continuous problems. All discrete solutions that do not correspond to solutions of the continuous problem and disappear for $h \to 0$, h the discretization parameter, are called spurious or numerically irrelevant solutions. If we consider the simple model equation

$$-\Delta u = \lambda f(u) \tag{4.1}$$

then there are essentially two types of those solutions. One type bifurcating from $\lambda = \pm \infty$ in case $f(u)$ has zeros and the other bifurcating from $\|u\| = \infty$ if the growth of f is sufficiently strong. It might be argued that not too much effort should be spent on investigating those spurious solutions if the main interest is that of approximately solving the continuous problem. There are, however, at least two counter arguments.

First, in a multi-grid context where one hopes to be able to use rather crude coarse grids on which the continuation will be done, one has to be aware of the spurious solutions since for 'large' h they are present for 'small' values of $\|u\|$ respectively λ and thus may be more easily encountered. Second, it has been shown recently ([9]) that for the second type of spurious solutions some of these may bifurcate from a relevant branch. Thus they definitely will be encountered and it becomes necessary to distinguish between spurious and relevant solutions.

If we specialize (4.1) to

$$-\Delta u = \lambda e^{u} \quad \text{in } \Omega = [0,1]^n,\ u = 0 \text{ on } \partial\Omega \tag{4.2}$$

then the proof in [9] covers the case n = 1 and a uniform grid of an even number of points. The midpoint of [0,1] where the relevant solution attains its maximum is then not a grid point. The discrete solution attains its maximum in the two neighboring points and from this branch bifurcate two branches attaining their maximum in either of these points.

We had observed these solutions and their bifurcation previously when numerically solving (4.2) for n = 2 by discretizing with the standard difference method on a corresponding grid. We also have applied a multi-grid method to compute spurious solutions ([14]). Since our goal is not to contribute to the knowledge about the spurious solutions but rather to provide a general purpose and if possible foolproof program to solve a general class of problems, it is natural to investigate the above situation with the program PLTMG.

Our expectation is that the implemented continuation method allows us to follow the spurious solution branches because these are genuine solutions of the discrete problems. On the other hand the multi-grid process using several discretizations simultaneously could help to identify spurious solutions. But additionally all the other features offered by the program as the adaptive refinement and the error estimators will have their impact on the solution process respectively may give other hints about which solutions are relevant.

Since the program uses a finite element discretization with piecewise linear elements we try to come close to the above discretization. The unit square is subdivided into nine subsquares of length h = 1/3 and then all these squares are divided into two triangles by the SW - NE diagonal. This way the Laplace operator will be discretized in the same way as by the five point star difference method. The right-hand side, however, in the finite element context discretized by a quadrature formula will be different from the pointwise

discretization of the difference method. This will introduce a bias in the SW-NE direction and we do not expect the relevant solution to attain its maximum in the four interior nodes.

While the numerical results presented in the following, of course, make no statements about the mathematical properties of the problem they provide some valuable insight into the practical use of the program and the numerical solution in the presence of spurious solutions.

The branch starting at $\lambda = 0$, $r = \|u\| = 0$ for the above discretization with four nodes attains its maximum, as expected, in the points (1/3, 1/3), (2/3, 2/3) and has two equal but smaller values at (1/3, 2/3), (2/3, 1/3). When following this curve beyond the limit point another sign-change of det (G_u) is encountered. Even a tracing of the neighborhood of this point with small continuation steps shows an abrupt change of the sign. When asked to locate this point the program ends up on a different branch and even a subsequent branch switching does not lead back to the relevant branch. This suggests that instead of bifurcation a perturbed bifurcation seems to be present. This is not so relevant here, since the fact remains that a sign-change seems to indicate bifurcation and naturally a spurious branch is found. Table 4.1 shows the continuation to and the location of the 'bifurcation' point. In the tables NI denotes the number of corrector iterations and δ is a quantity that changes sign at simple bifurcation points (cf. [2]).

In the following we present multi-grid results for four different cases. First we have followed the relevant branch up to $r = 4.5$ respectively $r = 5$ and computed a 3-grid solution for fixed norm with uniform and adaptive refinement. Table 4.2 contains only the refinement results.

NI	λ	"u"	$\dot{\lambda}$	$\dot{r}$	$\det(G_u)$	δ
Continue to λ = .1 then to r = 1, r = 3, r = 3.44, r = 3.46.						
1	0.000E+00	0.000E+00	0.995E+00	0.000E+00	0.492E 5	0.200E+01
1	0.100E+01	0.116E+00	0.992E+00	0.128E+00	0.276E 5	0.180E+01
1	0.390E+01	0.100E+01	0.244E+00	0.970E+00	0.190E 3	0.129E+00
2	0.156E+01	0.300E+01	-0.756E+00	0.653E+00	-0.623E 3	0.592E+00
2	0.111E+01	0.344E+01	-0.669E+00	0.741E+00	-0.185E 2	0.136E-01
1	0.109E+01	0.346E+01	-0.664E+00	0.745E+00	0.176E 2	-0.130E-01
Determinant changes sign; find bifurcation point by secant method on δ.						
1	0.110E+01	0.345E+01	-0.121E-01	0.130E-01	-0.113E -1	0.809E-05
1	0.110E+01	0.345E+01	0.192E+00	-0.215E+00	0.123E 0	-0.883E-04
1	0.110E+01	0.345E+01	0.497E-06	-0.591E-03	0.482E -2	-0.369E-09

Table 4.1 Continuation along relevant branch, detection of spurious branch.

A posteriori error estimates based on solving local problems with higher-order elements [5] are evaluated by the program and in Table 4.4 we give the number of estimated correct digits in three different norms.

Following the irrelevant branch that attains its maximum at (1/3, 1/3) to r = 5.5 and again refining uniformly and adaptively yielded the results in Table 4.3. We see that for the adaptive refinement it seems impossible to conclude from the error estimations alone that the solution is irrelevant, while for the uniform refinement the error was estimated as bigger than the solution resulting in a negative number of estimated correct digits. In this latter case also the solution on the finest grid did not attain its maximum at (1/3, 1/3) but closer to the center of the domain; i.e. the refinement had led onto a different spurious branch.

NI	λ	$\|u\|$	$\dot{\lambda}$	$\dot{r}$	$\det(G_u)$		δ
			Level 1 solution at r = 5.				
2	0.264E+00	0.500E+01	-0.255E+00	0.962E+00	0.865E	4	0.891E+00
			Refine uniformly.				
			Level 2 solution, no. of vertices 49.				
6	0.296E-01	0.500E+01	-0.566E-01	0.993E+00	0.530E	5	0.885E+00
			Level 3 solution, no. of vertices 169.				
8	0.836E-02	0.500E+01	-0.230E-01	0.990E+00	0.626E	5	0.251E+00
			Level 1 solution at r = 4.5.				
2	0.431E+00	0.450E+01	-0.379E+00	0.921E+00	0.393E	4	0.824E+00
			Refine adaptively.				
			Level 2 solution, no. of vertices 51.				
5	0.598E-01	0.450E+01	-0.115E+00	0.983E+00	0.334E	5	0.465E+00
			Level 3 solution, no. of vertices 265.				
9	0.402E-02	0.450E+01	-0.315E-01	0.862E+00	0.482E	5	0.504E-03

Table 4.2 Results of refinement on relevant branch.

One of the possible ways provided by the program to depict the results graphically is to generate a histogram. Over a square grid numbers from 0 to 9 are printed covering uniformly the range of the solution values from minimum to maximum. In Figures4.1 and 4.2 we see which solutions were computed on the coarsest and on the finest grid for uniform and adaptive refinement.

NI	λ	$\|u\|$	$\dot{\lambda}$	$\dot{r}$	det(G_u)		δ
			Level 1 solution at r = 5.5.				
2	0.560E-01	0.550E+01	-0.772E-01	0.994E+00	-0.839E	5	0.219E+01
			Refine uniformly.				
			Level 2 solution, no. of vertices 49.				
9	0.328E-01	0.550E+01	-0.367E-01	0.508E+00	-0.361E	5	-0.117E+00
			Level 3 solution, no. of vertices 169.				
10	0.474E-02	0.550E+01	0.112E-01	-0.990E+00	0.615E	5	0.182E+00
			Refine adaptively.				
			Level 2 solution, no. of vertices 41.				
4	0.968E-02	0.550E+01	-0.194E-01	0.994E+00	-0.824E	5	0.691E+00
			Level 3 solution, no. of verticies 174.				
7	0.111E-03	0.550E+01	-0.386E-03	0.999E+00	0.638E	5	0.776E-01

Table 4.3 Results of refinement on spurious branch.

Norm	Relevant Solution		Spurious Solution	
	Uniform	Adaptive	Uniform	Adaptive
H^1	0.526	0.593	-0.955	0.448
L^2	1.381	1.268	-0.312	1.198
L^∞	0.965	1.186	-1.099	1.183

Table 4.4 Numbers of estimated correct digits.

000000000000000	000000000000000	000000000000000
000000000000000	000000000000000	000000000000000
000011111111100	000011111100000	000000000000000
000122222222100	000112222111000	000011111000000
001233333332100	001123332211000	000111111110000
001234444432100	001234443321100	001122222110000
001246655432100	012345765321100	001233332211000
001367775432100	012357985432100	001244433211000
002578876432100	012358975422100	012357643211000
014689876432100	012356764321100	012359743211000
024688764332100	001234543221000	001345543211000
024666532221000	001223332211000	001233322110000
024444211110000	000111221110000	001112211100000
022221000000000	000000000000000	000001100000000
000000000000000	000000000000000	000000000000000

Figure 4.1 Histograms of the spurious solution on the coarsest grid and for uniform and adaptive refinement.

000000000000000	000000000000000	000000000000000
000001111122220	000000000000000	000000000000000
000023333444420	000011111110000	000000111100000
000234455666420	000111222211000	000011111111000
002356677886420	001122333221100	000112222211000
013467889986410	001123454322100	000122333221100
013468899975310	001234576432100	001123454321100
013578999875310	001235797532100	001223595321100
013579998864310	001234675432100	001123564321100
014689988764310	001223454321100	001123333221000
024688776653200	001122333221100	000112222211000
024666554432000	000112222111000	000111121110000
024444333320000	000011111110000	000001111000000
022221111100000	000000000000000	000000000000000
000000000000000	000000000000000	000000000000000

Figure 4.2 Histograms of the relevant solution on the coarsest grid and for uniform and adaptive refinement.

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