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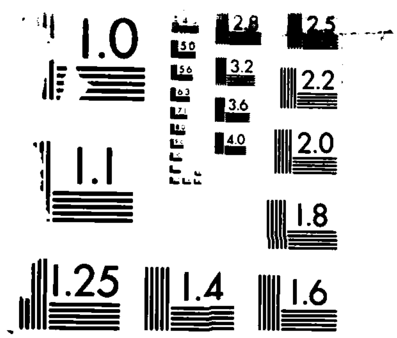

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CENTER FOR STOCHASTIC PROCESSES

Department of Statistics
University of North Carolina
Chapel Hill, North Carolina

CONTINUITY OF GAUSSIAN PROCESSES

by

Gennady Samorodnitsky

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CONTINUITY OF GAUSSIAN PROCESSES

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Abstract

~~We~~ give sufficient conditions for local continuity of the isonormal process L at some point of its parameter set. Since a Gaussian process, defined on a compact parameter space, that is a.s. continuous at each point is sample continuous, our result can be applied to the problem of general sample continuity of Gaussian processes. It is shown that ~~our~~ sufficient conditions are strictly weaker than the classical sufficient conditions for sample continuity.

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1. Introduction

Let $\{X(t), t \in C\}$ be a Gaussian process with a continuous covariance function over a compact subset C of a metric space (S,d). Such a process is called *sample-continuous* if there is a version of the process with continuous sample functions. Equivalently, $\{X(t), t \in C\}$ is sample-continuous if it is uniformly continuous for t restricted to a countable dense subset of C. The process is said to be *sample-bounded* if it has a version with bounded sample functions.

Let t_0 be a point of C. The process is said to be *continuous at* t_0 if there is a version of the process with sample functions that are continuous at t_0. Equivalently, the process is continuous at t_0 if $P(\lim_{\substack{t \to t_0 \\ t \in C^*}} X_t = X_{t_0}) = 1$, C^* being a countable dense subset of C.

Let H be a real, infinite-dimensional Hilbert space. A linear map L from H into real Gaussian variables with $ELx = 0$, $ELxLy = (x,y)$ for all $x,y \in H$ is called the *isonormal* Gaussian process on H. (As usual, $(\cdot,\cdot)$ denotes the inner product in H).

A modern approach to the study of sample function continuity and boundedness of Gaussian processes reduces this problem to the study of those sets $C \subset H$ on which the isonormal L has continuous or bounded sample functions, called GC-sets and GB-sets respectively (Dudley (1967, 1973), Feldman (1971), Sudakov (1969, 1971)). This approach relates GC and GB properties to a certain measure of the size of a set C in H, called *metric entropy*.

Let C be a subset of a metric space (S,d). Given $\varepsilon > 0$, let $N(C,\varepsilon) \equiv N_C(\varepsilon)$ be the minimal number of points $x_1, x_2, \ldots, x_n$ from C such that for any $y \in C$ there is an x_i such that $d(x_i,y) \le \varepsilon$. Then $H_C(\varepsilon) = \ell n N_C(\varepsilon)$ is called the metric entropy of C, and the *exponent of entropy* $r(C)$ is defined by

$$r(C) = \overline{\lim_{\varepsilon \to 0}} \frac{\ell n H_C(\varepsilon)}{|\ell n \varepsilon|}.$$

Dudley (1973) proved that $C \subset H$ is always a GC-set if

$$\int_0^1 H_C(x)^{1/2} dx < \infty. \tag{1.1}$$

This is an extremely sharp result, for it implies that C is a GC-set if $r(C) < 2$, and it is known that C cannot be a GB-set (and so not a GC-set) if $r(C) > 2$ (Sudakov (1969)). The case $r(C) = 2$ includes an ambiguous range, however, where $H_C(\varepsilon)$ cannot determine whether C is GB or GC (Dudley (1973)). In particular, there are GC-sets for which the integral (1.1) diverges.

In this paper we find conditions under which the isonormal process L on a set $C \subset H$ is a.s. continuous at some point $x_0 \in C$. This is closely related to the question of whether or not C is GC. Clearly, if C is GC, then the isonormal process is a.s. continuous at each point of C. Less evident is the converse statement: if the isonormal process is a.s. continuous at every point of C, then C is a GC-set. This important property of Gaussian processes was noted for Gaussian processes on [0,1] by Marcus and Shepp (1971), page 436, who give credit for the idea to R. Dudley, and can be extended as follows.

THEOREM 1.1.

Let C *be a compact subset of a metric space* (S,d), *and* $X(t)$ *a Gaussian process on* C. *If* $X(t)$ *is a.s. continuous at each point of* C *then it is sample continuous.*

The proof of Theorem 1.1 will be given in the Appendix at the end of the paper.

In the following section we give sufficient conditions for local continuity of the isonormal process. These conditions turn out to be strictly weaker than those obtainable from (1.1). Consequently, our result (Theorem 2.1) can be used to establish the GC property in situations in which the integral (1.1) diverges.

I am deeply indebted to Robert Adler and Stamatis Cambanis for their valuable remarks during the preparation of this paper.

2. Sufficient conditions for local continuity

We start with some additional definitions and preliminary results.

For a given set C in a Hilbert space H and a given point $x_0 \in C$, set

$$C_{x_0}(\delta) := \{x \in C : \|x - x_0\| \le \delta\}, \ \delta > 0. \tag{2.1}$$

$$N_{x_0}(\delta,\varepsilon) := N(C_{x_0}(\delta),\varepsilon), \ \delta > 0, \ \varepsilon > 0. \tag{2.2}$$

$$N_{x_0}(\delta_1,\delta_2,\varepsilon) := N(C_{x_0}(\delta_2) \cap \overline{C_{x_0}(\delta_1)},\varepsilon), \ \delta_2 > \delta_1 \ge 0, \ \varepsilon > 0. \tag{2.3}$$

A set C is said to satisfy *condition A* if

$$\sup_{\substack{x_1,x_2 \in C \\ x_1 \ne x_2}} \frac{|\|x_1\| - \|x_2\||}{\|x_1 - x_2\|^2} = M < \infty. \tag{2.4}$$

The isonormal process L restricted to such set C, satisfies the following bound, which is derived in Samorodnitsky (1986).

$$P\{\sup_{x \in C} Lx > \lambda_0\sigma + \sum_{j=1}^{\infty} \varepsilon_j\lambda_j\} \le N_C(\varepsilon_1) \cdot \frac{1}{\sqrt{2\pi}} \lambda_0^{-1} e^{-\frac{\lambda_0^2}{2}} \tag{2.5}$$

$$+ \frac{1}{\sqrt{2\pi}} \lambda_0^{-1} e^{-\frac{\lambda_0^2}{2}} \cdot \sum_{k=2}^{\infty} N_C(\varepsilon_k) \exp\{-\tfrac{1}{2}(\lambda_{k-1} - \rho_k^* \lambda_0)^2\}$$

for any positive sequences $\{\varepsilon_j\}_{j=1}^{\infty}$, $\{\lambda_j\}_{j=0}^{\infty}$, $\varepsilon_j \to 0$ as $j \to \infty$, satisfying the following conditions for all $k \ge 2$.

$$\lambda_{k-1} \ge 1, \tag{2.6}$$

$$\rho_k^* < \frac{\lambda_{k-1}}{\lambda_0} < \frac{1}{\rho_k^*}. \tag{2.7}$$

Here

$$\rho_k^* := \frac{2M\sigma + 1}{2\underline{\sigma}} \varepsilon_{k-1} \tag{2.8}$$

and

$$\sigma := \sup_{x \in C} \|x\|, \ \underline{\sigma} := \inf_{x \in C} \|x\| > 0$$

The proof of Theorem 2.1 is based on the bound (2.5).

Remark 2.1. The following property of metric entropy is used in the sequel. For any $C_1 \subset C_2$, any $\varepsilon > 0$

(2.9) $$N_{C_1}(\varepsilon) \le N_{C_2}(\tfrac{\varepsilon}{2}).$$

This relation is semi-evident; details can be found in Samorodnitsky (1986).

THEOREM 2.1.

Suppose there exists a function $H(s,t)$ *such that the following two conditions hold*

(2.10) $$\ell n N_{x_0}(\delta_1,\delta_2,\varepsilon) \le H(\delta_1,\varepsilon)$$

for any $\varepsilon > 0$, *any* $0 < \delta_1 < \delta_2 \le \theta$, θ *is a fixed constant.*

(2.11) $$\lim_{s\to 0} \int_0^s H(s,t)^{1/2} dt = 0.$$

Then the isonormal process L *restricted to* C *is continuous at* x_0.

Remark 2.2. An alternative sufficient condition for continuity of the isonormal process at x_0 that follows directly from (1.1) is

(2.12) $$\int_0^1 H_{C_{x_0}(\delta)}(t)^{1/2} dt < \infty \quad \text{for all } \delta > 0 \text{ small enough.}$$

It is easily seen that the condition given by Theorem 2.1 is strictly weaker than (2.12). (Suppose that (2.12) holds for all $\delta \le \theta$. Then take $H(s,t) := \ell n N_{x_0}(s,\theta,\frac{t}{2})$. Then (2.10) holds via (2.9), while (2.12) and (2.9) together imply (2.11). The examples given in the end of this section represent situations in which Theorem 2.1 works while (2.12) fails.)

<u>Proof of the theorem</u>.

Note that $\tilde{H}(s,t) := \ln N_{x_0}(s,\theta,\frac{t}{2})$ also possesses properties (2.10) and (2.11). Consequently we may and will assume that $H(s,t)$ is nonincreasing in t for each s. Fix $0<u<1$ and define

$$(2.13)\qquad H_u(s,t) := H(s,\tfrac{u}{2}t)$$

$$(2.14)\qquad H_1(s,t)^{1/2} := H_u(s,t)^{1/2} + |\ln s| + \ln|\ln t|$$

$$(2.15)\qquad H_2(s,t)^{1/2} := H_1(s,t)^{1/2} + \frac{1}{s}[\int_0^s H_1(s,u)^{1/2}du]^{1/2}.$$

These functions still possess properties (2.11) and (at least for small t) (2.10). Set

$$(2.16)\qquad I(s) := \int_0^s H_2(s,t)^{1/2}dt$$

and

$$(2.17)\qquad g(s) := [\sup_{u\le s}\{I(u)+I(u)^{1/2}\}]^{1/2}.$$

Then $g(s)\downarrow 0$ as $s\to 0$. We are going to show that for an appropriate version of the process

$$(2.18)\qquad \limsup_{\delta\to 0}\{\frac{|Lx-Lx_0|}{g(\|x-x_0\|)} : x\in C_{x_0}(\delta)\} \le 1 \quad \text{a.s.}$$

This implies, of course, that the isonormal process L restricted to C is continuous at x_0. As the supremum in (2.18) is taken over a decreasing family of sets, it is enough to prove that for any $\alpha>0$

$$(2.19)\qquad \lim_{\delta\to 0} P[\sup\{\frac{|Lx-Lx_0|}{g(\|x-x_0\|)} : x\in C_{x_0}(\delta)\} > 1+\alpha] = 0.$$

Set $\delta_i := \delta u^i$, $i=0,1,2,\ldots$. Then $\delta_i\downarrow 0$ as $i\to\infty$. Consequently the monotonicity of $g(s)$ implies that

$$(2.20)\qquad P[\sup\{\frac{|Lx - Lx_0|}{g(\|x - x_0\|)} : x \in C_{x_0}(\delta)\} > 1 + \alpha]$$

$$\le \sum_{i=0}^{\infty} P[\sup\{\frac{|Lx - Lx_0|}{g(\|x - x_0\|)} : x \in C_{x_0}(\delta_i) \cap \overline{C_{x_0}(\delta_{i+1})}\} > 1 + \alpha]$$

$$\le \sum_{i=0}^{\infty} P[\sup\{|Lx - Lx_0| : x \in C_{x_0}(\delta_i) \cap \overline{C_{x_0}(\delta_{i+1})}\} > (1+\alpha)g(\delta_{i+1})].$$

We are going to estimate the probabilities in the last sum in (2.20). Denote

$$C_i := \{x - x_0,\ x \in C_{x_0}(\delta_i) \cap \overline{C_{x_0}(\delta_{i+1})}\}.$$

We obtain by the linearity of the isonormal process that

$$(2.21)\quad P[\sup\{|Lx - Lx_0| : x \in C_{x_0}(\delta_i) \cap \overline{C_{x_0}(\delta_{i+1})}\} > (1+\alpha)g(\delta_{i+1})]$$

$$= P\{\sup_{x \in C_i} |Lx| > (1+\alpha)g(\delta_{i+1})\}.$$

We would like to apply the bound (2.5) to the last probability on (2.21). However, C_i may not satisfy condition A, thus we first define

$$\hat{C}_i := \{x \cdot \frac{\delta_i}{\|x\|},\ x \in C_i\}.$$

Clearly,

$$\sup_{x \in \hat{C}_i} |Lx| = \sup_{x \in C_i} \{|Lx| \cdot \frac{\delta_i}{\|x\|}\} \ge \sup_{x \in C_i} |Lx|.$$

Thus

$$(2.22)\quad P\{\sup_{x \in \hat{C}_i} |Lx| > (1+\alpha)g(\delta_{i+1})\} \le P\{\sup_{x \in C_i} |Lx| > (1+\alpha)g(\delta_{i+1})\}$$

Moreover, the points in $\hat{C}_i$ have equal norms (δ_i), and so $\hat{C}_i$ satisfies condition A with $M = 0$. We apply the bound (2.5) to $\hat{C}_i$, taking $\sigma = \underline{\sigma} = \delta_i$, $M = 0$. For any two sequences $\{\varepsilon_j^{(i)}\}_{j=1}^{\infty}$, $\{\lambda_j^{(i)}\}_{j=0}^{\infty}$ satisfying (2.6) and (2.7) we thus obtain

$$(2.23)\quad P\{\sup_{x\in\hat{C}_i}|Lx| > \lambda_0^{(i)}\delta_i + \sum_{j=1}^{\infty}\varepsilon_j^{(i)}\lambda_j^{(i)}\} \le \sqrt{\frac{2}{\pi}}(\lambda_0^{(i)})^{-1}\exp\{-\frac{1}{2}(\lambda_0^{(i)})^2\}N_{\hat{C}_i}(\varepsilon_1^{(i)})$$

$$+\sqrt{\frac{2}{\pi}}(\lambda_0^{(i)})^{-1}\exp\{-\frac{1}{2}(\lambda_0^{(i)})^2\}\sum_{j=2}^{\infty}N_{\hat{C}_i}(\varepsilon_j^{(i)})\exp\{-\frac{1}{2}(\lambda_{j-1}^{(i)}-\rho_j^*\lambda_0^{(i)})^2\}.$$

Note that for any $x_1, x_2 \in C_i$

$$\|x_i\cdot\frac{\delta_i}{\|x\|}-x_2\cdot\frac{\delta_i}{\|x_2\|}\| = \delta_i\|\frac{x_1}{\|x_1\|}-\frac{x_2}{\|x_1\|}+\frac{x_2}{\|x_1\|}-\frac{x_2}{\|x_2\|}\|$$

$$\le \delta_i(\frac{1}{\|x_1\|}\|x_1-x_2\|+\|x_2\|\cdot|\frac{1}{\|x_1\|}-\frac{1}{\|x_2\|}|)$$

$$\le \frac{2\delta_i}{\|x_1\|}\cdot\|x_1-x_2\| \le \frac{2\delta_i}{\delta_{i+1}}\|x_1-x_2\|$$

$$= \frac{2}{u}\|x_1-x_2\|.$$

Consequently for any $\varepsilon > 0$

$$(2.24)\qquad N_{\hat{C}_i}(\varepsilon) \le N_{C_i}(\varepsilon\cdot\frac{u}{2}) = N(C_{x_0}(\delta_i)\cap\overline{C_{x_0}(\delta_{i+1})},\ \varepsilon\cdot\frac{u}{2})$$

Thus, (2.23) can be rewritten as

$$(2.25)\quad P\{\sup_{x\in C_i}|Lx| > \lambda(i)\delta_i + \sum_{j=1}^{\infty}\varepsilon_j^{(i)}\lambda_j^{(i)}\}$$

$$\le\sqrt{\frac{2}{\pi}}\lambda(i)^{-1}\exp\{-\frac{1}{2}\lambda(i)^2\}N_{x_0}(\delta_{i+1},\delta_i,\frac{\varepsilon_1^{(i)}u}{2})$$

$$+\sqrt{\frac{2}{\pi}}\lambda(i)^{-1}\exp\{-\frac{1}{2}\lambda(i)^2\}\sum_{j=2}^{\infty}N_{x_0}(\delta_{i+1},\delta_i,\frac{\varepsilon_j^{(i)}u}{2})\exp\{-\frac{1}{2}(\lambda_{j-1}^{(i)}-\rho_j^*\lambda(i))^2\},$$

where $\lambda(i)$ denotes $\lambda_0^{(i)}$. Now we specify the $\varepsilon_j^{(i)}$ and $\lambda_j^{(i)}$ sequences. Set

$$\text{(2.26)} \qquad \lambda(i) = \lambda_0^{(i)} := \frac{g(\delta_{i+1})}{\delta_i} ,$$

$$\text{(2.27)} \qquad \varepsilon_j^{(i)} := B_1 \delta_i 2^{-j}, \ j = 1,2,\ldots,$$

$$\text{(2.28)} \qquad \lambda_j^{(i)} := B_2 \lambda(i) \frac{H_2(\delta_{i+1}, 2^{-j}\delta_{i+1})^{1/2}}{H_2(\delta_{i+1}, 2^{-1}\delta_{i+1})^{1/2}} , \ j = 1,2,\ldots,$$

for some positive constants B_1, B_2 to be specified later in order to meet certain conditions. Note that $g(s) \geq s^{1/2} |\ell ns|^{1/2}$, consequently, at least for small values of δ, $\lambda_j^{(i)} \geq 1$ for all i and j. Furthermore, the condition $\rho_j^* < \frac{\lambda_{j-1}^{(i)}}{\lambda(i)}$ holds for all $i \geq 0$, $j \geq 2$ if and only if

$$\text{(2.29)} \qquad B_2 > \frac{B_1}{4}.$$

The second part of condition (2.7), namely $\frac{\lambda_{j-1}^{(i)}}{\lambda(i)} < \frac{1}{\rho_j^*}$, becomes, after a substitution ,

$$\text{(2.30)} \qquad \frac{H_2(\delta_{i+1}, 2^{-1}\delta_{i+1})^{1/2}}{B_2 H_2(\delta_{i+1}, 2^{-(j-1)}\delta_{i+1})^{1/2}} > B_1 2^{-j}.$$

Note that for any $0 < d < 1$ we have

$$\text{(2.31)} \qquad \frac{H_2(t,dt)^{1/2}}{H_2(t,t)^{1/2}} = \frac{H_1(t,dt)^{1/2} + \frac{1}{t}[\int_0^t H_1(t,u)^{1/2} du]^{1/2}}{H_1(t,t)^{1/2} + \frac{1}{t}[\int_0^t H_1(t,u)^{1/2} du]^{1/2}}$$

$$\leq \frac{\frac{1}{dt}\int_0^{dt} H_1(t,u)^{1/2} du + \frac{1}{t}[\int_0^t H_1(t,u)^{1/2} du]^{1/2}}{\frac{1}{t}[\int_0^t H_1(t,u)^{1/2} du]^{1/2}}$$

$$\leq \frac{1}{d}[\int_0^t H_1(t,u)^{1/2} du]^{1/2} + 1 \leq \frac{K}{d} ,$$

where $K < \infty$ is a constant that does not depend on t and d. Thus, the following condition

$$\frac{2^{-(j-2)}}{B_2 K} > B_1 2^{-j},$$

which is equivalent to

(2.32) $$B_1 B_2 < \frac{4}{K},$$

implies (2.30) for all $i \geq 0$, $j \geq 2$. Consequently, if we choose B_1 and B_2 to satisfy the restrictions (2.29) and (2.32), the condition (2.7) will be satisfied. Furthermore,

$$\lambda(i)\delta_i + \sum_{j=1}^{\infty} \varepsilon_j^{(i)} \lambda_j^{(i)} = \lambda(i)\delta_i \Big[1 + \frac{B_1 B_2}{H_2(\delta_{i+1}, 2^{-1}\delta_{i+1})^{1/2}} \sum_{j=1}^{\infty} 2^{-j} H_2(\delta_{i+1}, 2^{-j}\delta_{i+1})^{1/2}\Big]$$

$$\leq \lambda(i)\delta_i \Big[1 + 2B_1 B_2 \frac{\int_0^{2^{-1}\delta_{i+1}} H_2(\delta_{i+1}, u)^{1/2} du}{\delta_{i+1} H_2(\delta_{i+1}, 2^{-1}\delta_{i+1})^{1/2}}\Big].$$

Here the monotonicity of $H_2(s,t)$ in t has been used. An argument similar to that in (2.31) yields that for $0 < d < 1$

(2.33) $$\frac{\int_0^{dt} H_2(t,u)^{1/2} du}{dt H_2(t,dt)^{1/2}} \leq \frac{K}{d}$$

where K is the same constant as in (2.31), independent of t and d. Consequently

$$\lambda(i)\delta_i + \sum_{j=1}^{\infty} \varepsilon_j^{(i)} \lambda_j^{(i)} \leq \lambda(i)\delta_i (1 + 2B_1 B_2 K).$$

If we also choose B_1 and B_2 in such a way that

(2.34) $$2B_1 B_2 K \leq \alpha$$

then we get

$$\lambda(i)\delta_i + \sum_{j=1}^{\infty} \varepsilon_j^{(i)}\lambda_j^{(i)} \le (1+\alpha)\lambda(i)\delta_i = (1+\alpha)g(\delta_{i+1}).$$

Consequently, (2.25) implies that

$$(2.35) \quad P\{\sup_{x\in \tilde{C}_i} |Lx| > (1+\alpha)g(\delta_{i+1})\} \le \sqrt{\tfrac{2}{\pi}}\,\lambda(i)^{-1}\exp\{-\tfrac{1}{2}\lambda(i)^2\}N_{x_0}(\delta_{i+1},\delta_i,\tfrac{\varepsilon_1^{(i)}u}{2})$$

$$+ \sqrt{\tfrac{2}{\pi}}\,\lambda(i)^{-1}\exp\{-\tfrac{1}{2}\lambda(i)^2\}\sum_{j=2}^{\infty} N_{x_0}(\delta_{i+1},\delta_i,\tfrac{\varepsilon_j^{(i)}u}{2})\exp\{-\tfrac{1}{2}(\lambda_{j-1}^{(i)} - \rho_j^*\lambda(i))^2\}.$$

For every fixed $i = 0,1,2,\ldots$ let a_i and b_i be the first and the second terms, respectively, in the right hand side of (2.35). Then (2.20), (2.21), (2.22) and (2.35) imply that

$$(2.36) \qquad P[\sup\{\frac{|Lx - Lx_0|}{g(\|x - x_0\|)} : x \in C_{x_0}(\delta)\} > 1+\alpha] \le \sum_{i=0}^{\infty} a_i + \sum_{i=0}^{\infty} b_i.$$

We are going to estimate each of these sums separately. Letting c be a finite positive constant that may vary from line to line we obtain

$$\sum_{i=0}^{\infty} a_i \le c\sum_{i=0}^{\infty}\exp\{-\frac{g(\delta_{i+1})^2}{2\delta_i^2} + \ln N_{x_0}(\delta_{i+1},\delta_i,\frac{\varepsilon_1^{(i)}u}{2})\}$$

$$\le c\sum_{i=0}^{\infty}\exp\{-\frac{g(\delta_{i+1})^2}{2\delta_i^2} + H(\delta_{i+1},\frac{\varepsilon_1^{(i)}u}{2})\}$$

$$\le c\sum_{i=0}^{\infty}\exp\{-\frac{g(\delta_{i+1})^2}{2\delta_i^2} + H_2(\delta_{i+1},\varepsilon_1^{(i)})\}$$

$$\le c\sum_{i=0}^{\infty}\exp\{-\frac{1}{2}\frac{I(\delta_{i+1})}{\delta_i^2} + H_2(\delta_{i+1},B_1 2^{-1}\delta_i)\}$$

$$\leq c\sum_{i=0}^{\infty}\exp\{-\frac{u}{2}\frac{H_2(\delta_{i+1},\delta_{i+1})^{1/2}}{\delta_i}+H_2(\delta_{i+1},B_1 2^{-1}\delta_{i+1})\}$$

$$= c\sum_{i=0}^{\infty}\exp\{-\frac{u}{2}\frac{H_2(\delta_{i+1},\delta_{i+1})^{1/2}}{\delta_i}[1-\frac{2\delta_i}{u}\cdot\frac{H_2(\delta_{i+1},B_1 2^{-1}\delta_{i+1})}{H_2(\delta_{i+1},\delta_{i+1})^{1/2}}]\}$$

for δ small enough. Note that (2.31) implies that

$$\frac{H_2(\delta_{i+1},B_1 2^{-1}\delta_{i+1})^{1/2}}{H_2(\delta_{i+1},\delta_{i+1})^{1/2}}\leq \max\{1,\frac{2K}{B_1}\}$$

while

$$\delta_i H_2(\delta_{i+1},B_1 2^{-1}\delta_{i+1})^{1/2}\leq u^{-1}\max(1,\frac{2K}{B_1})\delta_{i+1}H_2(\delta_{i+1},\delta_{i+1})^{1/2}$$

and the last expression converges uniformly in i to zero as $\delta\to 0$. Consequently, for δ small

$$\sum_{i=0}^{\infty}a_i\leq c\sum_{i=0}^{\infty}\exp\{-\frac{u}{4\delta_i}H_2(\delta_{i+1},\delta_{i+1})^{1/2}\}\leq c\sum_{i=0}^{\infty}\exp\{-|\ell n\delta_{i+1}|\}=c\sum_{i=0}^{\infty}\delta_{i+1}$$

$$=\frac{c\delta u}{1-u}\to 0$$

as $\delta\to 0$. Next we consider the sum $\Sigma_{i=0}^{\infty}b_i$. Note that for all $j\geq 2$

$$\lambda_{j-1}^{(i)}-\rho_j^*\lambda(i)=\lambda(i)[B_2\frac{H_2(\delta_{i+1},2^{-(j-1)}\delta_{i+1})^{1/2}}{H_2(\delta_{i+1},2^{-1}\delta_{i+1})^{1/2}}-B_1 2^{-j}] \tag{2.34}$$

$$\geq B_2\lambda(i)\frac{H_2(\delta_{i+1},2^{-(j-1)}\delta_{i+1})^{1/2}}{H_2(\delta_{i+1},2^{-1}\delta_{i+1})^{1/2}}(1-\frac{B_1}{4B_2})$$

$$\geq\frac{1}{2}B_2\lambda(i)\frac{H_2(\delta_{i+1},2^{-(j-1)}\delta_{i+1})^{1/2}}{H_2(\delta_{i+1},2^{-1}\delta_{i+1})^{1/2}},$$

whenever B_1 and B_2 are chosen to satisfy

(2.38) $$\frac{B_2}{B_1} \geq \frac{1}{2}.$$

Note that (2.38) implies (2.29). Using (2.37) we get that for small δ

$$\sum_{j=2}^{\infty} N_{x_0}(\delta_{i+1},\delta_i,\frac{\varepsilon_j^{(i)}u}{2})\exp\{-\frac{1}{2}(\lambda_{j-1}^{(i)}-\rho_j^*\lambda(i))^2\}$$

$$\leq \sum_{j=2}^{\infty}\exp\{+H_2(\delta_{i+1},B_1\delta_i 2^{-j}) - \frac{B_2^2}{8}\ \frac{g(\delta_{i+1})^2}{\delta_i^2}\cdot\frac{H_2(\delta_{i+1},2^{-(j-1)}\delta_{i+1})}{H_2(\delta_{i+1},2^{-1}\delta_{i+1})}\}$$

$$\leq \sum_{j=2}^{\infty}\exp\{-\frac{B_2^2}{8\delta_i}\cdot\frac{H_2(\delta_{i+1},2^{-(j-1)}\delta_{i+1})}{H_2(\delta_{i+1},2^{-1}\delta_{i+1})}\cdot\frac{I(\delta_{i+1})}{\delta_i}$$

$$\times [1-\frac{8}{B_2^2}\max(1,\frac{4K^2u^2}{B_1^2})\frac{H_2(\delta_{i+1},2^{-1}\delta_{i+1})\delta_i^2}{I(\delta_{i+1})}]\}$$

$$\leq \sum_{j=2}^{\infty}\exp\{-\frac{B_2^2u}{8\delta_i}\cdot\frac{H_2(\delta_{i+1},2^{-(j-1)}\delta_{i+1})}{H_2(\delta_{i+1},2^{-1}\delta_{i+1})}\cdot H_2(\delta_{i+1},\delta_{i+1})^{1/2}$$

$$\times [1-\frac{8}{B_2^2u}\cdot\max(1,\frac{4K^2u^2}{B_1^2})\frac{H_2(\delta_{i+1},2^{-1}\delta_{i+1})}{H_2(\delta_{i+1},\delta_{i+1})^{1/2}}\cdot\delta_i]\}$$

$$\leq \sum_{j=2}^{\infty}\exp\{-\frac{B_2^2u}{8\delta_i}\cdot\frac{H_2(\delta_{i+1},2^{-(j-1)}\delta_{i+1})^{1/2}}{H_2(\delta_{i+1},2^{-1}\delta_{i+1})^{1/2}}\cdot H_2(\delta_{i+1},\delta_{i+1})^{1/2}$$

$$\times [1-\frac{8}{B_2^2u}\cdot\max(1,\frac{4K^2u^2}{B_1^2})\frac{H_2(\delta_{i+1},2^{-1}\delta_{i+1})^{1/2}}{H_2(\delta_{i+1},\delta_{i+1})^{1/2}}\cdot\delta_i H_2(\delta_{i+1},2^{-1}\delta_{i+1})^{1/2}]\}$$

$$\leq \sum_{j=2}^{\infty} \exp\{- \frac{B_2^2 u}{8} \cdot \frac{1}{2K} \cdot \frac{H_2(\delta_{i+1}, 2^{-(j-1)}\delta_{i+1})^{1/2}}{\delta_i}$$

$$\times [1 - \frac{8}{B_2^2 u} \cdot \max(1, \frac{4K^2 u^2}{B_1^2}) \cdot (2K)^2 \cdot \delta_i H_2(\delta_{i+1}, \delta_{i+1})^{1/2}]\}.$$

But $\delta H_2(\delta,\delta)^{1/2} \to 0$ as $\delta \to 0$. Consequently for small δ, uniformly over i

$$\sum_{j=2}^{\infty} N_{x_0}(\delta_{i+1}, \delta_i, \frac{\varepsilon_j^{(i)} u}{2}) \exp\{- \frac{1}{2}(\lambda_{j-1}^{(i)} - \rho_j^* \lambda(i))^2\}$$

$$\leq \sum_{j=2}^{\infty} \exp\{- \frac{B_2^2 u}{32K} \cdot \frac{H_2(\delta_{i+1}, 2^{-(j-1)}\delta_{i+1})^{1/2}}{\delta_i}\}$$

$$\leq \sum_{j=2}^{\infty} \exp\{-2\ell n |\ell n 2^{-(j-1)}\delta_{i+1}|\}$$

$$\leq \sum_{j=2}^{\infty} ((j-1)\ell n 2)^{-2} < \infty$$

independently of i. Consequently, for small δ

$$\sum_{i=0}^{\infty} b_i \leq c \sum_{i=0}^{\infty} a_i.$$

Thus $\Sigma_{i=0}^{\infty} b_i$ is finite for small δ, and $\Sigma_{i=0}^{\infty} b_i \to 0$ as $\delta \to 0$. Consequently, (2.19) is proven and then (2.18) follows. To finish the proof we have to show that it is possible to choose B_1 and B_2 to satisfy conditions (2.32), (2.34), (2.38). This is simple. For any $B_2 > 0$ set $B_1(B_2) = 2B_2$. This satisfies (2.38) independently of B_2. Then $B_2 \cdot B_1(B_2) = 2B_2^2$. Consequently taking B_2 small enough we get both (2.32) and (2.34) satisfied. This completes the proof.

The following two examples are taken from Dudley (1973).

Example 2.1. Let $\{a_k\}$ be a sequence of positive numbers with $a_k \downarrow 0$. For $k = 1,2,\ldots$, let C_k be a cube of dimension k^2 and side $2a_k/k^2$ centered at 0. Let the cubes C_k lie in orthogonal subspaces. Let $C = \bigcup_{k=1}^{\infty} C_k$.

Consider the origin $x_0 = 0$. Letting $a_k \downarrow 0$ slowly, we can make $\varepsilon^2 H_{C_0(\delta)}(\varepsilon) \to 0$ as slowly as desired, thus the integral (2.12) can diverge for all $\delta > 0$. Nevertheless, the isonormal process is continuous a.s. at $x_0 = 0$. We show that Theorem 2.1 works here.

For every fixed k and ε

$$N(C_k,\varepsilon) \le \max\{(\frac{2a_k}{k\varepsilon})^{k^2},1\}$$

and for fixed $0 < \delta_1 < \delta_2$, $\varepsilon > 0$

$$N_0(\delta_1,\delta_2,\varepsilon) \le N(\overline{C_0(\delta_1)},\frac{\varepsilon}{2}) \le \sum_{k=1}^{n(\delta_1)} (\frac{4a_k}{k\varepsilon})^{k^2} + 1, \tag{2.39}$$

where

$$n(s) := \max\{k : \frac{a_k}{k} \ge s\} .$$

Define also

$$m(s) := \max\{k : \frac{a_k}{k} \ge s^{1/4}\}$$

Both n(s) and m(s) increase to infinity as $s \downarrow 0$. Define

$$H(s,t) := \ell n[\sum_{k=1}^{n(s)} (\frac{4a_k}{kt})^{k^2} + 1]. \tag{2.40}$$

We have to show that

$$\lim_{s\to 0} \int_0^s \{\ell n[\sum_{k=1}^{n(s)} (\frac{4a_k}{kt})^{k^2}]\}^{1/2} dt = 0. \tag{2.41}$$

The obvious relations $\ell n(x+y) \le \ell n 2x + \ell n 2y$, $x,y \ge 1$ and $(x+y)^{1/2} \le x^{1/2} + y^{1/2}$ yield that

$$(2.42)\qquad \int_0^s\{\ell n[\sum_{k=1}^{n(s)}(\frac{4a_k}{kt})^{k^2}]\}^{1/2}dt$$

$$\le \int_0^s\{\ell n[\sum_{k=1}^{m(s)}(\frac{4a_k}{kt})^{k^2}]\}^{1/2}dt + \int_0^s\{\ell n[\sum_{k=m(s)+1}^{n(s)}(\frac{4a_k}{kt})^{k^2}]\}^{1/2}dt + s(2\ell n2)^{1/2}.$$

Denote by I_1 and I_2 the first and second integrals in the right hand side of (2.42). We are going to show that

$$\lim_{s\to 0} I_1 = \lim_{s\to 0} I_2 = 0.$$

We have, for small s,

$$I_1 \le \int_0^s\{\sum_{k=1}^{m(s)}[k^2\ell n(\frac{4a_k}{kt}) + \ell n2]\}^{1/2}dt$$

$$\le s(m(s)\ell n2)^{1/2} + \int_0^s\sum_{k=1}^{m(s)}k[\ell n(\frac{4a_k}{kt})]^{1/2}dt$$

$$\le s(m(s)\ell n2)^{1/2} + \sum_{k=1}^{m(s)}k\int_0^s[\ell n(\frac{4a_1}{t})]^{1/2}dt$$

$$\le s(m(s)\ell n2)^{1/2} + \frac{m(s)(m(s)+1)}{2}\cdot 2s[\ell n(\frac{4a_1}{s})]^{1/2}$$

and this goes to zero as $s\to 0$ because $m(s)\le a_1 s^{-1/4}$. Further, for every $m(s) < k \le n(s)$

$$(2.43)\qquad (\frac{4a_k}{kt})^{k^2} = [(\frac{4a_k}{kt})^{\frac{kt}{4a_k}}]^{\frac{4ka_k}{t}} \le (e^{\frac{1}{e}})^{\frac{4ka_k}{t}} \le (e^{\frac{1}{e}})^{\frac{4a_k^2}{st}} \le \exp\{\frac{4a^2_{m(s)+1}}{e\cdot s\cdot t}\},$$

where the first inequality in (2.43) follows from the fact that

$$\max_{x>0} x^{\frac{1}{x}} = e^{\frac{1}{e}}$$

and the second inequality follows from the fact that $a_k/k \ge s$. We conclude that

$$(2.44)\qquad I_2 \le \int_0^s \{\ell n[n(s)\exp(\frac{4a_{m(s)+1}^2}{e\cdot s\cdot t})]\}^{1/2}dt$$

$$\le s[\ell nn(s)]^{1/2} + 2e^{-1/2}a_{m(s)+1}\cdot s^{-1/2}\cdot\int_0^s t^{-1/2}\cdot dt$$

$$= s[\ell nn(s)]^{1/2} + 4e^{-1/2}\cdot a_{m(s)+1}.$$

The first term in (2.44) converges to zero because $n(s) \le s^{-1}a_{n(s)}$, while the second term converges to zero because $m(s) \uparrow \infty$ as $s \downarrow 0$. Therefore, the conditions of Theorem 2.1 hold.

Remark 2.3. Note that the GC property of C follows. We have just proved that the isonormal process is continuous at $x_0 = 0$. Certainly, for any $x_0 \in C$ other than the origin, for any sufficiently small $\delta > 0$ the metric entropy of $C_{x_0}(\delta)$ is bounded by a logarithmic function of ε, and so the integral (2.12) is finite. In this example, consequently, we have proved that C is a GC class by proving that the isonormal process is continuous at the (only) "difficult" point $x_0 = 0$.

Example 2.2. Again, let $\{a_k\}$ be a sequence of positive numbers with $a_k \downarrow 0$. Let

$$C = \{\phi_n\cdot a_n(\ell nn)^{-1/2}, n \ge 2\} \cup \{0\},$$

ϕ_n orthonormal. Consider $x_0 = 0$. If $a_k \downarrow 0$ slowly ($a_k = (\ell n\ell nk)^{-1/2}$ is slow enough) then the integral (2.12) diverges for all $\delta > 0$. Let us show that Theorem 2.1 can be applied to this example as well. Set

$$M(s) := \min\{n : \frac{a_n^2}{\ell nn} \le s^2\}.$$

Then, for any $0 < \delta_1 < \delta_2$, $\varepsilon > 0$

$$(2.45)\qquad N_0(\delta_1,\delta_2,\varepsilon) = [\min(M(\varepsilon), M(\delta_1)) - M(\delta_2)]_+ + 1.$$

This implies that

(2.46) $$N_0(\delta_1,\delta_2,\varepsilon) \le 2M(\delta_1).$$

Set

(2.47) $$H(s,t) := \ln 2 + \ln M(s).$$

Then

$$\int_0^s H(s,t)^{1/2}dt = s[\ln 2 + \ln M(s)]^{1/2}$$

and we have to show that

$$\lim_{s\to 0} s^2 \ln M(s) = 0.$$

But this is clear, since

$$\frac{a^2_{M(s)-1}}{\ln(M(s)-1)} > s^2,$$

so

$$M(s) < 2\exp\{s^{-2}\cdot a_{M(s)-1}\},$$

and also $M(s)\uparrow\infty$ as $s\downarrow 0$.

Remark 2.4. Again, we have proved the GC property of C, since at any point $x_0 \in C$ other than the origin the isonormal process cannot have a discontinuity.

Appendix

Theorem 1.1 is proven here. The idea of the proof is the same as that of Marcus and Shepp (1971) in the case of Gaussian processes on [0,1].

Let C^* be a countable dense subset of C. For any $I \subset C$, any $\varepsilon > 0$, define

$$A_\varepsilon(I) = \left\{ \omega \,\middle|\, \begin{array}{l} \text{there is an } r > 0 \text{ such that for every } \delta > 0 \text{ there} \\ \text{are } t_1 \in C^* \cap I \text{ and } t_2 \in C^* \text{ such that } d(t_1,t_2) < \delta \\ \text{and } |X(t_1) - X(t_2)| \geq \varepsilon + r. \end{array} \right\}$$

Lemma A.1.

$$P(A_\varepsilon(I)) = 0 \text{ or } 1.$$

Proof of the lemma.

Let $s_1, s_2, \ldots$ be a numeration of the points of C^*. Then there exists a sequence of orthonormal Gaussian variables $Y_1, Y_2, \ldots$ and real numbers $\{a_{ij}\}$, $i \leq j$, such that for each i

$$X(s_i) = \sum_{j=1}^{i} a_{ij} Y_j. \tag{A.1}$$

Let $a_{ij} = 0$ for $j > i$, and let for $t_1, t_2 \in C^*$ i_1 and i_2 denote the places of t_1 and t_2 correspondingly in the fixed numeration of C^*. Then

$$\sigma^2(t_1,t_2) = E[X(t_1) - X(t_2)]^2 = \sum_{j=1}^{\infty} (a_{i_1 j} - a_{i_2 j})^2. \tag{A.2}$$

Note that the covariance function of the process, $R(s,t)$, is continuous on $C \times C$, since $X(t)$ is a.s. continuous at each point of C. Then the function $\sigma^2(s,t)$ is also continuous on $C \times C$, and, because of compactness, it is uniformly continuous. Consequently, for every $\theta > 0$ there is a $\eta = \eta(\theta) > 0$ such that $d(t_1,t_2) \leq \eta$ implies that $\sigma^2(t_1,t_2) \leq \theta^2$. If also $t_1, t_2 \in C^*$, then (A.2) implies that $|a_{i_1 j} - a_{i_2 j}| \leq \theta$ for every j. Let us rewrite the event $A_\varepsilon(I)$ as

$$A_\varepsilon(I) = \left\{\omega \,\middle|\, \begin{array}{l}\text{there is an } r>0 \text{ such that for every } \delta>0 \text{ there} \\ \text{are } t_1 \in C^* \cap I \text{ and } t_2 \in C^* \text{ such that } d(t_1,t_2) < \delta \\ \text{and } \left|\sum_{j=1}^{\infty} Y_j(\omega)(a_{i_1j} - a_{i_2j})\right| \geq \varepsilon + r.\end{array}\right\}$$

We claim that $A_\varepsilon(I)$ is a tail event for the sequence $Y_1, Y_2, \ldots$. It is sufficient to show that if ω_1 and ω_2 are such that for some finite m $Y_n(\omega_1) = Y_n(\omega_2)$ for all $n > m$, then $\omega_1 \in A_\varepsilon(I)$ implies $\omega_2 \in A_\varepsilon(I)$.

Suppose the contrary, i.e. $\omega_1 \in A_\varepsilon(I)$ and $\omega_2 \notin A_\varepsilon(I)$. Then for some positive $r(\omega_1)$, for all sufficiently small $\delta > 0$ there are $t_1 \in C^* \cap I$ and $t_2 \in C^*$ such that $d(t_1,t_2) < \delta$ while

$$\left|\sum_{j=1}^{\infty} Y_j(\omega_1)(a_{i_1j} - a_{i_2j})\right| \geq \varepsilon + r(\omega_1)$$

$$\left|\sum_{j=1}^{\infty} Y_j(\omega_2)(a_{i_1j} - a_{i_2j})\right| < \varepsilon + \frac{1}{2}r(\omega_1).$$

(Here, as before, i_1 and i_2 denote the places of t_1 and t_2 respectively in the fixed numeration of C^*). We conclude (recall that all sums have finite numbers of terms) that

$$\begin{aligned} \frac{1}{2}r(\omega_1) &< \left|\sum_{j=1}^{\infty} Y_j(\omega_1)(a_{i_1j} - a_{i_2j})\right| - \left|\sum_{j=1}^{\infty} Y_j(\omega_2)(a_{i_1j} - a_{i_2j})\right| \\ &\leq \left|\sum_{j=1}^{\infty} Y_j(\omega_1)(a_{i_1j} - a_{i_2j}) - \sum_{j=1}^{\infty} Y_j(\omega_2)(a_{i_1j} - a_{i_2j})\right| \qquad \text{(A.3)} \\ &= \left|\sum_{j=1}^{m} (Y_j(\omega_1) - Y_j(\omega_2))(a_{i_1j} - a_{i_2j})\right|. \end{aligned}$$

The inequality (A.3), however, cannot hold for all positive δ, since its left-hand side is positive, while the right-hand side goes to zero as $\delta \to 0$. This contradiction shows that $A_\varepsilon(I)$ is a tail event. Consequently, Kolmogorov's zero-one law implies that $P(A_\varepsilon(I)) = 0$ or 1.

We return now to the proof of Theorem 1.1. For any $I \subset C$, any $\varepsilon > 0$ define

$$B_\varepsilon(I) = \left\{\omega \,\middle|\, \begin{array}{l}\text{for any } \delta > 0 \text{ there are } t_1 \in C^* \cap I \text{ and } t_2 \in C^* \text{ such that} \\ d(t_1,t_2) < \delta \text{ and } |X(t_1) - X(t_2)| \geq \varepsilon.\end{array}\right\}$$

Then $B_\varepsilon(I) \subset A_{\varepsilon/2}(I)$. If $X(t)$ is not sample continuous then $P(B_\varepsilon(C)) > 0$ for some $\varepsilon > 0$. Then $P(A_{\varepsilon^*}(C)) > 0$ for some $\varepsilon^* > 0$. This implies that $P(A_{\varepsilon^*}(C)) = 1$.

Let $d_0 = \sup_{t_1,t_2 \in C} d(t_1,t_2) < \infty$. The compactness of C implies that we can cover it by a finite number of compact sets $C_1^{(1)}, C_2^{(1)}, \ldots, C_{k_1}^{(1)}$ of diameter at most $d_0/2$ each. Since

$$\text{(A.4)} \qquad A_{\varepsilon^*}(C) = \bigcup_{i=1}^{k_1} A_{\varepsilon^*}(C_i^{(1)})$$

we conclude, that for some i_1, $1 \leq i_1 \leq k_1$, $P(A_{\varepsilon^*}(C_{i_1}^{(1)})) > 0$. Thus $P(A_{\varepsilon^*}(C_{i_1}^{(1)})) = 1$. We divide now $C_{i_1}^{(1)}$ into compact subsets of diameter at most $d_0/4$, and so on. We obtain a sequence of nested compact non-empty sets

$$C = C_1^{(0)} \supset C_{i_1}^{(1)} \supset C_{i_2}^{(2)} \supset \ldots$$

with the following two properties: for each $k = 1,2,\ldots$ $P(A_{\varepsilon^*}(C_{i_k}^{(k)})) = 1$ and the diameter of $C_{i_k}^{(k)}$ is at most $d_0/2^k$. This sequence has to converge to a point $t_\infty \in C$. Then, by the definition of $A_{\varepsilon^*}(I)$,

$$P\Big(\overline{\lim_{\substack{t \to t_\infty \\ t \in C^*}}} X(t) - \underline{\lim_{\substack{t \to t_\infty \\ t \in C^*}}} X(t) \geq \varepsilon^*\Big) \geq P\Big(\bigcap_{k=0}^{\infty} A_{\varepsilon^*}(C_{i_k}^{(K)})\Big) = 1.$$

This contradicts the assumption that $X(t)$ is a.s. continuous at t_∞. This contradiction shows that $X(t)$ is sample continuous.

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