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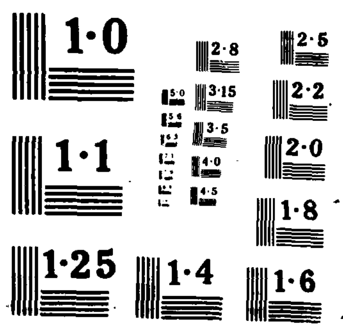
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Relationships between Volume, Surface and Line Distributions of Vorticity, Source and Doublicity.

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Summary

Several mathematical theorems are derived which demonstrate the equivalence of continuous volume distributions of doublicity, vorticity and source, and show furthermore that their influence may be expressed purely in terms of continuous surface distributions of these quantities over the closed boundary of the volume.

These general theorems may then be particularised to 'sheets' of singularities distributed over non-closed surfaces; amongst a number of examples, the special cases of the velocity fields induced by source, vortex and doublet sheets are considered, which under certain circumstances are equivalent to each other and reduce to simple line integrals.

These theorems are expected to have some application in aerodynamic problems involving the interaction between irrotational incompressible flow regions and regions of more general flow such as those arising in aerodynamic wakes and in the jet-in-crossflow problem, and to be of assistance in the development of improved surface singularity techniques.

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1. INTRODUCTION

It is well known that if the potential in a region obeys Laplace's equation everywhere within that region, then that potential can be expressed entirely in terms of surface integrals involving the value of the potential and its normal derivative everywhere on the closed boundaries of that region. This fact is exploited in practical aerodynamic calculation methods ('panel methods') which allow the potential and the velocity everywhere on and outside the surface of a configuration to be determined nominally exactly by evaluating one or the other of these functions at the configuration surface, such that the appropriate boundary conditions are satisfied.

However, in cases where the governing equations in the region of interest do not reduce to Laplace's equation (such as flows which are compressible or rotational) the expression for the velocity at any point in such a region involves not only surface integrals but also volume integrals which in general extend over the entire non-Laplacian volume. Consequently schemes treating such cases involve a significantly greater amount of computation and currently are restricted to geometries much simpler than those which 'panel methods' are able to handle.

Problems in which it is required to compute the flow inside regions where volume sources or vorticity may be considered to be present (e.g. compressible or rotational flow) will not be considered here. In many problems of interest, however, such regions may be embedded in a flow which is otherwise source- and vorticity-free; examples include the flow outside a thick wake (and its 'rolled up' core of rotational fluid) behind a lifting wing, or the flow outside a jet issuing at some angle into an otherwise irrotational stream.

In such cases it is often required (sometimes as part of an iterative scheme) to compute the effect produced in the outer (Laplacian) region by the sources and/or vorticity used to model these embedded regions. It is shown herein that this external effect may instead be computed from surface distributions over the boundary of these embedded regions, and that in certain special cases, when the embedded region may be modelled as an infinitesimally thin volume (i.e. a sheet), line distributions may be used. The theorems allow the singularities (vortices, sources, doublets) to be chosen which are computationally the most convenient for the problem in hand.

2. BASIC CONVENTIONS AND INFLUENCE EXPRESSIONS

In the following, a general vector $\bar{A}$ denotes the quantity $(A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k})$ where $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ are the unit vectors of the cartesian axes x, y, z. In particular the vector $\bar{r}$ denotes the vector drawn from a point Q to a point P.

i.e. $$\bar{r} = (x_P - x_Q)\hat{\imath} + (y_P - y_Q)\hat{\jmath} + (z_P - z_Q)\hat{k}$$

or $$r^2 = (x_P - x_Q)^2 + (y_P - y_Q)^2 + (z_P - z_Q)^2$$

The operators curl_Q (or $\nabla_Q\times$), div_Q (or $\nabla_Q\cdot$) and grad_Q (or ∇_Q) denote the usual vector differential operators, the differentations being defined with respect to the coordinates of the point Q. The operator grad_P (or ∇_P) implies differentiation with respect to the coordinates of P.

It can easily be seen in particular that:

$$\text{grad}_P\left(\frac{1}{r}\right) = -\text{grad}_Q\left(\frac{1}{r}\right) = -\frac{\bar{r}}{r^3} \tag{1}$$

Use will be made of the standard vector identities:

$$\text{div}\,(s\bar{A}) \equiv \bar{A}\,.\,\text{grad}\,s + s\,\text{div}\,\bar{A} \tag{2}$$

$$\text{curl}\,(s\bar{A}) \equiv (\text{grad}\,s)\times A + s\,\text{curl}\,\bar{A} \tag{2a}$$

and

$$(\bar{B}\times\bar{C})\times\bar{D} \equiv (\bar{B}.\bar{D})\,\bar{C} - (\bar{C}.\bar{D})\,\bar{B} \tag{3}$$

where s denotes a distributed scalar function, $\bar{A}$ a distributed vector function and $\bar{B}$, $\bar{C}$ and $\bar{D}$ are general vectors; Gauss' divergence theorem will also be used:

$$\iiint_{\Omega} \text{div}_Q\,\bar{A}\;d\Omega = \iint_S \bar{A}\,.\,\hat{n}\;dS \tag{4}$$

where $\bar{A}$ is a continuous vector function defined at every point Q within the arbitrary volume Ω and $\hat{n}$ denotes the outward normal to the surface S bounding Ω.

Consider the effects induced at a point P by an elemental singularity (source, doublet or elemental vortex filament) located at a point Q; this may be an element of a continuous line, surface or volume distribution.

The elemental potential $d\Phi_s$ and velocity $d\bar{V}_s$ induced at P by an elemental <u>source</u> of strength $d\sigma$ located at Q are given by:

$$d\Phi_s = -\frac{1}{4\pi r}\,d\sigma \quad \text{and} \quad d\bar{V}_s = \text{grad}_P\,(d\Phi_s) = \frac{\bar{r}}{4\pi r^3}\,d\sigma \tag{5}$$

The elemental potential $d\Phi_d$ and velocity $d\bar{V}_d$ induced at P by an elemental <u>doublet</u> $d\bar{\mu}$ located at Q (with $d\bar{\mu} = \hat{a}\,d\mu$, where $\hat{a}$ is a unit vector along the doublet axis, the positive direction of the axis being from the 'negative' to the 'positive' end of the doublet) are given by:

$$\left\{\begin{aligned} d\Phi_d &= \frac{1}{4\pi}\,\text{grad}_P\left(\frac{1}{r}\right)\cdot d\bar{\mu} = -\frac{1}{4\pi}\,\frac{\bar{r}.d\bar{\mu}}{r^3} \\ &\text{and} \\ d\bar{V}_d &= \frac{1}{4\pi}\,\text{grad}_P\,(d\Phi_d) = -\frac{1}{4\pi}\,\text{grad}_P\left(\frac{\bar{r}.d\bar{\mu}}{r^3}\right) \end{aligned}\right. \tag{6}$$

The elemental velocity $d\bar{V}_v$ induced at P by an elemental <u>vortex</u> filament $d\bar{\Gamma}$ located at Q (the positive direction of the vector $d\bar{\Gamma}$ being associated with a clockwise rotation about the filament) is given by:

$$(7) \qquad d\bar{V}_v = -\frac{1}{4\pi} \frac{\bar{r} \times d\bar{\Gamma}}{r^3}$$

It may be noted by comparing (5) and (7) that the velocity field induced by a <u>unidirectional</u> vorticity distribution (line, surface or volume) may be derived from that induced by a source distribution with the same spatial density variation, simply by computing the vector cross-product of the source-induced velocity with the unit vector defining the vorticity direction. The effect of vorticity distributions of variable direction may simply be constructed by compounding three different unidirectional vorticity distributions (even though each of these in isolation may be physically impossible). This subject is discussed in depth in ref. 1.

3. INTEGRATION OF ELEMENTARY CONTRIBUTIONS

3.1 Distribution of Vorticity on a Closed Surface

3.1.1 *Velocity Induced at an External Point*

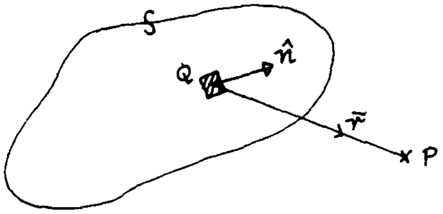


Fig. 1 (a)

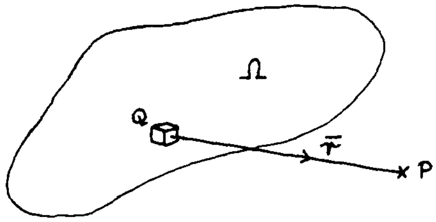


Fig. 1 (b)

Consider the velocity $\overline{V}_P$ induced at a point P due to a continuous surface vorticity distribution of variable density $\bar{\gamma}_s$ lying everywhere on the closed surface S of an arbitrary volume Ω (see Fig. 1 (a)); P lies outside this volume. Since $\bar{\gamma}_s$ at any point of the surface must be perpendicular to the local surface unit outward normal $\hat{n}$, we can write $\bar{\gamma}_s = \bar{\mu} \times \hat{n}$, where $\bar{\mu}$ is some variable vector field function defined at the surface S.

Using (7) the velocity induced at P is given by:

$$(8) \qquad 4\pi \overline{V}_P = \iint_S (\bar{\mu} \times \hat{n}) \times \bar{b} \; dS$$

where $\bar{b}$ is an abbreviation for the variable vector $\bar{r}/r^3$ (i.e. $-\mathrm{grad}_P(\frac{1}{r})$).

Now, using (3):

$$(\bar{\mu} \times \hat{n}) \times \bar{b} \equiv (\bar{\mu}.\bar{b})\,\hat{n} - (\hat{n}.\bar{b})\,\bar{\mu}.$$

Inserting this in (8) and considering only the component in the direction x:

$$4\pi \overline{V}_P . \hat{\imath} = \iint_S \left[(\bar{\mu}.\bar{b})\, n_x - (\hat{n}.\bar{b})\, \mu_x \right] dS$$

$$= \iint_S [\hat{\imath}\,(\bar{\mu}.\bar{b})].\hat{n}\; dS - \iint_S (\mu_x \bar{b}).\hat{n}\; dS$$

which becomes, using (4):

$$(9) \qquad 4\pi \overline{V}_P . \hat{\imath} = \iiint_\Omega \mathrm{div}_Q \left[\hat{\imath}\,(\bar{\mu}.\bar{b})\right] d\Omega - \iiint_\Omega \mathrm{div}_Q \left[\mu_x \bar{b}\right] d\Omega$$

the integration extending over all points Q within the volume Ω.

Now using (2):

$$\mathrm{div}_Q\,[\hat{\imath}\,(\bar{\mu}.\bar{b})] \equiv \hat{\imath}.\mathrm{grad}_Q(\bar{\mu}.\bar{b}) + (\bar{\mu}.\bar{b})\,\mathrm{div}_Q\,\hat{\imath}$$
$$= \hat{\imath}.\mathrm{grad}_Q(\bar{\mu}.\bar{b}) \qquad \text{since } \mathrm{div}_Q\,\hat{\imath} \equiv 0$$

and

$$\mathrm{div}_Q\,[\mu_x \bar{b}] \equiv \bar{b}.\mathrm{grad}_Q\,\mu_x + \mu_x\,\mathrm{div}_Q\,\bar{b}$$
$$= \bar{b}.\mathrm{grad}_Q\,\mu_x$$

since $div_Q \bar{b} = \nabla_Q^2(\frac{1}{r}) = 0$ for all $r \neq 0$, i.e. when P lies outside Ω.

Substituting these expressions in (9):

$$(10) \qquad 4\pi \bar{V}_P \cdot \hat{\imath} = \iiint_\Omega [\hat{\imath} \cdot grad_Q(\bar{\mu}.\bar{b}) - \bar{b} \cdot grad_Q \mu_x]\, d\Omega.$$

Expanding the operator $grad_Q$ gives:

$$(11) \quad \left\{ \begin{aligned} \hat{\imath}.\, grad_Q(\bar{\mu}.\bar{b}) &= \frac{\partial}{\partial x_Q}(\mu_x b_x + \mu_y b_y + \mu_z b_z) \\ &= \mu_x \frac{\partial b_x}{\partial x_Q} + \mu_y \frac{\partial b_y}{\partial x_Q} + \mu_z \frac{\partial b_z}{\partial x_Q} \\ &\quad + b_x \frac{\partial \mu_x}{\partial x_Q} + b_y \frac{\partial \mu_y}{\partial x_Q} + b_z \frac{\partial \mu_z}{\partial x_Q} \end{aligned} \right.$$

Now $\bar{\mu}$ is invariant with respect to the position of the point P, so that using (1) we obtain:

$$\mu_x \frac{\partial b_x}{\partial x_Q} = -\mu_x \frac{\partial b_x}{\partial x_P} = -\frac{\partial}{\partial x_P}(\mu_x b_x), \text{ etc.}$$

so that (11) becomes:

$$\hat{\imath}.\, grad_Q(\bar{\mu}.\bar{b}) = -\frac{\partial}{\partial x_P}(\mu_x b_x + \mu_y b_y + \mu_z b_z) + b_x \frac{\partial \mu_x}{\partial x_Q} + b_y \frac{\partial \mu_y}{\partial x_Q} + b_z \frac{\partial \mu_z}{\partial x_Q}$$

Also: $\bar{b}.\, grad_Q \mu_x = b_x \frac{\partial \mu_x}{\partial x_Q} + b_y \frac{\partial \mu_x}{\partial y_Q} + b_z \frac{\partial \mu_x}{\partial z_Q}$

Inserting these last two equations into (10) and rearranging gives:

$$4\pi \bar{V}_P \cdot \hat{\imath} = \iiint_\Omega \Big[-\frac{\partial}{\partial x_P}(\bar{\mu}.\bar{b}) + b_x\Big(\frac{\partial \mu_x}{\partial x_Q} - \frac{\partial \mu_x}{\partial x_Q}\Big) + b_y\Big(\frac{\partial \mu_y}{\partial x_Q} - \frac{\partial \mu_x}{\partial y_Q}\Big) + b_z\Big(\frac{\partial \mu_z}{\partial x_Q} - \frac{\partial \mu_x}{\partial z_Q}\Big)\Big]\, d\Omega$$

which by inspection is equivalent to:

$$(12) \qquad 4\pi \bar{V}_P \cdot \hat{\imath} = \iiint_\Omega -\hat{\imath} \cdot [grad_P(\bar{\mu}.\bar{b})]\, d\Omega + \iiint_\Omega \hat{\imath}.[\bar{b} \times curl_Q \bar{\mu}]\, d\Omega.$$

If the process following equation (8) is repeated for the components in the y and z directions, it follows that we finally obtain from (12):

$$(13) \qquad 4\pi \bar{V}_P = \iiint_\Omega -grad_P(\bar{\mu}.\bar{b})\, d\Omega - \iiint_\Omega -\bar{b} \times curl_Q \bar{\mu}\, d\Omega$$

By comparing with equations (6) and (7) and remembering that $\bar{b} = \bar{r}/r^3$ it can be seen that:

$$\frac{1}{4\pi}\iiint_{\Omega} -\,\mathrm{grad}_P\,(\bar{\mu}\cdot\bar{b})\,d\Omega$$

is the velocity induced at P by a volume doublet distribution throughout the volume Ω, of variable density $\bar{\mu}$, (see Fig. 1 (b)), and that

$$\frac{1}{4\pi}\iiint_{\Omega} -\bar{b}\times \mathrm{curl}_Q\,\bar{\mu}\;\; d\Omega$$

is the velocity induced at P by a volume vorticity distribution of variable density $\bar{\gamma}_v = \mathrm{curl}_Q\,\bar{\mu}$. Since the equations are derived for an arbitrary volume doublet distribution $\bar{\mu}$ and an arbitrary external point P, it follows that these relationships between the equivalent doublet and vorticity densities apply in a local sense for any point Q within Ω or on S, and not only in an integral sense.

The volume vorticity distribution $\bar{\gamma}_v$ is thus everywhere solenoidal since $\mathrm{div}_Q\;\bar{\gamma}_v = \mathrm{div}_Q\;\mathrm{curl}_Q\,\bar{\mu} = 0$ (i.e. vorticity is neither created nor destroyed at any point within Ω). It can furthermore easily be shown by constructing a local cartesian system (ξ, η, n) at any point on S that the vorticity flux $\hat{n}.\mathrm{curl}\,\bar{\mu}\;d\xi\,d\eta$ entering an elemental area $dS = d\xi\,d\eta$ from the interior of Ω is exactly equal to the flux of the surface vorticity leaving the edges of dS. It follows that the surface vorticity $\bar{\mu}\times\hat{n}$ is physically meaningful <u>in isolation</u> only in the special case $\mathrm{curl}\,\bar{\mu} = 0$; unless this is the case, the surface vorticity cannot be replaced by surface doublicity in the manner discussed later.

By equating the right-hand sides of equations (8) and (13) and rearranging the terms, we obtain the following theorem:

> <u>*Theorem 1*</u> *"The velocity induced by a volume distribution of doublets of arbitrary volume density $\bar{\mu}$, at a point lying outside that volume, is identical to the velocity induced at that point by a volume distribution of vorticity of volume density $\bar{\gamma}_v = \mathrm{curl}\,\bar{\mu}$ throughout that same volume, together with a surface vorticity distribution $\bar{\gamma}_s$ on the surface of that volume, whose surface density and direction are given by $\bar{\gamma}_s = \bar{\mu}_e\times\hat{n}$ where $\bar{\mu}_e$ denotes the local value of $\bar{\mu}$ infinitesimally inside that surface, and $\hat{n}$ denotes the local outward unit normal to that surface".*

It may be noted that, if the volume doublet distribution is known, the velocity induced at an external point may be evaluated explicitly either directly in terms of the volume doublet distribution or in terms of the equivalent, volume-plus-surface vorticity distribution. In the special case where the volume vorticity has a density of zero (i.e. $\mathrm{curl}\,\bar{\mu} = 0$), this velocity may be obtained purely from a surface integral. This statement requires modifying when the point at which the velocity is required lies inside the volume Ω.

3.1.2 *Velocity Induced at an Internal Point*

In the case where the point P lies within the volume Ω the value of $\frac{1}{r}$ approaches infinity for points Q in the immediate vicinity of P; at P, $\nabla_Q^2(\frac{1}{r})$ is no longer defined, and the steps leading from equation (9) to equation (10) which assume that $\nabla_Q^2(\frac{1}{r}) = 0$, are no longer valid. This problem may be circumvented by constructing a small sphere of radius ε about P (its surface being denoted by Σ and its volume by v) and excluding this sphere from the region of integration; the point P may then be considered to lie outside the volume $(\Omega - v)$, and the above equations are then still applicable for this volume and the associated surface $(S + \Sigma)$. The equations relating to the complete volume Ω may then be obtained by augmenting the right-hand sides of equations (10) and (12) by the expression

$$\underset{\varepsilon \to 0}{\text{Limit}} \left[- \iiint_v \mu_x \, \mathrm{div}_Q \bar{b} \, d\Omega \right]$$

which was zero for the case where P lay outside Ω. Now as $\varepsilon \to 0$ the function becomes progressively more uniform within v and in the limit the above expression may be written $-\mu_x \iiint \mathrm{div}_Q \bar{b} \, d\Omega$. By Gauss' theorem (equation (4)) this then becomes $-\mu_x \iint \bar{b} \cdot \hat{n}' \, d\Sigma$ with $\hat{n}'$ the outward unit normal to the surface Σ. Since $\bar{r}$ is defined positive from Q to P (i.e. pointing towards the centre of this sphere), the quantity $\bar{b}.\hat{n}'$ is equal to $-1/\varepsilon^2$ for all points Q on Σ so that the above surface integral reduces to $4\pi\mu_x$. Consequently equation (13) becomes for a point P inside Ω:

$$(14) \qquad 4\pi \bar{V}_P = \iiint_\Omega - \mathrm{grad}\, (\bar{\mu} \cdot \bar{b}) \, d\Omega \; - \iiint_\Omega - \bar{b} \times \mathrm{curl}_Q \, \bar{\mu} \; d\Omega \; + 4\pi \bar{\mu} .$$

It can thus be seen that for the case where the point P lies inside the volume Ω , the following theorem is obtained:

> <u>*Theorem 1a*</u> *"The velocity induced by a volume distribution of doublets of arbitrary volume density $\bar{\mu}$, at a point which lies within that volume, is identical to the velocity induced at that point by a volume distribution of vorticity (of volume density $\bar{\gamma}_v = \mathrm{curl}\, \bar{\mu}$ throughout that volume), together with a surface vorticity distribution $\bar{\gamma}_s$ on the surface of that volume, (whose surface density and direction are given by $\bar{\gamma}_s = \bar{\mu}_e \times \hat{n}$, where $\bar{\mu}_e$ denotes the value of $\bar{\mu}$ infinitesimally inside that surface, and $\hat{n}$ denotes the outward unit normal to that surface), plus a term which is locally equal to the vector value $-\bar{\mu}$".*

It may be noted that if the volume distribution $\bar{\mu}$ is known, then the velocity may be evaluated either directly in terms of the volume doublet distribution or in terms of the equivalent volume-plus-surface vorticity distribution, augmented by the local term $-\bar{\mu}$. In the special case where the volume vorticity has a density of zero (i.e. $\mathrm{curl}\,\bar{\mu} = 0$), this velocity may be obtained purely from a surface integral plus the local term $-\bar{\mu}$.

3.2 <u>Source Distribution on a Closed Surface</u>

Referring again to Fig. 1a, consider the potential ϕ_P induced at a point P due to a continuous surface source distribution of variable density σ_s on the closed surface S. Suppose that the scalar function σ_s is expressed as $\sigma_s = \bar{\mu} \cdot \hat{n}$ where $\bar{\mu}$ is again some vector field function defined at the surface S.

Using (5) the potential induced at P is given by:

(15) $$4\pi\Phi_P = -\iint_S \frac{\bar{\mu}.\hat{n}}{r}\,dS$$

which, using (4) may be expressed:

$$4\pi\Phi_P = -\iiint_\Omega \operatorname{div}_Q\left(\frac{\bar{\mu}}{r}\right)d\Omega$$

and which, using (2), further reduces to:

$$4\pi\Phi_P = -\iiint_\Omega \bar{\mu}.\operatorname{grad}_Q\left(\frac{1}{r}\right)d\Omega - \iiint_\Omega \frac{1}{r}\operatorname{div}_Q\bar{\mu}\,d\Omega$$

or, using (1), to:

(16) $$4\pi\Phi_P = \iiint_\Omega \operatorname{grad}_P\left(\frac{1}{r}\right)\cdot\bar{\mu}\,d\Omega - \iiint_\Omega \frac{1}{r}\operatorname{div}_Q\bar{\mu}\,d\Omega.$$

By comparing with equations (6) and (5) it can be seen that

$$\frac{1}{4\pi}\iiint_\Omega \operatorname{grad}_P\left(\frac{1}{r}\right)\cdot\bar{\mu}\,d\Omega$$

is the potential induced at P by a volume doublet distribution of variable volume density $\bar{\mu}$ throughout Ω and that

$$\frac{1}{4\pi}\iiint_\Omega \frac{1}{r}\operatorname{div}_Q\bar{\mu}\,d\Omega$$

is the potential induced at P by a volume source distribution of variable volume density $\sigma_v = -\operatorname{div}_Q\bar{\mu}$ throughout Ω.

It can be seen that none of the above steps precludes the case where P lies inside Ω, i.e. equation (16) is equally valid for points lying inside or outside Ω.

Once again, the equations are valid for an arbitrary volume doublet distribution and for any point P; consequently the relationships between the equivalent source and doublet distributions apply in a local sense for any point Q in the respective distributions, and not only in an integral sense.

The volume source distribution is everywhere irrotational since $\operatorname{curl}_Q\sigma_v = \operatorname{curl}_Q\operatorname{div}_Q\bar{\mu} = 0$ (i.e. vorticity is neither created nor destroyed at any point within Ω). Furthermore it follows immediately from Gauss' theorem (equation (4)) that the total volume integral of the source density $-\operatorname{div}\bar{\mu}$ is equal and opposite to the total surface integral of the source density $\bar{\mu}.\hat{n}$.

By applying the operator grad_P to both sides of equations (15) and (16), the corresponding equations for the velocity $\bar{V}_P$ are obtained. By equating the right-hand sides of these equations and rearranging the terms, the following theorem is obtained:

<u>Theorem 2</u> *"The potential and velocity induced at any point P by a volume distribution of doublets of arbitrary volume density $\bar{\mu}$ are identical to those induced at the same point by a volume distribution of sources of volume density $\sigma_v = -\operatorname{div}\bar{\mu}$ throughout that same volume, together with a surface source distribution $\sigma_s = \bar{\mu}_i.\hat{n}$ on the surface of that volume, where $\bar{\mu}_i$ denotes the local value of $\bar{\mu}$ infinitesimally inside that surface, and $\hat{n}$ denotes the local outward unit normal to that surface".*

Since Theorem 1 (and 1a) and Theorem 2 are stated for arbitrary volume doublet distributions, it follows that the velocity field due to any doublet distribution may be replaced by that due <u>either</u> to the source distributions of Theorem 2 <u>or</u> to the vorticity distributions of Theorem 1 (plus the local term $\bar{\mu}$ for points lying inside the influencing volume). By rearranging the various effects, the following theorem can be obtained:

<u>*Theorem 3*</u> *"The velocity field due to a combined volume distribution of sources of density ($\sigma_v = div\,\bar{\mu}$) and vorticity of density ($\bar{\gamma}_v = curl\,\bar{\mu}$) throughout a volume Ω is identical to that due to a combined distribution, on the surface of that volume, of surface sources of density ($\sigma_s = \bar{\mu}_e \cdot \hat{n}$) and surface vorticity of density ($\bar{\gamma}_s = -\bar{\mu}_e \times \hat{n}$), together with a local velocity increment equal to the local value of $\bar{\mu}$ for points which lie inside the volume Ω".*

3.3 <u>Normal Doublet Distribution on a Closed Surface</u>

Referring again to Fig. 1a, consider the velocity induced at a point P due to a continuous surface distribution of doublets, of density μ_s, normal to the closed surface S.

Using (6) and (1) the potential induced at P is given by:

$$(17) \qquad 4\pi\,\phi_P = -\iint_S \mu_s\, grad_Q\left(\frac{1}{r}\right)\cdot\hat{n}\; dS$$

which by virtue of Gauss' theorem (4), becomes:

$$(18) \qquad 4\pi\,\Phi_P = -\iiint_\Omega div_Q\left(f\, grad_Q\left(\frac{1}{r}\right)\right) d\Omega$$

where f is interpreted as a scalar function which varies continuously throughout Ω, but which has the scalar value μ_s at the surface S.

Now using (2):

$$\iiint_\Omega div_Q\left(f\, grad_Q\left(\frac{1}{r}\right)\right) = \iiint_\Omega \left(grad_Q\left(\frac{1}{r}\right)\right)\cdot\left(grad_Q f\right) d\Omega \; + \iiint_\Omega f\, \nabla_Q^2\left(\frac{1}{r}\right) d\Omega.$$

By an argument similar to that used in 3.1.2, the last term is equal to zero for points P which lie outside Ω, and equal to the local value $-4\pi f$ for points within Ω.

Thus for points outside Ω, (18) becomes:

$$(19) \qquad 4\pi\,\phi_P = -\iiint_\Omega \left(grad_Q\left(\frac{1}{r}\right)\right)\cdot\left(grad_Q f\right) d\Omega$$

whilst for points within Ω it becomes:

$$(19a) \qquad 4\pi\,\Phi_P = -\iiint_\Omega \left(grad_Q\left(\frac{1}{r}\right)\right)\cdot\left(grad_Q f\right) d\Omega \; + 4\pi f_P.$$

By comparing with equation (6) it can be seen that the volume integral in equations (19) and (19a) is equal to the potential induced at P by a volume doublet distribution $\bar{\mu}_v$, given by $\bar{\mu}_v = grad_Q f$; this doublet distribution is irrotational since $curl_Q grad_Q f = 0$. We thus obtain the following theorem:

Theorem 4 "*The potential and velocity induced at any point by an irrotational volume distribution of doublets of arbitrary density $\bar{\mu}_v$ = grad f (where f is an arbitrary scalar function in that volume) is identical to that induced by a surface distribution of normal doublets on the surface of that volume, given by $\bar{\mu}_s = f_e \hat{n}$ (where f_e denotes the value of f infinitesimally inside S and $\hat{n}$ denotes the unit outward normal to that surface), augmented when P lies inside that volume, by the local values of $-f$ and of $-$grad f, respectively*".

By comparing the above Theorem with Theorem 1a for the case where curl $\bar{\mu}_v$ is zero (as above) the following Theorem is obtained:

Theorem 5 "*The velocity induced at any point by a continuous surface distribution of normal doublets $\mu_s \hat{n}$ on any closed surface is identical to that induced by a surface vorticity distribution $\bar{\gamma}_s$ on that same surface given by $\bar{\gamma}_s = (\mathrm{grad}_2\, \mu_s) \times \hat{n}$ where $\hat{n}$ denotes the local unit normal to the surface and the operator grad$_2$ indicates that only surface derivatives are required (the normal component of grad μ contributes nothing to the vector product with $\hat{n}$)*".

3.4 General Statement of Equivalent Distributions

By comparing Theorems 2, 4 and 5 in the special case where the volume doublet distribution $\bar{\mu}_v$ throughout the volume Ω is such that $\mathrm{div}_Q\, \bar{\mu}_v = 0$ and $\mathrm{curl}_Q\, \bar{\mu}_v = 0$, and is defined by $\bar{\mu}_v$ = grad f, the following general theorem is obtained:

Theorem 6 "*The following distributions produce identical velocity fields:*

(i) The arbitrary doublet distribution $\bar{\mu}_v$ = grad f such that $\nabla^2 f = 0$ throughout Ω;

(ii) The surface source distribution $\sigma_s = \hat{n} \cdot$ grad f on the closed surface S of Ω, grad f being evaluated infinitesimally inside S;

(iii) The surface normal doublet distribution $\bar{\mu}_s = f\hat{n}$ on the closed surface S of Ω, f being evaluated infinitesimally inside S; for points lying inside Ω the velocity due to this $\bar{\mu}_s$ must be augmented by the local value of $-$ grad f;

(iv) The surface vorticity distribution $\bar{\gamma}_s$ = grad $f \times \hat{n}$, grad f being evaluated infinitesimally inside S; for points lying inside Ω the velocity due to this $\bar{\gamma}_s$ must be augmented by the local value of $-$grad f ".

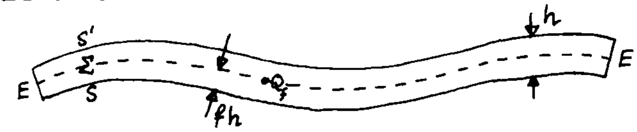
Note that the alternative surface distributions may lie on <u>different</u> surfaces of the volume. For example volume doublicity parallel to the axis of a prism may be replaced by surface source distributions on the end-faces normal to that axis, or by surface vorticity distributions on the remaining surfaces.

4. APPLICATION TO CERTAIN SIMPLE CASES

Some results will now be derived which are of relevance to surface singularity methods.

4.1 Doublet Distribution on a non-closed Surface

Consider an arbitrary surface S with continuously varying tangent plane. S has the perimeter L. Consider the volume Ω swept out by displacing every point of S by a constant distance h in the direction of the normal to the local tangent plane of S. It is assumed that h is less than the smallest radius of concave curvature of S so that normals to S do not intersect within Ω. The boundary of Ω can then be defined by the displaced surface S' (which also has continuously varying tangent plane), by the 'closed' edge E, and by the original surface S.



Consider now the surface Σ passing through all points Q_f which are contained within this volume at a constant normal distance fh from S ($0 < f < 1$); suppose that Σ is defined by the equation $F_f(x,y,z) = 0$ where (x,y,z) are the cartesian coordinates of a point on this surface, and that E is defined by $G(x,y,z) = 0$. At any point Q_f the vector $\overline{N} = grad\ F_f(x,y,z)$ defines a vector which is locally normal to Σ (and parallel to the unit vector $\hat{n}$ at the point where $\overline{N}$ intersects S). In general $\overline{N}$ will not be a unit vector; its magnitude will vary over Σ and indeed for points lying along the same line $\overline{N}$' but with different values of f.

Suppose no that the above volume Ω contains a doublet distribution which at any point is defined by $\overline{\mu}_v = m\overline{N}$ where m is a scalar function which is constant along any normal to the surface S but is otherwise variable throughout the volume Ω. At some external point P which is sufficiently far removed from Ω, this volume doublet distribution will produce approximately the same potential and velocity as a surface doublet distribution of variable surface density $mh\tilde{N}\hat{n}$ on some 'mean' surface between S and S', where $\tilde{N}$ is the mean magnitude of $\overline{N}$ along the local normal to S, and $\hat{n}$ is the local unit normal to S.

Now, according to Theorem 1, this volume doublet distribution $\overline{\mu}_v$ will produce exactly the same velocity at any external point P as a volume vorticity distribution of density $\overline{\gamma}_v = \mathrm{curl}_Q\,\overline{\mu}_v$ throughout Ω, together with a surface vorticity of density $\overline{\gamma}_s = \overline{\mu}_v \times \hat{n}$ on the boundaries S, S' and E, $\hat{n}$ being the local outward unit normal for each of these surface. Since $\overline{\mu}_v$ is defined to be normal to S and S', the equivalent $\overline{\gamma}_s$ on these surfaces is zero. The equivalent volume vorticity distribution $\overline{\gamma}_v$ can be written, using equation (2a):

$$\overline{\gamma}_v = \mathrm{curl}_Q\,\overline{\mu}_v = \mathrm{curl}_Q\, m\overline{N} = (\mathrm{grad}_Q\, m)\times\overline{N} + m\,\mathrm{curl}_Q\,\overline{N}$$
$$= (\mathrm{grad}_Q\, m)\times\overline{N}$$

since $\mathrm{curl}_Q\,\overline{N} = \mathrm{curl}_Q\,\mathrm{grad}_Q\,F_f(x,y,z) = 0$.

It may be noted that $\mathrm{grad}_Q m$ will have no component in the direction of $\overline{N}$ since m is defined to be constant in that direction. Also, since h is constant, we can write $(\mathrm{grad}_Q\, m) = \frac{1}{h}(\mathrm{grad}_Q\, mh)$ and this vector will be constant for all points which lie on a particular normal to S. Thus at an external point sufficiently far removed from Ω, the volume vorticity distribution will produce approximately the same velocity as a surface vorticity distribution of variable surface density $\overline{\gamma}_s = (\mathrm{grad}_2\, mh\tilde{N}) \times \hat{n}$ on some 'mean' surface; the operator grad_2 excludes derivatives normal to this surface; $\tilde{N}$ is again the mean magnitude of $\bar{N}$ along the local normal to S, and $\hat{n}$ the local unit normal to S.

The surface vorticity distribution $\bar{\gamma}_s$ on the 'edge' surface E can be written:

$$\bar{\gamma}_s = \bar{\mu}_v \times \hat{n} = (m_e \bar{N}_e) \times \hat{n}_e = \frac{1}{h}(m_e h \tilde{N})\,\hat{n} \times \hat{n}_e$$

where $\hat{n}_e$ is the local normal to E and m_e and $\bar{N}_e$ are the local values of M and N at the surface E.

At an external point sufficiently far removed from Ω this vorticity distribution on the edge surface E produces approximately the same velocity as a line vortex distribution of variable line density $\bar{\Gamma}$ defined by $\bar{\Gamma} = m_e h \tilde{N} \hat{t}$ where $\hat{t} = \hat{n} \times \hat{n}_e$ is the unit vector along the perimeter of the 'mean' surface.

In the limit as h decreases to zero, whilst the function m is increased such that the function $mh\tilde{N}$ remains finite (μ_s say), the 'mean' surface becomes coincident with S and $\tilde{N}$ becomes the vector grad $F_o(x, y, z)$ pertaining to S.

Consequently the above 'approximate' statements reduce to the following exact statement:

> *"The velocity field induced by a distribution of normal doublets of surface density $\bar{\mu}_s$, on any bounded surface S with continuously turning tangent plane, is identical to that induced by a surface vorticity distribution γ_s on that same surface, defined by $\bar{\gamma}_s = \text{grad}_2\, \mu_s \times \hat{n}$ (where $\hat{n}$ is the local unit normal to S), together with a line vortex $\bar{\Gamma}$ along the perimeter L of S, defined by $\bar{\Gamma} = \mu_e \hat{t}$, where μ_e is the magnitude of $\bar{\mu}_s$ at that perimeter, and $\hat{t}$ is the unit vector along L such that the vector $\hat{t} \times \bar{\mu}_s$ points away from S".*

This statement forms an extension to Theorem 5, which was derived for a closed surface only.

Note that the above statement has been derived for any general surface S with continuously turning tangent plane; however, it can be seen that it is also valid for the case of surfaces containing lines at which the tangent plane is discontinuous, provided that the magnitude of the doublet density is continuous across such a line; similarly the perimeter L need not form a line with continuously turning tangent.

In the special case of surface doublicity normal to S and of uniform density μ over S, the equivalent surface vorticity density is zero (since $\text{grad}_2\, \mu_s = 0$) so that this doublet distribution may be replaced by the concentrated edge vortex $\bar{\Gamma} = \mu\hat{t}$ alone, this line vortex being of constant line density. Conversely, it can be seen that a constant-strength line vortex along any arbitrary closed contour produces a velocity field identical to that produced by a uniform normal doublet distribution on any arbitrary surface having that contour as perimeter.

It is also worth noting that a surface containing doublicity whose axis is everywhere <u>tangential</u> to that surface and of density μ is equivalent, by a similar argument, to a surface <u>source</u> distribution of density $\sigma_s = -\text{div}_2\, \bar{\mu}$ together with a line source of density $\sigma_e = \bar{\mu}\cdot\hat{s}$ where $\hat{s}$ denotes the local unit normal to the perimeter of S, lying in the tangent plane to S at that perimeter and orientated away from S.

In the special case of surface doublicity having its axis everywhere tangential to S and unidirectional and its density μ uniform over S, the equivalent surface source density is zero, so that this doublet distribution may be replaced by a concentrated edge source of variable line density $\sigma_e = \bar{\mu}.\hat{s}$. Advantage will be taken of this fact in section 4.2.

4.2 Velocity Induced by a Uniform Source Distribution on a Planar Polygon

Consider a planar polygon S carrying a tangential surface doublet distribution $\bar{\mu}$ which is unidirectional and which has the uniform density K. Suppose that a cartesian coordinate system (x,y,z) is defined such that the polygon lies in z = 0 and that the x axis is parallel to the axis of the doublicity. Thus we can write $\bar{\mu} = K\hat{\imath}$. As indicated in section 4.1, this doublet distribution produces the same potential and velocity field as a concentrated line source of density ($K\hat{s}.\hat{\imath}$) along the edges of the polygon. On each straight edge of the polygon, the vector $\hat{s}$ is constant, so that the equivalent line source along that edge is uniform in density.

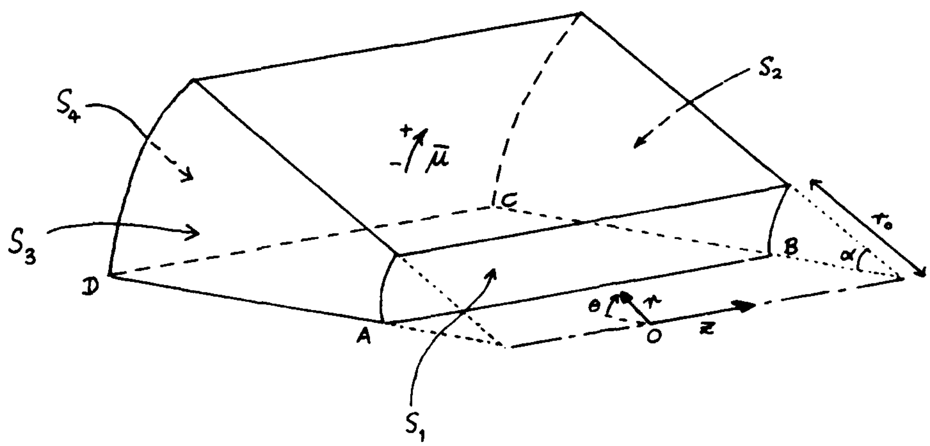
It can be seen by comparing equations (5) and (6) that the potential induced at any point by the doublet distribution $K\hat{\imath}$ on S is the same as the negative of the x-component of the velocity induced at that point by a source distribution of uniform density K on S.

It follows that the x-component of the velocity induced at any point by a uniform surface source distribution of density K on S is identical to the sum of the potentials induced at that point by a line source along each edge of the polygon of density $-K\hat{s}.\hat{\imath}$ (constant along each edge). Similarly the y-component can be obtained from the sum of the potentials due to line sources of density $-K\hat{s}.\hat{\jmath}$. These potentials are obtained very simply by integrating the elemental expression given in equation (5).

The z-component of the velocity may be derived by considering the potential due to a uniform doublet distribution defined by $\bar{\mu} = K\hat{k}$ (i.e. doublets with axes normal to the plane S). In this case the equivalent system consists of a line vortex of constant density K on each edge of the polygon; the potential at P due to this closed ring vortex may be obtained by integrating (from infinity to the point P, with respect to z) the z-component of the velocity induced by this vortex. This latter function is obtained by integrating equation (7), (or alternatively from the velocity induced by a line source of density K, using the equivalence expressed after equation (7)).

It can thus be seen that the velocity induced by a uniform surface source distribution on a plane polygon may be obtained purely from line integrals evaluated along the edges of that polygon; this fact is utilised in 'panel methods' which employ such source distributions. Similar arguments may be used to simplify the integrals required in higher-order methods employing non-uniform source distributions or non-planar panels.

4.3 Source distribution on the Surface of a Wedge



Consider the volume formed by rotating the rectangle ABCD through an angle α about the z axis which is parallel to AB and a distance r_0 from it. Points within this wedge will be defined in terms of the cylindrical polar coordinates (r, θ, z)

Suppose that the volume contains a doublet distribution $\bar{\mu}$ of volume density μ_0/r where μ_0 is constant, the axis of each elemental doublet being normal to the plane containing itself and Oz (with the positive end uppermost). It can easily be shown that both curl $\bar{\mu}$ and div $\bar{\mu}$ are zero throughout the volume.

Using Theorem 1, the volume doublet distribution produces the same external velocity as a surface vorticity distribution $\bar{\gamma}_s$ on the surfaces of that volume, described by $\bar{\gamma}_s = \bar{\mu} \times \hat{n}$ (the volume vorticity has zero density since curl $\bar{\mu} = 0$). Since the doublicity is locally normal to the rectangular faces, the value of $\bar{\gamma}_s$ on those faces is zero, and we are left with vorticity which has surface density μ_0/r on the remaining faces.

On the surface S_1 this vorticity vector points in the direction of the z axis whereas on S_4 it points in the opposite direction, on S_2 it points radially away from the z axis, and on S_3 radially towards it. In the limit as $r \rightarrow 0$, the surface vorticity distribution on S_1, of density μ_0/r_0 , distributed over an arc length of αr_0, produces the same velocity field as a line vortex of line density $\alpha\mu_0$, along the line AB which now lies on the z axis.

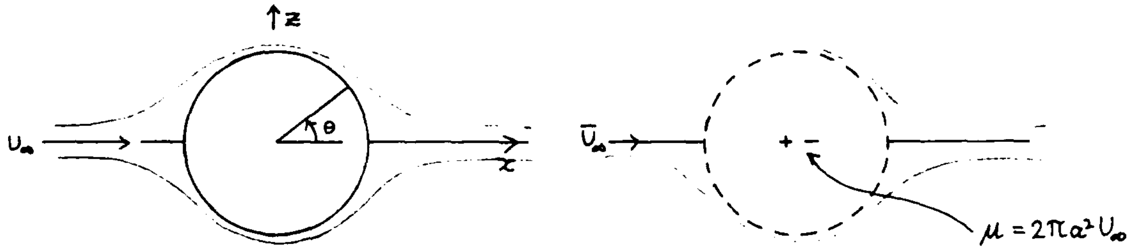
Using Theorem 2, on the other hand, the volume doublet distribution produces the same external velocity as a surface source distribution σ_s on the surfaces of that volume, of surface density $\sigma_s = \bar{\mu}.\hat{n}$ (the volume source density is zero since div $\bar{\mu} = 0$). This surface source density is zero except on the two rectangular plane faces, on which it has the variable magnitude μ_0/r. The source density is positive on the upper surface and negative on the lower surface.

Considering the case where ABCD is of indefinite extent and $r_0 = 0$, so that the volume corresponds to an infinite wedge of included angle α, it can be seen that the velocity field induced outside the wedge by a line vortex of density Γ along its ridge corresponds to that induced by surface source and sink distributions on its two faces, of surface density $\sigma = \Gamma/\alpha r$; this density clearly approaches infinity near the ridge (i.e. as $r \rightarrow 0$).

By adopting the results derived in section 4.1, the line vortex may be replaced by a uniform surface distribution of *normal* doublets over any indefinitely large surface which terminates at the ridge of the wedge.

These results (or modified versions of them, using line vortices which are not of constant strength, or surfaces more complex than an infinite wedge) are of relevance in problems where wakes, modelled by doublet sheets, arise from wing trailing edges or from smooth surfaces (wedge angle $\alpha = \pi$) which are modelled by surface source distributions.

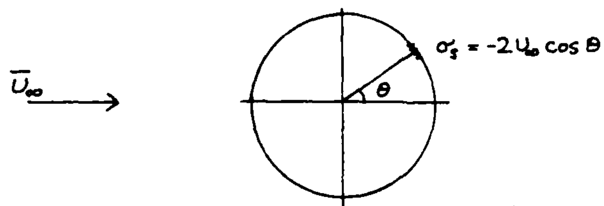
4.4 Modelling of Flow Past an Infinite Circular Cylinder; Jets



solid cylinder (a) 2-D Doublet

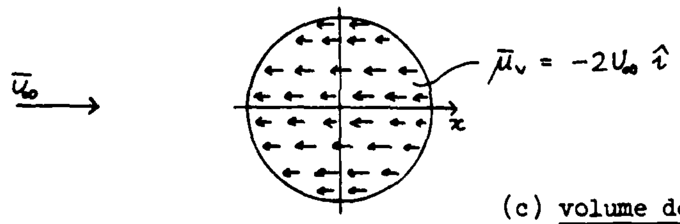
It is well known that the potential flow outside a solid 2-D circular cylinder of radius a installed perpendicular to a uniform free stream $\overline{U}_\infty$ can be modelled by the interaction between $\overline{U}_\infty$ and a 2-D line doublet of line density $2\pi a^2 U_\infty$, the doublet axis being opposite in direction to $\overline{U}_\infty$ (as sketched above); $\overline{U}_\infty$ is assumed parallel to the x axis with x = 0 at the centre of the cylinder.

It is also easy to demonstrate that the cylinder may be replaced by a surface source distribution of surface density $\sigma_s = -2U_\infty \cos\theta$:



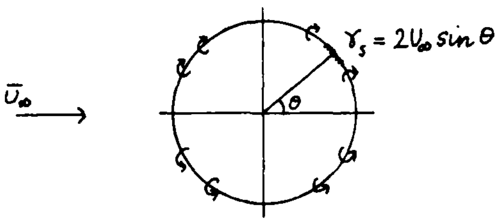
(b) Surface sources

By Theorem 6 it can then be seen that this source distribution will induce the same external velocity as a volume doublet distribution given by $\overline{\mu}_v = \text{grad}\,(-2U_\infty x) = -2U_\infty \hat{\imath}$, i.e. a uniform volume distribution of doublets with the doublet axis pointing in the -x direction:

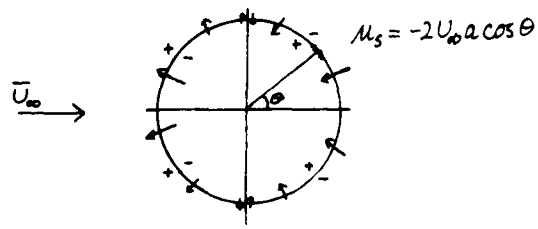


(c) volume doublicity

This volume doublet distribution is also equivalent to either a surface distribution of vorticity $\gamma_s = 2U_\infty \sin\theta$ or of normal doublets of surface density $\mu_s = -2U_\infty a \cos\theta$:



(d) <u>Surface vorticity</u>



(e) <u>Surface normal doublets</u>.

In this special case, the volume doublet distribution (c) is equivalent to either any one of the surface distributions (b, d, e) or the <u>line</u> distribution (a).

Similar arguments may be extended to cases in which the cylinder is not 'solid' or 'straight' or 'circular'; such cases arise in the modelling of jets issuing obliquely into a uniform stream. Such jets take up a curved shape along their axis, and have a non-circular cross-section; fluid is entrained into the jet through its boundary; the inside of the jet contains a pair of contra-rotating vortices.

The optimum modelling of this jet is a current subject of research and will not be considered in detail here, but it is worth noting that the equivalent representations of Theorem 6 may allow certain of the models under development to be considerably reduced in terms of their computational requirement.

5. CONCLUSIONS

Certain theorems have been derived which state the equivalence of particular line, surface and volume distributions of sources, doublets and vorticity. These theorems allow the influence of singularity distributions employed, for example, in 'panel methods', to be expressed in terms of the influence of more elementary distributions. It is possible that these results will be of some value in the computation of compressible and/or viscous flows and in the modelling of non-linear problems involving, for example, thick wakes or lifting jets.

Reference

1. A Note on the Relationship of the Influences of Sources and Vortices in Incompressible and Linearised Compressible Flow.

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