

AD-A173 748

SOLUTION TO THE ALGEBRAIC RICCATI EQUATION FOR
PARABOLIC SYSTEMS(U) JOHNS HOPKINS UNIV BALTIMORE MD
DEPT OF ELECTRICAL ENGINEERIN. L R RIDDLE ET AL

1/1

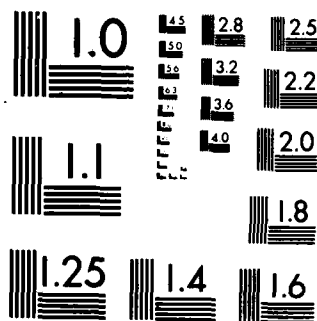
UNCLASSIFIED

22 OCT 86 JHU/EE-86/21 N00014-85-K-0255

F/G 12/1

NL





MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

Solution of the Algebraic Riccati Equation
for Parabolic Systems

By
J. K. S. CHAN

UD-6473 748

RECEIVED
JAN 10 1974
LIBRARY
OF THE
JOHNS HOPKINS UNIVERSITY

**Solution to the Algebraic Riccati Equation
for Parabolic Systems**

Laurence R. Riddle and Howard L. Weinert

Report JHU/EE-86/21

DTIC
ELECTE
S OCT 31 1986 D
B

This work supported by the Office of Naval Research under Contract N00014-85-K-0255.

DISTRIBUTION STATEMENT A
Approved for public release
Distribution Unlimited

Solution to the Algebraic Riccati Equation for Parabolic Systems *

Laurence R. Riddle and Howard L. Weinert

Department of Electrical Engineering

The Johns Hopkins University

Baltimore, MD 21218

ABSTRACT

This paper presents an analytical solution to the operator algebraic Riccati equation (ARE) for selfadjoint parabolic systems. The solution to the operator ARE is important in the design of the steady-state, on-line filter for estimating the system's states. This analytical solution is derived by considering the operator analog of Potter's method of using the Hamiltonian system's eigenvectors and eigenvalues to solve a finite-dimensional ARE. As an example of using this analytical solution, the steady-state filtering error covariance for the 2-D heat equation is studied.

* This work supported by the Office of Naval Research under Contract N00014-85-K-0255.



1. Introduction

In this paper, we derive an analytical solution to the operator algebraic Riccati equation (ARE) on a Hilbert space H :

$$AP + PA^* + Q = PRP \quad (1.1)$$

when $-A$ is a strongly positive (coercive) selfadjoint operator that generates a continuous semigroup, Q is positive-definite, bounded and commutes with A , and R is bounded and nonnegative-definite. We also assume that the domain of A^2 is dense in H , and is contained in the domain of A , and that the positive square root of Q commutes with A . In the context of distributed parameter filtering and control, one needs a solution to Eq. (1.1) that is bounded, nonnegative, selfadjoint and that maps the domain of A into itself. Gibson [3] has shown that under the above assumptions, Eq. (1.1) has a unique solution with these properties.

An example where one would solve Eq.(1.1) with the above assumptions on A , Q and R is the design of a steady-state filter to estimate a process governed by the heat equation:

$$\frac{\partial}{\partial t} u(x,t) = \nabla^2 u(x,t) + \epsilon(x,t), \quad x \in \Omega \subset \mathbb{R}^3, \quad t \geq 0 \quad (1.2)$$

$$\frac{\partial}{\partial n} u(x,t) = 0, \quad x \in \partial\Omega$$

where $\frac{\partial}{\partial n}$ denotes the outward normal derivative, and the observations are

$$y_j(t) = \int_{\Omega} C_j(x) u(x,t) dx + w_j(t), \quad j = 1, \dots, N$$

In Eq (1.2), $\epsilon(x,t)$ is a random process that is white in space and time with

constant intensity Q , $w_j(t)$ is a scalar white noise process with intensity R^{-1} , and $A = \nabla^2$. Several applications using this model to describe the temperature distribution of heated metals have been reported [1],[9]. Numerical examples of calculating the steady-state filtering error covariance P for this model are given in [11].

2. Solving the ARE

In this section we will solve Eq. (1.1) by using the operator analog to the Potter method [10] of solving the ARE in finite dimensions. The Potter method of solving the ARE is summarized as follows: let the Hamiltonian matrix L be defined as

$$L = \begin{bmatrix} A & Q \\ R & -A^* \end{bmatrix}$$

and diagonalize L via

$$LM = M\Lambda$$

where

$$\Lambda = \text{diag}(\lambda_1, \dots, \lambda_{2n}), \lambda_1, \dots, \lambda_n > 0$$

and

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

Then

$$P = M_{11}M_{21}^{-1}$$

is a nonnegative symmetric solution to the ARE, assuming M_{21} is invertible.

Using this approach in an operator setting, we will consider first a formal solution to the operator ARE. Let $Q = I$ and consider the following operator equation:

$$\begin{bmatrix} A & I \\ R & -A \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \lambda \quad (2.1)$$

where m_1, m_2 and λ are operators. It is easy to verify that the choices

$$m_1 = I, m_2 = -A + (A^2 + R)^{\frac{1}{2}}$$

$$\lambda = (A^2 + R)^{\frac{1}{2}}$$

satisfy Eq. (2.1) and thus

$$P_1 = (-A + (A^2 + R)^{\frac{1}{2}})^{-1} \quad (2.2)$$

satisfies Eq. (1.1) in a formal sense. If $Q \neq I$ then

$$P = S(-A + (A^2 + R)^{\frac{1}{2}})^{-1}S \quad (2.3)$$

satisfies Eq. (1.1), where $S^2 = Q$ and S is positive-definite.

To justify the formal solution given by Eq. (2.3) we need to show that P is bounded, nonnegative, selfadjoint, and maps the domain of A into itself. Since $-A > 0$ it follows that $A^2 + R$ has a unique positive square root that is selfadjoint and that the domain of A^2 is dense in the domain of $(A^2 + R)^{\frac{1}{2}}$ [7, p. 281]. Therefore $-A + (A^2 + R)^{\frac{1}{2}}$ is strongly positive and selfadjoint with dense domain:

$$D(-A + (A^2 + R)^{\frac{1}{2}}) = D(-A) \cap D((A^2 + R)^{\frac{1}{2}}) \subset D(A^2) \subset D(A) \quad (2.4)$$

where $D(\cdot)$ denotes domain; so that P_1 as defined in Eq. (2.2) is bounded [4, p. 209]. Furthermore, P_1 is selfadjoint and nonnegative because $(-A + (A^2 + R)^{\frac{1}{2}})$ is [7, p. 272]. We will prove that P_1 maps the domain of A into the domain of

A.

The range of P_1 is contained in the domain of A by Eq. (2.4), and so it is sufficient to show that the domain of P_1 contains the domain of A . Since P_1 is bounded and necessarily closed (being selfadjoint), it follows [7, p. 269] that the domain of P_1 is H , and hence contains the domain of A .

Since we have assumed that the operator S commutes with A , if $x \in D(A)$ then $Sx \in D(A)$, hence P in Eq. (2.3) also maps $D(A)$ into $D(A)$. We have thus shown that the operator defined in Eq. (2.3) is the correct solution to Eq. (1.1) needed for filtering and control applications.

3. Numerical Considerations and Example

The analytical approach to solving the operator ARE presented in this chapter provides a computationally faster way of implementing the optimal gain for filtering and control applications. In our approach, finite-dimensional approximations to the operators in Eq (1.1) are done after the analytical solution has been obtained, resulting in a computational complexity of $O(6n^3)$, where n is the size of the matrix that approximates the operator A , whereas in other techniques [2] one approximates Eq (1.1) by a matrix ARE and then solves this equation using algorithms developed for the finite-dimensional case which require approximately $O(75n^3)$ operations [8]. We remark that a similar approach (with the same computational complexity $O(6n^3)$) to the finite-dimensional ARE has been considered [5],[6].

Another important difference between the analytical approach used in this chapter and those using a high order finite-dimensional approximation to Eq.

(1.1), is that the implementation of Eq. (2.3) calls for approximations of A and A^2 separately, whereas in approximating Eq. (1.1) one is implicitly using $\bar{A}^2$, where $\bar{A}$ is a finite-dimensional approximation to A , as an approximation to A^2 . The fact that $\bar{A}^2$ may not be a good approximation to A^2 suggests that our approach may be more accurate.

The steady-state filtering error covariance for estimating the temperature profile of a heated square aluminum slab was calculated using the results obtained in the previous section. These calculations were based on the following model for the variation in temperature $u(x,y,t)$ above an assumed known ambient temperature T_0 :

$$\frac{\partial}{\partial t} u(x,y,t) = \alpha \nabla^2 u(x,y,t) + \epsilon(x,y,t) \quad 0 \leq x, y \leq L$$

with boundary conditions

$$u(0,y,t) = u(L,y,t) = u(x,0,t) = u(x,L,t) = 0$$

where

$$L = 1 \text{ meter}, \alpha = 5 \times 10^{-5} \text{ meter}^2/\text{second}$$

and the observations are

$$y(t) = u(.5,.5,t) + w(t)$$

The input $\epsilon(x,y,t)$ is assumed to be white in space and time with intensity

$$Q = 1(\text{degreeC})^2/(\text{meter}^2) \text{ second}$$

and the observation noise $w(t)$ is white with intensity

$$R^{-1} = 10^{-2}(\text{degreeC})^2/\text{second}$$

In Eq. (2.3), a 10×10 finite-difference approximation to the operators A and

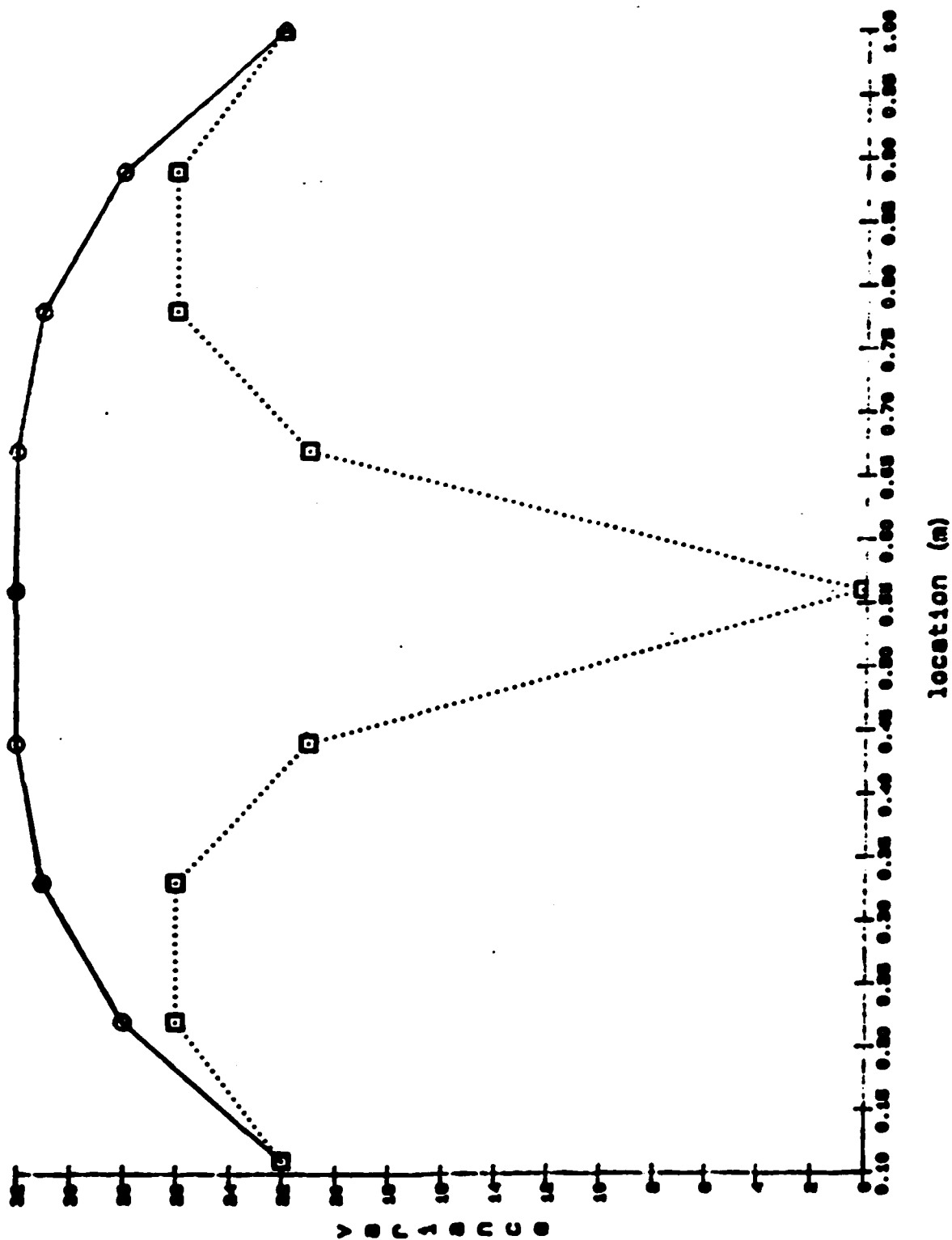
A^2 was used. Figure 1 shows a cross-section of the variances of the temperature variations $u(x,5,t)$ and the filtering error $\bar{u}(x,5,t)$. It is evident from Figure 1 that the unobserved temperatures more than 10 cm away from the sensor are not being estimated very effectively. This result places some doubt on the possibility of estimating, with few sensors, the unobserved temperature distribution of heated metals in the presence of spatially white process noise.

References

- [1] M. B. Ajinkya, M. Köhne, M. F. Mäder, and W. H. Ray, 'The experimental implementation of a distributed parameter filter,' *Automatica*, vol 11, pp. 571-577, 1975.
- [2] H.T. Banks and K. Kunisch, 'The linear regulator problem for parabolic systems,' *SIAM J. Contr. Opt.*, vol. 22, pp 684-695, 1984.
- [3] J.S. Gibson, 'The Riccati integral equations for optimal control problems on Hilbert spaces,' *SIAM J. Contr. Opt.* vol. 17, pp 537-565, 1979.
- [4] K. E. Gustafson, *Introduction to partial differential equations and Hilbert space methods*, Wiley, New York, 1980.
- [5] M. Hamidi, 'A method of optimal actuator and sensor placement', *Proc. Fourth IFAC Symp. Contr. Dist. Parm. Syst.*, Los Angeles, 1986.
- [6] F. Incertis, 'An extension on a new formulation of the algebraic Riccati equation problem', *IEEE Trans. Auto. Contr.*, vol AC-28, pp. 235-237, 1983.

- [7] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, New York, 1966.
- [8] A.J. Laub, 'A Schur method for solving algebraic Riccati equations,' *IEEE Trans. Auto. Contr.*, vol. AC-24, pp. 913-922, 1979.
- [9] G. K. Lausterer, W. H. Ray and H. R. Martens, 'The real time application of distributed parameter state estimation theory to a two dimensional heated ingot,' pp. 451-467 of *Proc. Second IFAC Symp. Contr. Dist. Parm. Syst.*, S. P. Banks, A. J. Pritchard Eds., Pergamon Press, Oxford, 1978.
- [10] J.E. Potter, 'Matrix quadratic solutions,' *SIAM J. Appl. Math.*, vol. 14, pp 496-501, 1966.
- [11] K. Watanabe, M. Iwasaki, 'A fast computational approach in optimal distributed parameter state estimation,' *J. Dynamical Systems, Measurement and Contr.*, vol. 105, pp. 1-10, 1983.

Figure 1
cross section of variances for 2-D Heat Eq



UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED			1b. RESTRICTIVE MARKINGS		
2a. SECURITY CLASSIFICATION AUTHORITY			3. DISTRIBUTION/AVAILABILITY OF REPORT UNLIMITED		
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE					
4. PERFORMING ORGANIZATION REPORT NUMBER(S) JHU/EE-86/21			5. MONITORING ORGANIZATION REPORT NUMBER(S)		
6a. NAME OF PERFORMING ORGANIZATION THE JOHNS HOPKINS UNIVERSITY		6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION OFFICE OF NAVAL RESEARCH		
6c. ADDRESS (City, State and ZIP Code) CHARLES AND 34TH STREETS BALTIMORE, MARYLAND 21218			7b. ADDRESS (City, State and ZIP Code) 800 N. QUINCY ST. ARLINGTON, VA 22217		
8a. NAME OF FUNDING/SPONSORING ORGANIZATION		8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER N00014-85-K-0255		
8c. ADDRESS (City, State and ZIP Code)			10. SOURCE OF FUNDING NOS.		
			PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO. NR661-019
11. TITLE (Include Security Classification) SOLUTION TO THE ALGEBRAIC RICCATI EQUATION FOR PARABOLIC SYSTEMS (UNCLASSIFIED)					
12. PERSONAL AUTHOR(S) RIDDLE, L.R. AND WEINERT, H.L.					
13a. TYPE OF REPORT INTERIM		13b. TIME COVERED FROM 5/1/85 TO 10/17/86		14. DATE OF REPORT (Yr., Mo., Day) OCTOBER 22, 1986	
15. PAGE COUNT 9					
16. SUPPLEMENTARY NOTATION					
17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)		
FIELD	GROUP	SUB. GR.	OPERATOR RICCATI EQUATION, PARABOLIC SYSTEMS, DISTRIBUTED PARAMETER FILTERING AND CONTROL.		
19. ABSTRACT (Continue on reverse if necessary and identify by block number) This paper presents an analytical solution to the operator algebraic Riccati equation (ARE) for selfadjoint parabolic systems. The solution to the operator ARE is important in the design of the steady-state, on-line filter for estimating the system's states. This analytical solution is derived by considering the operator analog of Potter's method of using the Hamiltonian system's eigenvectors and eigenvalues to solve a finite-dimensional ARE. As an example of using this analytical solution, the steady-state filtering error covariance for the 2-D heat equation is studied.					
20. DISTRIBUTION/AVAILABILITY OF ABSTRACT UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS ☐			21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED		
22a. NAME OF RESPONSIBLE INDIVIDUAL DR. NEIL L. GERR			22b. TELEPHONE NUMBER (Include Area Code) (202)696-4321		22c. OFFICE SYMBOL

END

12-86

DTIC