

AD-A174 418

A-OPTIMAL BLOCK DESIGNS FOR COMPARING TEST TREATMENTS

1/1

WITH A CONTROL (U) ILLINOIS UNIV AT CHICAGO CIRCLE

STATISTICAL LAB J STUFKEN 22 JUL 86 TR-86-10

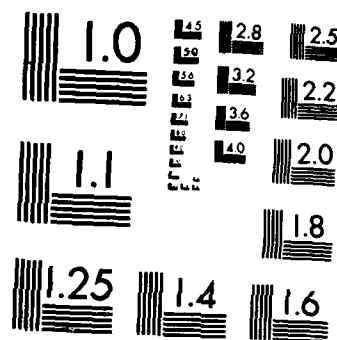
UNCLASSIFIED

AFOSR-TR-86-1092 AFOSR-85-0320

F/G 12/1

NL





MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

AD-A174 418

UIC FILE COPY



**THE UNIVERSITY OF ILLINOIS
AT CHICAGO**

AFOSR-TR. 86-1092

2

DTIC
ELECTE
NOV 26 1986
S D

Approved for public release;
distribution unlimited.

AIR FORCE OFFICE OF SCIENTIFIC RESEARCH (AFSC)
NOTICE OF TRANSMITTAL TO DTIC
This technical report has been reviewed and is
approved for public release IAW AFR 190-12.
Distribution is unlimited.
MATTHEW J. KERFER
Chief, Technical Information Division

**DEPARTMENT OF MATHEMATICS,
STATISTICS, AND COMPUTER SCIENCE**

86 11 25 363

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED			1d. RESTRICTIVE MARKINGS										
2a. SECURITY CLASSIFICATION AUTHORITY N/A			3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for Public Release; Distribution unlimited										
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE N/A			5. MONITORING ORGANIZATION REPORT NUMBER(S) AFOSR-TR-85-0320										
4. PERFORMING ORGANIZATION REPORT NUMBER(S)			7a. NAME OF MONITORING ORGANIZATION AFOSR/NM										
6a. NAME OF PERFORMING ORGANIZATION University of Illinois at Chicago		6b. OFFICE SYMBOL (If applicable)	7b. ADDRESS (City, State and ZIP Code) Bldg. 410 Bolling AFB, D.C. 20332-6448										
6c. ADDRESS (City, State and ZIP Code) P.O. Box 4348 Chicago, IL 60680		9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER AFOSR 85-0320											
8a. NAME OF FUNDING/SPONSORING ORGANIZATION AFOSR		8b. OFFICE SYMBOL (If applicable) NM	10. SOURCE OF FUNDING NOS.										
8c. ADDRESS (City, State and ZIP Code) Bldg. 410 Bolling AFB, D.C. 20332-6448		<table border="1"> <tr> <td>PROGRAM ELEMENT NO.</td> <td>PROJECT NO.</td> <td>TASK NO.</td> <td>WORK UNIT NO.</td> </tr> <tr> <td>6.1102F</td> <td>2304</td> <td>A5</td> <td></td> </tr> </table>				PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.	6.1102F	2304	A5	
PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.										
6.1102F	2304	A5											
11. TITLE (Include Security Classification) A-optimal Block Designs for Comparing Test Treatments with a Control													
12. PERSONAL AUTHOR(S) J. Stufken													
13a. TYPE OF REPORT Technical		13b. TIME COVERED FROM _____ TO _____		14. DATE OF REPORT (Yr., Mo., Day) July 22, 1986									
15. PAGE COUNT 9													
16. SUPPLEMENTARY NOTATION													
17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)										
FIELD	GROUP	SUB. GR.	Control, Test Treatments, A-optimality, R-type designs, S-type designs										
19. ABSTRACT (Continue on reverse if necessary and identify by block number)													
We consider the problem of comparing test treatments with a control in a proper block design. We derive sufficient conditions for the A-optimality of both R-type and S-type designs, and demonstrate how these conditions can be used to obtain families of optimal designs. We give an example for the construction of the desired S-type designs. A table with optimal R-type designs ($3 \leq k \leq 10$, $k \leq v \leq 30$) is also given.													
20. DISTRIBUTION/AVAILABILITY OF ABSTRACT UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ OTIC USERS ☐			21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED										
22a. NAME OF RESPONSIBLE INDIVIDUAL Major Brian W. Woodruff			22b. TELEPHONE NUMBER (Include Area Code) (202) 767-5011		22c. OFFICE SYMBOL NM								

DTIC
SELECTE
NOV 26 1986
S D

A-OPTIMAL BLOCK DESIGNS FOR COMPARING
TEST TREATMENTS WITH A CONTROL

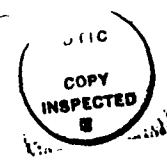
by

JOHN STUFKEN

Department of Mathematics, Statistics, and Computer Science
University of Illinois at Chicago
Chicago, Illinois 60680

July, 1986

Statistical Laboratory Technical Report No. 86-10



Research is sponsored by Grant AFOSR 85-0320

Accession For	
NTIS CRA&I	☒
DTIC TAB	☐
Unannounced	☐
Justification	
By	
Distribution/	
Availability Codes	
Dist	Avail and/or Special
A-1	

A-optimal block designs for comparing test treatments with a control

JOHN STUFKEN*

Department of Mathematics, Statistics, and Computer Science
University of Illinois at Chicago, Chicago, Illinois 60680

ABSTRACT

We consider the problem of comparing test treatments with a control in a proper block design. We derive sufficient conditions for the A-optimality of both R-type and S-type designs, and demonstrate how these conditions can be used to obtain families of optimal designs. We give an example for the construction of the desired S-type designs. A table with optimal R-type designs ($3 \leq k \leq 10$, $k \leq v \leq 30$) is also given.

1. INTRODUCTION

In this paper we study the problem of comparing a set of test treatments with a control under the assumption that the experimental units can be arranged in a proper block design. For a brief review on the available literature in this area we refer the reader to the introduction of Hedayat and Majumdar (1985). We will use the notation $D_0(v, b, k)$ for the collection of all connected block designs with b blocks of size k each, and based on v test treatments (labeled $1, \dots, v$) and a control (labeled 0). An observation $Y_{ij\ell}$, obtained by applying treatment i ($0 \leq i \leq v$) on the experimental unit in block j ($1 \leq j \leq b$) in plot ℓ ($1 \leq \ell \leq k$), will be assumed to follow the usual additive linear model without interactions:

$$Y_{ij\ell} = \mu + \tau_i + \beta_j + \epsilon_{ij\ell}.$$

We refer to μ as the general mean, to τ_i as the effect by treatment i and to β_j as the effect by block j . The error terms $\epsilon_{ij\ell}$ are assumed to be uncorrelated, and have a common mean 0 and a common variance σ^2 . The objective of the experiment is to estimate the test treatment-control contrasts $\tau_i - \tau_0$, $1 \leq i \leq v$. We assume that this is done by using their least square estimates, which we will denote by $\hat{\tau}_i - \hat{\tau}_0$. These estimates will obviously be design dependent and the problem that remains is thus, for given v , b and k , to select a design $d \in D_0(v, b, k)$ that, in some sense, gives us good estimates for the contrasts of interest. In this paper we will judge the performance of a design by the so-called A-criterion. A design $d^* \in D_0(v, b, k)$ is called A-optimal (in its class) if it minimizes

$$\sum_{i=1}^v \text{Var}_d(\hat{\tau}_i - \hat{\tau}_0)$$

over all designs $d \in D_0(v, b, k)$. Majumdar and Notz (1983) obtained a sufficient condition for a design to be A-optimal. Utilizing this condition we will search for families of A-optimal designs, both R-type designs (Section 2) and S-type designs (Section 3).

*Research is sponsored by Grant AFOSR 85-0320.

2. A-OPTIMAL R-TYPE DESIGNS

The following definition first appeared in Bechhofer and Tamhane (1981).

DEFINITION 2.1. A design $d \in D_0(v, b, k)$ is called a *Balanced Test treatment Incomplete Block Design (BTIBD)* if the following conditions are satisfied:

1. d is incomplete.
2. There are constants λ_0 and λ_1 such that $\sum_{j=1}^b n_{doj} n_{dij} = \lambda_0$, $1 \leq i \leq v$, and $\sum_{j=1}^b n_{di_1j} n_{di_2j} = \lambda_1$, $1 \leq i_1 \neq i_2 \leq v$.

The number n_{dij} , $0 \leq i \leq v$, $1 \leq j \leq b$, is the number of times that treatment i appears in block j under design d .

Such a design possesses a great amount of symmetry with respect to the test treatments, and it can indeed be shown that $\text{Var}_d(\hat{r}_i - \hat{r}_0)$ is independent of i , $1 \leq i \leq v$. Moreover the information matrix for estimating the test treatment control contrasts under such a design is a completely symmetric matrix. (See Bechhofer and Tamhane (1981)). This symmetry was utilized by Majumdar and Notz (1983) to obtain their main result. Before stating it we need one more definition.

DEFINITION 2.2. By a $BTIB(v, b, k; t, s)$ we denote a design $d \in D_0(v, b, k)$ with the following properties:

1. d is a BTIBD.
2. There are s blocks in d with $t+1$ replications of the control, while the remaining $b-s$ blocks contain t replications of the control.
3. d is binary in the test treatments.

THEOREM 2.1. (Majumdar-Notz (1983)). If $3 \leq k \leq v$ then a $BTIB(v, b, k; t, s)$ is A-optimal if

$$g(t, s) = \min\{g(x, z) : (x, z) \in \Lambda\},$$

where

$$g(x, z) := (v-1)^2(bvk(k-1) - (bx+z)(vk-v+k) + bx^2 + 2xz + z))^{-1} \\ + ((bx+z)k - (bx^2 + 2xz + z))^{-1},$$

and

$$\Lambda := \{(x, z) : x \in \{0, 1, \dots, [k/2] - 1\}, z \in \{0, 1, \dots, b\} \text{ and } z > 0 \text{ if } x = 0\}.$$

($[\cdot]$ denotes the largest integer function).

In the remainder of this paper we will assume that the condition $3 \leq k \leq v$ is satisfied. We will also adopt the terminology from Hedayat and Majumdar (1984), where a $BTIB(v, b, k; t, s)$ is called a *Rectangular-type (R-type)* design if $s = 0$ or b and a *Step-type (S-type)* design if $s \in \{1, \dots, b-1\}$. Through a more detailed study of the function g in Theorem 2.1 Hedayat and Majumdar (1985) obtained the following result on optimal R-type designs.

THEOREM 2.2. A BTIB($v, b, k; 1, 0$) is A-optimal if

$$(k-2)^2 + 1 \leq v \leq (k-1)^2.$$

The main result of this section is the following generalization of Theorem 2.2.

THEOREM 2.3. A BTIB($v, b, k; t, 0$) is A-optimal if

$$(2.1) \quad (k-t-1)^2 + 1 \leq t^2 v \leq (k-t)^2.$$

PROOF: By Theorem 2.1 it suffices to show that (2.1) implies that

$$(2.2) \quad g(t, 0) = \min\{g(x, z) : (x, z) \in \Lambda\}.$$

From Hedayat and Majumdar (1985) it follows that (2.2) holds if the following two conditions are satisfied.

$$(2.3) \quad \begin{aligned} b(k-t)((k-2t-1)(v(k-1)-t)^2 - at^2(p-2t-1)) \\ \leq v(k-2t-1)(p-2t-1)(k-1+(v-2)t) \end{aligned}$$

and

$$(2.4) \quad \begin{aligned} b(k-t)(at^2(p-2t+1) - (k-2t+1)(v(k-1)-t)^2) \\ \leq v(k-2t+1)(p-2t+1)(k-1+(v-2)t), \end{aligned}$$

where $a = (v-1)^2$ and $p = v(k-1) + k$. Since the right hand sides in (2.3) and (2.4) are non-negative these inequalities are obviously satisfied if

$$(2.5) \quad (k-2t-1)(v(k-1)-t)^2 - at^2(p-2t-1) \leq 0$$

and

$$(2.6) \quad at^2(p-2t+1) - (k-2t+1)(v(k-1)-t)^2 \leq 0.$$

The left hand side in (2.5) equals $-vq_2(v)$, while that of (2.6) equals $vq_1(v)$, where

$$\begin{aligned} q_1(v) := (k-1)t^2v^2 - ((k-t-1)(k+t-1)(k-2t+1) + 2t^2(k-1))v \\ + 2t(k-t-1)(k-2t+1) + t^2(k-1) \end{aligned}$$

and

$$\begin{aligned} q_2(v) := (k-1)t^2v^2 - ((k-t-1)(k+t-1)(k-2t-1) + 2t^2(k-1))v \\ + 2t(k-t-1)(k-2t-1) + t^2(k-1). \end{aligned}$$

Hence (2.5) and (2.6) are equivalent to $q_2(v) \geq 0$ and $q_1(v) \leq 0$. Since both q_1 and q_2 are convex functions in v , to establish the validity of Theorem 2.3 it suffices to show that

- (i) $q_1(1) \leq 0$
- (ii) $q_2(1) \leq 0$
- (iii) $q_1((k-t)^2/t^2) \leq 0$
- (iv) $q_2(((k-t-1)^2+1)/t^2) \geq 0$.

A simple evaluation gives us that

$$q_1(1) = -(k-t-1)^2(k-2t+1) \leq 0,$$

which shows (i). Also

$$q_2(1) = -(k-t-1)^2(k-2t-1) \leq 0.$$

This shows (ii). Further we obtain that

$$\begin{aligned} q_1((k-t)^2/t^2) &= ((-2t+1)k^3 + (6t^2-1)k^2 \\ &\quad - (t^4 + 6t^3 + 3t^2 - 2t)k + 2t^5 + 3t^4 - t^2)/t^2. \end{aligned}$$

Since $k \geq 2t$ and since the right hand side in the above equality, viewed as a function of k , is decreasing on $(2t, \infty)$ it follows by replacing k by $2t$ that

$$q_1((k-t)^2/t^2) \leq -(t-1)^2 \leq 0,$$

establishing (iii).

Finally

$$q_2(((k-t-1)^2+1)/t^2) = ((k-2t-1)((k-1)^2 - t^4 + t^2) + k-1)/t^2.$$

To see that this is non-negative we merely notice that $k \leq v \leq (k-t)^2/t^2$ implies that $k \geq t^2 + t$. This shows (iv) and completes the proof of Theorem 2.3. □

We would like to point out that it can be shown that for fixed k and t at most one value of v which satisfies (2.3) and (2.4) for all $b \geq v$ is not covered by the relation (2.1).

The result of Theorem 2.3 enables us also to obtain large families of A-optimal R-type designs from families of BIB designs. We formalize this in Corollary 2.1 and give an example of its use in Corollary 2.2. Other examples can be found in Stufken (1986a).

COROLLARY 2.1. *If there are a BIBD(v, b, k) and an integer $t \in \{1, 2, \dots, \lfloor k/2 \rfloor - 1\}$ such that*

(i) $v \geq k + t$, and

(ii) $(k-1)^2 + 1 \leq t^2 v \leq k^2$,

then an A-optimal BTIB($v, b, k+t; t, 0$) exists.

The proof is a simple verification of (2.1). For parameters satisfying the conditions in this corollary the result gives a formal justification for an idea of Cox (1958), who first recommended the use of R-type designs.

COROLLARY 2.2. If λ is a perfect square and $v \geq k + \lambda^{1/2}$, then a symmetric BIBD(v, k, λ) can be embedded in a A-optimal BTIB($v, v, k + \lambda^{1/2}; \lambda^{1/2}, 0$).

PROOF: We verify the conditions in Corollary 2.1. It is easy to verify that $\lambda^{1/2} < [k/2]$ under the assumptions in Corollary 2.2. Condition (i) is satisfied by assumption, while (ii) can be seen as follows:

$$\lambda v = \lambda(v - 1) + \lambda = k(k - 1) + \lambda \leq k^2,$$

and

$$\lambda v \geq k(k - 1) \geq (k - 1)^2 + 1.$$

□

We conclude this section by giving a table with all A-optimal BTIB($v, b, k; t, 0$) designs, $3 \leq k \leq 10$, $k \leq v \leq 30$, whose A-optimality follows from Theorem 2.3. The given values of b in the table may be replaced by any multiple of the given number. The BIB($v, b, k-t$) needed for the construction of the given BTIB exists for all parameters. Most of them, or their complements, can be found in Table 1.1 in Hall (1986).

Table 2.1. A-optimal BTIB($v, b, k; t, 0$) designs $3 \leq k \leq 10$, $k \leq v \leq 30$

No	v	b	k	t	No	v	b	k	t
1	3	3	3	1	20	15	15	10	2
2	4	6	3	1	21	15	105	5	1
3	5	10	4	1	22	16	20	5	1
4	6	10	4	1	23	16	30	10	2
5	7	7	4	1	24	17	68	6	1
6	8	28	8	2	25	18	306	6	1
7	8	56	4	1	26	19	171	6	1
8	9	12	4	1	27	20	76	6	1
9	9	12	8	2	28	21	21	6	1
10	10	15	5	1	29	22	462	6	1
11	10	30	9	2	30	23	253	6	1
12	11	55	5	1	31	24	552	6	1
13	11	55	9	2	32	25	30	6	1
14	12	33	5	1	33	26	65	7	1
15	12	132	9	2	34	27	117	7	1
16	13	13	5	1	35	28	126	7	1
17	13	39	10	2	36	29	406	7	1
18	14	91	5	1	37	30	145	7	1
19	14	91	10	2					

3. A-OPTIMAL S-TYPE DESIGNS

Families of A-optimal S-type designs are, with one exception in Cheng et al. (1986), not available in the literature. The reason for this is twofold. Partly it is due to the fact that a characterization of families of parameters corresponding to S-type designs whose A-optimality can be concluded from Theorem 2.1 is harder than the analogue for optimal R-type designs. In addition, once a family of such parameters has been established, we will still have to answer the often difficult combinatorial question regarding their existence by giving a method for their construction. In the case of R-type designs we could refer to the available literature on BIB designs, for S-type designs such an easy way out is not available. Cheng et al. (1986) proved a useful result for the determination of families of parameters of A-optimal S-type designs. A formulation of their result is:

THEOREM 3.1. *With $g(x, z)$ and Λ as in Theorem 2.1, if for some $t \in \{0, 1, \dots, [k/2] - 1\}$, $s \in \{1, 2, \dots, b - 1\}$*

$$(3.1) \quad g(t, s) \leq \min\{g(t, s - 1), g(t, s + 1)\}$$

then

$$g(t, s) = \min\{g(x, z) : (x, z) \in \Lambda\}.$$

This result can be used if we determine parameters for which (3.1) is satisfied and for which we can then show the existence of the corresponding S-type design. Cheng et al. (1986) showed proceeding in this manner that an A-optimal BTIB($k^2 - 1, \gamma(k + 2)(k^2 - 1), k; 0, \gamma(k + 1)(k^2 - 1)$) exists if k is a prime or power of a prime, where γ is any positive integer. In this section we will derive families of parameters which satisfy (3.1) for the case $t = 0$. They will include the above result. We will then conclude this section with a patchwork construction for designs with the obtained parameters. Let us start by looking at $g(0, z)$, $z \in (0, b]$. It is not hard to show that this function attains its minimum either at b or, assuming $v \geq 4$, at

$$(3.2) \quad z_0 := bk((v - 1)(v + 1)^{1/2} - (v + 1))/(v + 1)(v - 3)$$

if $z_0 < b$.

We are interested in the latter case. It can be verified that $z_0 < b$ if and only if $v \geq (k - 1)^2 + 1$. Thus if this condition holds we define $s = [z_0]$ or $s = [z_0] + 1$ depending on which of the two gives a smaller value for $g(0, z)$. With this choice of s , condition (3.1) will be satisfied, implying that a BTIB($v, b, k; 0, s$) is A-optimal, if it exists. This observation is useful if a particular choice of v , b and k is under consideration. It is then easy to obtain s and an attempt can be made to construct the desired optimal S-type design. A useful first step in such an attempt is to verify whether the necessary conditions for the existence of a BTIB($v, b, k; 0, s$) are satisfied. These are (see also Hedayat and Majumdar (1984)):

$$(3.3) \quad (k-1)s \equiv 0 \pmod{v}$$

$$(3.4) \quad k(b-s) \equiv 0 \pmod{v}$$

$$(3.5) \quad (k-2)(k-1)s/v + (k-1)k(b-s)/v \equiv 0 \pmod{v-1}.$$

If one or more of these conditions are violated, or the non-existence of the desired design is obtained through some other argument, considerations as in Cheng et al. (1986) may lead to an A-optimal or highly efficient design in $D_0(v, b, k)$.

For the purpose of deriving families of optimal S-type designs the form of s is rather complicated. The problem is made more tractable if we make the additional assumption that z_0 is an integer, and thus $s = z_0$. A design that will be obtained under this assumption has the extra feature that any number of copies of it forms again on A-optimal design. It is unknown whether this is true in general. The following result characterizes all parameters of interest for which z_0 is an integer in $\{1, \dots, b-1\}$.

THEOREM 3.2. *The parameters for which $z_0 \in \{1, \dots, b-1\}$ and for which the necessary conditions (3.3), (3.4) and (3.5) are satisfied are given by*

$$v = \alpha^2 - 1, k = \beta, b = \gamma\alpha(\alpha+2)(\alpha^2-1)/\delta\epsilon,$$

where $\alpha \geq \beta \geq 3$, $\delta = \gcd(\beta, \alpha(\alpha+2))$, $\epsilon = \gcd(\beta\delta^{-1}, \alpha^2-1)$ and γ is an arbitrary positive integer.

PROOF: For z_0 to be an integer $v+1$ must be a perfect square, say $v = \alpha^2 - 1$. Let $k = \beta \geq 3$. For $z_0 < b$ it must be that $v \geq (k-1)^2 + 1$, or $\alpha \geq \beta$. From (3.2) we obtain now $z_0 = b\beta(\alpha+1)/\alpha(\alpha+2)$. Since $\gcd(\alpha+1, \alpha(\alpha+2)) = 1$ we see that this is only an integer if $b \equiv 0 \pmod{\alpha(\alpha+2)/\delta}$. If we set $b = \gamma_1\alpha(\alpha+2)/\delta$ we obtain with $s = z_0$ from (3.3) and (3.4) (which imply $kb - s \equiv 0 \pmod{v}$) that $\gamma_1\beta(\alpha^2 + \alpha - 1)/\delta \equiv 0 \pmod{\alpha^2 - 1}$, or equivalently $\gamma_1\beta/\delta \equiv 0 \pmod{\alpha^2 - 1}$. Therefore $\gamma_1 = \gamma(\alpha^2 - 1)/\epsilon$, for some positive integer γ . This gives the parametrization for b as asserted in the theorem. It can indeed easily be verified that (3.3), (3.4) and (3.5) are now satisfied, while $z_0 = s = \gamma\beta(\alpha+1)(\alpha^2-1)/\delta\epsilon$, which is indeed an integer. $\square$

This result tells us thus that a $\text{BTIB}(\alpha^2-1, \gamma\alpha(\alpha+2)(\alpha^2-1)/\delta\epsilon, \beta; 0, \gamma\beta(\alpha+1)(\alpha^2-1)/\delta\epsilon)$ is A-optimal for any $\alpha, \beta, \gamma, \delta$ and ϵ as in Theorem 3.2. Although the conditions (3.3), (3.4) and (3.5) are also satisfied, this does unfortunately not guarantee the existence of the design. Patchwork constructions may resolve this problem, either for individual designs or for certain families of designs of the above form. We give here one example of such a construction. Let $\alpha = \beta = k$. This results in a $\text{BTIB}(k^2-1, \gamma(k+2)(k^2-1), k; 0, \gamma(k+1)(k^2-1))$, the existence of which was shown for k a prime or power of a prime, as referred to earlier. We add to this the following result.

THEOREM 3.3. *An A-optimal $\text{BTIB}(k^2-1, \gamma(k+2)(k^2-1), k; 0, \gamma(k+1)(k^2-1))$ exists if $k+1$ is a prime or power of a prime.*

PROOF: It suffices to show the existence for $\gamma = 1$. The layout of the desired design is as follows:

	$(k+1)(k^2-1)$		k^2-1
1	control		
$k-1$	A	B	k

We have to determine the parts labeled by A and B . Start by partitioning the $k^2 - 1$ test treatments into $k - 1$ groups $G_1, \dots, G_{k-1}$ of cardinality $k + 1$ each. For each of these groups form the trivial BIBD, $(k+1)\Sigma k$. This gives a total of $k^2 - 1$ blocks of size k , blocks that we use to form B . Next form an orthogonal array $OA((k+1)^2, k-1, k+1, 2)$ of index unity. The construction of this array is well known if $k+1$ is a prime or power of a prime, which is just our assumption. Use this orthogonal array to construct a group divisible design with $k^2 - 1$ treatments in blocks of size $k - 1$ in which the treatments are partitioned in $k - 1$ groups and a pair of treatments from the same group does not appear at all, while treatments from distinct groups appear exactly once as a pair. Clearly we can arrange this such that the groups are again $G_1, \dots, G_{k-1}$. Take $k - 1$ replications of this group divisible design and let this be A . It is easy to verify that the design constructed in this way is the desired S-type design. $\square$

Several other patchwork constructions for individual designs can be found in Stufken (1986a).

4. CONCLUDING REMARKS

The results in Section 2 extend the main results of Hedayat and Majumdar (1985). Theorem 2.3 and Corollary 2.1 provide a sufficient condition for the A-optimality of R-type designs, a condition that can easily be verified. Large families of optimal R-type designs can be obtained from it. It is unfortunately known that for many families of parameters v , b and k the best design is not a R-type design. An alternative would then be to search among S-type designs. Section 3 gives some families of optimal S-type designs and some ideas for finding optimal designs for specified parameters. The construction of these S-type designs is usually a non-trivial problem. Patching BIB designs and/or group divisible designs together in an appropriate way may resolve problems of this kind. After these considerations there will still be many parameters v , b , and k for which no optimal design has been found. This will be the case if optimal designs are neither of the R-type nor S-type or if Theorem 2.1 is not strong enough to determine their optimality. In such cases one could either search for highly efficient R-type designs (see Stufken (1986b)) or give consideration to the ideas suggested in Cheng et al. (1986).

We finally point out that all the optimal designs derived in this paper are not only A-optimal, but also MV-optimal.

Acknowledgement: This paper is based on parts of my Ph.D. dissertation. Thanks are due to my advisor, Professor A.S.Hedayat, as well as to Professor D. Majumdar for their support and helpful discussions.

REFERENCES

- [1] BECHHOFFER, R.E. AND TAMHANE, A.C., Incomplete block designs for comparing treatments with a control: General Theory, *Technometrics* **23**, (1981), 45-57.
- [2] CHENG, C.S., MAJUMDAR, D., STUFKEN, J. AND TURE, T.E., Optimal step type designs for comparing test treatments with a control, Technical Report 62, Department of Statistics, University of California, Berkeley, (1986).
- [3] COX, D.R., "Planning of experiments," Wiley, New York (1958).
- [4] HALL, M., JR., "Combinatorial Theory," Second edition, Wiley, New York (1986).
- [5] HEDAYAT, A.S. AND MAJUMDAR, D., A-optimal incomplete block designs for control-test treatment comparisons, *Technometrics* **26**, (1984), 363-370.
- [6] HEDAYAT, A.S. AND MAJUMDAR, D., Families of A-optimal block designs for comparing test treatments with a control, *Ann. Statist.* **13**, (1985), 757-767.
- [7] MAJUMDAR, D. AND NOTZ, W.I., Optimal incomplete block designs for comparing treatments with a control, *Ann. Statist.* **11**, (1983), 258-266.
- [8] STUFKEN, J., On optimal and highly efficient block designs for comparing test treatments with a control, Ph.D. dissertation, University of Illinois at Chicago, (1986a).
- [9] STUFKEN, J., On bounds for the efficiency of block designs for comparing test treatments with a control, Technical Report, 86-05, University of Illinois at Chicago, (1986b).

END

1-87

DTIC