AD-A174 030 A REFINEMENT OF SARKOVSKII'S THEOREM(U) ARMY BALLISTIC RESEARCH LAB ABERDEEN PROVING GROUND MD N P BHATIA ET AL. SEP 86 BRL-TR-2762 1/1

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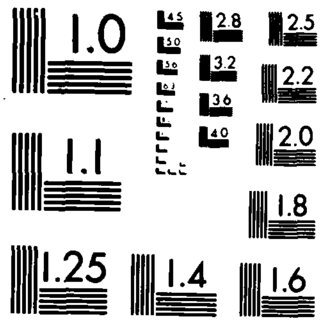

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TECHNICAL REPORT BRL-TR-2762

AD-A174 030

A REFINEMENT OF SARKOVSKII'S THEOREM

Nam P. Bhatia
Walter O. Egerland

September 1986

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US ARMY BALLISTIC RESEARCH LABORATORY
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86 11 14 068

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER Technical Report BRL-TR-2762	2. GOVT ACCESSION NO. AD A174 630	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A REFINEMENT OF SARKOVSKII'S THEOREM		5. TYPE OF REPORT & PERIOD COVERED 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Nam P. Bhatia Walter O. Egerland		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS US Army Ballistic Research Laboratory ATTN: SLCBR-SE- Aberdeen Proving Ground, MD 21005-5066		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS RDT&E 1L162618AH43
11. CONTROLLING OFFICE NAME AND ADDRESS US Army Ballistic Research Laboratory ATTN: SLCBR-DD-T Aberdeen Proving Ground, MD 21005-5066		12. REPORT DATE September 1986 13. NUMBER OF PAGES 14
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES The first author is Professor of Mathematics at the Univ of Maryland, Baltimore County. The results were partly established under the US Army Summer Faculty Research & Engineering Program 1985. A paper with the same title was summitted for publication in the Proceedings of the American Mathematical Society.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Periodic orbits	Infinite Loops
Periodic loops	Sarkovskii's Theorem

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Let f:R→R be continuous. If f has an orbit of period n, the question of which other periods f must necessarily have was answered by A.N. Sarkovskii by giving a total ordering of the natural numbers, now called the "Sarkovskii Ordering." The ordering does not take into account the period types and examples show that depending on the type of the period other periods than those implied by the Sarkovskii ordering are present. Introducing the concepts of a periodic loop (a periodic orbit of a certain type) and infinite loop, we give a total ordering of loops and obtain, as a consequence, a refinement of the theorem of Sarkovskii.

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I. INTRODUCTION

Let $f\colon R \to R$ be continuous and $x_0 \varepsilon R$. The orbit of x_0 under f is defined as the set $\{x\colon x = f^n(x_0),\ n = 0,1,\ldots\}$, where, for every positive integer n, f^n is the n-th iterate of f, $f^1 = f$, and $f^0(x_0) = x_0$. We shall write $x_n := f^n(x_0)$ for a given $x_0 \varepsilon R$ and call $x_1, x_2, \ldots$ the successors of x_0. A pre-orbit of a given $x_0 \varepsilon R$ is any (finite or infinite) sequence $x_0, x_{-1}, x_{-2}, \ldots$ such that $f(x_{-n}) = x_{-(n-1)}$ for all n for which x_{-n} is defined. The points $x_{-1}, x_{-2}, \ldots$ in any such sequence are called predecessors of x_0. A point c_0 is called critical if $f(c_0) = c_0$, i.e., a critical point of f is a fixed point of f. A periodic point x_0 of period $p > 1$ (p a positive integer) is a point for which the relations $f^p(x_0) = x_0$, $f^k(x_0) \neq x_0$, $1 \leq k < p$, hold. If x_0 is a periodic point of period p, its orbit is denoted by $(x_0, x_1, \ldots, x_{p-1})$. We shall denote the k^{th} iterate of x_0 under the function f^m by x_k^m, $k = 0,1,\cdots$. Thus $x_k^m := (f^m)^k(x_0) = x_{mk}$, and, in particular, $x_0^m = x_k^0 = x_0$ for all nonnegative integers k and m.

<u>Definition.</u> Let $f\colon R \to R$ be continuous and $x_0 \varepsilon R$. f has a loop of order n if x_0 has a pre-orbit $(x_0, x_{-1}, \ldots, x_{-n})$ such that either

$$x_0 < x_{-n} < x_{-(n-1)} < \ldots < x_{-2} < x_{-1}$$

or

$$x_0 > x_{-n} > x_{-(n-1)} > \ldots > x_{-2} > x_{-1}.$$

f has an infinite loop if x_0 has an infinite pre-orbit $(x_0, x_{-1}, \ldots, x_{-n}, \ldots)$ such that either

$$x_0 < \ldots < x_{-n} < x_{-(n-1)} < \ldots < x_{-2} < x_{-1}$$

or

$$x_0 > \ldots > x_{-n} > x_{-(n-1)} > \ldots > x_{-2} > x_{-1}.$$

A loop of order $(n - 1)$ is called an n-periodic loop if $x_0 = x_{-n}$.

We adopt the following concise notation: we say property $P(k)$ holds if f has a periodic orbit of period k. Thus $P(1)$, $L(k)$, $L(\infty)$ mean that f has a critical point, a periodic loop of period k, an infinite loop, respectively. Similarly, $P^n(k)$, $L^n(k)$, $L^n(\infty)$ shall mean that f^n has a k-periodic orbit, k-periodic loop, an infinite loop, respectively.

In this notation, Sarkovskii's theorem and our refinement read as follows.

Theorem (Sarkovskii). Let $f: R \to R$ be continuous. Then

$$
\begin{array}{l}
P(3) \Longrightarrow P(5) \Longrightarrow P(7) \Longrightarrow \ldots \Longrightarrow \\
P(2\cdot 3) \Longrightarrow P(2\cdot 5) \Longrightarrow P(2\cdot 7) \Longrightarrow \ldots \Longrightarrow \\
P(2^2\cdot 3) \Longrightarrow P(2^2\cdot 5) \Longrightarrow P(2^2\cdot 7) \Longrightarrow \ldots \Longrightarrow \\
\ldots \qquad\qquad\qquad\qquad\qquad\qquad \Longrightarrow \\
P(2^3) \Longrightarrow P(2^2) \Longrightarrow P(2) \Longrightarrow P(1).
\end{array}
$$

Theorem (SR). Let $f: R \to R$ be continuous. Then

$$
\begin{array}{l}
L(\infty)\ldots \Longrightarrow L(5) \Longrightarrow L(4) \Longrightarrow L(3) \Longleftrightarrow \\
P(3) \Longrightarrow P(5) \Longrightarrow P(7) \Longrightarrow \ldots \Longrightarrow \\
L^2(\infty)\ldots \Longrightarrow L^2(5) \Longrightarrow L^2(4) \Longrightarrow L^2(3) \Longleftrightarrow \\
P(2.3) \Longrightarrow P(2.5) \Longrightarrow P(2.7) \Longrightarrow \ldots \\
L^{2^2}(\infty)\ldots \Longrightarrow L^{2^2}(5) \Longrightarrow L^{2^2}(4) \Longrightarrow L^{2^2}(3) \Longleftrightarrow \\
P(2^2\cdot 3) \Longrightarrow P(2^2\cdot 5) \Longrightarrow P(2^2\cdot 7) \Longrightarrow \ldots \\
\ldots\ldots \\
\ldots \Longrightarrow P(2^3) \Longrightarrow P(2^2) \Longrightarrow P(2) \Longrightarrow P(1).
\end{array}
$$

A. N. Sarkovskii obtained the fundamental result which bears now his name in his seminal paper[1] of 1964. The purpose of this paper is to prove Theorem (SR).

II. ELEMENTARY LEMMAS

It follows from the definition of a periodic loop that every three-periodic orbit is a three-periodic loop and that an (n + 1)-periodic loop implies the existence of a loop of order n.

Lemma 2.1. If f has a critical point c_0 such that $c_0 < c_{-2} < c_{-1}$, f has an infinite loop satisfying

$$c_0 < \ldots < c_{-n} < \ldots < c_{-2} < c_{-1}.$$

The same statement holds with all inequalities reversed.

Proof. Since $f(c_{-2})=c_{-1}$ and $c_0 < c_{-1}$, there exists $c_{-3} \varepsilon (c_0, c_{-2})$. Repeating this argument establishes the lemma.

Lemma 2.2. If f has a critical point c_0 such that $c_{-1} < c_{-3} < c_0 < c_{-2}$, c_0 has

[1] A. N. Sarkovskii, "Coexistence of cycles of a continuous map of a line into itself," Ukrain. Mat. Zh. 16 (1964), 61-71.

an infinite pre-orbit satisfying

$$c_{-1} < c_{-3} < \dots < c_0 < \dots < c_{-4} < c_{-2}.$$

In particular, f^2 has two infinite loops. The same statement holds with all inequalities reversed.

Proof. Since $f(c_{-2})=c_{-1} < c_{-3}$ and $c_0 > c_{-3}$, there exists $c_{-4}\varepsilon(c_0, c_{-2})$, and since $f(c_{-3})=c_{-2} > c_{-4} > c_0$, there exists $c_{-5}\varepsilon(c_{-3}, c_0)$. Repeating this argument proves the lemma.

Lemma 2.3. $P^{2^k}(n) \Longleftrightarrow P(2^k\cdot n)$, $n, k = 1,2,\cdots$.

Proof. It suffices to show that $P^2(n) \Longleftrightarrow P(2\cdot n)$. If $(x_0, x_1,\dots, x_{2n-1})$ is a 2n-periodic orbit of f, $(x_0^2,x_1^2,\dots,x_{n-1}^2)$ is an n-periodic orbit of f^2. Hence $P(2\cdot n) \Longrightarrow P^2(n)$. If $(x_0^2,x_1^2,\dots,x_{n-1}^2)$ is an n-periodic orbit of f^2, we consider the set $\{x_0,x_1,\dots,x_{2n-1}\}$, where $x_{2n} = x_n^2 = x_0^2 = x_0$. If $x_0 \neq x_k$, $k=1,2, \dots ,2n-1$, then $C=(x_0,x_1,\dots,x_{2n-1})$ is a 2n-periodic orbit of f. Otherwise, there is a smallest odd k, $1 < k < 2n$, such that $x_0 = x_k$, i.e., x_0 is an odd-periodic point of f. But then, by Sarkovskii's theorem, f has periodic orbits of every even period and, therefore, in particular, a 2n-periodic orbit. Hence $P^2(n) \Longrightarrow P(2\cdot n)$ and the proof of the lemma is complete.

III. PRINCIPAL RESULTS

Let $C=(x_0,x_1,\dots,x_{n-1})$ be any n-periodic orbit of f. We define the subsets

$$C^+ = \{x_i\varepsilon C\colon x_{i+1} > x_i\},\quad C^- = \{x_i\varepsilon C\colon x_{i+1} < x_i\},$$
$$D^+ = \{x_i\varepsilon C\colon x_{i+2} > x_{i+1} > x_i\},\quad D^- = \{x_i\varepsilon C\colon x_{i+2} < x_{i+1} < x_i\}.$$

The sets C^+ and C^- are non-empty since min C ε C^+ and max C ε C^-. Letting further $a_0^+ = \min C^+$ (= min C), $b_0^+ = \max C^+$, $a_0^- = \min C^-$, and $b_0^- = \max C^-$ (= max C), it is clear that either $a_0^+ \le b_0^+ < a_0^- \le b_0^-$ or $a_0^+ < a_0^- < b_0^+ < b_0^-$.

Theorem 3.1. If $a_0^+ \le b_0^+ < a_0^- \le b_0^-$ and $D^+ \cup D^- \neq \phi$, f has a critical point c_0 such that f^2 has two infinite loops $(d_0^2, d_{-1}^2,d_{-2}^2,\dots)$ and $(c_0^2, c_{-1}^2, c_{-2}^2,\dots)$ satisfying

$$d_{-1}^2 < d_{-2}^2 < \dots < d_0^2 = c_0 = c_0^2 < \dots < c_{-2}^2 < c_{-1}^2 .$$

In particular, $L^2(*)$ holds.

Proof. It is sufficient to assume that $D^+ \neq \phi$. Then, if we let $\beta_0^+ = \max D^+$, we have

$$a_0^+ \leq \beta_0^+ < \beta_1^+ \leq b_0^+ < a_0^- \leq \beta_2^+ \leq b_0^-,$$

and conclude the existence of a critical point c_0 and a predecessor c_{-1} such that

$$a_0^+ \leq \beta_0^+ < c_{-1} < \beta_1^+ \leq b_0^+ < c_0 < a_0^- \leq b_0^-.$$

We consider now the set $E^- = \{x_i \in C^-: x_{i+1} < c_{-1}\}$. E^- is non-empty since $a_{n-1}^+ \in C^-$ and $a_n^+ = a_0^+ < c_{-1}$. Letting $r_0^- = \min E^-$, we have

$$a_0^+ \leq r_1^- < c_{-1} < b_0^+ < c_0 < a_0^- \leq r_0^- \leq b_0^-.$$

This shows that, since $c_0 > c_{-1}$ and $r_1^- < c_{-1}$, there exists a predecessor c_{-2} such that

$$a_0^+ < c_{-1} < b_0^+ < c_0 < c_{-2} < r_0^- \leq b_0^-.$$

Our construction implies that

(i) if $x_i \in C^+$ and $x_i > c_{-1}$, then $x_{i+1} \in C^-$

(ii) if $x_i \in C^-$ and $x_i < r_0^-$, then $x_{i+1} > c_{-1}$.

Hence, there is an $x_i \in C^+$, $x_i > c_{-1}$ such that $x_{i+1} \geq r_0^-$. For otherwise we would have $b_i^+ \in (c_{-1}, r_0^-)$ for all i, contradicting the fact that C is the orbit of b_0^+ (thus $b_i^+ = a_0^+$ for some $i > 1$). We now choose $\delta_0^+ \in C^+$ such that $\delta_0^+ > c_{-1}$ and $\delta_1^+ \geq r_0^-$ to obtain

$$a_0^+ < c_{-1} < \delta_0^+ \leq b_0^+ < c_0 < c_{-2} < r_0^- \leq \delta_1^+ \leq b_0^-.$$

But this implies that we may choose a predecessor c_{-3} in the interval (c_{-1}, δ_0^+), and hence that c_0 and its predecessors c_{-1}, c_{-2}, and c_{-3} satisfy the inequality $c_{-1} < c_{-3} < c_0 < c_{-2}$. Appeal to Lemma 2.2 completes the proof.

Theorem 3.2. If $a_0^+ < a_0^- < b_0^+ < b_0^-$, there exist critical points d_0, c_0 of f and two infinite loops $(d_0, d_{-1}, d_{-2}, \ldots)$ and $(c_0, c_{-1}, c_{-2}, \ldots)$ of f satisfying

$$d_{-1} < d_{-2} < \ldots < d_0 \leq c_0 < \ldots < c_{-2} < c_{-1}.$$

In particular, L(*) holds.

Proof. We note first that there is $\alpha_0^- \in C^-$ and $\beta_0^+ \in C^+$ such that

(i) $a_0^+ < a_0^- \leq \alpha_0^- < \beta_0^+ \leq b_0^+ < b_0^-$

(ii) if $x_i \in C$, then $x_i \leq \alpha_0^-$ or $x_i \geq \beta_0^+$

(iii) if $x_i \in C$ and $\alpha_0^- < x_i \leq b_0^+$, then $x_i \in C^+$

(iv) if $x_i \in C$ and $b_0^+ < x_i \leq b_0^-$, then $x_i \in C^-$.

We now show that there are predecessors c_{-1} and c_{-2} of the critical point c_0 $\in (\alpha_0^-, \beta_0^+)$ that satisfy the inequality $c_0 < c_{-2} < c_{-1}$. The set $A^- = \{x_i \in C^- : x_i > b_0^+ \text{ and } x_{i+1} \leq \alpha_0^-\}$ is non-empty (otherwise $\beta_n^+ \geq \beta_0^+$ for all integers $n \geq 0$, a contradiction). Let $r_0^- = \min A^-$. We have $r_0^- > b_0^+$ and observe that the set $A^+ = \{x_i \in C^+ : \beta_0^+ \leq x_i \leq b_0^+ \text{ and } x_{i+1} \geq r_0^-\}$ is non-empty (since otherwise β_n^+ will satisfy $\beta_0^+ \leq \beta_n^+ < r_0^-$ for $n \geq 0$, a contradiction). We choose any $y_0^+ \in A^+$ and have

$$\alpha_0^- < c_0 < \beta_0^+ \leq y_0^+ \leq b_0^+ < r_0^- \leq b_0^-.$$

Hence

$$c_0 < c_{-2} < y_0^+ \leq b_0^+ < c_{-1} < r_0^- \leq b_0^-,$$

where the existence of c_{-1} follows from $b_1^+ > c_0$ and $r_1^- < c_0$ and that of c_{-2} from $c_0 < c_{-1}$ and $y_1^+ \geq r_0^- > c_{-1}$. The infinite loop $(c_0, c_{-1}, c_{-2}, \ldots)$ satisfying $c_0 < \ldots < c_{-2} < c_{-1}$ follows from Lemma 2.1. An analogous procedure locates a critical point d_0 and predecessors d_{-1}, d_{-2} such that $d_{-1} < d_{-2} < d_0 \leq c_0$, and hence an infinite loop $(d_0, d_{-1}, d_{-2}, \ldots)$ satisfying $d_{-1} < d_{-2} < \ldots < d_0 \leq$

c_0. This completes the proof.

<u>Theorem 3.3.</u> If f has a loop of order $n \geq 3$, f has two distinct n-periodic loops. In particular, $L(n)$ holds.

<u>Proof.</u> Let $(x_0, x_{-1}, \ldots, x_{-n})$ be a loop of order $n \geq 3$ of f such that

$$x_0 < x_{-n} < \ldots < x_{-2} < x_{-1}.$$

Since there is a critical point $c_0 \varepsilon (x_{-2}, x_{-1})$, there are predecessors c_{-1}, $c_{-2}, \ldots, c_{-(n-2)}$ such that

$$x_0 < x_{-n} < c_{-(n-2)} < x_{-(n-1)} < \ldots < c_{-1} < x_{-2} < c_0 < x_{-1}.$$

We consider now the set

$$S = \{y_0 \varepsilon R: y_n < y_0 < c_{-(n-2)} < y_1 < \ldots < y_{n-3} < c_{-1} < y_{n-2} < c_0 < y_{n-1}\}.$$

The set S is non-empty since $x_{-n} \varepsilon S$ and open since f is continuous. Let (a_0, b_0) be the component of S such that $x_{-n} \varepsilon (a_0, b_0)$. Since $c_{-(n-2)} \not\varepsilon S$ and $y_0 \varepsilon (a_0, b_0)$ implies $y_0 < c_{-(n-2)}$, we must have

$$-\infty \leq a_0 < b_0 < c_{-(n-2)}.$$

We first note that $a_0 > -\infty$. This is so because for every $y_0 \varepsilon (a_0, b_0)$ we have $y_n < y_0$ and $y_n \varepsilon f^2([c_{-1}, c_0])$, which is a compact set. Thus $y_0 \geq \min f^2([c_{-1}, c_0]) > -\infty$. This implies $a_0 \geq \min f^2([c_{-1}, c_0]) > -\infty$. Since $a_0, b_0 \not\varepsilon S$ and $y_0 \varepsilon (a_0, b_0)$ implies $y_0 \varepsilon S$, we conclude by the continuity of f that

$$a_n = a_0 < c_{-(n-2)} < a_1 < \ldots < a_{n-3} < c_{-1} < a_{n-2} < c_0 < a_{n-1}$$

and

$$b_n = b_0 < c_{-(n-2)} < b_1 < \ldots < b_{n-3} < c_{-1} < b_{n-2} < c_0 < b_{n-1}.$$

Hence both a_0 and b_0 are n-periodic points, and since $a_0 < b_0 < b_1 < \ldots < b_{n-1}$, the orbits of a_0 and b_0 are distinct. This completes the proof of the theorem.

<u>Corollary 3.3.</u> If f has an $(n + 1)$-periodic loop, $n \geq 3$, f has two distinct n-periodic loops.

<u>Theorem 3.4.</u> $L(\infty) \Longrightarrow \ldots \Longrightarrow L(4) \Longrightarrow L(3) \Longrightarrow P(5) \Longrightarrow P(7) \Longrightarrow \ldots \Longrightarrow L^2(\infty)$.

<u>Proof.</u> If f has an infinite loop satisfying

$$x_0 < \dots < x_{-2} < x_{-1},$$

the subset $\{x_0, x_{-1}, \dots, x_{-n}\}$, $n \geq 3$, satisfies

$$x_0 < x_{-n} < \dots < x_{-2} < x_{-1},$$

and is, therefore, a loop of order n. By Theorem 3.3 , L(n) holds. Hence L(•) ==> L(n). By Corollary 3.3, L(n) ==> L(n-1). The implications L(3) ==> P(5) ==> P(7) ==> ... follow from Sarkovskii's theorem. Finally, to prove the implication P(2n+1) ==> L^2(•) for every $n \geq 1$, we note that if $C = (x_0, x_1, \dots, x_{2n})$ is a $(2n + 1)$-periodic orbit, then $n(C^+) \neq n(C^-)$, so that the hypothesis of either Theorem 3.1 or Theorem 3.2 is satisfied. In the first case L^2(•) holds by Theorem 3.1. In the second case, Theorem 3.2 implies that L(•) holds, and hence that L(3) holds. Now for any three-periodic orbit, the hypothesis of Theorem 3.1 holds trivially. Hence L^2(•) holds. This completes the proof.

<u>Corollary 3.4.</u> $L^{2^k}(\bullet) \dots \Longrightarrow L^{2^k}(5) \Longrightarrow L^{2^k}(3) \Longrightarrow P(2^k \cdot 5) \Longrightarrow P(2^k \cdot 7) \Longrightarrow \dots L^{2^{k+1}}(\bullet)$.

<u>Proof.</u> This follows from Theorem 3.4 and Lemma 2.3.

<u>Proof of Theorem (SR).</u> Theorem (SR) follows by combining Theorem 3.4, Coroll--y 3.4, Lemma 2.3 and Sarkovskii's theorem.

IV. REMARKS

1. Theorem (SR) is a step in the direction of obtaining a complete refinement of Sarkovskii's theorem that takes into account the orbit types of each period n. A periodic loop is only one of the orbit types of a given period. That certain orbit types imply the existence of infinite loops is implicit in Theorem 3.2 and is strikingly illustrated by the example $f(x) = ax(1 - |x|)$. f has the four-periodic orbit $\left(\frac{1}{2}, \frac{a}{4}, -\frac{1}{2}, -\frac{a}{4}\right)$, where $a \approx 4.411138875$ is given by

$$a^{-1} = \frac{1}{2}\left[\left(\frac{1}{2} + \frac{1}{2}\sqrt{\frac{59}{27}}\right)^{1/3} + \left(\frac{1}{2} - \frac{1}{2}\sqrt{\frac{59}{27}}\right)^{1/3}\right]$$

This orbit satisfies the conditions of Theorem 3.2 and hence guarantees the existence of two infinite loops as well as two periodic loops of each period $n \geq 3$ by Theorem (SR). The results of[5] ensure for this example merely the existence of a three-periodic orbit.

2. The results in this paper offer a novel approach to detecting chaos. Most practical methods for detecting chaos rely, either implicitly or explicitly, on the existence of odd periodic orbits[2,3,4,5] However, lemmas 2.1 and 2.2 can be used by finding only a few predecessors of a critical point. Lemma 2.2, in particular, is independent of odd periods. A notable illustration is the example $f(x) = x^2 - s$, for which at

$$s^* = \frac{1}{3}\left\{2 + \left(3\sqrt{33} + 17\right)^{1/3} - \left(3\sqrt{33} - 17\right)^{1/3}\right\}$$

$L^3(\bullet)$ holds, with no odd period $\neq 1$ being present.

---

[2] Gyorgy Targonski, "Topics in Iteration Theory," Studia Mathematica, Skript 6, Vandenhock and Ruprecht, Göttingen and Zürich (1981).

[3] Nam P. Bhatia and Walter O. Egerland, "On the Existence of Li-Yorke Points in the Theory of Chaos," Mathematics Research Report No. 84-4, Department of Mathematics, UMBC, July 1984. (To appear in Vol. 9, No. 10 of Nonlinear Analysis.)

[4] Nam P. Bhatia and Walter O. Egerland, "Non-periodic conditions for Chaos and Snap-Back Repellers," Transactions of the Second Army Conference on Applied Mathematics and Computing, ARO Report 85-1 159-164 (Also Mathematics Research Report No. 84-5, Department of Mathematics, UMBC, September 1984).

[5] Tien-Yien Li, Michal Misiurewicz, Giulio Pianigiani and James A. Yorke, "No Division Implies Chaos," Transactions of the American Mathematical Society, 273(1) (1982), 191-199.

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