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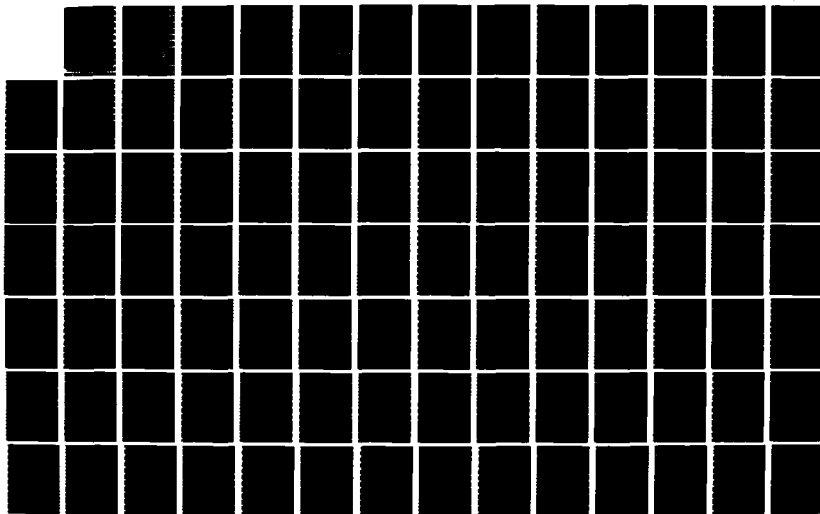
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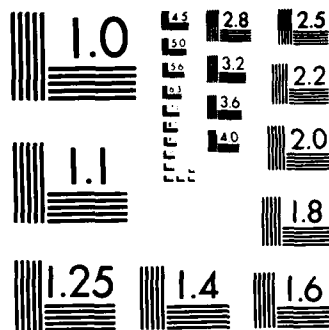
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MECHANICAL BEHAVIOR OF SATURATED SOILS - A REVIEW
Geotechnical Engineering Report No. 10

Ranbir S. Sandhu
Department of Civil Engineering

DEPARTMENT OF THE AIR FORCE
Air Force Office of Scientific Research
Bolling Air Force Base, D.C. 20332

Grant No. AFOSR-83-0055

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FOREWORD

The investigation reported herein is a part of the Research Project at The Ohio State University, Columbus, Ohio supported by the Air Force Office of Scientific Research grant 83-00-55. Lt. Col. Lawrence D. Hokanson is the Program Manager. The present report documents part of the work done from February 1, 1983 to January 31, 1985. At The Ohio State University, the Project is being supervised by Dr. Ranbir S. Sandhu, Professor, Department of Civil Engineering.



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ABSTRACT

Theories of mechanical behaviour of saturated soils are reviewed. The discussion spans the classical approaches to quasi-static and dynamic response of deformable porous media as well as theories of mixtures and their application in geomechanics. Different approaches to description of motion of superposed continua, derivation of balance laws and development of constitutive relationships are compared. The notion of motion of the mixture as a single body is questioned. Alternative concepts of behaviour of liquid-solid mixtures are considered. The role of thermodynamic principle in development of constitutive relationships is examined.

TABLE OF CONTENTS

FOREWORD	i
ABSTRACT	ii
TABLE OF CONTENTS	iii
NOTATION AND SYMBOLS.....	vii
I. INTRODUCTION	1
II. EARLIER THEORIES.....	7
2.1 SLOW FLOW OF A FLUID THROUGH A SOLID.....	7
2.1.1 d'Arcy's Law and Its Generalization.....	7
2.1.2 Mass Continuity.....	10
2.1.3 Analysis of Deformation.....	10
2.1.3.1 Scheidegger's correction.....	10
2.1.3.2 Terzaghi's theory of one-dimensional consolidation.....	11
2.1.3.3 Biot's theory of three-dimensional consolidation.....	12
2.1.3.4 Gibson's theory.....	16
2.2 WAVE PROPAGATION IN SATURATED SOILS.....	18
2.2.1 Preliminaries.....	18
2.2.2 The Engineering Approach.....	18
2.2.3 Biot's Theory.....	20
2.2.4 Fukuo's Theory.....	25
2.2.5 Comments.....	25
III. THEORY OF MIXTURES.....	27

TABLE OF CONTENTS (CONTINUED)

3.1	INTRODUCTION.....	27
3.2	KINEMATICS.....	28
3.2.1	Preliminaries.....	28
3.2.2	Density of Constituents and the Mixture.....	28
3.2.3	Motion of Constituents.....	30
3.2.4	Some Measures of Deformation of Saturated Soils..	34
3.3	BALANCE LAWS.....	37
3.3.1	Introduction.....	37
3.3.2	Truesdell's Theory.....	38
3.3.3	Green and Naghdi's Theory.....	46
3.4	OTHER WORK.....	56
3.4.1	Introduction.....	56
3.4.2	Continuity Equation Using Relative Velocity.....	56
3.4.3	Mass Balance in Terms of Porosity (Volume Fraction Theory).....	57
3.4.4	Energy Balance in Terms of Porosity (Volume Fraction Theory).....	59
3.4.5	Alternative Form of Linear Momentum Balance Equation.....	60
3.4.6	Other Forms of the Energy Balance Equation.....	63
3.5	COMMENTS.....	66
IV.	CONSTITUTIVE RELATIONSHIPS.....	69
4.1	INTRODUCTION.....	69
4.2	DIFFUSIVE RESISTANCE.....	74
4.3	STRESS-STRAIN RELATIONS.....	81
4.3.1	Effective Stress.....	81
4.3.2	Pore-pressure Parameters.....	85

TABLE OF CONTENTS (CONTINUED)

4.3.3	Stress-strain Relations.....	86
4.3.3.1	Phenomenological and micromechanical approaches.....	86
4.3.3.2	Thermodynamic considerations.....	94
4.3.3.3	Fluid flow through an elastic solid....	100
4.3.3.4	Linear theory.....	103
4.4	CONSTITUTIVE EQUATIONS FOR POROSITY.....	106
4.5	CONSTITUTIVE EQUATIONS FOR SPECIFIC VOLUME.....	109
4.5.1	Kenyon's Theory.....	109
4.6	COMMENTS.....	110
V.	DISCUSSION.....	111
VI.	REFERENCES.....	117

NOTATION AND LIST OF SYMBOLS

Both the standard indicial notation and matrix notation have been used in this report. Indices in Roman characters take on the range of values 1, 2, and 3. Repeated indices imply summation over the range of index values. Pairs of indices in parentheses represent the symmetric part of the tensor, while square brackets denote the anti-symmetric part. A superposed dot is used to indicate differentiation with respect to the time variable. A subscripted comma indicates differentiation with respect to the coordinates defined by the subscripts following the comma. δ_{ij} is Kronecker's delta and e_{ijk} denotes the permutation symbol. Indices used elsewhere as mathematical symbols and summation parameters have significance evident from the context.

For matrix notation, $\{\cdot\}$ denotes a column vector and $[\cdot]$ a rectangular matrix. Diagonal matrices are indicated by $[\cdot]_{\cdot}$. Superscript T is used to denote 'transpose'.

Generally speaking, symbols have been defined as they appear in the text. The list given below is not exhaustive.

a	Constant
$\bar{a}$	Constant
a_i	Constants, Initial compressibility
a_v	Coefficient of compressibility
A	Surface, Helmholtz free energy
A_{ji}	Components of the matrix of phenomenological coefficients
A_{ijkl}	Soil-water interaction tensor
b	Constant, total body force
b_1	Constant
b_i	Components of the body force vector
B	Body
c	Constant, Mass production factor
c_1 to c_6	Constants
c_v	Coefficient of consolidation
$[C]$	Viscous damping matrix
C_{ji}	Components of the flow resistivity tensor

C_{ijkl}	Components of the compliance tensor
$C_{ijkl}^{(1)*}$	Components of the intrinsic compliance tensor of solid
dW	Reversible work of fluid phase.
d_{ij}	Components of rate of deformation tensor
D	Phenomenological coefficient
D_i	Components of the diffusive force vector
D_{ij}	Components of the spatial rate of deformation tensor
D_{ij}^*	Components of the intrinsic rate of deformation tensor
e_{ij}	Infinitesimal strain tensor
e_{kk}	Volumetric strain
E	Young's modulus
E_r	One-dimensional rebound modulus
$E_{kl ij}$	Components of the elasticity tensor for dry solid
f_i	Components of the acceleration vector
F	Deformation gradient
F_j	Prigogine forces
F_{ij}	Components of the material deformation tensor
F_{iN}	Total force function
h_i	Components of the heat flux vector.
H	Constant
H_1	Constant
J	Jacobian, Bulk volume measure
K	Equilibrated inertia, Bulk compressibility, Bulk modulus, Consolidation factor, Constant
K_1	Intrinsic bulk modulus for solid
K_2	Intrinsic bulk modulus for fluid
K_s	Intrinsic compressibility of solid
$K_D^{(1)*}$	Bulk modulus of rock
$K^{(i)}$	Compressibility
l	External equilibrated body force
L_{ij}	Components of the spatial velocity gradient
$L_{ij}^{(k)}$	Scalar coefficients
m_i	Momentum supply density
M	Material parameter, Constant

M_{ij}	Components of the material parameter tensor, Components of the skew-symmetric matrix
$[M]$	Diagonal mass matrix
n	Porosity, Volume fraction
$n^{(k)}$	Volume fraction of constituent $s^{(k)}$
n_0	Initial porosity
N	Constant
p	Fluid pressure
$\{P(t)\}$	Excitation vector
P_0	Axial compressive load
P_i	Components of the interaction force or diffusive resistance
$\bar{q}$	Surface flux of energy
$q_i^{(k)}$	heat flux vector of constituent $s^{(k)}$
R	Constant, Unitary operator
$s^{(k)}$	k th constituent of mixture
S	Entropy per unit mass of mixture
S_j	Components of the equilibrated body force vector
S_{ij}	Components of the surface force tensor
t	Total traction
t_{ij}	Total stress tensor
t'_{ij}	Effective stress tensor
$t_{ij}^{(k)}$	Partial stress tensor for constituent $s^{(k)}$
$t_{ij}^{(k)*}$	Intrinsic stress tensor for $s^{(k)}$ th constituent
T	Total kinetic energy, Temperature of mixture
$\{u\}$	Displacement vector
$\{\dot{u}\}$	Velocity vector
$\{\ddot{u}\}$	Acceleration vector
$v_i^{(k)}$	Diffusive velocity
U	Molecular energy, Specific total internal energy of mixture
U^*	Weighted average of energies of constituent
v	Relative fluid velocity
v_i^{1k}	Components of the velocity vector
v_j	Components of barycentric velocity
$v^{(k)}$	Partial volume of constituent $s^{(k)}$
w_i	Components of the nominal relative velocity

$z^{(k)}$	Surface interaction force of constituent $s^{(k)}$
α	Constant
α_i	Components of internal force
γ	Constant
Γ_{ij}	Components of the vorticity tensor
ΔV	Differential change in volume
$\Delta \pi$	Incremental pore pressure
ζ	Relative strain
θ	Change of water content
$\theta_i^{(k)}$	Interaction force of constituent $s^{(k)}$
λ	Constant, Lamé's constant
λ_1 to λ_5	Constants
$\lambda_i^{(k)}$	Components of entropy flux
λ_{ij}	Coupling tensor due to interaction
Δ_{ij}	Difference in velocity tensor for a binary mixture
μ	Bulk shear modulus, Chemical potential, Constant, Lamé's constant
μ_1 to μ_5	Constants
ν	Poisson's Ratio
π	Bulk fluid pressure
π^*	Intrinsic fluid pressure
ρ	Mass density of the saturated soil, Density of the mixture
ρ^*	Intrinsic fluid mass density
ρ_0	Initial mass density
ρ^{12}	Mass coupling between solid & fluid
$\rho^{(k)}$	Mass density of constituent $s^{(k)}$
$\rho^{(k)*}$	Intrinsic fluid density of constituent $s^{(k)}$
ϕ	Potential
$\phi^{(k)}$	Volume interaction term for constituent $s^{(k)}$

SECTION I

INTRODUCTION

Study of response of fluid-saturated porous solids to applied loading/unloading or fluid injection/withdrawal is of considerable and growing interest not only in geotechnical engineering but also in underground disposal of liquid wastes, geothermal energy extraction, petroleum reservoir engineering, and to evaluate the hazard associated with a seismic or other significant dynamic event (e.g., a blast or explosion). It is necessary to investigate the motion of the soil mass in the vicinity of the source of the disturbance and near any significant structures whether above or under-ground. This investigation is made difficult by the presence of fluids in the pores of the soil mass. In the case of water-bearing deposits, loading results in a build-up of pore fluid pressures. Subsequent dissipation associated with gradual fluid flow leads to time-dependent deformation. Incidence of earthquake motion or other dynamic loading on fluid-saturated geological deposits results in changes in the fluid pressure as well as the stress field in the solid matrix. This phenomenon, in some cases, can lead to instability of the soil matrix resulting in soil liquefaction and disastrous consequences to the structures that may depend upon that soil for support. Also, the energy dissipation associated with the relative oscillatory motion between the fluid and the solid may introduce significant attenuation of the propagating wave. This effect would be in addition to the energy dissipation in any inelastic deformation of the soil matrix. Inasmuch as the transmissibility of motion through soil layers depends upon the soil characteristics, the soil deposit may act as a selective filter/amplifier. The importance of transmission characteristics and stability of the soil matrix in the study of earthquake hazard and risk of structural damage has long been recognized. For seismograph installations located on soil deposits, the records include local site effects which have to be allowed for. For reliable prediction of stability and time-dependent deformation of the system and for evaluation of various remedial measures to control or alleviate any

adverse conditions that might arise due to natural causes or certain engineering or other human activity, it is necessary that adequate yet simple mathematical simulation of events be available. Various theories and methods of analysis, including numerical procedures, have been developed to calculate the surface motion response and histories of stress in fluid-saturated geologic deposits.

Terzaghi [187] was the first to study the problem of development and dissipation of pore-water pressures in embankments and in foundation soils in the vicinity of structural foundations. His theory of one-dimensional consolidation was extended by Gibson [75]; Schiffman [162]; Barden [10,11]; and Fukuo [68] to viscoelastic soils. Gibson [77] developed a one dimensional consolidation theory admitting finite strains. To use Terzaghi's one-dimensional theory for studying three-dimensional problems, semi-empirical relationships were proposed by Skempton [179,180]. This approach was inapplicable to cases of repeated loading and needed modification [113]. Moore [129] developed a theory applicable to one-dimensional consolidation of soils consisting of spherical or cylindrical particles which are themselves fluid-saturated and consolidate radially. Treatment of consolidation on the basis of rate-process theory was proposed by Wu [199]. De Jong [50] proposed a statistical theory of consolidation. Various extensions of these basic approaches to allow for nonlinear material properties etc. were proposed by several investigators [e.g., 48,121]. In all this development, primary interest was in surface settlements.

Biot [15] was the first to propose a general theory of three-dimensional deformation of porous fluid-saturated solids. First proposed for linear elastic isotropic soils, this theory was extended in a sequence of papers [18-20,24,25] to anisotropic elastic and viscoelastic soils and to systems under initial stress and finite strain. This theory has been extensively used [e.g., 16,17,76,119,163] to obtain solutions to specific problems. A simplified theory, known as the Terzaghi-Rendulic theory, based on a direct generalization of Terzaghi's theory to three dimensions, has had some advocates because of its simplicity. These theories have

been documented in geotechnical engineering literature. As an example one may cite Zaretsky [201]'s monograph, and Schiffman's review paper [164]. Biot [21-23] extended his theory to wave propagation in saturated geological materials. His work has been the basis for several exact [e.g., 21,23,38,52,71] as well as numerical [e.g., 72-74,148] solution procedures.

Balance laws for mass, momentum and energy and constitutive laws for the constituents had been traditionally proposed on essentially intuitive, ad hoc basis to explain phenomenological observations. Typical of this approach of writing the governing equations 'by inspection' are the several theories which are used widely in soil mechanics. Theories of mechanical behavior of mixtures based on basic principles of mechanics were developed by Truesdell and Toupin [189], Truesdell [190] and Green [81-88].

Truesdell [189], Adkins [1], and Green and Adkins [81] proposed the equations of linear momentum and continuity as postulates. Green and Naghdi [83], following Green and Rivlin's [82] work, derived the equations of balance of momentum, mass continuity and heat balance from the frame invariance of a rate of energy equality. The theory was restated [87] and compared with the classical viewpoint. Crochet and Naghdi [45] specialized the theory to the case of fluid flow through an elastic solid. Green and Naghdi [83], postulated an energy equality for each constituent admitting at first all dependent variables. Local equations of mass and momentum balance were derived by repeated use of Noll's [137] principle of frame indifference. Beatty [12] showed that Green and Rivlin's method is essentially based upon Noll's [137] principle of frame indifference of mechanical power and does not require the energy equation at all. Essentially, both the concepts are correct and closely related. Tabaddor [184,185] followed the line of thought presented by Green and Naghdi [83] and Crochet and Naghdi [45]. Shi, Rajagopal and Wineman [177] applied Green and Naghdi's theory to flow of a fluid through a nonlinear elastic solid. Schiffman [165] used Green and Steel's approach [84] to develop equations of soil consolidation and to define the components of stress of a porous solid. In application to soil

mechanics, Sandhu [156,158] sought to introduce Terzaghi's effective stress concept and to regard the soil as buoyant in the fluid. It was noticed by Green and Naghdi [83] that even though the total stress had to be symmetric in the absence of body couples applied to the mixture, the partial stress for individual constituents need not be symmetric. Other theories, based essentially on the pioneering work of Truesdell and Green, have been proposed in recent years [e.g., Schiffman [165], Morland [130-132], Aifantis [4], Prevost [147-148], Vardoulakis [194], Katsube [102], among others]. An excellent review of various theories was recently presented by Bedford and Drumheller [14].

The existing theories may be divided into two groups. Within each group there is considerable divergence of opinion. One group uses equations of balance and constitutive relationships based on purely phenomenological observations. The other class of theories would use basic principles of mechanics and thermodynamics to develop the governing equations. Biot's theory is the one most widely used in soil mechanics. However, it is based on several ad hoc assumptions. Theories of mixtures, proposed by various investigators, are based on different assumptions.

Reliable analysis of the motion and stability of fluid-saturated solid deposits subjected to dynamic loading involves three steps:

- i. Formulation of the equations of dynamic equilibrium.
- ii. Correct representation of material behavior.
- iii. Exact or approximate solution of the problem.

In view of the apparent diversity of opinions, postulates, and assumptions made in setting up various approaches to the problem, it seems necessary to examine the available theories to determine their theoretical foundation and the assumptions made in their development. All the three components listed above need to be addressed. The present report contains a review of the theoretical formulation of the problem of quasi-static as well as dynamic behavior of fluid-saturated soils. In Section II we review earlier theories of

flow through porous deformable media including dynamic effects. Sections III and IV cover theories of mechanical behavior of mixtures. The basic assumptions of these theories are reviewed in the context of their application to the problem of fluid-saturated solids. Section III contains the various approaches used to describe the motion of superposed continua and to set up the equations of balance of mass, linear momentum, angular momentum, and energy. Section IV is devoted to review of the procedures for setting up constitutive relationships for fluid-saturated porous media including phenomenological approaches as well as those based on theories of mixtures.

SECTION II

EARLIER THEORIES

2.1. SLOW FLOW OF A FLUID THROUGH A SOLID.

2.1.1. d'Arcy's Law and its Generalizations.

Linear dependence between nominally one-dimensional water flow and pressure gradient was observed by d'Arcy [47]. Classical approach to analysis of fluid flow through porous media uses, as the basic equation of flow, a generalization of this observation to two and three dimensions as well to coupled flow and diffusion phenomena. Symbolically, in one-dimensional flow, d'Arcy equation is:

$$v_x = -K \phi_{,x} \quad (\text{II-1})$$

where v_x denotes the macroscopic or nominal fluid velocity and $\phi_{,x}$ is the gradient, along the direction of the mean flow, of the potential causing the flow. The constant of proportionality K has been found to vary with the size, shape, distribution, and orientation of pores and with temperature, viscosity and velocity of the fluid. Also, nonlinearities in the relationship have been noticed and empirical attempts made to devise formulas fitting the observed data. Comprehensive reviews are available in the literature [e.g.,112].

In extension of d'Arcy's observation to three-dimensional flow, the constant K was enlarged to have the nature of a second rank tensor transforming the potential gradient to the flux vector. Thus, using a cartesian reference frame.

$$v_i = -K_{ji} \phi_{,j} \quad (\text{II-2})$$

Here, and in the sequel, the indicial notation is used. Summation on repeated indices is implied except where indicated otherwise by an explicit statement to the effect or by enclosing each of the indices in parentheses. An index following a subscripted comma indicates

differentiation with respect to the spatial coordinate represented by the index. The range of indices is one to three representing spatial coordinates except when indicated otherwise by the context.

The permeability tensor satisfies the symmetry requirements

$$K_{ij} = K_{ji} \quad (\text{II-3})$$

Nonhomogeneity of the solid and spatial variations in fluid properties have been allowed for by assuming the components K_{ij} to be spatially varying. Their tensorial character also permits consideration of hydraulic anisotropy. In case where the solid as well as the fluid are in motion, d'Arcy's law is taken to apply to the relative velocity of the fluid with respect to the solid matrix [e.g., 18,77].

In the so-called thermoelastic theory of consolidation, Schiffman [166] wrote the following equations for coupled mass and heat flow

$$\begin{Bmatrix} h_i \\ v_i \\ T \end{Bmatrix} = \begin{bmatrix} -C_{ji} & -D_{ji} \\ -J_{ji} & -K_{ji} \end{bmatrix} \begin{Bmatrix} \frac{1}{T^2} T_{,j} \\ D_j \end{Bmatrix} \quad (\text{II-4})$$

Mokadam [125-127] studied the thermodynamics of d'Arcy's law under multicomponent flow. Setting up an expression for rate of increase of entropy, Mokadam wrote, for the flow of n fluids through a rigid solid:

$$\begin{Bmatrix} h_i \\ m_i \\ v_i \\ T \end{Bmatrix} = \begin{bmatrix} -C_{ji} & -A_{ji}^m & -D_{ji} \\ -B_{ji} & -E_{ji}^m & -F_{ji} \\ -J_{ji} & -G_{ji}^m & -K_{ji} \end{bmatrix} \begin{Bmatrix} \frac{1}{T^2} T_{,j} \\ \frac{\mu^m}{T} T_{,j} \\ D_j \end{Bmatrix} \quad (\text{II-5})$$

where h_i, m_i, v_i, D_i denote, respectively, components of the heat flux, the diffusive flux, the mass velocity, and the diffusive force vector. μ^m is the chemical potential of the m th constituent. For isothermal flow in the absence of chemical reactions, this equation merely indicates the temperature dependence of the permeability tensor.

The d'Arcy fluid flow equation has been generalized still further [79,80], and indeed forms part of the general phenomenological equations of force-flux given by Onsager [143]. These express the effect of simultaneous presence of fields of mechanical pressure, electric potential, temperature and chemical concentration. The general linear relationship is expressed as

$$J_i = L_{ji} F_j \quad (\text{II-6})$$

where J_i are the fluxes viz., mass flow, heat flow, electric current, chemical diffusion, and F_j are the Prigogine forces [79,143]. L_{ij} are components of a positive definite symmetric tensor. Onsager [143] expected the relationship to be symmetric i.e., $L_{ij} = L_{ji}$. Evidently, the Onsager equation, proposed originally for small perturbations on an equilibrium state, is a restricted type of relationship, assuming a quadratic form for the entropy rate function. In general, J_i can be treated as functionals, and may depend linearly or nonlinearly on the spatial gradients, of all orders, of the potential fields and their history. This has been discussed by Coleman and Mizel [44] in the case of heat conduction. Same type of reasoning would apply to other flux phenomena, and the only thermodynamic restriction is that the scalar product of flux and force be non-negative [190]. Truesdell [191] discussed the limitations of the Onsager relationships.

To allow for compressibility of the fluid, d'Arcy's rule was further generalized, to the form

$$n \rho^* v_i = -K \phi_{,i} \quad (\text{II-7})$$

where ρ^* is the intrinsic fluid mass density and n is the porosity equal to volume fraction of the fluid.

2.1.2. Mass Continuity.

For conservation of mass in arbitrary fixed volumes, in the steady state of saturated flow in rigid porous media,

$$(\rho^* v_i)_{,i} + n \dot{\rho}^* = 0 \quad (\text{II-8})$$

where a superposed dot signifies partial differentiation with respect to the time variable. Substituting for velocity from the generalized d'Arcy equation (II-2),

$$(-\rho^* K_{ji} \phi_{,j})_{,i} + n \dot{\rho}^* = 0 \quad (\text{II-9})$$

For isotropic flow of an incompressible fluid through a homogeneous solid, the density is invariant over time and space and also

$$K_{ij} = K \delta_{ij} \quad (\text{II-10})$$

In this case (II-9) reduces to

$$\phi_{,ii} = 0 \quad (\text{II-11})$$

This is the Laplace equation governing d'Arcy isotropic flow of an incompressible fluid through a homogeneous rigid porous solid.

2.1.3. Analysis of Deformation.

2.1.3.1. Scheidegger's correction.

To extend (II-8) to the case of compressible media, Scheidegger [160] introduced a correction treating porosity as a function of time. The corrected equation was referred to the deformed solid. Thus

$$(\rho^* v_i)_{,i} = - \frac{\partial}{\partial t} [\rho^* n(1+k)] \quad (\text{II-12})$$

where $(1+k)$ is the 'consolidation factor' for the porous medium. If the solid phase is relatively incompressible and the change in volume

is entirely due to change in pore volume, in the above equation we can write

$$n(1+k) = n_0 + k \quad (\text{II-13})$$

where n_0 is the porosity at time zero and k is the volume strain of the solid. Scheidegger was interested primarily in the flow of the fluid, and made the simplifying assumption that solid deformation could be defined by a single variable k .

2.1.3.2. Terzaghi's theory of one-dimensional consolidation.

Drainage and consolidation of a saturated column of soil under constant load was investigated by Terzaghi [187]. Assuming d'Arcy flow and linear elastic saturated solid, he arrived at the equation

$$c_v \pi_{,zz} = \dot{\pi} \quad (\text{no sum}) \quad (\text{II-14})$$

governing dissipation of fluid pressure π . Here c_v , the coefficient of consolidation, is defined as

$$c_v = K/(a_v(1-n)) \quad (\text{II-15})$$

where a_v is the coefficient of compressibility and K the permeability constant.

The only deformation of the porous medium that interested Terzaghi was the displacement of the top surface of the vertical column. This was obtained by defining the degree of consolidation as a linear function of the excess water pressure over the equilibrium (steady state) value, and integrating it over the height of the column, after obtaining the solution to the diffusion equation by Fourier type analysis. Terzaghi's theory was extended to the case of varying permeability and compressibility by several investigators [e.g., 9,48,77,100,121]. The governing equation was written in terms of uniaxial strain e_z as

$$(c_v e_z)_{,z} = \dot{e}_z \quad (\text{no sum}) \quad (\text{II-16})$$

In this manner, a non-constant c_v could be considered. Actually variations in permeability and compressibility become significant only if the void ratio changes and strains are appreciable. However, all these theories assumed small strains. Gibson [77] extended the theory to finite strain. Davis [48] and Simons [178] extended the theory to nonlinear compressibility.

A noteworthy aspect of Terzaghi's work was his definition of fluid pressure and effective stress. For the static case, the fluid pressure in a saturated porous medium is measured by means of piezometers and the measured pressure is assumed to act over 100% of the area of internal surfaces. Water pressure measurements [5,33,92,103,111,188,200] in soils, rocks, and concrete support this assumption. Indeed, stability calculations allowing for uplift in hydraulic structures are based on this assumption. It essentially amounts to regarding the solid particles as having only point contacts with one another and treating the solid as buoyant in the fully connected fluid. This assumption is not valid for very dense rocks where the fluid may not completely enclose each particle of the solid.

2.1.3.3. Biot's theory of three-dimensional consolidation.

Assuming water to be incompressible and d'Arcy flow through an isotropic porous soil, Biot [15] wrote the equation for mass continuity of the fluid in terms of the intrinsic fluid pressure π^* as

$$K \pi_{,ij}^* = \dot{\theta} \quad (\text{II-17})$$

where θ is the change of water content in a fixed volume of the soil expressed as the difference from the initial state, and equals the inflow rate given by the left hand side of the equation. The superposed $*$ indicates an 'intrinsic' rather than a 'bulk' quantity. Biot [15] regarded the total stress t_{ij} and the fluid pressure as the mechanical variables. Assuming that the hydrostatic fluid pressure causes volumetric strains, in the soil mass, which can be superposed on the elastic equilibrium strains for no 'excess' pressure; and that the water content per unit volume of the soil would depend upon the hydrostatic components of stresses in the solid and the fluid, Biot

[15] proposed the following relationships for homogeneous, linear elastic isotropic soil undergoing small strains:

$$e_{ij} = \frac{(1 + \nu)}{E} t_{ij} - \frac{\nu}{E} t_{kk} \delta_{ij} + \frac{1}{3H} \pi^* \delta_{ij} \quad (II-18)$$

and

$$\theta = \frac{1}{3H_1} t_{kk} + \frac{\pi^*}{R} \quad (II-19)$$

Here e_{ij} are components of the strain tensor for the soil, H , H_1 , and R are material constants, E , ν , are, respectively, Young's modulus and Poisson's ratio. Effectively, this amounted to introducing θ as an additional kinematic quantity. Further assuming the existence of an energy function for the saturated soil in terms of e_{ij} and θ , Biot [15] set $H_1 = H$. The inverse form of (II-19) was, for that case,

$$t_{ij} = 2\mu e_{ij} + \lambda \delta_{ij} e_{kk} - \alpha \pi^* \delta_{ij} \quad (II-20)$$

and

$$\theta = \alpha e_{ii} + \frac{\pi^*}{Q} \quad (II-21)$$

where λ , μ are Lamé's constants and α is related to Q , R , and H by the following equations

$$\alpha = \frac{E}{3(1-2\nu)H} \quad (II-22)$$

and

$$\frac{1}{Q} = \frac{1}{R} - \frac{\alpha}{H} \quad (II-23)$$

Combining (II-17) and (II-21), Biot [15] obtained

$$K\pi_{,ii}^* = \dot{\theta} = \alpha \dot{e}_{ii} + \frac{\dot{\pi}^*}{Q} \quad (II-24)$$

(II-20), (II-21) can be rewritten as

$$t_{ij} = 2\mu e_{ij} + (\lambda + \alpha^2 Q) e_{kk} \delta_{ij} - \alpha Q \theta \delta_{ij} \quad (II-25)$$

and

$$\pi^* = -\alpha Q e_{kk} + Q \theta \quad (II-26)$$

Physically, the constant $1/R$ represents change in fluid content per unit 'excess' fluid pressure for no total stress on the system. To define α Biot [15] considered the deformation of a uniaxially loaded system under lateral restraint i.e., $e_{11} = e_{22} = 0$. The initial condition, $\theta = 0$ gave

$$\pi^* = -\alpha Q e_{33} \quad (II-27)$$

and the total stress

$$\begin{aligned} -P_0 &= 2\mu e_{33} + (\lambda + \alpha^2 Q) e_{33} \\ &= (\lambda + 2\mu) e_{33} + \alpha^2 Q e_{33} \end{aligned} \quad (II-28)$$

Here P_0 is the axial compressive load on the unit column. The initial compressibility a_i of the soil column is given by

$$\frac{1}{a_i} = \frac{-P_0}{e_{33}} = \lambda + 2\mu + \alpha^2 Q \quad (II-29)$$

At the equilibrium condition when complete consolidation has occurred, the excess pore water pressure vanishes, i.e., $\pi^* = 0$. For this case, $\theta = \alpha e_{kk}$ and

$$-P_0 = (\lambda + 2\mu) e_{33} \quad (II-30)$$

Thus, the final compressibility a_f of the soil column is given by

$$\frac{1}{a_f} = \frac{-P_0}{e_{33}} = \lambda + 2\mu \quad (II-31)$$

Subtracting (II-31) from (II-29) to eliminate Lamé's constants, and rearranging

$$a_i = \frac{a_f}{1 + \alpha^2 a_f Q} \quad (II-32)$$

Biot [15] combined the equations of equilibrium, mass continuity and the stress-strain relations to set up Navier type equations governing

consolidation of linear elastic soils undergoing small deformations:

$$\mu u_{i,jj} + (\lambda + \mu) u_{j,ij} - \alpha \pi^*_{,i} = 0 \quad (II-33)$$

Biot used this analysis to obtain solutions for the standard consolidation test-constant axial load on a saturated laterally restrained column of soil-and specialized it for the case of saturated clay taking a_i very small in comparison with a_f , the final compressibility of soil. This implied $Q \rightarrow \infty$, $\alpha \rightarrow 1$ and $H = R = \frac{E}{3(1-2\nu)}$. The analysis was later used [16,17] to find solutions for two-dimensional cases of strip loading of half-space.

While extending the analysis to compressible fluids and anisotropic elastic [18] or viscoelastic [19] solids, Biot introduced the volumetric strain of the fluid as the additional strain parameter instead of the change in water content used in the earlier theory. Biot [24] later explained that the two variables were essentially the same. Also, the reference to the stationary state used in the earlier theory was omitted and the force balance equation written, ignoring inertia effects, as

$$t_{ji,j}^{(1)} + \pi_{,i} + \rho b_i = 0 \quad (II-34)$$

Here b_i are components of the body force vector acting on unit mass of the saturated soil. Superscript 1 refers to the soil. The fluid pressure here is the 'bulk' value equal to $n^{(2)} \pi^*$. Superscript 2 refers to the fluid. The flow equations were based on d'Arcy rule applied to relative velocity of the fluid with respect to the solid. Symmetry of the permeability tensor was based upon the assumed existence of a dissipation function. The resulting equations were, in the isotropic case, the same as those obtained earlier. The theory was further extended [19] to the case of viscoelastic solids and to allow for initial stress [24]. For linear elastic solids [18] the constitutive relationship was written as

$$t_{ij}^{(1)} = C_{kl ij} e_{kl}^{(1)} + M_{ij} e_{kk}^{(2)} \quad (II-35)$$

$$\pi^* = M_{ij} e_{ij}^{(1)} + M e_{kk}^{(2)} \quad (\text{II-36})$$

where $e_{kk}^{(2)}$ is the volumetric strain in the fluid and M_{ij} , C_{klij} , M are material parameters. In [24] the quantity $e_{kk}^{(2)}$ was again replaced by a fluid content variable. The same concept was extended to the case of finite elastic deformations [24]. The inflow into an elementary volume was regarded as a vector and its gradient regarded as the quantity conjugate to the fluid pressure in making a contribution to the energy of the system. This would correspond to the velocity gradient-fluid stress pair used by Adkins [2] and Green [81]. Westmann [196] discussed this noticing that while writing mass continuity relations care must be exercised because the volume of each continuum phase is not the same as the true volume of each material. For the case of initial stresses, equilibrium had to be satisfied in the initial as well as the deformed configuration realized after incrementation of stress. This necessitated introduction [24] of incremental stress and body force intensities in local coordinates after deformation. Constitutive laws for incremental stresses and strains were expected to have the same form as (II-35), (II-36).

2.1.3.4. Gibson's Theory.

Gibson [77] removed the limitation of small strains. The relative velocity was expected to follow d'Arcy rule. The theory allowed for nonhomogeneity, rate dependence of materials, and compressibility of the pore fluid and the solid. Consolidation associated with the self-weight of the soil was allowed for. In the one-dimensional theory developed, the soil had a reference configuration and motion of the fluid was considered relative to the current configuration of the solid. The equation of equilibrium of vertical forces associated with the reference volume of the solid in the reference configuration is

$$\frac{\partial s}{\partial X} + \rho \frac{\partial x}{\partial X} = 0 \quad (\text{II-37})$$

where X , x are, respectively, the reference and the place coordinates, s is the vertical total stress and ρ is the density of the saturated

soil in the current configuration. This density is related to densities of the constituents by the relation

$$\rho = n^{(1)} \rho^{(1)*} + n^{(2)} \rho^{(2)*} = \rho^{(1)} + \rho^{(2)} \quad (\text{II-38})$$

Here $n^{(k)}$ is the volume fraction of the constituent indicated by the superscript. As the description of the solid phase is unaffected by deformation, the equation of mass continuity for the solid is simply

$$\rho_0^{(1)*} n_0^{(1)} = \rho^{(1)*} n^{(1)} \frac{\partial x}{\partial X} \quad (\text{II-39})$$

or,

$$\rho_0^{(1)} = \rho^{(1)} \frac{\partial x}{\partial X}$$

where the subscript 0 refers to the initial value, in the reference configuration, of the quantity subscripted. The equation of mass continuity of the fluid, equating the inflow into an elementary volume of the solid with the increase in fluid content, is

$$\frac{\partial}{\partial X} (n^{(2)} \rho^{(2)*} v) + \frac{\partial}{\partial t} (n^{(2)} \rho^{(2)*} \frac{\partial x}{\partial X}) = 0 \quad (\text{II-40})$$

or,

$$\frac{\partial}{\partial X} (\rho^{(2)} v) + \frac{\partial}{\partial t} (\rho^{(2)} \frac{\partial x}{\partial X}) = 0 \quad (\text{II-41})$$

Here v is the velocity of the fluid relative to the soil. Allowing for compressibility of the pore fluid, Gibson [77] used Scheidegger's formulation of d'Arcy law (II-7). In this equation, the differentiation is with respect to the 'place' or the current configuration. Keeping these considerations in view, the flow equation has the form

$$n^{(2)} \rho^{(2)*} v \frac{\partial x}{\partial X} = -K \left[\frac{\partial \pi}{\partial X} + \rho^{(2)*} \frac{\partial x}{\partial X} \right] \quad (\text{II-42})$$

The permeability coefficient K may be a function of porosity and location of the point under consideration.

Gibson expected the effective stress to depend upon the deformation or the rate and history of deformation of the solid skeleton. The fluid pressure was expected to depend upon the fluid

density in an isothermal system.

2.2. WAVE PROPAGATION IN SATURATED SOILS.

2.2.1. Preliminaries.

Presence of water in soil strongly influences its static and dynamic response as well as its strength. In addition to the effect on dynamic amplification factors, continued oscillatory motion can result in undesirable build-up of fluid pressure in the pores leading, in some cases, to "liquefaction" and "soil mobility". This has led to interest in development of methods to predict liquefaction and soil mobility during seismic and other dynamic events, and to define the liquefaction potential of soils. Several different formulations of the problem and solution procedures have been used. Herein we review some of the approaches.

2.2.2. The Engineering Approach.

Seed and Idriss [96-99,170] used methods of structural dynamics to solve the problem of shear wave propagation in soils. They used the finite element method, originally developed by Clough and Chopra [42] and extended to oblique incidence of seismic waves by Dibaj and Penzien [58], to examine the effect of the nature of the soil deposits upon surface motion. These methods had the capability to account for irregular surface geometry as well as nonlinear material behavior.

To allow for pore-water pressure build-up during dynamic loading, Seed and his co-workers developed a methodology and examined several case histories [107,110,153,167-176]. Essentially, this approach consisted of a finite element analysis of the system, under an acceleration history applied to a number of points on the system, to evaluate the stress history. This was followed by a laboratory study of the material behavior under cyclic stress conditions equivalent to those determined from the finite element analysis. This approach was used, with apparent success, by Seed and his co-workers to analyze a number of case histories. However, the method was deficient in

several respects. It could predict the initiation of liquefaction at different points in the system but could not predict the distribution of pore water pressures after the initiation of liquefaction; nor was it able to estimate the extent of the resulting displacements in the system. Furthermore, in laboratory testing under cyclic loads, pore pressures were observed to accumulate. However, in the initial analysis used to determine the stress history under input excitation, variations in pore water pressure were ignored. Dissipation effects could only be included through appropriate choice of damping ratios. The soil properties were assigned at the beginning of the analysis and assumed to represent soil behavior throughout a wide range of pore water pressures. Setting up an "equivalent" number of cycles to reproduce the effect of a non-cyclic stress history was based on "judgment" and would be suspect. Subsequent work has been concerned with improvements in the method leading up to the formulation described by Finn [66]. This is essentially a periodic updating of material behavior to allow for the strain history as well as pore-water pressure build-up and dissipation. In this approach, the equations of motion, after finite element discretization and lumped mass idealization, are

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = \{p(t)\} \quad (II-43)$$

where $[M]$ is the diagonal mass matrix, $[C]$ the viscous damping matrix, $[K]$ the nonlinear stiffness matrix, and $p(t)$ the excitation. $\{u\}$, $\{\dot{u}\}$, $\{\ddot{u}\}$ are, respectively, the displacements, the velocities and the accelerations of the masses $[M]$. The stiffness was assumed to be a function of the volumetric strain and the effective stress in the soil [66,117]. Generation of pore water pressure was assumed to be related to volume changes in water and soil. Assuming water to be incompressible, an incremental relationship was written as

$$\Delta\pi = E_r \Delta e_{kk} \quad (II-44)$$

where $\Delta\pi$ is the incremental pore-water pressure and E_r is the one-dimensional rebound modulus of the material at the effective stress

applicable to the increment. For one-dimensional idealization of a horizontally layered system subjected to shear wave at the base, dissipation of the pore water pressure was expected to be governed by the equation

$$\dot{\pi} = E_r \left(\frac{K}{w} \pi_{,z} \right)_{,z} + E_r \dot{e}_{kk} \quad (\text{II-45})$$

where w is the unit weight of water and K the permeability. Finn called this the coupled theory of liquefaction. The sequence of occurrences is assumed to be as follows: shearing stresses cause volume changes, volume changes result in pore-water pressure changes, pore-pressure dissipation follows (II-45), pore-pressures determine effective stresses and effective stresses along with the cumulative shearing strain define the effective shear modulus to be used for determination of displacements and stresses for the next time step.

2.2.3. Biot's Theory.

Biot [18-23] extended his theory of consolidation of soils to wave propagation in saturated soils. For a statistically isotropic saturated material, the kinetic energy was expected to have the form

$$T = \frac{1}{2} (\rho^{11} v_i^{(1)} v_i^{(1)} + 2 \rho^{12} v_i^{(1)} v_i^{(2)} + \rho^{22} v_i^{(2)} v_i^{(2)}) \quad (\text{II-46})$$

where ρ^{12} is the mass coupling between the fluid and the solid $v_i^{(k)}$ denotes velocity of the constituent $s^{(k)}$. For no relative velocity, $v_i^{(1)} = v_i^{(2)}$; $2T = (\rho^{11} + 2 \rho^{12} + \rho^{22}) v_i^{(1)} v_i^{(1)}$. Hence

$$\rho^{11} + 2 \rho^{12} + \rho^{22} = \rho \quad (\text{II-47})$$

Biot also showed that, assuming gradient of fluid pressure is the only force causing acceleration of the fluid for no relative velocity, Lagrange equation applied to T gives

$$\rho^{(2)} = \rho^{22} + \rho^{12} \quad (\text{II-48})$$

Similarly

$$\rho^{(1)} = \rho^{11} + \rho^{12} \quad (\text{II-49})$$

Also

$$\rho^{12} < 0 \quad (\text{II-50})$$

and the effective mass of soil in motion is

$$\rho^{11} = \rho^{(1)} - \rho^{12} \quad (\text{II-51})$$

Similarly,

$$\rho^{22} = \rho^{(2)} - \rho^{12} \quad (\text{II-52})$$

The above formulation was assumed to be non-dissipative. For the dissipative case, a dissipation function quadratic in relative velocity, was introduced. This led to the equations of equilibrium in the following form

$$t_{ji,j}^{(1)} = (\rho^{11} u_i^{(1)} + \rho^{12} u_i^{(2)}) \dots + b(u_i^{(1)} - u_i^{(2)}) \cdot \quad (\text{II-53})$$

and

$$\pi_{,i} = (\rho^{12} u_i^{(1)} + \rho^{22} u_i^{(2)}) \dots - b(u_i^{(1)} - u_i^{(2)}) \cdot \quad (\text{II-54})$$

Here $u_i^{(1)}$, $u_i^{(2)}$ are the components of the displacement vectors associated with the soil and the pore water respectively, and the superposed dot denotes material time derivative. The coefficient b was related to d'Arcy's coefficient of permeability. For low frequencies, it was [21,69,71]

$$b = \frac{\mu(n^{(2)})^2}{K} \quad (\text{II-55})$$

where μ is the fluid viscosity and K the permeability. Biot's formulation of the coupled problem was the basis of theoretical solutions developed, among others, by Biot [21-23], Deresiewicz [52],

Garg [71], and Chakraborty and Dey [38].

While developing finite element solution procedures for the problem, Ghaboussi [73], following Biot [19,22,23], introduced relative volumetric strain in the formulation. Defining relative strain as

$$\zeta = n^{(2)} (e_{kk}^{(2)} - e_{kk}^{(1)}) \quad (\text{II-56})$$

Biot [19,20] wrote the constitutive equations for the flow of a compressible fluid through a porous saturated linearly elastic anisotropic medium as

$$t_{ij} = E_{kl ij} e_{kl}^{(1)} + M_{ij} \zeta \quad (\text{II-57})$$

$$\pi^* = M_{ij} e_{ij}^{(1)} + M \zeta \quad (\text{II-58})$$

Introducing d'Arcy's law for relative velocity of the fluid with respect to the solid explicitly in the form

$$n^{(2)} (u_i^{(2)} - u_i^{(1)})^* = K \pi^*_{,i} \quad (\text{II-59})$$

and assuming that the order of temporal and spatial differentiation can be interchanged, i.e.,

$$\dot{\zeta} = [n^{(2)} (u_{i,i}^{(2)} - u_{i,i}^{(1)})]^* \quad (\text{II-60})$$

$$= [n^{(2)} (u_i^{(2)} - u_i^{(1)})^*]_{,i} \quad (\text{II-61})$$

Biot obtained

$$\dot{\zeta} = K \pi^*_{,ii} \quad (\text{II-62})$$

Biot [20,22,23] assuming $M_{ij} = \alpha M \delta_{ij}$, wrote (II-58) in an alternative form as

$$\pi^* = \alpha M e_{kk}^{(1)} + M \zeta \quad (\text{II-63})$$

where α is a measure of compressibility the solid particles. Further assuming $t_{ij} = t_{ij}^* + \alpha \pi^* \delta_{ij}$, where t_{ij}^* is the effective stress and substituting for π^* , he got

$$t_{ij} = E_{klij} e_{kk}^{(1)} + \alpha M \delta_{ij} (\alpha e_{kk}^{(1)} + \zeta) \quad (II-64)$$

This form was used by Ghaboussi and Wilson [73,74] and others for development of finite element procedures. Garg [71] wrote the constitutive relations in the form

$$t_{ij}^{(1)} = \bar{a} e_{kk}^{(1)} \delta_{ij} + c e_{kk}^{(2)} \delta_{ij} + 2 \mu (e_{ij}^{(1)} - \frac{1}{3} \delta_{ij} e_{kk}^{(1)}) \quad (II-65)$$

$$\pi = n^{(2)} \pi^* = c e_{kk}^{(1)} + b e_{kk}^{(2)} \quad (II-66)$$

Here μ is the bulk shear modulus of the porous solid and a, b, c are functions of the volume fractions of the constituents, the bulk modulus of the porous solid, and the intrinsic bulk moduli of the fluid and the solid. Specifically comparing Garg's [71] equation of equilibrium along with the constitutive relations (II-65), (II-66), with Biot's equations (II-53), (II-54),

$$b = M(n^{(2)})^2 \quad (II-67)$$

$$c = M(\alpha - n^{(2)}) n^{(2)} \quad (II-68)$$

$$\bar{a} = \lambda + 2/3 \mu + M(\alpha - n^{(2)})^2 \quad (II-69)$$

where λ, μ are Lamé's constants for the porous elastic solid. Of course, the interpretation of the constants is different. In Garg's theory, the constants are related to the properties of the constituents and the volume fractions as follows:

$$\bar{a} = n^{(1)} K_1 (1 + b_1) \quad (II-70)$$

$$b_1 = a_1 n^{(2)} (n^{(2)} - (n^{(1)} K_2 / K_1) a_1)^{-1} \quad (II-71)$$

$$b = (n^{(2)} - n^{(1)} b_2) K_2 \quad (\text{II-72})$$

$$c = n^{(1)} K_1 b_2 = - n^{(1)} K_2 b_1 \quad (\text{II-73})$$

$$a_1 = K/(n^{(1)} K_1) - 1 \quad (\text{II-74})$$

Here, K_1 , K_2 are the intrinsic bulk moduli of the solid and the fluid; and K , μ are the bulk and the shear modulus of the porous solid. In Biot's theory, the coefficients are phenomenological coefficients directly measured in the laboratory.

Ghaboussi and Wilson [74] stated Biot's balance equations as

$$t_{ji,j} + \rho b_i = \rho \ddot{u}_i^{(1)} - \rho^{(2)} [\ddot{u}_i^{(1)} - \ddot{u}_i^{(2)}] = \rho^{(1)} \ddot{u}_i^{(1)} + \rho^{(2)} \ddot{u}_i^{(2)} \quad (\text{III-75})$$

and

$$\begin{aligned} \pi_{,i} + \rho^{(2)} b_i &= \rho^{(2)} \ddot{u}_i^{(1)} - \rho^{(2)} (\ddot{u}_i^{(1)} - \ddot{u}_i^{(2)}) - \frac{1}{K} (\dot{u}_i^{(1)} - \dot{u}_i^{(2)}) \\ &= \rho^{(2)} \ddot{u}_i^{(2)} - \frac{1}{K} (\dot{u}_i^{(1)} - \dot{u}_i^{(2)}) \end{aligned} \quad (\text{II-76})$$

where K is a measure of permeability of the soil. Combining (II-60), (II-62) and (II-63), Ghaboussi wrote

$$\int_0^t K(\pi_{,ii}^* + \rho^{(2)} b_{i,i}) dt + \zeta_0 + \alpha u_{i,i}^{(1)} - \frac{1}{M} \pi^* = 0 \quad (\text{II-77})$$

The momentum balance equations (II-75) and (II-76) were written in terms of the six displacement components $u_i^{(1)}$ and $(u_i^{(2)} - u_i^{(1)})$. There is no dissipative term in (II-75). For obtaining numerical solutions to the problem, Ghaboussi [73] introduced an arbitrary viscous damping term proportional to $u_i^{(1)}$. A Rayleigh type damping was assumed. Regarding (II-75) as the equation of motion for the soil and (II-76) as the one for relative motion of the fluid, we may consider the damping in (II-75) to represent intrinsic damping of the soil and the relative velocity term in (II-76) to represent the damping associated with relative motion.

Garg [71] used Biot's equations of motion, (II-53), [II-54] in the following form:

$$\tau_{ij,i}^{(1)} = \rho^{(1)} \ddot{u}_j^{(1)} + b(\dot{u}_j^{(1)} - \dot{u}_j^{(2)}) \quad (\text{II-78})$$

$$\pi_{,j} = \rho^{(2)} \ddot{u}_j^{(2)} - b(\dot{u}_j^{(1)} - \dot{u}_j^{(2)}) \quad (\text{II-79})$$

The second term on the right hand side of these equations represents the viscous coupling between the solid and the fluid. Ghaboussi and Wilson presented variational formulations for the problem along with a finite element solution procedure.

2.2.4. Fukuo's Theory.

Fukuo [68] proposed a dynamic theory of deformation of a fluid-saturated solid. He assumed the fluid to be Newtonian, the fluid pressure to be the thermodynamic pressure determined by its density and temperature, d'Arcy law to hold for relative motion of the fluid with respect to the solid, and the fluid to be incompressible. The solid was assumed to be linear viscoelastic. Values of displacement, stress, velocity, were averaged over elementary volumes. Strains in the solid were assumed to be infinitesimal. In writing the equation of motion for the solid, the effect of pressure of the fluid as well as the 'drag' force of flowing viscous fluid upon the solid was taken into account. Equations of mass continuity were written considering reference volumes of the solid in the reference or original configuration. This is parallel to the development, proposed by Gibson [77], for the one dimensional quasi-static problem. The theory is essentially similar to Biot's.

2.2.5. Comments

We note that the 'Engineering' approach is cumbersome. The physical properties can only be determined as a function of the complete number of cycles of stress at a certain amplitude. Thus, in an explicit type solution scheme, it is possible to reduce the time

interval for updates to only as low as the time period of vibration. Implicit schemes can use larger intervals and are expected to be more reliable. However, it has been difficult to show convergence of results. Under harmonic loading, the time to liquefaction and the location of the layer that liquefies have been correctly determined in some cases. However, the acceleration and displacement response calculation is not reliable. Post-liquefaction distribution of pore pressures cannot be correctly determined in this theory. The approach has only been used for one-dimensional wave propagation. It cannot be extended to two and three-dimensional cases. Use of this theory requires considerable experience and 'judgment', in addition to extensive laboratory testing program, to get useful results.

Considering Biot's theories, we note several different forms. Adding (II-53) and (II-54), noting (II-48), (II-49), and adding the body force on the mixture, we get (II-75). However, (II-76) is different from (II-54) not only in that (II-75) includes body force effects, but also in that the inertia terms are different. Of course, the coefficient b in (II-54) may be identified with $1/K$ in (II-76). Again, (II-78) and (II-79) can be identified with (II-53) and (II-54) only if the quantities ρ^{12} , ρ^{21} vanish; i.e., there is no inertial coupling between the fluid and the porous solid.

In writing the kinetic energy of the mixture (II-46) Biot included a term involving product of the velocities of the constituents. This implies that the kinetic energies of the constituents are not additive. This assumption will be examined further in Chapter III in the context of Truesdell's [189-191] and Green's [83-88] work.

SECTION III

THEORY OF MIXTURES

3.1. INTRODUCTION.

In this section we outline various schemes used to describe the motion of a multi-constituent material and to set up equations of balance of mass, linear momentum, angular momentum and energy. The relationship of quantities associated with the constituents and with the mixture as a whole is reviewed. Truesdell's [189,191] and Green's [83-88] theories are compared. Other relevant work is mentioned as needed. Some of the related work is summarized towards the end of the section.

Truesdell and Toupin [189] wrote the local form of the equations of balance of mass, linear momentum, angular momentum, and energy for the constituents and for the mixture. They introduced the notion of a diffusive force. They also wrote the local form of an entropy inequality for the mixture. Kelly [104] presented integral forms of the balance equations and derived the local forms from them.

In developing a rational theory of mixtures, Truesdell [191] laid down the following "three metaphysical principles".

1. All properties of the mixture must be mathematical consequences of properties of the constituents.
2. So as to describe the motion of a constituent, we may in imagination allow properly for the actions of the other constituents upon it.
3. The motion of the mixture is governed by the same equations as is a single body.

To ensure satisfaction of these principles, the total stress had to be so defined that it did not equal the sum of partial stresses in the constituents. Green and Naghdi [83,85,87] proposed an alternative approach assuming that the total stress must equal the sum of partial

stresses in the constituents. A deficiency in the theory, pointed out by Müller [136], was corrected [88] to admit interacting partial stresses in the constituents which may not contribute to the equations of motion or the total stress.

Other theories of mixtures have been proposed. Our interest is primarily in mixtures of immiscible constituents so that the results might be applicable to the problem of fluid-saturated soils. Of immediate interest would be the so-called volume fraction theories.

3.2. KINEMATICS.

3.2.1. Preliminaries.

To apply the principles of continuum mechanics, it is customary to regard a fluid-saturated solid as superposed continua. We note immediately that averaging of various quantities, kinematic as well as mechanical, associated with the various constituents, is inherent in this assumption. The mixture is defined by the current coincident configuration of the constituents. It is assumed that, in the current configuration, each point of space is occupied by a particle of each of the constituents. This necessitates the introduction of 'bulk' description of the material instead of the 'intrinsic' description which would apply if the material were the single constituent of a body. In any theory of mixtures, it would be necessary that as the volume fraction of one of the constituents approaches unity, and the remaining constituents disappear, the theory for a single constituent be realized as a limiting case.

3.2.2. Density of Each Constituent and of the Mixture.

If $n^{(k)}$ is the volume fraction of constituent $s^{(k)}$ in the mixture, its effective (we shall also refer to it as partial or bulk) density, when it is regarded as one of the continua occupying every point in the spatial region of interest, is related to its intrinsic density $\rho^{(k)*}$ as

$$\rho^{(k)} = n^{(k)} \rho^{(k)*} \quad (\text{III-1})$$

Density ρ of a mixture of n constituents is defined as the sum of their partial densities, i.e.,

$$\rho = \sum_{k=1}^n \rho^{(k)} \quad (\text{III-2})$$

We note that other definitions for partial densities are possible. For instance, Terzaghi's theory of consolidation and seepage assumes the effective density of water to have the intrinsic value and the soil is regarded as buoyant in the water. In a binary mixture of a fluid and a solid, the conventional theory of mixture definitions given by (III-1) and (III-2) would imply

$$\rho = n^{(1)} \rho^{(1)*} + n^{(2)} \rho^{(2)*} \quad (\text{III-3})$$

Here, the superscript 1 refers to the solid and 2 to the fluid. Using Terzaghi's definition of bulk density for the solid and intrinsic density for the fluid, the total density of the saturated soil is given by

$$\rho = \rho^{(1)} + \rho^{(2)*} \quad (\text{III-4})$$

and, consequently, the buoyant bulk density of the solid is

$$\rho^{(1)} = \rho - \rho^{(2)*} = n^{(1)} (\rho^{(1)*} - \rho^{(2)*}) \quad (\text{III-5})$$

Incompressibility of $s^{(k)}$ implies $\dot{\rho}^{(k)*}$ equal to zero. It is important to note here that the total density of a mixture is the weighted sum of the densities of the constituents. However, the mixture cannot be regarded as a continuum, the particles of which are constructed by superposition of constituent particles, except in the relatively uninteresting case of no relative motion between the constituents. Atkin [7] pointed out that the mixture assigned the mass density calculated according to (III-2) consists only of centers of mass. Sandhu [158,159] pointed out that the mixture cannot be

treated as a continuum as it does not satisfy the impenetrability postulate except in the case of no relative motion. It should, therefore, be kept in mind that the total density introduced above is a purely mathematical entity having no physical interpretation except in a special case.

3.2.3. Motion of the Constituents.

Several approaches have been used to describe the motion of the constituents of a multicomponent mixture. One is to refer the motion of each constituent to its reference configuration. Another is to refer the motion of all constituents to the reference configuration of one of them. Yet another is to refer all motion to the current configuration, which is the same for all constituents. Superposition of relative diffusive motion of the constituents upon the mean motion of the mixture as a whole is also used. It has been pointed out in the previous paragraph that the mixture consisting of coincident current configurations of all constituents has only momentary existence as a continuum. It does not deform or move as a body, in general. For a binary mixture of a solid and a fluid, some investigators describe the motion of the solid with respect to its reference configuration but the motion of the fluid is described as relative to the solid. Another procedure refers to a material region consisting of the same set of particles of one of the constituents so that the bounding surface of this constituent varies with time. If, in this case, the description of the material particles is the same during motion as in the initial configuration, this procedure would be the convected coordinate description for the reference constituent. The following quantities and notations are often employed. Using a fixed rectangular cartesian reference frame, let $x_i^{(k)}(\tau)$ be the place occupied at time τ by the material particle described $x_i^{(k)}$ in the reference configuration for the constituent $s^{(k)}$. Then

$$x_i^{(k)}(\tau) = x_i^{(k)}(x_i^{(k)}, \tau) \quad \text{for } -\infty < \tau < t \quad (\text{III-6})$$

For the position in the current configuration, i.e. for $\tau = t$,

following Green [83], we use the notation

$$x_i^{(k)} = x_i^{(k)}(t) \quad (\text{III-7})$$

Then, the position at any time τ can be expressed in terms of the current position in the form

$$x_i^{(k)}(\tau) = x_i^{(k)}(x_j^{(k)}, t, \tau) \quad (\text{III-8})$$

Material deformation $F_{ij}^{(k)}$ is defined as the partial derivative of the place coordinates $x_i^{(k)}$ with respect to the material coordinates $x_j^{(k)}$, i.e.,

$$F_{ij}^{(k)} = \frac{\partial}{\partial x_j^{(k)}} x_i^{(k)}(x_j^{(k)}, t) \quad (\text{III-9})$$

For the deformation to be possible in a real material,

$$J = \det [F_{ij}^{(k)}] \neq 0 \quad (\text{III-10})$$

At a given time t , consider the mixture consisting of particles of all constituents having the same configuration, i.e.,

$$x_i^{(k)}(t) = x_i^{(k)}(t) \quad (\text{III-11})$$

The velocity vectors at the point $x_i^{(k)}$ are defined as the material rate

$$v_i^{(k)} = \frac{D^{(k)}}{Dt} x_i^{(k)} = \frac{\partial}{\partial t} x_i^{(k)}(x_j^{(k)}, t) \quad (\text{III-12})$$

Here the differentiation is carried out keeping $x_j^{(k)}$ fixed. If the position x_i is treated as function of the position in the current configuration, the material rate operator is written as

$$\frac{D^{(k)}}{Dt} = \frac{\partial}{\partial t} + v_i^{(k)} \frac{\partial}{\partial x_i^{(k)}} \quad (\text{III-13})$$

The acceleration vector at time t for constituent $s^{(k)}$ is denoted by

$$f_i^{(k)} = \frac{D^{(k)}}{Dt} v_i^{(k)} = \frac{\partial}{\partial t} v_i^{(k)} (x_j^{(k)}, t) \quad (\text{III-14})$$

Components of the rate of deformation tensor, for the material $s^{(k)}$, at time t are defined by, [45],

$$d_{ij}^{(k)} = \frac{D^{(k)}}{Dt} F_{ij}^{(k)} = \frac{D^{(k)}}{Dt} x_{i,j}^{(k)} \quad (\text{III-15})$$

Interchanging the order of spatial and temporal differentiation,

$$d_{ij}^{(k)} = (v_i^{(k)})_{,j} = L_{im}^{(k)} (F_{jm}^{(k)})^{-1} \quad (\text{III-16})$$

where $L_{ij}^{(k)}$ are components of the spatial velocity gradients. These relations hold for each constituent but not for the mixture as a whole, because the mixture does not have a reference configuration. The symmetric part of $L_{ij}^{(k)}$ is the spatial rate of deformation and the antisymmetric part is the vorticity tensor whose components are given by

$$\Gamma_{ij}^{(k)} = L_{[ij]}^{(k)} \quad (\text{III-17})$$

where the square brackets around the pair of indices denote 'anti-symmetric part'. The spatial rate of deformation is

$$D_{ij}^{(k)} = L_{(ij)}^{(k)} \quad (\text{III-18})$$

Most investigators, following Truesdell [189] and Green and Naghdi [83] also introduce a barycentric velocity for the mixture and a material rate for the mixture as a whole. The velocity is defined as

$$v_i = \sum_{k=1}^n \rho^{(k)} v_i^{(k)} / \rho \quad (\text{III-19})$$

where ρ has already been defined by (III-2) as the density of the mixture. A material rate, executed on the mixture, is defined by the following argument:

$$\begin{aligned}
\sum_{k=1}^n \rho^{(k)} \frac{D^{(k)}}{Dt} &= \sum_{k=1}^n \rho^{(k)} \left[\frac{\partial}{\partial t} + v_i^{(k)} \frac{\partial}{\partial x_i^{(k)}} \right] \\
&= \rho \frac{\partial}{\partial t} + \rho v_i \frac{\partial}{\partial x_i} \\
&= \rho \frac{D}{Dt}
\end{aligned} \tag{III-20}$$

We note, however, that this material rate has, in Atkin and Craine's [7] words, 'no particular physical significance'. It is so because the rate is executed not on a material particle but on a center of mass. Another quantity, found convenient, is the diffusive velocity. This is defined as the velocity of a constituent relative to that of the mixture, i.e.,

$$u_i^{(k)} = v_i^{(k)} - v_i \tag{III-21}$$

Clearly the sum of $u_i^{(k)}$ over all the constituents vanishes. The relative velocity of constituent $s^{(1)}$ with respect to constituent $s^{(k)}$ is

$$v_i^{(1k)} = v_i^{(1)} - v_i^{(k)} \tag{III-22}$$

Biot [22], for the case of binary mixture of a solid and a fluid, introduced a nominal relative velocity

$$w_i = n^{(2)} (v_i^{(2)} - v_i^{(1)}) \tag{III-23}$$

This and its variants were used by Ghaboussi and Wilson [73,74], Krause [109] and Kenyon [105,106], among others. It can be easily shown that

$$n^{(2)} \rho u_i^{(2)} = \rho^{(1)} w_i \tag{III-24}$$

$$n^{(2)} \rho u_i^{(1)} = - \rho^{(2)} w_i \tag{III-25}$$

3.2.4. Some Measures of Deformation of Saturated Soils.

Noting that the rate of deformation $d_{ij}^{(k)}$ is uniquely related to the deformation gradient $F_{ij}^{(k)}$ and the spatial velocity gradient $L_{ij}^{(k)}$, and $L_{ij}^{(k)}$ can in turn be expressed in terms of $D_{ij}^{(k)}$ and $\Gamma_{ij}^{(k)}$, these last two quantities can be used to completely define rates of deformation. Superposed rigid body angular velocity Ω would change $\Gamma_{ij}^{(k)}$ to $\Gamma_{ij}^{(k)} + \Omega$ leaving the other quantities unchanged. Thus, for material objectivity to be satisfied, the vorticity tensors, in binary mixtures, must occur as the difference $\Delta_{ij} = \Gamma_{ij}^{(1)} - \Gamma_{ij}^{(2)}$.

The deformation gradient F_{ij} requires a reference configuration. For single constituent solids, this quantity is generally used to introduce an objective measure of strain. For a fluid-saturated configuration of the solid. Garg [69], Garg and Nur [70] and Morland [130,131] among others, used as measures of deformation of the solid, the quantity

$$e_{ij}^{(1)} = u_{(i,j)}^{(1)} \quad (\text{III-26})$$

where $u_i^{(1)} = x_i^{(1)} - \chi_i^{(1)}$. Thus $e_{ij}^{(1)}$ is the symmetric part of $F_{(ij)}^{(1)} - \delta_{ij}$. This measure of deformation characterizes the linear theory. In the general case, other measures of deformation are used [e.g. 130]. The deformation gradient F_{ij} satisfies the polar decomposition theorem. Using direct notation, this means that there exist unique second rank tensors M and R such that $F = R M$ where M is symmetric and R is a unitary operator such and $R^T R$ equals identity and the determinant of R equals one. For an initially isotropic solid, the isotropy group would include R , and it is possible to define as a measure of deformation the quantity

$$2e_{ij} = F_{mi} F_{mj} - \delta_{ij} \quad (\text{III-27})$$

Here $F_{mi} F_{mj}$ are components of the right Cauchy-Green deformation tensor [137]. Often, the only deformation of interest is the volume change. In that case, the change in density is an adequate measure of

deformation. Considering the deformation of the solid constituent $s^{(1)}$ in a binary fluid-solid mixture,

$$\rho^{(1)} \det [F_{ij}^{(1)}] = \rho_0^{(1)} \quad (\text{III-28})$$

where the subscript zero denotes the initial configuration. For small deformations

$$\det[F_{ij}^{(1)}] \approx 1 + e_{jj}^{(1)} \quad (\text{III-29})$$

where $e_{jj}^{(1)}$ is the volumetric strain. The change in density, therefore, for small strain is

$$\rho^{(1)} - \rho_0^{(1)} = - \rho^{(1)} e_{jj}^{(1)} \quad (\text{III-30})$$

For the fluid constituent $s^{(2)}$, the deformation is completely represented by its density. Following (III-30),

$$e_{jj}^{(2)} = \frac{\rho_0^{(2)}}{\rho^{(2)}} - 1 = \frac{n_0^{(2)} \rho_0^{(2)*}}{n^{(2)} \rho^{(2)*}} - 1 \quad (\text{III-31})$$

Introducing an intrinsic measure of deformation [130]

$$F_{ij}^{(2)*} = [n^{(2)}/n_0^{(2)}]^{1/3} F_{ij}^{(2)} \quad (\text{III-32})$$

Morland [131] also proposed, for infinitesimal strain theory,

$$e_{jj}^{(1)*} = e_{jj}^{(1)} - [n^{(2)} - n_0^{(2)}]/[1 - n_0^{(2)}] \quad (\text{III-33})$$

$$- e_{jj}^{(1)} = \rho^{(1)} / \rho_0^{(1)} - 1 \quad (\text{III-34})$$

$$e_{ij}^{(1)*} - 1/3 e_{mm}^{(1)*} \delta_{ij} = e_{ij}^{(1)} - 1/3 e_{mm}^{(1)} \delta_{ij} \quad (\text{III-35})$$

and [133]

$$e_{jj}^{(2)*} = e_{jj}^{(2)} + [n^{(2)} - n_0^{(2)}]/n_0^{(2)} \quad (\text{III-36})$$

For a fluid-saturated solid, in addition to $e_{ij}^{(1)}$, $e_{jj}^{(2)}$ the porosities $n^{(1)}$, $n^{(2)}$ have been used as state variables. An intrinsic rate of strain of the fluid

$$D_{ij}^{(2)*} = D_{ij}^{(2)} + \frac{1}{3} \left(\frac{n_0^{(2)}}{n^{(2)}} \right) \frac{D^{(2)}}{Dt} \left(\frac{n_0^{(2)}}{n^{(2)}} \right) \delta_{ij} \quad (\text{III-37})$$

was introduced by Morland [131] assuming the deviation parts of $D_{ij}^{(2)*}$, and $D_{ij}^{(2)}$ to be the same.

Another measure of relative deformation sometimes used is, e.g., Krause [109] the change in fluid content of a fixed volume in space. This amounts to

$$n = \rho^{(2)} - \rho_0^{(2)} = (n^{(2)} - n_0^{(2)}) \rho^{(2)*} \quad (\text{III-38})$$

Carroll [34] considered a fixed volume of the solid in the reference configuration following Gibson's [77] approach. The total deformation was considered to be the sum of the strains of the solid particles and the change in geometry of the pores. Writing volume of the reference set of particles as

$$V = V^{(1)} + V^{(2)} = n^{(1)} V + n^{(2)} V \quad (\text{III-39})$$

the deformation (change in volume) is given, for constituent $s^{(k)}$, by

$$\Delta V^{(k)} = n^{(k)} \Delta V + \Delta n^{(k)} V \quad (\text{III-40})$$

Also, for a binary mixture,

$$\Delta V = \Delta V^{(1)} + \Delta V^{(2)} = \Delta V^{(1)} + n^{(2)} \Delta V + \Delta n^{(2)} V$$

Hence

$$(1 - n^{(2)}) \frac{\Delta V}{V} = \Delta n^{(2)} + \frac{\Delta V^{(1)}}{V} = \Delta n^{(2)} + \frac{n^{(1)} \Delta V^{(1)}}{V^{(1)}} \quad (\text{III-41})$$

i.e., the bulk volume strain is

$$\frac{\Delta V}{V} = \frac{\Delta n^{(2)}}{n^{(1)}} + \frac{\Delta v^{(1)}}{v^{(1)}} \quad (\text{III-42})$$

For anisotropic case, Carroll and Katsube [36] wrote

$$e_{ij} = n^{(1)} e_{ij}^{(1)*} + n^{(2)} e_{ij}^{(2)*} = e_{ij}^{(1)*} + n^{(2)} (e_{ij}^{(2)*} - e_{ij}^{(1)*}) \quad (\text{III-43})$$

as the strain of the reference material volume of the bulk solid. Carroll, Schatz and Yamada [35] assumed the solid intrinsic deformations to be reversible under solid pressure and the pore volume change to be irreversible under effective pressure. This would explain cumulative volume change under cycling of load. Kenyon [106] used the specific volume of the solid defined as the reciprocal of the intrinsic mass density as a state variable related to the pore fluid pressure π^* and the solid deformation gradient $F_{ij}^{(1)}$. Assuming specific volumes to be independent of the bulk strain deviation, $F_{ij}^{(1)}$ would be replaced by the volumetric strain $e_{jj}^{(1)}$.

Mindlin [123], considering the microdisplacement of the solid particles of a porous solid, introduced relative deformation r_i and microdeformation gradient g_{ij} as kinematic variables in addition to the macroscopic deformation e_{ij} . Introducing an energy function, quadratic in these three variables, corresponding mechanical quantities were defined.

3.3. BALANCE LAWS.

3.3.1. Introduction.

Discussing the thermodynamics of d'Arcy's law, Mokadam [126] wrote the equations of mass, momentum and energy balance for a fluid flowing through a porous rigid solid medium. Truesdell [189] postulated equations of balance of mass, linear momentum, moment of momentum and energy such that the form of the equations was the same for each constituent and for the mixture. Apparently, the existence of the mixture as a continuum in motion was implied in this line of

thought. To accomplish this identity of form, the total stress tensor, the total heat flux vector, and the specific energy supply had to be specially defined. These quantities did not equal the sum of the corresponding quantities for the constituents. The energy of the mixture was, however, equal to the sum of energies of the constituents. On the other hand, Green and Naghdi [83,85] postulated the additive property of stress, heat flux, and the energy supply and derived the balance laws from frame invariance of a rate of energy equality. The energy density of the mixture was seen to be different from the sum of energy densities of the constituents. This was attributed to interaction between the constituents. Green and Naghdi [85] compared the two viewpoints. In the following section, we summarize both. Recent work by Gurtin [90,91], Oliver [140,141], Sampaio [154] and Williams [197] uses a different approach to the theory of mixtures. The equations of balance of momentum and energy they obtain differ from those of Truesdell [189] and Kelly [104]. In addition to the partial stress for each constituent, they obtained embedding stresses governed by additional balance of force equations. Beatty [12] defined a dynamical process as one in which the power is frame independent and derived the balance laws from the frame indifference of mechanical power.

3.3.2. Truesdell's Theory.

For each constituent $s^{(k)}$ of the mixture, Truesdell [189,191] postulated the balance laws in the point form. This was, using the notation of [85],

i. Continuity of Mass.

$$\frac{\partial}{\partial t} \rho^{(k)} + (\rho^{(k)} v_i^{(k)})_{,i} = \rho c^{(k)} \quad (\text{III-44})$$

where $\rho^{(k)}$, $c^{(k)}$ are, respectively, the density and the mass production fraction of the constituent $s^{(k)}$. As

$$\begin{aligned}\frac{D^{(k)}}{Dt} \rho^{(k)} + \rho^{(k)} v_{i,i}^{(k)} &= \frac{\partial \rho^{(k)}}{\partial t} + v_i^{(k)} \rho_{,i}^{(k)} + \rho^{(k)} v_{i,i}^{(k)} \\ &= \frac{\partial}{\partial t} \rho^{(k)} + (\rho^{(k)} v_i^{(k)})_{,i}\end{aligned}$$

(III-44) can also be written in the form

$$\frac{D^{(k)}}{Dt} \rho^{(k)} + \rho^{(k)} v_{i,i}^{(k)} = \rho^{(k)} c^{(k)} \quad (\text{III-45})$$

Alternatively [29], (III-45) is given by

$$\frac{D}{Dt} (\rho^{(k)} \det F_{ij}^{(k)}) = \rho^{(k)} c^{(k)} \det [F_{ij}^{(k)}] \quad (\text{III-46})$$

For no chemical reaction i.e., $c^{(k)} = 0$, this gives (III-28).

ii. Balance of Linear Momentum.

$$\frac{\partial}{\partial t} (\rho^{(k)} v_i^{(k)}) + (\rho^{(k)} v_i^{(k)} v_j^{(k)})_{,j} - t_{ji,j}^{(k)} = \rho^{(k)} m_i^{(k)} + \rho^{(k)} b_i^{(k)} \quad (\text{III-47})$$

Here $t_{ji}^{(k)}$, $\rho^{(k)} b_i^{(k)}$, $\rho^{(k)} m_i^{(k)}$ are, respectively, components of the partial Cauchy stress tensor, the body force vector per unit mass of the constituent, and the partial momentum supply density for the constituent. Using (III-44), (III-47) can be rewritten as

$$\rho^{(k)} f_i^{(k)} = t_{ji,j}^{(k)} + \rho^{(k)} m_i^{(k)} + \rho^{(k)} b_i^{(k)} - \rho^{(k)} c^{(k)} v_i^{(k)} \quad (\text{III-48})$$

Bowen [29] proposed to replace the momentum supply term $m_i^{(k)}$ by $m_i^{(k)} + c^{(k)} v_i^{(k)}$ in (III-47) to get

$$\rho^{(k)} f_i^{(k)} = t_{ji,j}^{(k)} + \rho^{(k)} m_i^{(k)} + \rho^{(k)} b_i^{(k)} \quad (\text{III-49})$$

Of course, $\rho^{(k)} m_i^{(k)}$ in (III-49) is different from the quantity represented by the same symbol in (III-45) and (III-47).

iii. Balance of Moment of Momentum.

$$t_{ij}^{(k)} - t_{ji}^{(k)} = \rho M_{ij}^{(k)} \quad (\text{III-50})$$

Here $\rho M_{ij}^{(k)}$ are components of the skew-symmetric tensor describing the partial production density of the moment of momentum of the constituent $s^{(k)}$. Bowen [29] derived (III-50) from the axiom of balance of momentum applied to a fixed volume in space.

iv. Balance of Energy Rates.

$$\rho^{(k)} \frac{D^{(k)}}{Dt} (U^{(k)}) + q_{i,i}^{(k)} - \rho^{(k)} r^{(k)} = t_{ij}^{(k)} v_{(i,j)}^{(k)} + \rho \Delta U^{(k)} \quad (\text{III-51})$$

where

$$\begin{aligned} \Delta U^{(k)} = & e^{(k)} - (U^{(k)} - \frac{1}{2} v_i^{(k)} v_i^{(k)})_c^{(k)} - v_i^{(k)} m_i^{(k)} \\ & - \frac{1}{2} M_{ij}^{(k)} v_{j,i}^{(k)} \end{aligned} \quad (\text{III-52})$$

Here $U^{(k)}$, $q_i^{(k)}$, $r^{(k)}$, $e^{(k)}$ are respectively, the specific internal energy, components of the partial heat flux vector, the partial energy supply and the partial energy production density of the constituent $s^{(k)}$. These equations represent the balance of elementary volumes in the current configuration.

Bowen [29] wrote the energy rate balance equation, for a fixed volume in space of the constituent $s^{(k)}$, in the form

$$\begin{aligned} & \frac{\partial}{\partial t} \int_V \rho^{(k)} (U^{(k)} + \frac{1}{2} v_i^{(k)} v_i^{(k)}) dV + \\ & \int_A \rho^{(k)} (U^{(k)} + \frac{1}{2} v_i^{(k)} v_i^{(k)}) v_j^{(k)} n_j dA \end{aligned}$$

$$\begin{aligned}
&= \int_A (t_{ij}^{(k)} v_i^{(k)} - q_j^{(k)}) n_j dA \\
&+ \int_V [\rho^{(k)} r^{(k)} + \rho^{(k)} v_i^{(k)} b_i^{(k)} + \rho^{(k)} e^{(k)} + \rho^{(k)} v_i^{(k)} m_i^{(k)} \\
&+ \rho^{(k)} c^{(k)} (U^{(k)} + \frac{1}{2} v_i^{(k)} v_i^{(k)})] dV \quad (III-53)
\end{aligned}$$

Application of the divergence theorem and substitution of equations of balance of mass and linear momentum give the point form

$$\rho^{(k)} \frac{D^{(k)} U}{Dt} + q_{j,j}^{(k)} - \rho^{(k)} r^{(k)} = t_{i,j}^{(k)} v_{i,j}^{(k)} + \rho^{(k)} e^{(k)} \quad (III-54)$$

Several alternative forms of the above equation can be stated [29,189].

Truesdell postulated balance equations for the mixture as well. These equations can be derived by summing over the equations for the constituents. Truesdell introduced the following quantities to ensure that the resulting equations for the motion of the mixture have the same form as the equations of motion of a single constituent. We particularly note that the sum of the specific internal energy and the kinetic energy per unit volume of the mixture is assumed to be equal to the sum of corresponding energies of the constituents contained in the unit volume. The quantity $u_i^{(k)}$ is defined by (III-21).

i. Specific internal energy of the mixture

$$U = \frac{1}{\rho} \sum_{k=1}^n \rho^{(k)} (U^{(k)} + \frac{1}{2} u_i^{(k)} u_i^{(k)}) \quad (III-55)$$

Or,

$$\rho (U + \frac{1}{2} v_i v_i) = \sum_{k=1}^n \rho^{(k)} (U^{(k)} + \frac{1}{2} v_i^{(k)} v_i^{(k)}) \quad (III-56)$$

ii. Total stress tensor

$$t_{ij} = \sum_{k=1}^n (t_{ij}^{(k)} - \rho^{(k)} u_i^{(k)} u_j^{(k)}) \quad (III-57)$$

$$\text{Or,} \quad t_{ij} - \rho v_i v_j = \sum_{k=1}^n (t_{ij}^{(k)} - \rho^{(k)} v_i^{(k)} v_j^{(k)}) \quad (\text{III-58})$$

iii. Total heat flux vector

$$q_i = \sum_{k=1}^n (q_i^{(k)} + \rho^{(k)} (U^{(k)} + \frac{1}{2} u_j^{(k)} u_j^{(k)}) u_i^{(k)} - t_{ji}^{(k)} u_j^{(k)}) \quad (\text{III-59})$$

$$\begin{aligned} \text{Or,} \quad q_i + \rho v_i [U + \frac{1}{2} v_j v_j] - t_{ij} v_j \\ = \sum_{k=1}^n [q_i^{(k)} + \rho^{(k)} v_i^{(k)} (U^{(k)} + \frac{1}{2} v_j^{(k)} v_j^{(k)}) - t_{ji}^{(k)} v_j^{(k)}] \end{aligned} \quad (\text{III-60})$$

and iv. Specific energy supply

$$r = \frac{1}{\rho} \sum_{k=1}^n \rho^{(k)} (r^{(k)} + b_i^{(k)} u_i^{(k)}) \quad (\text{III-61})$$

$$\text{Or,} \quad \rho (r + b_i v_i) = \sum_{k=1}^n \rho^{(k)} (r^{(k)} + b_i^{(k)} v_i^{(k)}) \quad (\text{III-62})$$

Further, it was assumed that the mixture is thermodynamically isolated, i.e.,

$$\sum_{k=1}^n c^{(k)} = 0 \quad (\text{III-63})$$

$$\sum_{k=1}^n m_i^{(k)} = 0 \quad (\text{III-64})$$

$$\sum_{k=1}^n M_{ij}^{(k)} = 0 \quad (\text{III-65})$$

$$\text{and} \quad \sum_{k=1}^n e^{(k)} = 0 \quad (\text{III-66})$$

Summing the equations of balance of the constituents over all of them gives, in view of the definitions introduced above for the total quantities:

i. Balance of Mass

$$\frac{\partial \rho}{\partial t} + (\rho v_i)_{,i} = \sum_{k=1}^n \rho c^{(k)} = 0 \quad (\text{III-67})$$

ii. Balance of Linear Momentum

$$\frac{\partial}{\partial t} (\rho v_i) + (\rho v_i v_j)_{,j} = t_{ji,j} + \sum_{k=1}^n \rho^{(k)} b_i^{(k)} \quad (\text{III-68})$$

Bowen [29] introduced a body force for the mixture through the definition

$$\rho b_i = \sum_{k=1}^n \rho^{(k)} b_i^{(k)} \quad (\text{III-69})$$

Then the equation of balance of linear momentum is

$$\frac{\partial}{\partial t} (\rho v_i) + (\rho v_i v_j)_{,j} = t_{ji,j} + \rho b_i \quad (\text{III-70})$$

An equivalent form, obtained using (III-67), is

$$\rho \frac{D}{Dt} v_i + v_i \sum_{k=1}^n \rho c^{(k)} = t_{ji,j} + \sum_{k=1}^n \rho^{(k)} b_i^{(k)} = t_{ji,j} + \rho b_i \quad (\text{III-71})$$

iii. Balance of Angular Momentum

$$t_{ij} = t_{ji} \quad (\text{III-72})$$

i.e., the total stress tensor is symmetric. As $v_i^{(k)} v_j^{(k)}$ is symmetric and so is $v_i v_j$, it follows from (III-58) that

$$\sum_{k=1}^n t_{ij}^{(k)} = \sum_{k=1}^n t_{ji}^{(k)} \quad (\text{III-73})$$

iv. Energy Balance

$$\rho \frac{DU}{Dt} + q_{i,i} - t_{ij} v_{i,j} = \rho r = \sum_{k=1}^n \rho^{(k)} (r^{(k)} + b_i^{(k)} u_i^{(k)}) \quad (\text{III-74})$$

Chao, Sha and Soo [39] noted that this approach of combining the balance equations of constituents to obtain the balance equations of the

mixture can lead to errors. As an example, they cited the absence of inertial coupling forces in the momentum equations of the constituents. However, their own work has been criticized by several investigators. The review in reference [14] contains a discussion of Chao et al.'s theories.

The analysis presented above gives the equations of balance of mass, linear momentum, and energy for the mixture as if it were a single continuum moving with acceleration derived from the barycentric velocity. We note that the analysis above is founded on the so-called fundamental identity [189] involving 'material derivatives of the mean value'. Whereas these can be accepted as hypothetical entities for simplification of analysis, it is difficult to assign a physical meaning to them. This is because the mixture consists of centers of mass and is defined only for the particular instant of time. It cannot be regarded as a continuum, consisting of a set of non-penetrating particles, in motion.

An alternative derivation of the equations of mass, momentum and energy balance for the mixture was presented by Kelly [104], Truesdell [190], Green and Naghdi [87] and Bowen [29]. They wrote the equations of mass, momentum and energy conservation for the mixture contained in an volume V bounded by an arbitrary fixed surface A . The conservation of mass was expressed by

$$\frac{\partial}{\partial t} \int_V \sum_{k=1}^n \rho^{(k)} dV + \int_A \sum_{k=1}^n n_j \rho^{(k)} v_j^{(k)} dA = 0 \quad (\text{III-75})$$

This equation valid for any V , upon use of the divergence theorem along with (III-2) and (III-19), implies (III-67). Thus mass conservation of a mixture is the same as if it was a single continuum of density ρ moving with the barycentric velocity v_i . The change of linear momentum of the mixture bounded by a fixed surface A is

$$\frac{\partial}{\partial t} \int_V \sum_{k=1}^n \rho^{(k)} v_i^{(k)} dV + \int_A \sum_{k=1}^n n_j \rho^{(k)} v_i^{(k)} v_j^{(k)} n_j dA \quad (\text{III-76})$$

Using (III-2), (III-21) along with (III-67), (III-76) may be rewritten [87] as

$$\int_V \rho \frac{D}{Dt} v_i dV + \int_A n_j \sum_{k=1}^n \rho^{(k)} u_i^{(k)} u_j^{(k)} dA \quad (\text{III-77})$$

and, using the divergence theorem,

$$= \int_V \left[\rho \frac{D}{Dt} v_i + \sum_{k=1}^n \rho^{(k)} u_i^{(k)} u_j^{(k)} \right]_{,j} dV \quad (\text{III-78})$$

Equating the linear change of momentum to the total force exerted on the material inside the region V, Green and Naghdi [87] obtained

$$\begin{aligned} & \int_V \left[\rho \frac{D}{Dt} v_i + \left(\sum_{k=1}^n \rho^{(k)} u_i^{(k)} u_j^{(k)} \right)_{,j} \right] dV \\ &= \int_V \sum_{k=1}^n \rho^{(k)} b_i^{(k)} dV + \int_A \sum_{k=1}^n n_j t_{ij}^{(k)} dA \end{aligned} \quad (\text{III-79})$$

Again using the divergence theorem, the change of linear momentum equals

$$\int_V \sum_{k=1}^n (\rho^{(k)} b_i^{(k)} + t_{ji,j}^{(k)}) dV \quad (\text{III-80})$$

Hence (III-79) can be rewritten as

$$\begin{aligned} & \int_V \left[\rho \frac{D}{Dt} v_i + \left(\sum_{k=1}^n \rho^{(k)} u_i^{(k)} u_j^{(k)} \right)_{,j} \right] dV \\ &= \int_V \sum_{k=1}^n (\rho^{(k)} b_i^{(k)} + t_{ji,j}^{(k)}) dV \end{aligned} \quad (\text{III-81})$$

This is the integral form of the equation of balance of momentum. In point form the balance of momentum is, [87],

$$\begin{aligned} & \rho \frac{D}{Dt} v_i + \sum_{k=1}^n (\rho^{(k)} u_i^{(k)} u_j^{(k)})_{,j} \\ &= \sum_{k=1}^n \rho^{(k)} b_i^{(k)} + \sum_{k=1}^n t_{ji,j}^{(k)} \end{aligned} \quad (\text{III-82})$$

Green and Naghdi [87] noted that with the definition (III-57) proposed

by Truesdell, (III-82) is the same as (III-68). However, if we assume that the total stress equals the sum of partial stresses, (III-82) reflects the fact that the total rate of increase of linear momentum is not equal to the barycentric rate of increase of momentum of a continuum of density ρ moving with velocity v_i . Bowen [29] wrote (III-47) in the equivalent form (III-49) and, for the mixture, simply regarded (III-71) as the equation of linear momentum balance. Similar formulation was used for the other balance equations.

3.3.3. Green and Naghdi's Theory

The theory for a mixture of two constituents originally presented by Green and Naghdi [83,85] was generalized by Mills [122] to the case of multicomponent mixtures. Green and Naghdi [83,85] considered the concepts of stress, heat flux, and energy supply assumed to be primitive to each constituent to be primitive to the mixture as a whole as well. It was proposed that the total stress and the total heat flux for the mixture should equal the sum of the corresponding quantities for the constituents. Thus

$$t_{ij} = \sum_{k=1}^n t_{ij}^{(k)} \quad (\text{III-83})$$

$$q_i = \sum_{k=1}^n q_i^{(k)} \quad (\text{III-84})$$

and

$$r = \sum_{k=1}^n \rho^{(k)} r^{(k)} / \rho \quad (\text{III-85})$$

Green and Naghdi [87] stated that Truesdell's equations were correct but they had difficulty accepting the interpretations associated with some of the quantities which occur in these equations. These interpretations are of special significance if boundary conditions on the surface of the mixture involve total stress and total heat flux. Morland [130] supported the additivity of partial stresses on the ground that tractions are additive and Cauchy's stress principle should hold for total stress and total traction as well as for the constituents. In Green and Naghdi's theory, the equations of mass and momentum balance are derived from the material frame invariance of a rate of energy equality. For a volume V enclosed by a fixed surface A , following Truesdell's contention that the rate of energy equality

have the same form as for a single constituent, Green and Naghdi [87] wrote,

$$\begin{aligned} & \frac{\partial}{\partial t} \int_V \rho \left(U + \frac{1}{2} v_i v_i \right) dV + \int_A \rho n_j v_j \left(U + \frac{1}{2} v_i v_i \right) dA \\ &= \int_V \sum_{k=1}^n \rho^{(k)} (r^{(k)} + b_i^{(k)} v_i^{(k)}) dV + \int_A (t_{ji} v_j - q_i) n_i dA \quad (\text{III-86}) \end{aligned}$$

Here the left hand side represents the rate of change in energy in volume V bounded by a fixed surface A plus the energy flux for the mixture across the boundary. U is the specific internal energy per unit mass of the mixture and is related to internal energies of the constituents by Truesdell's equation (III-55). The equality may be regarded as written for a surface moving with a continuum which has a velocity field equal to the barycentric velocity v_i . Then the left hand side is equal to the material rate, executed on the mixture, of the sum of the internal energy and the kinetic energy of the mixture. However, this line of thought is open to objection. It would assume the existence of mixture particles and the time rates executed on them. This is, in general, not correct as the barycentric velocity is not particle velocity in the ordinary sense. Green and Naghdi [87] accept the form of (III-86) and (III-74) but not the interpretation associated with some of the quantities occurring in them.

Green and Naghdi [83] proposed a rate of energy equality in the following form:

$$\begin{aligned} & \frac{\partial}{\partial t} \int_V \left(\rho U + \frac{1}{2} \sum_{k=1}^n \rho^{(k)} v_i^{(k)} v_i^{(k)} \right) dV + \int_A \left(U \sum_{k=1}^n \rho^{(k)} v_j^{(k)} \right. \\ & \quad \left. + \frac{1}{2} \sum_{k=1}^n \rho^{(k)} v_i^{(k)} v_i^{(k)} v_j^{(k)} \right) n_j dA \\ &= \int_V \left(\rho r + \sum_{k=1}^n \rho^{(k)} b_i^{(k)} v_i^{(k)} \right) dV + \int_A \sum_{k=1}^n (t_i^{(k)} v_i^{(k)} - q) dA \quad (\text{III-87}) \end{aligned}$$

Here, the heat fluxes and the energy supply are assumed to be additive. U is the internal energy per unit mass of the mixture

allowing for all interactions between the constituents. $t_i^{(k)}$ are components of tractions on the surface A. Essentially, (III-87) states

Rate of change of energy in a fixed volume V + energy loss due to mass flow across the boundary A of V = Rate of energy supply to the volume + rate of work supply to the volume + rate of work supply at the boundary A - heat loss across the boundary A.

In this theory, multipolar stresses and externally supplied multipolar body forces were excluded. Green and Naghdi [83] also made no attempt to define the internal energy and the energy supply for each constituent. It was considered unnecessary for a complete general theory. Bowen [29] also pointed out that for single-temperature mixtures, explicit use of an energy equality for the constituents is not needed.

In a later paper, [85], the role of interactions between constituents was made explicit by writing, for each constituent, the rate of energy equality

$$\begin{aligned} & \frac{\partial}{\partial t} \int_V \rho^{(k)} (U^{(k)} + \frac{1}{2} v_i^{(k)} v_i^{(k)}) dV + \int_A \rho^{(k)} n_j v_j^{(k)} \\ & (U^{(k)} + \frac{1}{2} v_i^{(k)} v_i^{(k)}) dA \\ & = \int_V \rho^{(k)} (r^{(k)} + b_i^{(k)} v_i^{(k)}) dV + \int_A (t_i^{(k)} v_i^{(k)} - q^{(k)}) dA \\ & + \int_V [(\alpha_i^{(k)} v_i^{(k)} + \lambda_{ij}^{(k)} \Gamma_{ij}^{(k)}) + \psi^{(k)}] dV - \int_A \bar{q}^{(k)} dA \quad (\text{III-88}) \end{aligned}$$

where $\alpha_i^{(k)}$, $\lambda_{ij}^{(k)}$ are the internal force and coupling acting on $s^{(k)}$ due to interactions and $\psi^{(k)}$, $\bar{q}^{(k)}$ represent contributions to the balance of energy arising from the interactions of the n constituents with $\psi^{(k)}$ being a volume contribution and $\bar{q}^{(k)}$ a surface flux of energy. Also

$$\sum_{k=1}^n \alpha_i^{(k)} = 0 \quad (\text{III-89})$$

$$\sum_{k=1}^n \lambda_{ij}^{(k)} = 0 \quad (\text{III-90})$$

$$\sum_{k=1}^n \bar{q}^{(k)} = 0 \quad (\text{III-91})$$

Without loss of generality, $\lambda_{ij}^{(k)}$ can be taken to be [85] antisymmetric as $\Gamma_{ij}^{(k)}$ is antisymmetric. Bowen [29] and Bedford and Drumheller [14] used essentially the same form in writing energy balance equality for each constituent. The rate of energy equality for the mixture, (III-86), can be rewritten using the following relationships:

$$\begin{aligned} \frac{D^{(k)}}{Dt} \rho^{(k)} + \rho^{(k)} v_{i,i}^{(k)} &= \frac{\partial}{\partial t} \rho^{(k)} + v_i^{(k)} \rho_{,i}^{(k)} + \rho^{(k)} v_{i,i}^{(k)} \\ &= \frac{\partial}{\partial t} \rho^{(k)} + (\rho^{(k)} v_i^{(k)})_{,i} \\ &= \rho c^{(k)} \quad \text{by (III-44)} \end{aligned} \quad (\text{III-92})$$

Also,

$$\begin{aligned} \frac{\partial}{\partial t} (\rho^{(k)} v_i^{(k)} v_i^{(k)}) &= \frac{D^{(k)}}{Dt} (\rho^{(k)} v_i^{(k)} v_i^{(k)}) - v_j^{(k)} (\rho^{(k)} v_i^{(k)} v_i^{(k)})_{,j} \\ &= v_i^{(k)} v_i^{(k)} \frac{D^{(k)}}{Dt} \rho^{(k)} + 2 \rho^{(k)} v_i^{(k)} \frac{D}{Dt} v_i^{(k)} \\ &\quad - (\rho^{(k)} v_i^{(k)} v_i^{(k)} v_j^{(k)})_{,j} + \rho^{(k)} v_i^{(k)} v_i^{(k)} v_{j,j}^{(k)} \\ &= v_i^{(k)} v_i^{(k)} \left(\frac{D^{(k)}}{Dt} \rho^{(k)} + \rho^{(k)} v_{j,j}^{(k)} \right) \\ &\quad + 2 \rho^{(k)} v_i^{(k)} f_i^{(k)} - (\rho^{(k)} v_i^{(k)} v_i^{(k)} v_j^{(k)})_{,j} \\ &= \rho c^{(k)} v_i^{(k)} v_i^{(k)} + 2 \rho^{(k)} v_i^{(k)} f_i^{(k)} - (\rho^{(k)} v_i^{(k)} v_i^{(k)} v_j^{(k)})_{,j} \end{aligned} \quad (\text{III-93})$$

Assuming the quantities ρ , U , v_i smooth in the space and time variables,

$$\frac{\partial}{\partial t} \int_V \rho U \, dV + \int_A U \left(\sum_{k=1}^n \rho^{(k)} v_j^{(k)} \right) n_j \, dA$$

$$\begin{aligned}
&= \frac{\partial}{\partial t} \int_V \rho U \, dV + \int_A \rho U v_j n_j \, dA \\
&= \frac{\partial}{\partial t} \int_V \rho U \, dV + \int_V (\rho U v_j)_{,j} \, dV \\
&= \int_V \left[\frac{\partial}{\partial t} (\rho U) + v_j (\rho U)_{,j} + \rho U v_{j,j} \right] \, dV \\
&= \int_V \left[\frac{D}{Dt} (\rho U) + \rho U v_{j,j} \right] \, dV \\
&= \int_V \left(\rho \frac{DU}{Dt} + U \frac{D\rho}{Dt} + \rho U v_{j,j} \right) \, dV \\
&= \int_V \left[\rho \frac{DU}{Dt} + U \frac{D\rho}{Dt} + U (\rho v_j)_{,j} \right] \, dV
\end{aligned}$$

and using (III-67),

$$= \int_V \left(\rho \frac{DU}{Dt} + \sum_{k=1}^n \rho U c^{(k)} \right) \, dV \quad (\text{III-94})$$

The rate of energy equality upon substitution of (III-84), (III-85), (III-93), (III-94), in (III-87) and noting the relationships (III-2), (III-19), and (III-44) is

$$\begin{aligned}
&\int_V \left[\rho \frac{DU}{Dt} - \rho r + \sum_{k=1}^n \rho^{(k)} v_i^{(k)} (f_i^{(k)} - b_i^{(k)}) \right] \, dV \\
&\quad + \int_V \sum_{k=1}^n \rho c^{(k)} \left(U + \frac{1}{2} v_i^{(k)} v_i^{(k)} \right) \, dV \\
&= \int_A \sum_{k=1}^n (t_i^{(k)} v_i^{(k)} - q) \, dA \quad (\text{III-95})
\end{aligned}$$

Invariance of the energy equality under superposed uniform translational velocities yields, for arbitrary V ,

$$\int_V \sum_{k=1}^n \rho c^{(k)} \, dV = 0 \quad (\text{III-96})$$

i.e., the mass elements of the mixture are conserved. By (III-92) and (III-20),

$$\frac{D\rho}{Dt} + \rho v_{i,i} = 0 \quad (\text{III-97})$$

Also

$$\int_V \sum_{k=1}^n [\rho^{(k)}(b_i^{(k)} - f_i^{(k)}) - \rho c^{(k)} v_i^{(k)}] dV + \int_A \sum_{k=1}^n t_i^{(k)} dA = 0 \quad (\text{III-98})$$

Applying (III-91) to an arbitrary tetrahedron, defining partial tractions on surfaces perpendicular to x_k axis as partial stresses $t_{ij}^{(k)}$, a generalization of Cauchy's stress principle is realized i.e.,

$$\sum_{k=1}^n t_i^{(k)} = \sum_{k=1}^n t_{ji}^{(k)} n_j \quad (\text{III-99})$$

Substituting (III-99) in (III-98) and using the divergence theorem, in point form

$$\sum_{k=1}^n [\rho^{(k)}(b_i^{(k)} - f_i^{(k)}) - \rho c^{(k)} v_i^{(k)} + t_{ji,j}^{(k)}] = 0 \quad (\text{III-100})$$

(III-100) is the same as (III-71), the equation of balance of linear momentum for the mixture postulated by Truesdell. Recalling (III-93) to replace $f_i^{(k)}$, (III-100) may be written in the alternative form [83]

$$\sum_{k=1}^n [\rho^{(k)}(b_i^{(k)} - \frac{\partial}{\partial t} \rho^{(k)} v_i^{(k)} - (\rho^{(k)} v_i^{(k)} v_j^{(k)})_{,j} + t_{ji,j}^{(k)}] = 0 \quad (\text{III-101})$$

This is the equation of linear momentum balance for an arbitrary fixed surface. Substituting (III-97), (III-99), and (III-101) in (III-87), Green and Naghdi [83] obtained, for a binary mixture,

$$\begin{aligned} & \int_V [\rho r - \rho \frac{DU}{Dt} + \frac{1}{2} \{ \rho^{(1)}(b_i^{(1)} - f_i^{(1)}) - \rho^{(2)}(b_i^{(2)} - f_i^{(2)}) \} (v_i^{(1)} - v_i^{(2)}) \\ & + \frac{1}{2} t_{ij} (v_i^{(1)} + v_i^{(2)})_{,j}] dV + \frac{1}{2} \int_A (t_i^{(1)} - t_i^{(2)}) (v_i^{(1)} - v_i^{(2)}) dA \\ & - \int_A q dA = 0 \end{aligned} \quad (\text{III-102})$$

Following Mills [122], define

$$p_i^{(k)} = t_i^{(k)} - t_{ji}^{(k)} n_j \quad (\text{III-103})$$

Then (III-99) implies

$$\sum_{k=1}^n p_i^{(k)} = 0 \quad (\text{III-104})$$

Let h_j be the heat flux across x_j planes at a point x_i . Applying (III-95) to a tetrahedron bounded by the coordinate planes at x_i and by a plane with unit normal n_j ,

$$\sum_{k=1}^n p_i^{(k)} v_i^{(k)} + (h_j n_j - q) = 0 \quad (\text{III-105})$$

From (III-104)

$$p_i^{(n)} = - \sum_{k=1}^{n-1} p_i^{(k)} \quad (\text{III-106})$$

Hence, from (III-105)

$$\begin{aligned} q - h_j n_j &= \sum_{k=1}^n p_i^{(k)} v_i^{(k)} \\ &= \sum_{k=1}^{n-1} p_i^{(k)} v_i^{(k)} + p_i^{(n)} v_i^{(n)} \\ &= \sum_{k=1}^{n-1} p_i^{(k)} v_i^{(k)} - v_i^{(n)} \sum_{k=1}^{n-1} p_i^{(k)} \\ &= \sum_{k=1}^{n-1} p_i^{(k)} (v_i^{(k)} - v_i^{(n)}) \end{aligned} \quad (\text{III-107})$$

For a binary mixture, $p_i^{(1)} = -p_i^{(2)}$. Hence

$$q - h_j n_j = p_i^{(1)} (v_i^{(1)} - v_i^{(2)}) \quad (\text{III-108})$$

$$= \frac{1}{2} (p_i^{(1)} - p_i^{(2)}) (v_i^{(1)} - v_i^{(2)}) \quad (\text{III-109})$$

the result obtained by Green and Naghdi [83]. In the special case when $q = h_j n_j$, $p_i^{(1)}$ and $p_i^{(2)}$ vanish except, possibly, in the case of no relative motion between the constituents. Substitution of (III-96), (III-103), (III-105) in (III-95) yields

$$\int_V \left[\rho \frac{DU}{Dt} - \rho r + h_{j,j} - \sum_{k=1}^n \theta_i^{(k)} v_i^{(k)} + \sum_{k=1}^n \frac{1}{2} \rho c^{(k)} v_i^{(k)} v_i^{(k)} - \sum_{k=1}^n t_{ji}^{(k)} v_{i,j}^{(k)} \right] dV = 0 \quad (\text{III-110})$$

where we define

$$\theta_i^{(k)} = t_{ji,j}^{(k)} + \rho^{(k)} (b_i^{(k)} - f_i^{(k)}) \quad (\text{III-111})$$

Inconsequence of (III-100)

$$\sum_{k=1}^n \theta_i^{(k)} = \sum_{k=1}^n \rho c^{(k)} v_i^{(k)} \quad (\text{III-112})$$

Invariance of (III-110) under a superposed uniform rigid body angular velocity gives

$$\sum_{k=1}^n t_{[ji]}^{(k)} = 0 \quad (\text{III-113})$$

i.e., the sum of the partial stresses is symmetric. The partial stresses do not have to be symmetric. In view of (III-67), (III-96), (III-112), and (III-113), application of (III-110) to arbitrary volumes gives

$$\begin{aligned} \rho \frac{DU}{Dt} - \rho r + h_{j,j} - \sum_{k=1}^n t_{(ji)}^{(k)} d_{ij}^{(k)} - \sum_{k=1}^{n-1} t_{[ji]}^{[k]} [\Gamma_{ij}^{(k)} - \Gamma_{ij}^{(n)}] \\ - \sum_{k=1}^{n-1} p_i^{(k)} (v_i^{(k)} - v_i^{(n)}) = 0 \end{aligned} \quad (\text{III-114})$$

where

$$p_i^{(k)} = \theta_i^{(k)} - \frac{1}{2} c^{(k)} (v_i^{(k)} - v_i^{(n)}) \quad (\text{III-115})$$

$$= t_{ji,j}^{(k)} + \rho^{(k)} (b_i^{(k)} - f_i^{(k)}) - \frac{1}{2} c^{(k)} (v_i^{(k)} - v_i^{(n)}) \quad (\text{III-116})$$

For a binary mixture, (III-114) is identical to the energy equality proposed by Green and Steel [84].

In setting up the rate of energy equality, it was stated that the internal energy per unit mass of the mixture may not be equal to the sum of internal energies of the constituents. We note that Shi, Rajagopal, and Wineman [177] assumed the internal energies, the entropies and the Helmholtz free energies of the constituents to add up to the corresponding quantities for the mixture. Green and Naghdi [87] established a relationship between these quantities. Following [85] we write the rate of energy equality in terms of the energies of the constituents, ignoring external body couples, as follows:

$$\begin{aligned} & \frac{\partial}{\partial t} \int_V \sum_{k=1}^n (\rho^{(k)} U^{(k)} + \frac{1}{2} \rho^{(k)} v_i^{(k)} v_i^{(k)}) dV \\ & + \int_A \sum_{k=1}^n n_j (\rho^{(k)} v_j^{(k)} U^{(k)} + \frac{1}{2} \rho^{(k)} v_i^{(k)} v_i^{(k)} v_j^{(k)}) dA \\ & = \int_V \sum_{k=1}^n \rho^{(k)} (r^{(k)} + b_i^{(k)} v_i^{(k)}) dV + \int_A \sum_{k=1}^n (t_{ij}^{(k)} v_j^{(k)} - q^{(k)}) dA \quad (\text{III-117}) \end{aligned}$$

Let ρU^* be the sum of the energies of the constituents, i.e.,

$$\rho U^* = \sum_{k=1}^n \rho^{(k)} U^{(k)} \quad (\text{III-118})$$

Also define, [87]

$$K = \sum_{k=1}^n (\rho^{(k)} u_i^{(k)} U^{(k)})_{,i} \quad (\text{III-119})$$

Then

$$\frac{\partial}{\partial t} \int_V \sum_{k=1}^n \rho^{(k)} U^{(k)} dV + \int_A \sum_{k=1}^n \rho^{(k)} v_j^{(k)} U^{(k)} n_j dA$$

$$\begin{aligned}
&= \frac{\partial}{\partial t} \int_V \sum_{k=1}^n (\rho U^*) dV + \int_A \sum_{k=1}^n \rho^{(k)} (v_i + u_i^{(k)}) U^{(k)} n_i dA \\
&= \int_V \left[\frac{\partial}{\partial t} (\rho U^*) + \sum_{k=1}^n (\rho^{(k)} U^{(k)} v_{i,i} + \rho^{(k)} u_i^{(k)} U^{(k)})_{,i} \right] dV \\
&= \int_V \left[\frac{\partial}{\partial t} (\rho U^*) + v_i (\rho U^*)_{,i} + \rho U^* v_{i,i} + K \right] dV \\
&= \int_V \left[\frac{D}{Dt} (\rho U^*) + \rho U^* v_{i,i} + K \right] dV \\
&= \int_V \left[\rho \frac{DU^*}{Dt} + K + U^* (\rho v_{i,i} + \frac{D\rho}{Dt}) \right] dV \\
&= \int_V \left[\rho \frac{DU^*}{Dt} + K + U^* \left(\frac{\partial \rho}{\partial t} + (\rho v_i)_{,i} \right) \right] dV
\end{aligned}$$

and in view of (III-67)

$$= \int_V \left[\rho \frac{DU^*}{Dt} + K \right] dV \quad (\text{III-120})$$

If we define the total energy, U , of the mixture so that

$$\begin{aligned}
\rho \frac{DU}{Dt} &= \rho \frac{DU^*}{Dt} + K \\
&= \rho \frac{D}{Dt} \left[\left(\sum_{k=1}^n \rho^{(k)} U^{(k)} \right) / \rho \right] + \sum_{k=1}^n (\rho^{(k)} u_i^{(k)} U^{(k)})_{,i} \quad (\text{III-121})
\end{aligned}$$

the rate of energy equality (III-117), using (III-93), becomes

$$\begin{aligned}
\int_V \left[\rho \frac{DU}{Dt} - \sum_{k=1}^n \rho^{(k)} (r^{(k)} + b_i^{(k)} v_i^{(k)}) + \sum_{k=1}^n \rho c^{(k)} \left(U + \frac{1}{2} v_i^{(k)} v_i^{(k)} \right) \right. \\
\left. + \sum_{k=1}^n \rho^{(k)} v_i^{(k)} f_i^{(k)} \right] dV = \int_A \sum_{k=1}^n (t_i^{(k)} v_i^{(k)} - q^{(k)}) dA \quad (\text{III-122})
\end{aligned}$$

(III-122) is identical to (III-95). Equation (III-121) defines the relationship between the specific energies of the constituents and that of the mixture for the two rate of energy equalities (III-95) and (III-117) to be equivalent.

3.4. OTHER WORK

3.4.1. Introduction

Several investigators have used the basic concepts introduced by Truesdell and by Green and Naghdi. These efforts aim at simplifying the description of motion for certain special cases. For instance, in granular porous media, the total deformation can be viewed as made up of two parts; one related to deformation of the solid particles and the other to their rearrangement i.e. changes in pore geometry. The volume fraction theories, for compressible materials, introduce the volume fractions as additional variables. Attempts have been also made to separate the role of molecular, kinetic and potential energies in setting up the balance equations and in entropy production inequalities. In this section we outline the essential features of some of these efforts.

3.4.2. Mass Continuity Equation Using Relative Velocity

Krause [109], referring to fixed volumes in space, for continued saturation, wrote

$$\sum_{k=1}^n n^{(k)} = 1 \quad (\text{III-123})$$

Mass continuity, (III-44), using the definition (III-1), for no mass production, is

$$\dot{n}^{(k)} \rho^{(k)*} + n^{(k)} \dot{\rho}^{(k)*} + (n^{(k)} \rho^{(k)*} v_j^{(k)})_{,j} = 0 \quad (\text{III-124})$$

If the materials are intrinsically incompressible, $\dot{\rho}^{(k)*} = 0$. Hence, if the intrinsic density is also spatially constant

$$\dot{n}^{(k)} + (n^{(k)} v_j^{(k)})_{,j} = 0 \quad (\text{III-125})$$

For a binary mixture, (III-123) and (III-125) give

$$[v_j^{(1)} + n^{(2)}(v_j^{(2)} - v_j^{(1)})]_{,j} = 0 \quad (\text{III-126})$$

Using (III-23), (III-126) can be rewritten as [106,109]

$$(v_j^{(1)} + w_j)_{,j} = 0 \quad (\text{III-127})$$

Hsieh and Yew [95] added the equations (III-67) for the constituents of a binary mixture and for no chemical reaction obtained

$$\dot{\rho} + (\rho v_i^{(1)})_{,i} + (\rho^{(2)} w_i)_{,i} = 0 \quad (\text{III-128})$$

as the mass continuity equation in terms of relative velocity. This is similar to Bowen [29]. Assuming small deformations, they also wrote the continuity equation for the fluid volume contained in a fixed volume in space as

$$\frac{\partial}{\partial t} (\rho^{(1)} - \rho_0^{(1)}) + \rho_0 e_{jj}^{(1)} + [\rho_0^{(2)} \frac{\partial}{\partial t} (u_i^{(2)} - u_i^{(1)})]_{,i} = 0 \quad (\text{III-29})$$

where $u_i^{(k)}$ are components of the displacement vector for the constituent $s^{(k)}$. Integrating over time, for compressible fluid and uniform $\rho_0^{(2)}$

$$\rho = \rho_0 (1 - e_{jj}^{(1)}) + \rho_0^{(2)} \int_0^t w_{i,i} dt \quad (\text{III-130})$$

3.4.3. Mass Balance in Terms of Porosity (Volume Fraction Theories)

Fukuo [68] used the equations of mass balance to set up equations in terms of volume fractions of the constituents. The mass balance equation, (III-45), assuming no chemical interaction, becomes

$$\dot{\rho}^{(k)} + (\rho^{(k)} v_i^{(k)})_{,i} = 0 \quad (\text{III-131})$$

Using intrinsic densities

$$\dot{n}^{(k)} \rho^{(k)*} + n^{(k)} \dot{\rho}^{(k)*} + (n^{(k)} \rho^{(k)*} v_i^{(k)})_{,i} = 0 \quad (\text{III-132})$$

If the solid skeleton is incompressible, $\dot{\rho}^{(1)*} = 0$, and $\rho_{,i}^{(1)*} = 0$. Hence, (III-132) gives

$$\dot{n}^{(1)} + (n^{(1)} v_i^{(1)})_{,i} = 0 \quad (\text{III-133})$$

If the fluid is incompressible,

$$\dot{n}^{(2)} + (n^{(2)} v_i^{(2)})_{,i} = 0 \quad (\text{III-134})$$

Fukuo [68] used Gibson's [77] approach of referring to the fixed set of particles in the reference configuration to get

$$n_0^{(1)} \rho_0^{(1)*} = n^{(1)} \rho^{(1)*} \det[F_{ij}] \quad (\text{III-135})$$

Thus, for small deformation, using (III-29),

$$n^{(1)} \rho^{(1)*} (1 + e_{jj}^{(1)}) = n_0^{(1)} \rho_0^{(1)*} \quad (\text{III-136})$$

If $\rho^{(1)*}$ is constant

$$n^{(1)} (1 + e_{jj}^{(1)}) = n_0^{(1)} \quad (\text{III-137})$$

Then (III-134) can be written as

$$-\dot{n}^{(2)} = - (1 - n^{(1)}) \dot{n}^{(1)} = n_0^{(1)} \frac{\partial}{\partial t} \left[\frac{1}{1 + e_{jj}^{(1)}} \right] = (n^{(2)} v_i^{(2)})_{,i} \quad (\text{III-138})$$

If $e_{jj}^{(1)} \ll 1$,

$$n_0^{(1)} \dot{e}_{jj}^{(1)} = (n^{(2)} v_i^{(2)})_{,i} \quad (\text{III-139})$$

This equation combined with d'Arcy's law may be regarded as a generalization of Gibson's equation (II-42). Hsieh and Yew [95] considered a porous solid saturated with an incompressible fluid and undergoing small deformations. For this condition, considering a unit volume in the undeformed state, they showed that the fluid content change is

$$n = - (n^{(2)} - n_0^{(2)}) - n^{(2)} e_{jj}^{(1)} \quad (\text{III-140})$$

The time derivative of (III-140), noting that for continued saturation the rate of change in fluid content must equal the divergence of relative velocity, is

$$-\frac{\partial}{\partial t} [n^{(2)} (1 + e_{jj}^{(1)})] = (w_i)_{,i} \quad (\text{III-141})$$

Hence

$$\frac{\partial}{\partial t} (n^{(2)} - n_0^{(2)}) + n_0^{(2)} \frac{\partial}{\partial t} e_{jj}^{(1)} + w_{i,i} = 0 \quad (\text{III-142})$$

This gives a relationship between rate of porosity change, the rate of volumetric strain and the relative velocity vector. These quantities, in this theory for incompressible fluids and no thermal or chemical effects cannot, therefore, be treated as independent variables. For compressible fluids, Morland [130] proposed constitutive equations for porosity. Hsieh and Yew [95] refer to (III-142) as the compatibility condition of volume change of a fluid-saturated porous medium.

3.4.4. Energy Balance in Terms of Porosity (Volume Fraction Theory).

Goodman and Cowin [78] postulated the equation of energy balance for a porous material with porosity n as

$$\begin{aligned} \frac{D}{Dt} \int_V n \rho^* \left[(U + \frac{1}{2} v_i v_i + \frac{1}{2} K (\frac{\partial n}{\partial t})^2) - b_i v_i - l \frac{\partial n}{\partial t} - r \right] dV \\ = \int_A (t_{ji} v_j + s_i \frac{\partial n}{\partial t} - q_i) n_i dA \end{aligned} \quad (\text{III-143})$$

Here K , s_j , l are, respectively, the equilibrated inertia, components

of the equilibrated stress vector, and the external equilibrated body force. This equation admits an additional degree of freedom, viz., the volume fraction. A kinetic energy term is associated with the rate of change of n . Similarly, rate of work terms are associated with the rate of change of n over volumes and surfaces using generalized forces s_j and l . Invariance of this equality, as in Green and Naghdi's theory, leads to the equations of linear momentum and angular momentum balance. The equation of linear momentum balance, as might have been directly written, is

$$n \rho^* (f_i - b_i) = t_{ji,j} \quad (\text{III-144})$$

Goodman and Cowin [78] also postulated an equation of motion, called the equation of balance of equilibrated forces, for the variable n , as

$$\frac{D}{Dt} \int_V n \rho^* [K \frac{\partial n}{\partial t} + l + g] dV = \int_A s_j n_j dA \quad (\text{III-145})$$

Here g is the intrinsic equilibrated body force. The local form of (III-145) is

$$n \rho^* K \ddot{n} = s_{j,j} + n \rho^* (l + g) \quad (\text{III-146})$$

Substituting (III-144) and (III-146) in (III-143), the local form of energy equality is

$$n \rho^* \frac{\partial}{\partial t} U = t_{ij} v_{(i,j)} + s_j (\dot{n})_{,j} - n \rho^* g \frac{\partial n}{\partial t} - h_{j,j} + n \rho^* r \quad (\text{III-147})$$

3.4.5. Alternative Form of Linear Momentum Balance Equation.

Hsieh and Yew [95] wrote the momentum balance, for no body forces, using (III-76) and (III-79) as

$$\frac{\partial}{\partial t} \int_V \sum_{k=1}^n \rho^{(k)} v_i^{(k)} dV + \int_A \sum_{k=1}^n \rho^{(k)} v_i^{(k)} v_j^{(k)} n_j dA = \int_A \sum_{k=1}^n t_{ji}^{(k)} n_j dA \quad (\text{III-148})$$

Using the divergence theorem, the point form of the equation is

$$\frac{\partial}{\partial t} \sum_{k=1}^n \rho^{(k)} v_i^{(k)} + \sum_{k=1}^n (\rho^{(k)} v_i^{(k)} v_j^{(k)})_{,j} = \sum_{k=1}^n t_{ji,j}^{(k)} \quad (\text{III-149})$$

Assuming additivity of stresses and introducing relative velocity, $v_i = v_i^{(2)} - v_i^{(1)}$, for a binary mixture, they obtained

$$\rho \frac{D}{Dt} v_i^{(1)} + \rho^{(2)} \left[\frac{\partial}{\partial t} v_i + v_j^{(2)} v_{i,j} + v_j v_{i,j} \right] = t_{ji,j} \quad (\text{III-150})$$

Williams [198] summarized the work by Gurtin and De La Penha [90], Oliver [140], and Sampaio [154]. Defining the force exerted by a volume C of constituent $s^{(j)}$ on volume D of constituent $s^{(i)}$ as $F_{ij}(D,C)$ for all D, C contained in the body B if i is different from j and for all disjoint D,C contained in B if $i=j$, if prefix ∂ denotes the boundary of any volume, continuity of stress implies

$$F_{ii}(D,C) = \int_{\partial D \cap \partial C} T_{ii} n \, dA \text{ for disjoint } D,C \quad (\text{III-151})$$

$$F_{ij}(D,C) + F_{ji}(D,C) = \int_{\partial D \cap \partial C} T_{ij} n \, dA \text{ for disjoint } D,C \quad (\text{III-152})$$

and

$$= \int_{\partial D \cap \partial C} -T_{ij} n \, dA + \int_D (p_{ij} + p_{ji}) \, dV \text{ for } D \text{ contained in } C \quad (\text{III-153})$$

and i different from j .

Also,

$$F_i(D) = \int_{\partial D \cap \partial B} t_i \, dA + \int_D b_i \, dV \quad (\text{III-154})$$

The equations of force balance were, for the constituent $s^{(i)}$ denoted by subscript i ,

$$\text{Div. } T_{ii} + \sum_{i \neq j} p_{ij} + b_i = 0 \quad (\text{III-155})$$

$$\text{Div. } T_{ij} - (p_{ij} + p_{ji}) = 0 \quad (\text{III-156})$$

and

$$T_{ii} n = t_i + \sum_{j \neq i} s_{ij} \quad (\text{III-157})$$

$$T_{ij} n = - (s_{ij} + s_{ji}) \quad (\text{III-158})$$

where s_{ij} is a surface force on ∂B_i exerted by ∂B_j e.g., a capillary force. The quantity b_i includes any inertia effects. For the global body, define

$$T = \sum_i \sum_j T_{ij} \quad (\text{III-159})$$

Then

$$\text{Div. } T + b = 0 \text{ in } B \quad (\text{III-160})$$

and

$$T \cdot n = t \text{ on } \partial B \quad (\text{III-161})$$

where b is the sum of all b_i and t is, likewise, the sum of all t_i i.e. the body forces and the tractions are additive. If partial stresses be defined by

$$T_i = T_{ii} + \sum_{j \neq i} \lambda_{ij} T_{ij} \quad (\text{III-162})$$

with $\lambda_{ji} = 1 - \lambda_{ij}$, then T is the sum of all T_i and the interaction force

$$p_i = \sum_{j \neq i} \frac{1}{2} (p_{ij} - p_{ji}) \quad (\text{III-163})$$

The equations of balance of the constituent $s^{(i)}$ are

$$\text{Div. } T_i + p_i + b_i = 0 \quad (\text{III-164})$$

$$\text{Div. } T_{ij} - (p_{ij} + p_{ji}) = 0 \quad (\text{III-165})$$

and sum of the forces p_i vanishes. The definition of the partial stress is thus variable. The traditional approach ignores (III-164) and constitutive equations are written for T_i and p_i . Gurtin [90] and

Sampaio [155] postulated these equations as constitutive identities. If $F_{iN}(D,C)$ is defined as the total force function,

$$F_{iN}(D,C) = \sum_{j=i}^n F_{ij}(D,C) + F_{ii}(D \underline{o} C, C \cap D) \quad (\text{III-166})$$

where $D \underline{o} C$ is the interior of the complement of D in C . Additivity of $F_{iN}(D,D)$, for a binary mixture leads to

$$F_{ij}(D,C) = - F_{ji}(C,D) \quad (\text{III-167})$$

which seems anti-intuitive [91,141]. Similarly, extension of the traditional theory for single materials to mixtures by simply replacing the forces by 'total force' is inadequate to express balance of forces for other than pure constituents. In the theory of balance of forces proposed for mixtures, continuity would apply to all force components $F_{ij}(D,C)$ and the balance of forces would require

$$F_{ij}(D,C) + F_{ji}(C,D) = 0 \quad (\text{III-168})$$

to insure balance of forces on any part of B .

3.4.6. Other Forms of the Energy Balance Equation.

Bowen [29] postulated the point form of the rate of energy equality as

$$\rho \frac{\partial}{\partial t} [U + \frac{1}{2} v_i v_i] = (t_{ji} v_j - q_i)_{,i} + \sum_{k=1}^n \rho^{(k)} v_i^{(k)} b_i^{(k)} \quad (\text{III-169})$$

For the case of no relative motion, $v_i^{(k)} = v_i$ for all the constituents $s^{(k)}$. Hence

$$\sum_{k=1}^n \rho^{(k)} v_i^{(k)} b_i^{(k)} = v_i \sum_{k=1}^n \rho^{(k)} b_i^{(k)} = \rho b_i v_i \quad (\text{III-170})$$

In the case when $b_i^{(k)}$ is the same for all the constituents $s^{(k)}$, we have a similar results, i.e.,

$$\sum_{k=1}^n \rho^{(k)} v_i^{(k)} b_i^{(k)} = b_i \sum_{k=1}^n \rho^{(k)} v_i^{(k)} = \rho b_i v_i \quad (\text{III-171})$$

Bowen pointed out the effect of certain approximations. If all second order terms in diffusion velocities are neglected, (III-58), (III-60), and (III-62) reduce, respectively, to (III-83), (III-84), and (III-85). If we also neglect products of the diffusion velocities with their time and space derivatives, the energy equality (III-169) becomes

$$\rho \frac{\partial U}{\partial t} = t_{ji} v_{j,i} - q_{i,i} + \rho r + \sum_{k=1}^n \rho^{(k)} u_i^{(k)} b_i^{(k)} \quad (\text{III-172})$$

We note that the interpretation of quantities appearing in (III-172) and (III-169) is quite different. Green and Naghdi [87] wrote the energy balance equation in the same form as (III-172). However, their formulation was based on different definitions for the quantities associated with the mixture in terms of those associated with the constituents rather than on any approximation.

Using an interaction force vector, Green and Naghdi [88] wrote the energy balance equation as

$$\begin{aligned} \rho \frac{DU}{Dt} = \rho r - q_{i,i} - \phi + \sum_{k=1}^n (\mu_i^{(k)} v_i^{(k)} + t_{ji}^{(k)} v_{i,j}^{(k)}) \\ + \frac{1}{2} \rho c^{(k)} u_i^{(k)} u_i^{(k)} \end{aligned} \quad (\text{III-173})$$

where

$$\phi = \sum_{k=1}^n (\rho^{(k)} u_i^{(k)} u_i^{(k)})_{,j} \quad (\text{III-174})$$

$$\rho U = \sum_{k=1}^n \rho^{(k)} U^{(k)} \quad (\text{III-175})$$

and

$$\mu_i^{(k)} = t_{ji,j}^{(k)} + \rho^{(k)} (b_i^{(k)} - f_i^{(k)}) - \rho c^{(k)} u_i^{(k)} \quad (\text{III-176})$$

In an earlier paper, [83], Green and Naghdi wrote for a binary

mixture.

$$\rho \frac{DU}{Dt} = \rho r - q_{i,i} + \sum_{k=1}^2 t_{ji}^{(k)} v_{i,j}^{(k)} + \frac{1}{2} (\theta_i^{(1)} - \theta_i^{(2)}) v_i^{(1)} - v_i^{(2)} \quad (\text{III-177})$$

The quantity $\theta_i^{(1)} - \theta_i^{(2)}$ represents the interaction force between the two constituents. Using (III-111), and (III-100), for a binary mixture

$$\theta_i^{(1)} + \theta_i^{(2)} - \sum_{k=1}^2 \rho c^{(k)} v_i^{(k)} = 0 \quad (\text{III-178})$$

Hence, the diffusive force is given by

$$\begin{aligned} D &= \frac{1}{2} (\theta_i^{(1)} - \theta_i^{(2)}) \\ &= \frac{1}{2} (\theta_i^{(1)} + \theta_1^{(1)} - \sum_{k=1}^n \rho c^{(k)} v_i^{(k)}) \\ &= \theta_i^{(1)} - \frac{1}{2} \sum_{k=1}^n \rho c^{(k)} v_i^{(k)} \end{aligned} \quad (\text{III-179})$$

and again using (III-110),

$$D_i = -\theta_i^{(1)} - \frac{1}{2} \sum_{k=1}^n \rho c^{(k)} v_i^{(k)} \quad (\text{III-180})$$

Morland [131] stated the energy balance equation for the mixture as

$$\sum_{k=1}^n [\rho^{(k)} \frac{D^{(k)}}{Dt} U^{(k)} - t_{ij}^{(k)} v_{i,j}^{(k)} + q_{i,i}^{(k)}] = 0 \quad (\text{III-181})$$

This is the same as (III-177) for no energy supply and no interaction force.

Mokadam [126] considered the total internal energy for the mixture to consist of three components,

$$E = U + T + L \quad (\text{III-182})$$

where U , T , L , are respectively, the molecular, the kinetic, and the

potential energies per unit mass of the mixture. Identifying the diffusive force as the body force causing mass flow, Mokadam postulated equations of balance of energy in the form

$$\frac{\partial \rho E}{\partial t} = - (\rho E v_j + q_j - t_{ij} v_i)_{,j} + D_j v_j \quad (\text{III-183})$$

where D_j are components of the diffusive force. Also

$$\frac{\partial \rho L}{\partial t} = - (\rho L v_j)_{,j} - \rho f_j v_j \quad (\text{III-184})$$

For no chemical reaction, (III-184) and mass conservation imply

$$\frac{DL}{Dt} = f_i v_i \quad (\text{III-185})$$

Noting that, for no chemical reaction, mass continuity (III-45) implies

$$\rho \frac{DT}{Dt} = \frac{\partial \rho T}{\partial t} - (\rho T v_i)_{,i} \quad (\text{III-186})$$

and also,

$$\frac{DT}{Dt} = (\rho f_i + D_i) v_i \quad (\text{III-187})$$

Mokadam wrote

$$\frac{\partial \rho U}{\partial t} = (t_{ij} v_i)_{,j} - (\rho t v_i + q_i)_{,i} \quad (\text{III-188})$$

3.5 COMMENTS

Various approaches to description of motion of the constituents and the mixture as well as to formulation of the equations of balance of mass, linear momentum, angular momentum, and rate of energy have been discussed. We note that in many theories of mixtures, deformation is referred to an initial configuration for each constituent and motion to the place coordinates. Also the equations of balance are written for a fixed volume in space. This appears to be the correct approach for mixtures of fluids but may not be

convenient for multiphase mixtures. The notion of motion of a mixture as a whole is often introduced. Indeed, Truesdell would require the form of the equations of balance for the mixture to be the same as for a single material. We note that for the case of no relative motion between the constituents, the mixture will have motion and deformation as a material body. For that case the development of equations of motion for the mixture is meaningful. However, if relative motion is present, the mixture does not satisfy the axiom of continuity and its corollary, the principle of impenetrability. Hence, the 'third metaphysical principle' stated by Truesdell [162] appears to be irrelevant. We note too that the sum of internal energies of the constituents does not equal the internal energy of the mixture. (c.f. Biot's theory where the kinetic energy of the saturated soil is expected to be a quadratic in the velocity of the fluid and the solid rather than the sum of kinetic energies of the constituents). The balance equations due to Truesdell and to Green and Naghdi have similar form and are essentially equivalent but the quantities appearing in the two sets have different interpretations based upon the relationships postulated between the quantities associated with the constituents and with the mixture. The equations of energy balance contain scalar products of corresponding quantities. This indicates the quantities for which constitutive relationships would be required. These will be discussed in the next section. Some investigators, considering the special problem of flow through deformable porous solids, have attempted to write the balance equations in terms of relative motion and porosity, which is essentially a measure of relative deformation. It would appear that a theory based upon balance equations written for a reference set of particles of the porous solid would be the most appropriate for this case. For the one-dimensional case, Gibson developed such a theory for the quasi-static problem. This theory needs to be developed further to include inertia effects and a generalization to three-dimensional problems. This would require a convected coordinate description for the stresses and deformations in the porous solid along with a referential description for the motion of the fluid relative to the solid.

SECTION IV

CONSTITUTIVE RELATIONSHIPS

4.1. INTRODUCTION.

In order to set up constitutive equations, it is necessary first to define the mechanical quantities for which such relationships are desired and to identify the kinematic or state variables on which these quantities might depend. For a mixture of several constituents, the stresses in the constituents are obviously the primary mechanical variables. Truesdell [189], following Maxwell [118], recognized diffusive resistance as another mechanical variable, reflecting the interaction between the constituents. Green and Naghdi [83,85] pointed out the necessity of postulating constitutive relationships for this quantity. The rate of energy equality, e.g., (III-114), contains scalar products of 'corresponding' quantities. This indicates the need for postulating constitutive relations for $t_{(ij)}^{(k)}$, $t_{[ij]}^{(k)}$, and $p_i^{(k)}$. Again, if the equations of mass, linear momentum, angular momentum, and energy balance for constituents be regarded as equations for $\rho^{(k)}$, $v_i^{(k)}$, $M_{ij}^{(k)}$, and $T^{(k)}$ respectively, constitutive equations are required for the other quantities appearing in those equations, viz., $c^{(k)}$, $t_{ij}^{(k)}$, $t_{[ij]}^{(k)}$, $p_i^{(k)}$, $u^{(k)}$, $q_i^{(k)}$ and also for the partial entropy $s^{(k)}$. These constitutive equations are subject to the balance equations for the mixture and to an appropriate entropy production inequality. There has been some difficulty in defining components of the partial stress tensors. Williams [198] claimed that the usual interpretation of the partial stress tensors is misleading and stated: "While it is indisputable that the partial stress tensor represents a force exerted per unit area, it is not always so clear exactly upon what the force is exerted and by what". Truesdell [189,190] required the equations of motion of the mixture to have the same form as those for the motion of a single constituent. In this theory, the sum of partial stresses was not equal to the total stress in the mixture except in the approximation where second order quantities are neglected. Green and Naghdi [83,85,87], on the other hand, started with the assumption that

the sum of partial stresses must equal the total stress. For fluid-saturated solids, the isotropic fluid stress is generally considered to be the stress variable in addition to the stresses in the solid. Biot [15] regarded the total stress and the fluid stress as the mechanical variables. The definition of pore-fluid pressure used by various investigators differs considerably. Traditionally, for a water-saturated soil, the pressure recorded by piezometers inserted into the water-filled pores has been assumed to be the fluid pressure acting over 100 percent of the area of internal surfaces. Biot [21] pointed out that the generalized forces defined by divergence of the stresses are correctly defined by the virtual work of microscopic stresses per unit value of the displacements of the constituents and not as the average of the microscopic stresses. However, according to Biot, for practical purposes, either definition would be acceptable. This is in line with Mokadam [125] who following Guggenheim [89], regarded the fluid pressure to be the thermodynamic property such that

$$\pi dV = dW \quad (IV-1)$$

where dV is the differential change in the intrinsic fluid volume and dW the reversible work of the fluid phase.

Terzaghi [187,188], as well as Green [82-88] assumed the partial stresses to act over the entire area of any surface element. Further, assuming partial stresses to be additive, the total stress

$$t_{ij} = t_{ij}^{(1)} + t_{ij}^{(2)} \quad (IV-2)$$

For isotropic fluid stress, $t_{ij}^{(2)} = \pi \delta_{ij}$ and

$$t_{ij} = t_{ij}^{(1)} + \pi \delta_{ij} \quad (IV-3)$$

Biot regarded $t_{ij}^{(1)}$ as the bulk stress acting over the entire area of internal surfaces. In his earlier work [15], there was no reference to the area over which the fluid pressure acts. Later [16], Biot assumed the fluid pressure to act only over the pore area. This corresponds to the notion of the fluid pressure being an intrinsic

quantity. Thus

$$t_{ij} = t_{ij}^{(1)} + n^{(2)} \pi^* \delta_{ij} \quad (\text{IV-4})$$

Garg [70], Morland [130-132], Pecker [145] and Carroll [34], among others, introduced the notion of intrinsic stresses $t_{ij}^{(k)*}$ for each constituent leading to

$$t_{ij}^{(k)} = n^{(k)} t_{ij}^{(k)*} \quad (\text{IV-5})$$

In reference [131], the coefficient $n^{(k)}$ being area fraction could be different from the volume fraction and was denoted by $m^{(k)}$.

Tsien [193] divided the total stress into stress deviation and a hydrostatic component viz. $t_{kk}/3$. The hydrostatic stress was expected to be distributed over the solid and the fluid in proportion to their volume fractions $n^{(k)}$, and the solid was expected to take the entire stress deviation. Carroll [34] and Morland [130-133], among others, also assumed that the shear traction is carried by the bulk solid and is related to changes in pore geometry. The hydrostatic stress components in the solid and the fluid were expected to cause intrinsic volume changes in the constituents.

In the theory of consolidation originally developed by Biot [15] assuming water to be incompressible, he regarded the change in water content of the soil expressed as difference from the initial state to be the kinematic variable for setting up constitutive equations. Later [16], while extending the theory to anisotropic solids and compressible fluids, the volumetric strain of the fluid was introduced as a kinematic variable in addition to the strains in the solid. In another work [24], the inflow into an elementary volume was regarded as a vector and its gradient was the quantity conjugate to the fluid pressure in making a contribution to the energy of the system. This would correspond to the velocity gradient-fluid stress pair used by Adkins [2] and Green [81]. Biot [24] explained that the several measures used by him from time to time were essentially the same quantity. In theories of mixtures, densities, deformations,

deformation rates, velocities, and vorticities of the constituents are taken to be the kinematic variables. Morland [130-132], Garg and Nur [70], and several investigators following them introduced intrinsic densities of the materials as well the porosity (relative volume fraction of the pores) as variables. Bowen [32] regarded volume fractions as internal state variables obeying rate type constitutive relations.

It has been difficult to define the dependence of components of the partial stress tensors upon the kinematic variables and densities of the constituents of a mixture. In his earlier work, Adkins [1,2] assumed that the stress in each constituent depended upon the density and the kinematic quantities associated with that constituent only. Green and Adkins [81] admitted interdependence of partial stresses of each constituent upon the kinematics of all. This was in line with the principle of equipresence, stated by Truesdell and Toupin [189], according to which the mechanical variables for any constituent are assumed to be influenced by the kinematics of all constituents unless precluded by considerations of thermodynamics or deliberate assumption to obtain simplicity. However, to make the independent variables distinct for various constituents, Green and Adkins [81] stipulated that the partial stresses for any constituent would depend upon the densities, velocities and antisymmetric deformation of all constituents but only on the symmetric part of the deformation gradient of the constituent itself. Invariance of stress under superposed rigid body motions and under superposed uniform rigid body angular velocities of the mixture as a whole showed that, for a binary mixture, the velocities and the rotation tensors must occur as difference terms of the type $v_i = v_i^{(1)} - v_i^{(2)}$ and $\Lambda_{ij} = \Gamma_{ij}^{(1)} - \Gamma_{ij}^{(2)}$ in the set of independent variables. Bowen [28] restated Truesdell's point of view and dropped partial stress tensors from the discussion. In fact, following classical thermodynamics, he did not introduce any 'partial' quantities. It was shown that in the case of dilute mixtures of ideal substances (dilute implying preponderance of a single constituent over all the others), the partial stresses can be defined such that they depend upon the density and kinematics of individual constituents and their sum is the

total stress tensor. This corresponds to Adkins' earlier view [1,2] and as, in general, mixtures may not be dilute, the principle of equipresence would appear to be the most logical basis for the development of constitutive theories. Morland [130] postulated that the relation of 'effective' stress to 'effective' deformation for each constituent should be its own constitutive law as a single continuum. The interaction of the partial constitutive laws would be included in the scaling functions relating the 'effective' and 'bulk' quantities. These scaling functions would require a constitutive description in terms of macroscopic variables. For a binary mixture of a liquid and solid, porosity was taken to be the scaling function. Mokadam [125] assumed the stress tensor to be a linear function of a velocity gradients. Truesdell [189] as well as Biot [21] considered the balance of momentum equation to be a generalized form of d'Arcy's rule.

Thermodynamics of d'Arcy's law, for fluid flow in a rigid porous medium, was discussed by Mokadam [125-127]. He pointed out that d'Arcy's law is valid only for isothermal flow in which the inertial and viscous effects are negligible. Also that the rate of entropy production must be nonnegative separately for terms involving quantities of different tensorial ranks. Thus, regarding stress as a flux it would need a constitutive equation in terms of velocity gradients and the phenomenological equations for heat, mass and molecular flow will be separately formulated as in (II-5). Truesdell and Noll [192] noted that a 'proper' generalization of the Clausius-Duhem inequality would be basic to development of constitutive equations for diffusion. Green and Naghdi [83,85] postulated an entropy production inequality for the entire continuum to define restrictions on the character of various constitutive relationships. This was applied to the case of mixture of Newtonian fluids [83]. Crochet and Naghdi [45] applied these considerations to the flow of a fluid through an elastic solid. Atkin [7] explicitly stated the form of these constitutive assumptions for fluid flow through an elastic solid. He also presented an alternative method of deriving the linearized theory of elastic solid-viscous fluid mixtures and the thermodynamic restrictions imposed on this theory by the entropy

production inequality. In later work, Green and Naghdi [88] based the thermodynamic restrictions on the behavior of each constituent on the requirement that suitable combinations of the equations for individual constituents should yield a single entropy production inequality for the mixture as a whole. Bowen [28] noted that these formulations lead to the result that, in equilibrium, the partial free energy density of a given constituent is independent of the deformations of the other constituents. Also that such independence fails to be confirmed by experiments on fluid mixtures. Müller [136] showed that if gradients of densities of the constituents were included among the constitutive variables, this difficulty would not arise. Some investigators have proposed use of an entropy production inequality for each constituent. Bowen [28] considers this to be too special.

In this section we describe various phenomenological approaches to development of constitutive relationships as well as those based on theories of mixtures.

4.2. DIFFUSIVE RESISTANCE.

Diffusive force was identified as the interaction force by Truesdell [189,190]. Green [83-88] and Crochet [45] refer to it as 'diffusive resistance'. Biot [22] described it as 'disequilibrium force'. The words 'friction' and 'drag' have also been used. Maxwell [118] defined it as a pair of equal and opposite forces acting on the two constituents in a binary mixture. The force was proportional to the densities of the constituents and to the relative velocity. For non-chemically reacting continua, in the absence of inertia effects, the equilibrium equations for the fluid-saturated solid are:

$$[t_{ij}^{(1)} + t_{ij}^{(2)}]_{,i} + [\rho^{(1)} b_i^{(1)} + \rho^{(2)} b_i^{(2)}] = 0 \quad (\text{IV-6})$$

where $b_i^{(k)}$ is the body force per unit mass associated with the constituent $s^{(k)}$. (IV-6) can be rearranged as

$$[t_{ij,i}^{(1)} + \rho^{(1)} b_i^{(1)}] = - [t_{ij,i}^{(2)} + \rho^{(2)} b_i^{(2)}] \quad (\text{IV-7})$$

Each side of the equality (IV-7) representing interaction between the constituents is set equal to the diffusive resistance D_j . A set of single-constituent stresses in equilibrium can be added to the stresses on either side without affecting the definition of D_j . For instance, let $\bar{t}_{ij}^{(k)}$, $k=1,2$ such that $\bar{t}_{ij,i}^{(k)} = 0$. Then, evidently, addition of $\bar{t}_{ij}^{(k)}$ to $t_{ij}^{(k)}$ in (IV-7) will not affect the value of the quantities on either side. However, superposition of a total stress state $\bar{t}_{ij}^{(1)} + \bar{t}_{ij}^{(2)}$ satisfying equilibrium will influence D_j . For hydrostatic fluid stress,

$$D_i = -(\pi_{,i} + \rho^{(2)} b_i^{(2)}) - \bar{t}_{ji,j}^{(2)} \quad (IV-8)$$

where

$$(\bar{t}_{ji}^{(1)} + \bar{t}_{ji}^{(2)})_{,j} = 0 \quad (IV-9)$$

If inertia effects are included, for a binary mixture, (III-116) gives $p_i^{(k)}$ the quantity conjugate to the relative velocity in the energy equality (III-114) as

$$-p_i = \rho^{(2)} (f_i^{(2)} - b_i^{(2)}) + \rho c^{(2)} v_i^{(2)} - t_{ji,j}^{(2)} - \bar{t}_{ji,j}^{(2)} \quad (IV-10)$$

This definition is somewhat more general than the one used by Green and Naghdi [85,86] where the term involving $c^{(2)}$ was ignored. In [87], however, following Mills [122], a general form of (IV-10) is stated for a mixture of $n-1$ fluids and a solid.

Truesdell [190] examined mathematical theories of diffusion and proposed a mechanical theory. Four existing theories viz. kinematical, hydrodynamical, kinetic, and thermodynamic were recognized. The kinematical theory leads to Fick's law. Based on analogy to the flow of heat, Fick [64] proposed that in a binary mixture at uniform total density

$$\rho^{(k)} u_i^{(k)} = -D \rho_{,i}^{(k)} \quad (IV-11)$$

where $\rho^{(k)}$ is the density of constituent $s^{(k)}$, $u_i^{(k)}$ its diffusive velocity and D is a phenomenological coefficient. Onsager [143] proposed a generalization of (IV-11) taking $\rho^{(k)} u_i^{(k)}$ as a linear

combination of the density gradients of all constituents. However, as pointed out by Truesdell, the purely kinematical presumptions leading to Fick's law are unsupported by any principle or method of mechanics or physics and cannot be expected to explain or predict motions except in specially simple circumstances. Fick's law is applicable for no mass exchange between constituents, constant pressure and total density and for no mean motion. Bowen [29] pointed out that (IV-11) only represents a special case of a more general mixture theory. He also showed that (IV-11) and the equation of mass continuity, (III-37) lead to Fick's second law which involves rate of change of constituent density. Stefan [182] generalized Maxwell's equation and proposed

$$\rho^{(k)} f_i^{(k)} = \rho^{(k)} b_i^{(k)} - \pi_{,i}^{(k)} - \sum_{j=1}^n A_{ji}^{(k)} \rho^{(k)} \rho^{(j)} (u_i^{(k)} - u_i^{(j)}) \quad (\text{IV-12})$$

for a mixture of n constituents. Here $\pi^{(k)}$ is the hydrostatic pressure (equal to partial pressure in condition of equilibrium). $A_{ji}^{(k)}$, for given i , are elements of a matrix of phenomenological coefficients assumed to be independent of the densities of the constituents. $A_{ji}^{(k)} > 0$ for k not equal to j was intended though not so stated. The concept of equal and opposite action implied $A_1^{(2)} = A_2^{(1)}$. For mixtures with more than two constituents, $A_{ji}^{(k)} = A_{ki}^{(j)}$ was assumed without comment. This is due to the assumption of pairwise interaction of the constituents. The argument of equal and opposite forces for a binary mixture does not extend to three or more components. When more than two constituents are present, the interaction of any pair may be affected by the presence of the others. Truesdell [189] cites results, for the Chapman-Enskog process in Maxwell's kinetic theory obtained for multicomponent mixtures by Hirschfelder, Curtiss and Bird [94]. Truesdell showed that the kinetic theory yields a quasi-static approximation to the Maxwell-Stefan theory, with the addition of a term representing thermal diffusion. Also that the coefficients $A_{ji}^{(k)}$ depend, in general, upon densities and molecular constants of all constituents and the matrix $A_{ji}^{(k)}$, for given i , is not necessarily symmetric. According to the kinetic theory, diffusion is a sum of binary phenomena in first approximation but not in a more accurate treatment. Truesdell [190]

noted that in the kinetic theory, accelerations are not considered and hence it is to be regarded as an approximation. The Maxwell-Stefan theory is more general in that accelerations are included. However, Stefan's relations regarding coefficients $A_{ji}^{(k)}$ arise in first approximation of the full results at the first stage of the Chapman-Enskog process. Each of the theories has aspects of generality wanting in the other. Eckert-Meixner thermodynamic theory implies, neglecting thermal diffusion,

$$\rho^{(k)} u_i^{(k)} = \sum_{j=1}^n L_{ji}^{(k)} F_j \quad (IV-13)$$

where $u_i^{(k)}$ is defined by (III-21) and F_j are 'forces' defined differently by different authors. An additional condition, sometimes imposed is

$$\sum_{j=1}^n F_j^{(k)} = 0 \quad (IV-14)$$

However, if no terms are neglected, there is no reason to believe (IV-14) will hold. However, from (III-19) and (III-21)

$$\sum_{j=1}^n \rho^{(k)} u_i^{(k)} = 0 \quad (IV-15)$$

Hence

$$\sum_{j=1}^n \sum_{k=1}^n L_{ji}^{(k)} F_j = 0 \quad (IV-16)$$

The coefficients $L_{ji}^{(k)}$ are not uniquely defined but must satisfy

$$\sum_{j=1}^n L_{ji}^{(k)} = L_i = \text{constant} \quad (IV-17)$$

Further, the Onsager relations are often assumed i.e. $L_{ji}^{(k)}$ is assumed to be symmetric in k, j . In fact, to satisfy the non-negative production of entropy all that is needed is that the matrix $L_{ji}^{(k)}$ for given i , be non-negative definite.

Truesdell [190] proposed a more general mechanical theory of diffusion. According to Truesdell [190], "diffusion, being a change of motion, arises from forces; the motions produced by these forces must conform to the principle of linear momentum applied to each

constituent and to the whole mixture". Thus, for constituent $s^{(k)}$, the supply of momentum is defined by, (III-48),

$$\rho m_i^{(k)} = \rho^{(k)} (f_i^{(k)} - b_i^{(k)}) - t_{ji,j}^{(k)} + \rho c^{(k)} \quad (\text{IV-18})$$

The sum of supplies of partial momenta to the constituents must vanish. The simplest constitutive equation for supply of momentum would be a linear dependence of the partial momentum upon the quantity 'corresponding' to it in the energy balance equation (III-51), viz., the relative velocity of the constituent with respect to the others. For an isotropic medium,

$$\rho m_i^{(k)} = \sum_{j=1}^n L_{ji}^{(k)} (v_j^{(k)} - v_j^{(j)}) \quad (\text{IV-19})$$

The scalar coefficients $L_{ji}^{(k)}$ are defined uniquely for k not equal to j and $L_{kk}^{(k)}$ must vanish. For each constituent as perfect fluid, (IV-18) and (IV-19) lead to Stefan-Maxwell equations. The determination of constitutive equations for materials has nothing to do with diffusion in linear theories. The restriction that the sum of partial moment supplies must vanish places a restriction upon the coefficients $L_{ji}^{(k)}$. Truesdell showed that this restriction leads to the necessary and sufficient condition

$$\sum_{j=1}^n (L_{ji}^{(k)} - L_{ki}^{(j)}) = 0 \quad (\text{IV-20})$$

For a binary mixture, this implies $L_{2i}^{(1)} = L_{1i}^{(2)}$. Considering the rate of work of diffusive drags, it is necessary that diffusion be a dissipative process. Inversion of relations (IV-19) gives

$$\rho^{(k)} v_i^{(k)} = \sum_{j=1}^n G_{ji}^{(k)} (m_i^{(k)} - m_i^{(j)}) \quad (\text{IV-21})$$

where

$$\sum_{j=1}^n (G_{ji}^{(k)} - G_{ki}^{(j)}) = 0 \quad (\text{IV-22})$$

For a mixture of perfect fluids, $t_{ij} = t\delta_{ij}$. Substituting in (IV-21)

and eliminating $v_i^{(k)}$ using the equation of conservation of mass of the constituent $s^{(k)}$, (III-45), Truesdell obtained an equation for rate of change of density of a constituent in terms of gradients of pressures. Theories of diffusion involve variables other than pressures. Assuming the existence of a thermal equation of state for each constituent, Truesdell showed that writing gradients of pressure in terms of gradients of density and temperature, neglecting accelerations, for a binary mixture, the theory reduces to Fick's law under isothermal conditions when body forces are absent and when the total density is uniform. However, the general form of the theory holds whether or not there exist thermal equations of state. Truesdell showed that the kinetic theory, the hydrodynamical theory and the thermodynamical theory were all specializations of the general theory of mechanical diffusion proposed.

Mokadam [125] proposed constitutive equations for heat and fluid flux. These essentially amount to regarding the diffusive force to have the form

$$p_i = -\frac{a}{T} v_i - \frac{b}{T^2} T_{,i} \quad (\text{IV-23})$$

Crochet [45] showed (we give a summary of the arguments later in this chapter) that under isothermal conditions and in the absence of chemical reactions, the constitutive relations will involve $F_{ij}^{(k)}$, $d_{ij}^{(k)}$, v_i and Δ_{ij} . For the linear theory of irrotational relative motion, in the absence of chemical reaction and inertia effects, and non-Newtonian behavior in a binary mixture of a fluid and a solid this immediately leads to an expression of the type

$$p_i = \pi_{,i} + \rho^{(2)} b_i^{(2)} = -C_{ji} v_j \quad (\text{IV-24})$$

The inverse form of this equation is the well-known d'Arcy flow rule [146, 160]. Biot [15, 18-25] and earlier investigators (e.g., Terzaghi) [187] had used this rule as the starting point for their theories. Biot [16] also assumed the existence of a dissipation function so that the permeability tensor was symmetric. Schiffman [166] extended d'Arcy's law to non-isothermal consolidation. Further

generalizations based on Onsager's principle have been proposed [e.g. 25, 142, 156, 157, 159, 186]. Generalization to nonlinear cases has been proposed [128] assuming C_{ij} to be functions of porosity $n^{(2)}$. Ghaboussi and Wilson [74] pointed out that Biot's [21, 22] formulation for momentum balance may be regarded as a generalization of (IV-24) to include inertia effects in the body force term giving, for no chemical reaction and isotropic fluid stress, a specialization of (IV-10)

$$p_i = \pi_{,i} + \rho^{(2)} (b_i^{(2)} - f_i^{(2)}) = - C_{ji} v_j \quad (\text{IV-25})$$

Regarding (III-116) as the definition of p_i , a general relationship based on the correspondence of $p_i^{(k)}$ and the relative velocity $v_i^{(1)} - v_i^{(2)}$ in (III-114) is

$$p_i = \pi_{,i} + \rho^{(2)} (b_i^{(2)} - f_i^{(2)}) - \frac{1}{2} \rho c^{(2)} v_i = - C_{ji} v_j \quad (\text{IV-26})$$

Morland [131] stated the total volume flux across a fixed surface as

$$\sum_{k=1}^n n^{(k)} v_i^{(k)} n_i$$

For a binary mixture of fluid and solid this reduces to

$$[v_i^{(1)*} + n^{(2)} (v_i^{(2)*} - v_i^{(1)*})] n_i \quad (\text{IV-27})$$

Thus, the volume flux of the fluid relative to the solid surface is

$$n^{(2)} (v_i^{(2)*} - v_i^{(1)*})$$

This was used as the justification for writing d'Arcy's law in terms of relative flux. We note, however, that (IV-27) does not follow from mass continuity except for incompressible constituents. [cf. (III-125)].

4.3. STRESS-STRAIN RELATIONS.

4.3.1. Effective Stress.

Terzaghi [187] introduced the concept of effective stress. This was defined to be the stress component causing deformations of the soil. For hydrostatic pore-water pressure acting on incompressible soil grains, the entire deformation of the soil was assumed to be due to changes in the pore volume and pore geometry. For this case, Terzaghi [187] termed $t_{ij}^{(1)}$ in (IV-2) as the effective stress related to deformation of the solid. It was found [e.g. 139] that the fluid pressure did in fact influence the effective stress-strain relationship when the solid grains had compressibility comparable to that of the matrix as a whole and the fluid was not incompressible. To allow for this the effective stress related to deformation was defined as

$$t'_{ij} = t_{ij} - c\pi\delta_{ij} \quad (IV-28)$$

$$= t_{ij}^{(1)} + (1 - c)\pi\delta_{ij} \quad (IV-29)$$

where $c = 1$ implies Terzaghi's definition and $c = 0$ would correspond to total stress being regarded as effective. Suklje [183] discussed selection of appropriate values of c . Schiffman [165] expected c to be between $n^{(2)}$ and 1. This was based on the assumption that the pore fluid pressure may not act over the entire area of surface elements but only over a part. This fractional area of action of the fluid pressure would be bounded below by the porosity and above by 1. It is relevant to note here that several investigators confuse the effective stress with the partial stress. This is partly due to the dual definition given by Terzaghi. As used in this review, the effective stress is the stress component causing deformations of the solid and is thus defined completely by these deformations. The partial stress in the solid could conceivably be related to quantities other than the deformations of the solid. Considering the solid grains to be compressible, Nur [139] derived the equation

$$c = 1 - K/K_s \quad (IV-30)$$

where K , K_s are, respectively, reciprocals of the 'bulk' and the intrinsic of the solid, For incompressible grains $K_s \rightarrow \infty$ and for highly deformable pore space $K \rightarrow 0$. Hence, for either of these cases $K/K_s = 0$, and Terzaghi's definition is recovered. Suklje [183] suggested, without proof

$$c = 1 - n^{(1)} \frac{K}{K_s} \quad (IV-31)$$

Schiffman [165] gave a more general form for (IV-31) allowing fluid pressure to be a second rank tensor and c a fourth rank tensor, i.e.,

$$t'_{ij} = t_{ij} - A_{klij} t_{kl}^{(2)} \quad (IV-32)$$

where the superscript 2 refers to the fluid. Carroll [34] carried out a similar development. These approaches were based on superposition of effects of the hydrostatic stress and the stress deviation. Carroll [34] determined, for the linear case, the relation

$$t'_{ij} = t_{ij} - E_{klij} C_{klmn}^{(s)} \pi \delta_{mn} \quad (IV-33)$$

where E_{klij} are components of the elasticity tensor for the dry solid material and $C_{ijkl}^{(s)}$ those of the intrinsic compliance under hydrostatic stress. For clays, Mitchell [124] would include attractive and repulsive forces within the material in defining the effective stress. Biot [22] defined effective stress as

$$t'_{ij} = t_{ij} - \alpha \pi \delta_{ij} \quad (IV-34)$$

and assumed $t_{ij}^{(1)}$ to be the quantity related to solid deformation. Morland [130] criticized Schiffman's work stating: 'Schiffman neglects the scaling of the partial solid stress to define an effective stress in the matrix. As a consequence, in his linear soil-water model with an interaction coefficient given by the water porosity ($n^{(2)}$, for example), the elastic soil and water responses are

independent, in contrast to the linearized elastic-water mixture equations derived later from the present model and to the Biot [1956] equations'. This criticism is based essentially on different notions of 'effective' stress. Morland defined 'effective' stress $t_{ij}^{(k)*}$ by the equation

$$t_{ij}'^{(k)} = m^{(k)} t_{ij}^{(k)*} \quad (IV-35)$$

and proposed constitutive equations for the scaling factors $m^{(k)}$. This is different from the definition in (IV-28) above. Later, Morland [131] used the term 'intrinsic' to replace 'effective' to avoid confusion.

Garg [70], following Haimson [92], proposed a dual definition for effective stress. For strength of rock, he would set $c=1$ in (IV-28) but for constitutive relations another value of c would be used.

Carroll, Schatz and Yamada [35] introduced intrinsic solid stress on the solid particles and an effective stress influencing deformation of the pore space. Using the equation

$$t_{jj} = n^{(1)} t_{jj}^{(1)*} + n^{(2)} t_{jj}^{(2)*} \quad (IV-36)$$

the solid stress

$$t_{jj}^{(1)} = (t_{jj} - n^{(2)} \pi^*) / n^{(1)} \quad (IV-37)$$

Thus, in general, $t_{ij}^{(1)}$ would be a function of the volume change of the solid and $t_{jj}' = t_{jj} - \pi$ would be a function of porosity. Kenyon [105,106] also considered the effect of grain and fluid compressibilities and introduced material parameters to characterize this dependence. Contact stress in the solid and the bulk stress independent of K_s were used.

For large deformation, as incremental form of the stress tensor was introduced by Biot [24]. Carter [37] and Prevost [147] used the Jaumann stress rate T_{ij} to ensure frame indifference. Its relation to the Cauchy stress $t_{ij}^{(1)}$ is

$$\dot{T}_{ij}^{(1)} = \dot{t}_{ij}^{(1)} - t_{ik}^{(1)} \Gamma_{kj}^{(1)} - t_{jk}^{(1)} \Gamma_{ki}^{(1)} \quad (\text{IV-38})$$

where $\Gamma_{ij}^{(1)}$ are components of the antisymmetric rate of rotation tensor for the solid in Eulerian description and the superposed $\dot{}$ denotes the material time derivative. The effective stress rate was

$$\dot{T}_{ij} = \dot{T}_{ij} - \pi \delta_{ij} \quad (\text{IV-39})$$

4.3.2. Pore Pressure Parameters

Skempton [179,180] proposed a simplification of the theory of consolidation. He suggested that the fluid pressure be expressed in terms of the total stress components through pore pressure parameters. A general statement of this type would be

$$\pi = b_1 J_1 + b_2 J_2^{1/2} + b_3 J_3^{1/3} \quad (\text{IV-40})$$

where b_i are material constants and J_i are the invariants of t_{ij} , the total stress tensor. In order that the formulation reflect the difference in fluid pressures observed during loading and unloading cycles, and their steady increment with plastic straining and at constant strain during failure, Lo [113] proposed use of strain invariants I_i of $e_{ij}^{(1)}$ instead of J_i . The coefficients b_i have been found [100] to depend upon stress. However, both these approaches are similar in character and as stated by Skempton [179], are semi-empirical. Nur [139] argued that the strains caused by the hydrostatic pore-water pressure and the total stress cancel each other i.e.,

$$M_{ij} \pi + C_{kl ij} t_{kl} = 0 \quad (\text{IV-41})$$

Consequently,

$$\pi = -(M_{ij})^{-1} C_{kl ij} t_{kl} \quad (\text{IV-42})$$

For isotropic behavior, this would be a special case of (IV-40).

4.3.3. Stress-Strain Relations.

4.3.3.1. Phenomenological and micromechanical approaches.

Tsien [193] proposed a linear elastic isotropic relation for $t_{ij}^{(1)}$ in terms of $e_{ij}^{(1)}$ using Terzaghi's definition i.e. $c=1$ in (IV-28). Biot [15] assumed a quadratic energy function in θ , the change in water content per unit volume of the solid, and $e_{ij}^{(1)}$ leading, for isotropic linear elastic soil and incompressible fluid, to

$$\pi = M e_{kk} + N\theta \quad (IV-43)$$

and

$$t_{ij} = 2 \mu e_{ij} + \lambda e_{kk} \delta_{ij} + M\theta \delta_{ij} \quad (IV-44)$$

In later work, [21], the total stress in (IV-44) was replaced by the solid stress $t_{ij}^{(1)}$. In extension to anisotropic elastic [16] solids and compressible fluid the relationship was stated as

$$t_{ij}^{(1)} = C_{klij} e_{kl}^{(1)} + M_{ij} e_{kk}^{(2)} \quad (IV-45)$$

$$\pi^* = M_{ij} e_{ij}^{(1)} + M e_{kk}^{(2)} \quad (IV-46)$$

Similar construction was used for viscoelastic soils [16]. In [24] the quantity $e_{kk}^{(2)}$ was again replaced by θ as used in (IV-43) and (IV-44). The same concept was extended to the case of finite elastic deformation [25]. Biot [22] defined effective stress through (IV-31) but the quantity related to solid deformations was taken to be $t_{ij}^{(1)}$ defined by (IV-3). This leads to

$$t_{ij}^{(1)} = t'_{ij} + (\alpha-1) \pi^* \delta_{ij} = E_{klij} e_{kl}^{(1)} \quad (IV-47)$$

Considering π^* to be given by (IV-46)

$$t'_{ij} = E_{klij} e_{kl}^{(1)} - (\alpha-1) [M_{pq} e_{pq}^{(1)} - M e_{mm}^{(2)}] \delta_{ij} \quad (IV-48)$$

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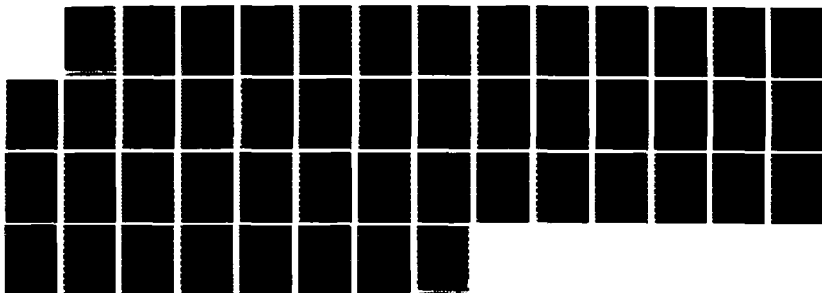
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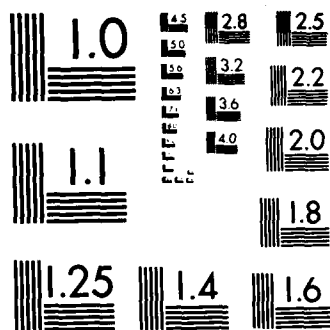
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For incompressible fluid and incompressible solid matrix, $e_{jj}^{(2)} = 0$ and α equals one. Hence $t_{ij}^{(1)}$ and $\hat{t}_{ij}$ coincide.

Lubinski [114] assumed that the total strain of a bulk porous solid can be expressed as a summation of the strains due to pore-water pressure and strains due to stress acting on the solid skeleton. He proposed relations of the type

$$t_{ij}^{(1)} = E_{klij} e_{kl}^{(1)} + (n^{(1)} - \gamma) \pi \delta_{ij} \quad (IV-49)$$

$$\pi^* = M e_{jj}^{(1)} + N (n^{(2)} - n_o^{(2)}) \quad (IV-50)$$

where γ , M , N , are constants to be determined experimentally. Krause [109] added terms to the right side of (IV-43) to reflect linear dependence of the fluid pressure on the deformation rate $d_{ij}^{(2)}$ of the fluid. This assumes a viscous component for fluid flow. Adkins, in his earlier theory [2], assumed that the stress in each constituent depended only on the density and the kinematic quantities associated with only that constituent. Nur [139] assumed the effective stress, given by (IV-28) along with (IV-30), to be related to $e_{ij}^{(1)}$. This admitted a certain dependence of $t_{ij}^{(1)}$ upon the fluid pressure. Explicitly,

$$\hat{t}_{ij} = t_{ij}^{(1)} + (1-c) \pi \delta_{ij} = C_{klij} e_{kl}^{(1)} \quad (IV-51)$$

Hence

$$t_{ij}^{(1)} = C_{klij} e_{kl}^{(1)} - (1-c) \pi \delta_{ij} \quad (IV-52)$$

These approaches were based on the superposition of effects of the hydrostatic stress and the shear stress. Carroll [34] determined, that for the linear case, (IV-52) would be

$$t_{ij}^{(1)} = E_{klij} e_{kl}^{(1)} - E_{klij} C_{klmn}^{(1)*} \pi^* \quad (IV-53)$$

where $E_{kl ij}$ are components of the elasticity tensor for the dry solid material and $C_{ijkl}^{(1)*}$ those of the intrinsic compliance under hydrostatic stress. For intrinsic hydrostatic fluid stress π^* acting on the solid particles, the strain in the solid phase is

$$e_{ij}^{(1)} = C_{kl ij}^{(1)*} \pi^* \delta_{kl} = C_{kk ij}^{(1)*} \pi^* \quad (IV-54)$$

The strain in the pore space is due to the bulk solid behavior, i.e.,

$$e_{ij}^{(2)} = C_{kl ij} (t_{kl} - \pi^* \delta_{kl}) \quad (IV-55)$$

Therefore, the total strain, adding (IV-54) and (IV-55), is

$$\begin{aligned} e_{ij} &= e_{ij}^{(1)} + e_{ij}^{(2)} = C_{kl ij} t_{kl} + (C_{kk ij}^{(1)*} - C_{kk ij}) \pi^* \\ &= C_{kl ij} (t_{kl} - \pi^* \delta_{kl}) + C_{kl ij}^{(1)*} \pi^* \\ &= C_{kl ij} \hat{t}_{kl} \end{aligned} \quad (IV-56)$$

where

$$\begin{aligned} \hat{t}_{kl} &= t_{kl} - \pi^* (\delta_{kl} - E_{ijkl} C_{ij mn}^{(1)*}) \\ &= t_{kl}^{(1)} + E_{ijkl} C_{ij mn}^{(1)*} \pi^* \end{aligned} \quad (IV-57)$$

Hence (IV-53). This formulation was specialized to allow for the presence of internal symmetries. For isotropy, the formulation reduces to Nur and Byerlee's [IV-29]. Schiffman [165] proposed a generalization of (IV-52) in the form

$$t_{ij}^{(1)} = E_{kl ij} e_{ij}^{(1)} - (\delta_{ik} \delta_{jl} - A_{kl ij}) t_{kl}^{(2)} \quad (IV-58)$$

The quantity A_{ijkl} was termed the soil water interaction tensor. Garg [70] obtained a relationship between the intrinsic and the bulk

behavior of rocks under hydrostatic stress. Interpretation of coefficients appearing in his equations were attempted by Biot [20]. Additions to this discussion have appeared from time to time [e.g. 105-106,139,150].

In extending the theory to the nonlinear case, Westmann [196] assumed $t_{ij}^{(1)}$ to be a function of the Cauchy deformation tensor for the solid and the rate of deformation (Eulerian description) of the fluid. The fluid stress was expected to consist of a hydrostatic component and another component depending upon the same quantities as $t_{ij}^{(1)}$. It was noted that in this formulation it would be difficult to design experiments to evaluate the parameters. A simplification proposed assumed the fluid pressure to be hydrostatic and related to the velocity field through Darcy's law. This is similar to Sandhu's [156-158] argument that the constitutive equation for diffusive resistance is a sufficient relationship between the fluid partial stress and kinematics of the mixture. Westmann [196] wrote relative velocity as a function of π , D_j , and the Cauchy deformation tensor for the solid. This would reflect, among other factors, the dependence of permeability on the porosity of the solid.

Bedford and Drumheller [14] have pointed out that fluid-saturated porous media fall in the class of immiscible mixtures. The constituents of such mixtures remain physically separate on a scale which is large in comparison with molecular dimensions. This immiscibility has two important consequences. Because of physical separation, in some local sense, each constituent will obey the constitutive relations for that constituent alone. Also, the constituents intrinsically have microstructure defined by the interfaces which separate the constituents. To set up macroscopic constitutive relations, one approach would be to postulate these relations directly as described above. Another alternative would be to relate macroscopic behavior to intrinsic properties of the constituents. The simplest theories involve the volume fraction of the constituents in addition to the usual variables. Morland [130] pointed out the meaning of deformation and stress associated with the continuum model of each constituent. In particular, the partial density variation is not the density variation of the constituent

since the mixture postulate eliminates reference to the actual volume occupied by each constituent in an immiscible mixture. Morland [130] and Garg [70] assumed the 'effective' (intrinsic) stress in each component to be a function of the deformation of that constituent only and having the same form as for a single material. Thus

$$t_{jj}^{(k)*} = f(e_{mm}^{(k)*}) \quad (IV-59)$$

The bulk stresses and deformations were expected to be related to the corresponding intrinsic quantities through scaling functions. Thus, the bulk stress in the constituent $s^{(k)}$

$$t_{jj}^{(k)} = m^{(k)} t_{jj}^{(k)*} \quad (IV-60)$$

For linear isotropic elastic rock

$$t_{jj}^{(1)} = m^{(1)} K_D^{(1)*} e_{jj}^{(1)*} \quad (IV-61)$$

where $K_D^{(1)*}$ denoted the bulk modulus of dry rock. The bulk deformation gradient was related to the effective deformation gradient F_{ij}^* as

$$F_{ij}^* = [n^{(1)}/n_0^{(1)}]^{1/3} F_{ij}^{(1)} \quad (IV-62)$$

The relation between $F_{ij}^{(1)}$ and $t_{ij}^{(1)}$ was expected to be the same function f as in (IV-59). For an ideal fluid, Morland [130] used

$$\pi = - \frac{\rho_0^{(2)}}{\rho^{(2)}} p \quad (IV-63)$$

where p is the fluid pressure, to write partial stress

$$t_{ij}^{(2)} = - m^{(2)} \left[\frac{n^{(2)}}{n_0^{(2)}} (\det(F_{pq}^{(2)})) \right] p \delta_{ij} \quad (IV-64)$$

For the solid constituent, he assumed the isotropic relation

$$t_{ij}^{(1)*} = a_0 \delta_{ij} + a_1 B_{ij}^* + a_2 B_{ik}^* B_{jk}^* \quad (IV-65)$$

where

$$B_{ij}^* = F_{ik}^* F_{jk}^* \quad (IV-66)$$

and a_i are functions of the invariants of B_{ij}^* . The corresponding partial stress tensor was

$$t_{ij}^{(1)} = n^{(1)} [b_0 \delta_{ij} + b_1 \left[\frac{n^{(1)}}{n_0^{(1)}} \right]^{\frac{2}{3}} B_{ij}^{(1)} + b_2 \left[\frac{n^{(1)}}{n_0^{(1)}} \right]^{\frac{4}{3}} B_{ik}^{(1)} B_{jk}^{(1)}] \quad (IV-67)$$

Further, it was assumed that the volume fraction $n^{(1)}$ is a dependent variable for which a constitutive equation is required to be postulated. For fluid stress, assuming ideal fluid, Morland [130] proposed

$$t_{ij}^{(2)} = - n^{(2)} \pi^* \left[\frac{n^{(2)}}{n_0^{(2)}} \right] J^{(2)} \delta_{ij} \quad (IV-68)$$

For a linearly viscous compressible fluid with zero bulk viscosity and a temperature dependent shear viscosity $\mu(T)$, Morland [131] wrote

$$t_{ij}^{(2)*} = - \pi^* \delta_{ij} + 2 \mu^{(2)} (D_{ij}^{(2)} - \frac{1}{3} v_{m,m}^{(2)} \delta_{ij}) \quad (IV-69)$$

Morland [131] assumed the stress deviation in the solid to be dependent only upon its strain deviation. Thus, for linearly elastic isotropic rock,

$$s_{ij}^{(1)*} = t_{ij}^{(1)*} - \delta_{ij} t_{mm}^{(1)*} / 3 = 2\mu n^{(1)} (e_{ij}^{(1)} - \delta_{ij} e_{mm}^{(1)} / 3) \quad (IV-70)$$

The stress-strain relations for the 'bulk' stresses were written as

$$t_{ij}^{(1)} = n^{(1)} \left[2\mu^{(1)} (e_{ij}^{(1)} - \frac{1}{3} e_{mm}^{(1)} \delta_{ij}) + \frac{1}{K^{(1)}} (e_{mm}^{(1)} - \frac{n^{(2)} - n_0^{(2)}}{1 - n_0^{(2)}} \delta_{ij}) \right] \quad (IV-71)$$

$$t_{ij}^{(2)} = \frac{n^{(2)}}{K^{(2)}} \left[e_{mm}^{(2)} + \frac{n - n_0^{(2)}}{1 - n_0^{(2)}} \right] \delta_{ij} + 2\mu^{(2)} n^{(2)} \left[D_{ij}^{(2)} - \frac{1}{3} v_{m,m}^{(2)} \delta_{ij} \right] \quad (IV-72)$$

where $\kappa^{(i)}$ is the compressibility of the constituent $s^{(i)}$. The isotropic pressures in the fluid and the solid were assumed to depend upon the volumetric strain of both the constituents, i.e.

$$t_{jj} = a e_{mm}^{(1)} + b e_{mm}^{(2)} \quad (\text{IV-73})$$

$$\pi = c e_{mm}^{(1)} + d e_{mm}^{(2)} \quad (\text{IV-74})$$

However, unlike Biot, the existence of an energy function was not postulated so that the constants b and c in (IV-73) and (IV-74) do not have to be equal. Morland expressed the coefficients in terms of compressibilities of the fluid and the solid, and the bulk shear modulus of the mixture.

Carroll and Katsube [36] postulated the following relations for a fluid-saturated solid:

i. Relation between stress in the mixture and stresses in the constituents:

$$\frac{1}{3} t_{jj} = \frac{1}{3} n^{(1)} t_{jj}^{(1)} + n^{(2)} \pi^* \quad (\text{IV-75})$$

ii. Solid stress-strain law:

$$\frac{1}{3} t_{jj}^{(1)} = -K_s \frac{\Delta V^{(1)}}{V^{(1)}} \quad (\text{IV-76})$$

where the symbol Δ indicates change in the quantity following it.

iii. Effective stress-strain law:

$$\frac{1}{3} t'_{jj} = \frac{1}{3} t_{jj} - \pi^* = -K \frac{\Delta n^{(2)}}{n^{(1)}} \quad (\text{IV-77})$$

Combining these relationships they obtained the bulk relations for the mixture as:

$$\frac{1}{3} t_{jj} - \left(1 - \frac{K}{K_s}\right) \pi^* = -K \frac{\Delta V}{V} \quad (\text{IV-78})$$

where

$$\frac{1}{K} = \frac{1}{K^*} + \frac{1}{n^{(1)}} K_s \quad (IV-79)$$

and

$$\frac{1}{3} t_{jj} - \left(1 - \frac{K^{(2)}}{K_s}\right) \pi^* = -K^{(2)} \frac{\Delta V^{(2)}}{V^{(2)}} \quad (IV-80)$$

where

$$\frac{1}{K^{(2)}} = \frac{1}{N^{(1)} K^*} + \frac{1}{n^{(1)}} K_s \quad (IV-81)$$

Here $K^{(2)}$ is the bulk modulus of the fluid.

4.3.3.2. Thermodynamic considerations.

Adkins [3] and Green [81] admitted interdependence of stress of each constituent upon the kinematics of all. This was in line with the principle of equipresence stated by Truesdell [189]. In application to elastic materials, the existence of an energy function for the mixture was assumed by Biot [15,18-20,23-25]. This has been consistently followed by numerous investigators [e.g. 7,8,32,45,51,81-88,177]. Sandhu [158, 159] pointed out that as the mixture could not be regarded as a continuum in motion, it was inappropriate to assume energy functions for it in the form that has been popular.

Morland [130-132] did not assume the existence of an energy function for the mixture but still admitted interdependence. This gave relations of the type postulated by Biot (e.g., equations (IV-45) and (IV-46) with M_{ij} in the second equation replaced by say $\bar{M}_{ij}$ not necessarily equal to M_{ij} in the first equation. This implies a nonsymmetric constitutive relationship of the type proposed earlier by Schiffman [165]. For $c = n^{(2)}$ the constitutive equations for $t_{ij}^{(1)}$ and π become uncoupled. Sandhu [156-159], Westmann [196] and Morland [130-132] have followed Adkins' [2] original idea that the stresses in each constituent depend upon the kinematics of only that constituent. However, Morland [130-132] used this for the intrinsic rather than the bulk stresses. This brings back some dependence of $t_{ij}^{(1)}$ upon the fluid pressure because the porosity was postulated to be a linear function of the partial stresses.

i. Entropy Production Inequality.

Mokadam [126,127] wrote the entropy production rate, per unit volume, of a multicomponent fluid as

$$\sigma = \frac{1}{T} (\pi \delta_{ij} - t_{ij}) v_{j,i} - \frac{1}{T^2} q_i T_{,i} - \frac{1}{T} m_i^{(k)} \mu_{,i}^{(k)} - \frac{1}{T} F_i v_i \quad (IV-82)$$

Here $m_i^{(k)}$ is the density of diffusive flow of the constituent $s^{(k)}$ and is given by

$$m_i^{(k)} = \rho^{(k)} u_i^{(k)} \quad (IV-83)$$

F_i is the total interaction force between the porous solid and the multicomponent fluid, and $\mu^{(k)}$ is the chemical potential of the constituent $s^{(k)}$. The multicomponent fluid and the porous solid were assumed to be at the same temperature. Irreversible changes occur in the system due to viscous flow, heat conduction, diffusion and movement of center of mass of the liquid phase in the presence of forces of friction. The first term on the right side of the equality (IV-82) is the scalar work of terms of second order while the last three terms are scalar works of tensors of the first order. Hence, there is no relation between the first and the other three terms. This argument of separability, of the first and the last three terms on the right side of (IV-82), was the basis for Mokadam's generalized d'Arcy flow law (II-5). Considerations of mass conservation and non-negative entropy production rates led to certain restrictions on the coefficients appearing in (II-5). (IV-82) was derived from Gibb's equation. Truesdell [190] objects to the use of this equation which essentially is the condition that the thermodynamic state be homogeneous. It is questionable to apply it to any problem involving departures from equilibrium.

Green and Naghdi [83,85] postulated an entropy production inequality for the entire continuum to define restrictions on the character of various constitutive relationships. Crochet and Naghdi [45] applied these considerations to the flow of a fluid through an

elastic solid. The starting point is the requirement that the Clausius-Duhem inequality be satisfied, i.e., [83]

$$\int_V \rho \left(\frac{DS}{Dt} - \frac{r}{T} \right) dV + \int_A \frac{q}{T} dA > 0 \quad (IV-84)$$

Assuming $q = h_j n_j$, (III-105), and $t_i = t_{ji} n_j$, (III-99), the above inequality upon use of the divergence theorem, gives

$$\int_V \left[\rho \left(\frac{DS}{Dt} - \frac{r}{T} \right) + \left(\frac{h_k}{T} \right)_{,k} \right] dV > 0 \quad (IV-85)$$

Here S is the entropy per unit mass and $T (>0)$ is the temperature of the mixture. Seeking constitutive equations for internal energy, entropy, heat flux vector, partial stresses and diffusive resistance vector, the point form of (IV-85) was written as, [84],

$$\rho \left(\frac{DS}{Dt} - \frac{r}{T} \right) + \frac{1}{T} h_{j,j} - \frac{1}{T^2} h_j T_{,j} > 0 \quad (IV-86)$$

Substituting for r from the energy equality (III-114), an alternative form of the entropy production inequality is,

$$\begin{aligned} \rho \left(T \frac{DS}{Dt} - \frac{DU}{Dt} \right) + \sum_{k=1}^n [t_{(ji)}^{(k)} d_{ji}^{(k)} + t_{[ji]}^{(k)} \Delta_{ji}^{(k)}] \\ + \sum_{k=1}^n p_i^{(k)} (v_i^{(k)} - v_i^{(k)}) \\ - \frac{1}{T} h_j T_{,j} > 0 \end{aligned} \quad (IV-87)$$

Shi [177], considering the binary mixture of a liquid and a solid, added the term

$$\frac{1}{\rho} \sum_{k=1}^2 (\rho^{(k)} u_j^{(k)} s^{(k)})_{,j} \quad (IV-88)$$

to the left hand side of (IV-86). Here $u_j^{(k)}$ is the velocity of the constituent $s^{(k)}$ relative to the barycentric velocity v_i of the mixture. This gave the local form

$$\frac{DS}{Dt} - \frac{r}{T} + \frac{1}{\rho T} h_{j,j} - \frac{1}{\rho T^2} h_j T_{,j} + \frac{1}{\rho} \sum_{k=1}^2 (\rho^{(k)} v_j^{(k)} s^{(k)})_{,j} > 0 \quad (IV-89)$$

Dunwoody and Müller [63] used the inequality

$$\begin{aligned} \int_V \left[\frac{\partial}{\partial t} (\rho S) - \sum_{k=1}^n \frac{1}{T^{(k)}} \rho^{(k)} r^{(k)} \right] dV \\ + \int_A \left[\rho S v_i + \sum_{k=1}^n \frac{1}{T^{(k)}} H_i^{(k)} \right] n_i dA > 0 \end{aligned} \quad (IV-90)$$

Here $H_i^{(k)}$ is a constitutive variable. For $H_i = h_i$ and $T^{(k)} = T$ for all k , the local form of (IV-90) is (IV-89). Truesdell [191] proposed, for the constituent $s^{(k)}$, the inequality

$$\begin{aligned} \int_V \sum_{k=1}^n \left[\frac{\partial}{\partial t} (\rho^{(k)} s^{(k)}) + c^{(k)} s^{(k)} - \frac{\rho^{(k)} r^{(k)}}{T^{(k)}} \right] dV \\ + \int_A \sum_{k=1}^n \left(\rho^{(k)} v_i^{(k)} s^{(k)} + \frac{1}{T^{(k)}} h_i \right) n_i dA > 0 \end{aligned} \quad (IV-91)$$

with the local form

$$\sum_{k=1}^n \left[\rho^{(k)} \frac{DS^{(k)}}{Dt} + \rho c^{(k)} s^{(k)} + \left(\frac{h_i}{T^{(k)}} \right)_{,i} - \rho^{(k)} \frac{r^{(k)}}{T^{(k)}} \right] > 0 \quad (IV-92)$$

Bowen [29] showed that (IV-92) and (IV-90) are the same if

$$\rho S = \sum_{k=1}^n \rho^{(k)} s^{(k)} \quad (IV-93)$$

and

$$H_i^{(k)} = h_i^{(k)} + \rho^{(k)} T^{(k)} s^{(k)} u_i^{(k)} \quad (IV-94)$$

Green and Naghdi [85] wrote the entropy flux balance equation for a fixed volume bounded by surface A as

$$\begin{aligned} \frac{\partial}{\partial t} \int_V \rho^{(k)} s^{(k)} dV - \int_V \left[\rho^{(k)} \frac{DS^{(k)}}{Dt} + \rho c^{(k)} s^{(k)} \right] dV \\ + \int_A (\rho^{(k)} s^{(k)} v_i^{(k)} n_i) dA = 0 \end{aligned} \quad (IV-95)$$

where $s^{(k)}$ is the entropy of the constituent $s^{(k)}$ per unit mass. The

entropy production inequality was assumed to be

$$\begin{aligned} \int_V \left[\rho^{(k)} \frac{DS^{(k)}}{Dt} + \rho c^{(k)} S^{(k)} \right] dV - \int_V \frac{1}{T^{(k)}} (\rho^{(k)} r^{(k)} + \phi^{(k)}) dV \\ + \int_A \frac{1}{T^{(k)}} (q^{(k)} + z^{(k)}) dA > 0 \end{aligned} \quad (IV-96)$$

where $\phi^{(k)}$, $z^{(k)}$ represent volume and surface interaction terms. The local form of the inequality, using results of [84], was shown to be

$$\begin{aligned} \rho^{(k)} T^{(k)} \frac{DS^{(k)}}{Dt} + \rho c^{(k)} T^{(k)} S^{(k)} - \rho^{(k)} r^{(k)} + h_{j,j}^{(k)} \\ - \frac{1}{T^{(k)}} h_j^{(k)} T_{,j}^{(k)} - \phi^{(k)} \\ + \frac{1}{T^{(k)}} g_j^{(k)} T_{,j}^{(k)} > 0 \end{aligned} \quad (IV-97)$$

where

$$q^{(k)} + z^{(k)} = n_j (n_j^{(k)} + g_j^{(k)}) \quad (IV-98)$$

Further, combining the inequalities for the two constituents in a binary mixture, (IV-91) was established. The entropy of the mixture was related to that of the constituents by the relation

$$\rho T \frac{DS}{Dt} = \sum_{k=1}^2 \left[\rho^{(k)} T^{(k)} \frac{DS^{(k)}}{Dt} + \rho c^{(k)} T^{(k)} S^{(k)} - \bar{\Phi}^{(k)} \right] \quad (IV-99)$$

where $\bar{\Phi}^{(k)}$ is related to $\phi^{(k)}$, $g_j^{(k)}$, $h_j^{(k)}$, $h_{j,j}$, and $\rho^{(k)}$. In [86], the Helmholtz free energy was introduced as

$$A = U - TS$$

Then

$$\dot{A} = \dot{U} - T\dot{S} - \dot{TS} \quad (IV-100)$$

(IV-100) was substituted in the energy equality to reduce the entropy inequality (IV-87), for a binary mixture, to the form

$$- \rho (\dot{A} + \dot{S}T) - p_i (v_i^{(1)} - v_i^{(2)}) + \sum_{k=1}^2 t_{ji}^{(k)} v_{i,j}^{(k)} - \frac{1}{T} h_j T_{,j} > 0 \quad (\text{IV-101})$$

ii. Constitutive Equations.

Seeking constitutive equations for internal energy, entropy, heat flux vector, partial stresses and diffusive resistance vectors p_i , θ_i and the quantity $q = h_j n_j$ in the case of mixture of two Newtonian fluids, Green and Naghdi [83] assumed these to depend upon the densities of the constituents, the velocities, the gradients of velocities, and the temperature. For the heat flux vector, the temperature gradient replaced the deformation gradients. For the case of fluid flow through an elastic solid, Crochet and Naghdi [45] started by admitting fairly general constitutive assumptions in line with Truesdell's [189] principle of equipresence, and then determined the restrictions placed upon these general constitutive relations by thermodynamic considerations. This approach is similar to that used by Noll [137] and Coleman and Mizel [44]. The following functional dependence was assumed.

$$U = U(T, F_{mn}^{(k)}, d_{mn}^{(k)}, v_m) \quad (\text{IV-102})$$

$$S = S(T, F_{mn}^{(k)}, d_{mn}^{(k)}, v_m) \quad (\text{IV-103})$$

$$t_{(ij)}^{(k)} = t_{(ij)}^{(k)}(T, F_{mn}^{(k)}, d_{mn}^{(k)}, v_m) \quad (\text{IV-104})$$

$$t_{[ij]} = t_{[ij]}(T, F_{mn}^{(k)}, d_{mn}^{(k)}, v_m) \quad (\text{IV-105})$$

$$p_i = p_i(T, F_{mn}^{(k)}, d_{mn}^{(k)}, v_m) \quad (\text{IV-106})$$

$$q_i = q_i(T, F_{mn}^{(k)}, d_{mn}^{(k)}, v_m, T_{,m}) \quad (\text{IV-107})$$

Here U , S , p_i , q_i are, respectively, the internal energy, the entropy, components of the diffusive resistance vector and the heat flux vector. This interaction force is equal and opposite for the two constituents in view of the symmetry of the total stress tensor for the mixture in the absence of body couples. T is the temperature field, $F_{mn}^{(k)}$ and $d_{mn}^{(k)}$ are components of the deformation gradient

and the deformation rate defined in section III, and v_i are components of relative velocity given by

$$v_i = v_i^{(2)} - v_i^{(1)} \quad (\text{IV-108})$$

Assuming U to be smoothly invertible in T , the variable T in (IV-102) to (IV-107) can be replaced by U . Crochet and Naghdi [45] showed that the restriction of nonnegative entropy production require that S and U be independent of the deformation rates, relative velocity and the temperature gradient. Thus

$$S = S(U, F_{ij}^{(k)}) \quad (\text{IV-109})$$

and as $U = U(T, F_{ij}^{(k)})$

$$T = T(U, F_{ij}^{(k)}) \quad (\text{IV-110})$$

Defining $t_{(ij)(0)}^{(k)}$, $t_{[ij](0)}^{(k)}$, $p_i(0)$ as values of $t_{(ij)}^{(k)}$, $t_{[ij]}^{(k)}$, and p_i for the equilibrium state of no relative motion and vanishing rates of deformation, i.e., $d_{ij}^{(k)} = 0$, $v_i = 0$, and assuming p_i continuous in v_i at $v_i = 0$ and $t_{(ij)}^{(k)}$, $t_{[ij]}^{(k)}$, p_i all continuous in $d_{ij}^{(k)}$ at $d_{ij}^{(k)} = 0$, Crochet and Naghdi [38] also established the following relationships:

$$t_{(ij)(0)}^{(k)} = \frac{1}{2} \rho F_{im}^{(k)} \left(\frac{\partial A}{\partial F_{jm}^{(k)}} + \frac{\partial A}{\partial F_{mj}^{(k)}} \right) \quad (\text{IV-111})$$

$$\begin{aligned} t_{[ij](0)} &= \frac{1}{2} \rho F_{im}^{(1)} \left(\frac{\partial A}{\partial F_{jm}^{(1)}} - \frac{\partial A}{\partial F_{mj}^{(1)}} \right) \\ &= -\frac{1}{2} \rho F_{im}^{(2)} \left(\frac{\partial A}{\partial F_{jm}^{(2)}} - \frac{\partial A}{\partial F_{mj}^{(2)}} \right) \end{aligned} \quad (\text{IV-112})$$

and

$$p_i(0) = \rho^{(1)} \frac{\partial A}{\partial F_{mn}^{(2)}} F_{mn,i}^{(2)} - \rho^{(2)} \frac{\partial A}{\partial F_{mn}^{(1)}} F_{mn,i}^{(1)} \quad (\text{IV-113})$$

where the Helmholtz free energy A is a function only of the temperature T and the deformation gradients $F_{ij}^{(k)}$ because U and S are dependent only upon T and $F_{ij}^{(k)}$. Equations (IV-110) thru (IV-112) are quite general and can be specialized to binary mixtures of various classes of materials.

4.3.3.3. Fluid flow through an elastic solid

Noting that the rate of deformation $d_{ij}^{(k)}$ is uniquely related to $F_{ij}^{(k)}$ and the spatial velocity gradient $L_{ij}^{(k)}$, and $L_{ij}^{(k)}$ can in turn be expressed in terms of $D_{ij}^{(k)}$ and $\Gamma_{ij}^{(k)}$, the variables $d_{ij}^{(k)}$ can be replaced by $D_{ij}^{(k)}$ and $\Gamma_{ij}^{(k)}$. Superposed rigid body angular velocity Ω would change $\Gamma_{ij}^{(k)}$ to $\Gamma_{ij}^{(k)} + \Omega$ leaving the other variables unchanged. Thus, for material objectivity to be satisfied, the vorticity tensors must occur as the difference $\Lambda_{ij} = \Gamma_{ij}^{(1)} - \Gamma_{ij}^{(2)}$.

For the fluid, the deformation can be represented by its density. For the solid, we can replace $F_{ij}^{(1)}$ by the strain measure, (III-27), e_{ij} as the independent variable in (IV-102) through (IV-107). In the light of the foregoing, and in view of the thermodynamic restrictions on the energy and the entropy functions, Crochet and Naghdi [45] reduced the general relationships admitted by (IV-102) to (IV-107) to the following form for application to fluid flow through a solid.

$$A = A(T, \rho^{(2)}, e_{ij}) \quad (IV-114)$$

$$t_{(ij)}^{(k)} = t_{(ij)}^{(k)}(T, \rho^{(2)}, e_{mn}, D_{mn}^{(1)}, D_{mn}^{(2)}, \Lambda_{mn}, v_m) \quad (IV-115)$$

$$p_i = p_i(T, \rho^{(2)}, e_{mn}, D_{mn}^{(1)}, D_{mn}^{(2)}, \Lambda_{mn}, v_m) \quad (IV-116)$$

$$t_{[ij]} = t_{[ij]}(T, \rho^{(2)}, e_{mn}, D_{mn}^{(1)}, D_{mn}^{(2)}, \Lambda_{mn}, v_m) \quad (IV-117)$$

$$q_i = q_i(T, \rho^{(2)}, e_{mn}, D_{mn}^{(1)}, D_{mn}^{(2)}, \Lambda_{mn}, v_m, T_{,m}) \quad (IV-118)$$

For the equilibrium state, i.e., vanishing $D_{ij}^{(k)}$, Λ_{ij} , w_i , $T_{,i}$.

keeping in view the restriction that for $\rho^{(1)} = 0$ the constitutive relation must reduce to that for $P_{ij}^{(2)}$ and for $\rho^{(2)} = 0$, $P_{ij}^{(1)}$ must reduce to the form for the solid only, Crochet and Naghdi [45] wrote

$$t_{(ij)}^{(1)} = \frac{1}{2} \rho (2e_{im} + \delta_{im}) \left(\frac{\partial A}{\partial e_{jm}} + \frac{\partial A}{\partial e_{mj}} \right) \quad (IV-119)$$

$$t_{(ij)}^{(2)} = - \rho \rho^{(2)} \frac{\partial A}{\partial \rho^{(2)}} \delta_{ij} \quad (IV-120)$$

$$P_{i(0)} = \rho^{(1)} \frac{\partial A}{\partial \rho^{(2)}} \rho_{,i}^{(2)} - \rho^{(2)} \frac{\partial A}{\partial e_{jm}} e_{jm,i} \quad (IV-121)$$

and

$$t_{[ij]}^{(0)} = 0 \quad (IV-122)$$

To allow for static equal and opposite stresses in the constituents of a binary mixture, it is only necessary, [86], to add a functional $F(\rho^{(k)}, T)(v_i^{(2)} - v_i^{(1)})_{,i}$ to the Helmholtz free energy. This will add $F(\rho^{(k)}, T)$ to $t_{(ij)}^{(2)}$ and subtract the same amount from $t_{(ij)}^{(1)}$. The diffusive resistance p_i is changed by $F_{,i}$. This additional term in the Helmholtz free energy is necessary only if equations for partial stresses and diffusive resistance are required. Kenyon [105] introduced, for a binary mixture of a liquid and a solid,

$$K_j = \sum_{k=1}^2 [h_i^{(k)} - T \lambda_i^{(k)} - \rho^{(k)} A^{(k)} u_i^{(k)}] \quad (IV-123)$$

where $h_i^{(k)}$, $\lambda_i^{(k)}$ are, respectively, the components of the heat flux and the entropy flux and $A^{(k)}$ is the Helmholtz free energy associated with the constituent $s^{(k)}$. Following Bowen [30], Kenyon assumed $h_i^{(k)} = T \lambda_i^{(k)}$ so that

$$K_j = \sum_{k=1}^2 \rho^{(k)} A^{(k)} u_i^{(k)} \quad (IV-124)$$

The additional term in the expression for stress was $\frac{\partial K_j}{\partial w_i}$ in $t_{ij}^{(1)}$ and an equal and opposite term in $t_{ij}^{(2)}$. It is easy to show that

$$\rho u_i^{(2)} = \rho (v_i^{(2)} - v_i) = \rho^{(1)} w_i \quad (IV-125)$$

$$\rho u_i^{(1)} = \rho (v_i^{(1)} - v_i) = \rho^{(2)} w_i \quad (\text{IV-126})$$

$$\text{and } \rho K_i = \rho^{(1)} \rho^{(2)} w_i (A^{(2)} - A^{(1)}) \quad (\text{IV-127})$$

Here $w_i = v_i^{(2)} - v_i^{(1)}$. Further, setting

$$A = \sum_{k=1}^2 \rho^{(k)} A^{(k)} \quad (\text{IV-128})$$

Kenyon [105] wrote

$$t_{ij(0)}^{(2)} = - \rho^{(2)} \left[\rho^{(2)} \frac{\partial A^{(2)}}{\partial \rho^{(2)}} + \rho^{(1)} \frac{\partial A^{(1)}}{\partial \rho^{(2)}} \right] \delta_{ij} \quad (\text{IV-129})$$

Here $t_{ij(0)}^{(2)}$ is the 'bulk area-averaged' fluid stress. Defining the intrinsic hydrostatic fluid stress π^* by the equation

$$n^{(2)} \pi^* = \rho^{(2)} \left[\rho^{(2)} \frac{\partial A^{(2)}}{\partial \rho^{(2)}} + \rho^{(1)} \frac{\partial A^{(1)}}{\partial \rho^{(2)}} \right] \quad (\text{IV-130})$$

Kenyon [105] wrote the chemical potential as

$$\mu = \frac{\pi^*}{\rho^{(2)}} + A^{(1)} \quad (\text{IV-131})$$

and the balance equation for the fluid as

$$\rho^{(2)} \mu_{,i} - \rho^{(2)} b_i^{(2)} = 0 \quad (\text{IV-132})$$

The stress in the solid

$$\begin{aligned} t_{ij}^{(1)} &= \text{contact stress} + n^{(1)} \pi^* \\ &= \rho^{(2)} \frac{\partial A^{(2)}}{\partial F_{ik}} F_{jk} + \rho^{(1)} \frac{\partial A^{(1)}}{\partial F_{ik}} F_{jk} - \frac{\pi^*}{\rho^{(2)}} \frac{\partial \rho^{(2)}}{\partial J} J \delta_{ij} \\ &= \rho^{(2)} \frac{\partial A^{(2)}}{\partial F_{ik}} F_{jk} + \rho^{(1)} \frac{\partial A^{(1)}}{\partial F_{ik}} F_{jk} - n^{(1)} \pi^* \delta_{ij} - \pi^* \kappa \delta_{ij} \end{aligned} \quad (\text{IV-133})$$

where

$$\kappa = J \sum_{k=1}^2 \left[\frac{n^{(k)}}{\rho^{(k)}} \frac{\partial \rho^{(k)}}{\partial J} \right] \quad (\text{IV-134})$$

For negligible adsorption, $A^{(1)}$, $\rho^{(2)*}$ are functions of the intrinsic fluid pressure π^* only. Hence

$$t_{ij}^{(1)} = \rho^{(1)} \frac{\partial A^{(1)}}{\partial F_{ik}} F_{jk} - n^{(1)} \pi^* \delta_{ij} - \pi^* \kappa \delta_{ij} \quad (IV-135)$$

For incompressible mixture $\kappa=0$ and the above equation simplifies to

$$t_{ij}^{(1)} = \rho^{(1)} \frac{\partial A^{(1)}}{\partial F_{ik}} F_{jk} - n^{(1)} \pi^* \delta_{ij} \quad (IV-136)$$

4.3.3.4. Linear theory

In the development so far, apart from the functional dependence of A , $t_{(ij)}^{(k)}$, p_i , $t_{[ij]}^{(k)}$, and q_i upon thermal and kinematic variables, there is no restriction as to the form of the relationship admissible. Thus, the relationship may be linear or nonlinear and the constituents may be rate-type materials or materials with memory. Smoothness and invertibility of U and S have been assumed as also the admissibility of e_{ij} as a measure of deformation. Saturated state, i.e., all points of the system being occupied by both the constituents is another assumption made throughout the theory of mixtures. The relationships for the linear theory can be directly obtained as a specialization.

For infinitesimal deformation, the strain measure for the solid constituent is, (III-26), $e_{ij} = u_{(i,j)}$. For the relationships for $t_{(ij)}^{(k)}$, $t_{[ij]}^{[k]}$, and p_i be linear, the Helmholtz free energy has to be quadratic in T and the measures of strain in the constituents. Thus for an initially isotropic solid, in the absence of body couples (i.e., symmetric total stress), the 'equilibrium' values of the mechanical variables are given by the following equations:

$$t_{(ij)}^{(1)}(0) = \lambda_1 e_{mm} \delta_{ij} + 2\mu_1 e_{ij} - \gamma \rho^{(2)} \delta_{ij} - \alpha T \delta_{ij} \quad (IV-137)$$

$$t_{(ij)}^{(2)}(0) = (-K \rho^{(2)} + \gamma e_{mm} - \rho T) \delta_{ij} \quad (IV-138)$$

$$p_i = 0 \quad (IV-139)$$

$$t_{[ij]}(0) = 0 \quad (IV-140)$$

where λ_1 , μ_1 , γ , and K are constants depending upon initial conditions and temperature. Symmetry of the constitutive relations for the stresses follows from the existence of the Helmholtz free energy function.

It may be noted here that the equations (IV-137) and (IV-138) are essentially the same as those postulated by Biot [15] using fluid strain as the additional kinematic parameter. The conceptual difficulties associated with Biot's theories are now obvious. Biot regarded the irreversible flow-deformation process as a series of equilibrium states, independent of all rate parameters. This would conflict with the constitutive equation of flow (d'Arcy's law) which relates the partial stress in the fluid to the relative motion rate of the solid and the fluid. In the theory proposed by Crochet and Naghdi [45], the mechanical component may depend upon all kinematic variables. For the linear theory this dependence becomes additive. The total values of these quantities would consist of an equilibrium and a non-equilibrium component. For the 'excess' stresses and other quantities in addition to the 'equilibrium' values, we would expect dependence only upon $D_{ij}^{(k)}$, Λ_{ij} , and v_j , which vanish at equilibrium. Thus, for isotropic materials,

$$t_{(ij)}^{(1)} = t_{(ij)}(0)^{(1)} + \lambda_3 D_{mm}^{(1)} \delta_{ij} + 2\mu_3 D_{ij}^{(1)} + \lambda_4 D_{mm}^{(2)} \delta_{ij} + 2\mu_4 D_{ij}^{(2)} \quad (IV-141)$$

$$t_{(ij)}^{(2)} = t_{(ij)}(0)^{(2)} + \lambda_5 D_{mm}^{(1)} \delta_{ij} + 2\mu_5 D_{ij}^{(1)} + \lambda_2 D_{mm}^{(2)} \delta_{ij} + 2\mu_2 D_{ij}^{(2)} \quad (IV-142)$$

$$p_i = c_3 w_i + c_4 e_{ijk} \Lambda_{jk} \quad (IV-143)$$

$$t_{[ij]} = c_1 \Lambda_{ij} + c_2 e_{ijk} w_k \quad (IV-144)$$

and

$$q_i = -KT_{,i} + c_5 v_i + c_6 e_{ijk} \Lambda_{jk} \quad (IV-145)$$

where e_{ijk} is the permutation symbol. For initially isotropic materials, c_2 , c_4 , c_6 vanish. To satisfy the Clausius-Duhem inequality for arbitrary velocities and velocity gradients, the following must hold [45]:

$$c_1, c_3, K, \mu_2, \mu_3 \text{ must be non-negative} \quad (\text{IV-146})$$

$$\lambda_2 + \frac{2}{3} \mu_2 > 0 \quad (\text{IV-147})$$

$$(\lambda_3 + \frac{2}{3} \mu_3) > 0 \quad (\text{IV-148})$$

$$(\mu_4 + \mu_5)^2 < 4 \mu_2 \mu_3 \quad (\text{IV-149})$$

$$c_5^2 < 4(KT_0)c_3 \quad (\text{IV-150})$$

and

$$[\lambda_4 + \lambda_5 + \frac{2}{3} (\mu_4 + \mu_5)]^2 < 4 (\lambda_2 + \frac{2}{3} \mu_2) (\lambda_3 + \frac{2}{3} \mu_3) \quad (\text{IV-151})$$

Combining the equation for resistance to diffusion, (IV-143), with the definition of diffusive resistance, (III-116), and ignoring chemical action, we obtain

$$c_3 w_i = \rho^{(2)} (f_i^{(2)} - b_i^{(2)}) - t_{ji,j}^{(2)} \quad (\text{IV-152})$$

Neglecting inertia effects and assuming $v_i^{(1)} \ll v_i^{(2)}$, the above equation reduces to

$$c_3 v_i^{(2)} = \rho^{(2)} b_i^{(2)} + t_{ji,j}^{(2)} \quad (\text{IV-153})$$

or,

$$v_i^{(2)} = (c_3)^{-1} (\rho^{(2)} b_i^{(2)} + t_{ji,j}^{(2)}) \quad (\text{IV-154})$$

which is d'Arcy's equation of fluid flow. In (IV-152), ignoring body forces $b_i^{(2)}$ and writing $f_i^{(2)}$ as $\ddot{u}_i^{(2)}$,

$$t_{ji,j}^{(2)} = \rho^{(2)} \ddot{u}_i^{(2)} - c_3 w_i \quad (\text{IV-155})$$

This equation, for isotropic fluid pressure, has the same forms as Garg's [71] statement of Biot's equation of momentum balance, allowing for interaction.

Green and Naghdi [87] extended Crochet and Naghdi's [45] work to anisotropic solids and to include initial stresses. In the absence of chemical action and for uniform temperature, the Clausius-Duhem inequality to be satisfied is, for a binary mixture,

$$\sum_{k=1}^2 [t_{(ij)}^{(k)} D_{ij}^{(k)} + t_{[ij]}^{(k)} \Delta_{ij}^{(k)} + p_j^{(k)} w_j - \theta_{,j} \rho^{(k)} S_j^{(k)} (v_i^{(k)} - v_i)] - \frac{1}{T} h_{j\theta,j} > 0 \quad (\text{IV-156})$$

where θ is the change in temperature.

The explicit form of the dependence of stress in the solid or the fluid upon the strains, strain rates and history of deformation will not be discussed here. For illustration, the materials have been assumed to be linear elastic. However, these relations may be nonlinear elastic, viscous, viscoelastic, hypoelastic or elasto-viscoplastic. A large volume of literature on the subject exists. As examples of recent work on a variety of models we cite Desai [53-55], Prevost [149], Digby [59], Dragson [60], Christofferson [40], Coon [41], and Mullenger [135]. In developing plasticity theories for saturated porous media, Carroll [35] has pointed out the difficulty in defining plastic strains. Shi [177] derived the explicit form of stress-strain relations following Crochet [45] and Green [87]. We note, however, that Shi assumed the Helmholtz free energy for the mixture to be the sum of Helmholtz free energies of the constituents. This would imply uncoupled constitutive equations for the 'equilibrium' stresses as contended by Sandhu [156-158]. However, this assumption could lead to other difficulties because the derivation of balance laws in Green and Naghdi's theory was based upon a different set of assumptions.

4.4. CONSTITUTIVE EQUATIONS FOR POROSITY.

Gibson [77] treated porosity as a function of effective stress

and proposed compliance relationships in the form

$$n^{(2)} = f(t_{ij}, x_j, t) \quad (\text{IV-157})$$

Walsh [195] regarded the pore space or the volume fraction of the pores to be a function of the solid stress. Thus, as implied in (III-39),

$$v^{(2)} = n^{(2)} v \quad (\text{IV-158})$$

Consequently,

$$\frac{dv^{(2)}}{dt_{jj}^{(1)}} = v \frac{dn^{(2)}}{dt_{jj}^{(1)}} + n^{(2)} \frac{dv}{dt_{jj}^{(1)}} \quad (\text{IV-159})$$

Hence

$$\frac{dn^{(2)}}{dt_{jj}^{(1)}} = \frac{1}{v} \frac{dv^{(2)}}{dt_{jj}^{(1)}} - \frac{n^{(2)}}{v} \frac{dv}{dt_{jj}^{(1)}} = \frac{1}{v} \frac{dv^{(2)}}{dt_{jj}^{(1)}} - \frac{v^{(2)}}{v^2} \frac{dv}{dt_{jj}^{(1)}} \quad (\text{IV-160})$$

For $v^{(2)} \ll v^2$,

$$\frac{dn^{(2)}}{dt_{jj}^{(1)}} = \frac{1}{v_0} \frac{dv^{(2)}}{dt_{jj}^{(1)}} \quad (\text{IV-161})$$

Garg and Nur [70] pointed out that Walsh's analysis was acceptable for very dense rock i.e., $n^{(1)}$ nearly equal to 1. For the general case, they introduced bulk and intrinsic solid compressibilities and set up more general expressions. They proposed constitutive equations for porosity based on the existence of an energy function.

Aifantis [4] assumed effects of changes in fluid pressure and the solid stress to be additive and proposed a compliance relationship

$$\Delta n^{(2)} = a \Delta \pi + b \Delta t_{kk}^{(1)} \quad (\text{IV-162})$$

Morland [130] introduced scaling functions for density and stress. Hence constitutive equations would be needed for these. These factors could, in general, depend upon the deformation history of all the constituents and thereby define an interdependence between the partial constitutive relations. However, a simplifying assumption was made that the area factors are uniquely related to volume scaling

factors i.e. volume fractions. Then only the constitutive equations for volume fractions are needed. Further assuming that the relationship is linear and the same for all constituents, Morland set the area fraction equal to the volume fraction, a result postulated directly by Biot [21], Morland [130] assumed the porosity $n^{(2)}$ to be a function of the current deformations of the constituents. Assuming a linear relationship (i.e. using only the first invariants of the infinitesimal strain tensor)

$$n^{(2)} = n_0^{(2)} (1 + c e_{jj}^{(1)} + d e_{jj}^{(2)}) \quad (\text{IV-163})$$

where c, d are of order unity or less. Equivalently, the relationship could be stated in terms of partial stresses instead of strains as

$$n^{(2)} = n_0^{(2)} (1 + a t_{jj}^{(1)} + b \pi) \quad (\text{IV-164})$$

To include dilatancy, the relationship was generalized further to

$$n^{(2)} = n_0^{(2)} (1 + a_{ij} t_{ij}^{(1)} + b \pi) \quad (\text{IV-165})$$

Of course, temperature dependence of porosity can be included. Also, to allow for the influence of shearing stresses in the solid on porosity, Morland [131] included the second invariant of the solid stress in the equation.

Noting the dependence of t_{jj}, π upon $e_{jj}^{(k)}$ Morland [110], following Garg and Nur [70], proposed a constitutive equation for the ratio of $n^{(1)}$ to $n_0^{(1)}$ as an expansion in $e_{jj}^{(1)}$ and $e_{jj}^{(2)}$. Garg and Nur [70] assumed the existence of an energy function in terms of porosity and the volumetric stress defined by (III-31) to get symmetric expressions for hydrostatic stresses in the fluid and the solid. Morland [133] extended this concept to cyclic loading using a hypoelastic model for the deviatoric response. The change in porosity was expected to have the form [133]

$$n^{(2)} = H(n^{(2)}, K) = H_1(n^{(2)}) H_2(K) > 0 \quad (\text{IV-166})$$

Morland [133] suggested a polynomial representation for H_2 . He also described experiments to determine the coefficients in (IV-164) and (IV-165).

4.5. CONSTITUTIVE EQUATIONS FOR SPECIFIC VOLUME.

4.5.1. Kenyon's Theory.

Kenyon [105] considered the fluid density to be a function of the intrinsic fluid pressure and the bulk volume measure J , (III-10). Thus

$$n^{(2)} = 1 - n^{(1)} = \frac{\rho^{(2)}}{\rho^{(2)*}} = n^{(2)}(\rho^{(2)*}, J) \quad (\text{IV-167})$$

would be a constitutive equation for $n^{(2)}$. For an incompressible mixture $n^{(2)}$ "must depend upon J as fluid must replace solid when the latter changes its macroscopic volume. In addition, at constant J , $n^{(2)}$ will generally depend on the amount of fluid present since fluid can be used to compress the cells or granules independent of any bulk deformation of the solid". In choosing the functional form for (IV-167), Kenyon assumed that the bulk strain deviation will not change pore volume i.e. only the bulk volume change of the solid represented by J can cause microscopic volume changes.

Kenyon [106] regarded velocity of the fluid to be a function of the fluid pressure and the velocity of the solid a function of the fluid pressure and the solid stress i.e.,

$$v_i^{(2)} = v_i^{(2)}(\pi^*) \quad (\text{IV-168})$$

and

$$v_i^{(1)} = v_i^{(1)}(\pi^*, t_{jj}^{(1)}) \quad (\text{IV-169})$$

Then, differentiation leads to the incremental relationships

$$\dot{v}_i^{(2)} = \pi \frac{\partial}{\partial \pi} v_i^{(2)} \quad (\text{IV-170})$$

and

$$\dot{v}_i^{(1)} = \pi \frac{\partial}{\partial \pi} v_i^{(1)} + \dot{t}_{jm}^{(1)} \frac{\partial}{\partial t_{jm}^{(1)}} v_i^{(1)} \quad (\text{IV-171})$$

Kenyon proposed using these incremental equations to identify the functional relationships (IV-168) and (IV-169) by experimental observations.

4.6. COMMENTS.

For fluid-saturated soils, identification of quantities for which constitutive equations are required along with various approaches to development of constitutive relationships has been reviewed. In addition to the partial stresses in solid and the fluid, constitutive relations have been proposed for diffusive resistance and for porosity. Micromechanical approaches have been concerned with relating the macroscopic behaviour of a saturated porous solid to the intrinsic behaviour of the solid matrix and the fluid. Thermodynamic considerations based on the Clausius-Duhem inequality and the existence of a free-energy function provide certain guidelines. However, when writing the specific form of the Helmholtz free energy function for a mixture, it is important to keep in view (III-121) which expresses the relationships between the internal energies of the constituents and the internal energy of the mixture.

It is clear from (III-42) that for a complete description of the kinematics, if the solid grains are compressible, it is necessary to introduce an additional quantity. This has led to the constitutive equations for porosity.

The difficulty with the constitutive modelling of saturated soils lies in design of procedures for determination of the actual form of the constitutive relationships. The mechanical quantities e.g. effective and partial stresses, diffusive resistance etc. need to be carefully defined. Morland's work appears to offer the most promising approach in this direction.

SECTION V

DISCUSSION

Theories of mechanical behaviour of saturated soils, including various approaches to description of motion, derivation of balance laws and development of constitutive relationships have been reviewed. These include early work on soil consolidation and wave propagation in saturated soils as well as modern approaches based on theories of mixtures.

Early work on quasi-static behaviour of saturated soils was based largely upon phenomenological observations, 'reasonable' extensions and generalizations and certain ad hoc assumptions. More recent work has examined the thermodynamic basis of the mechanics of mixtures.

Linear dependence between nominally one-dimensional waterflow and the fluid pressure gradient observed by d'Arcy was generalized to three-dimensional flow, to thermoelastic flow and to multicomponent flow in presence of thermal and chemical potential gradients. It was shown to be a special case of the more general Onsager relations between fluxes and Prigogine forces. Similar constitutive formulations were proposed considering relative motion in the case of a moving/deforming solid matrix.

Terzaghi introduced the notion of effective stress. The dual description he used quite appropriate to his problem has been the source of considerable confusion. He defined the effective stress as the stress related to soil deformation and also as the partial stress in soil. He assumed additivity of the partial stresses and considered the fluid pressure to be acting over 100 percent of the area of the soil. This assumption, combined with defining the pore pressure measured by a piezometer as the fluid pressure, in the case of zero fluid velocity leads to the soil being 'buoyant' in water with a reduced density and corresponding effective stress.

Biot [15] while developing the equations of three-dimensional consolidation, assumed the irreversible process of soil consolidation to be a sequence of reversible processes and assumed existence of an

energy function in terms of the soil strains and the fluid content or the change in fluid density. Existence of a dissipation function was assumed as well as the additivity of partial stresses. Gibson's theory [77] of one-dimensional consolidation considered a reference volume of the soil and fluid flow relative to it. This essentially amounts to use of convected coordinates for the solid and a 'referential description' of fluid kinematics.

For wave propagation in soils, two approaches have been used. The 'Engineering Approach' introduced by Seed was initially based upon the total stress analysis using methods of structural dynamics assuming the saturated soil to be a single material i.e. relating the total stress to kinematics of the soil skeleton. Later development used effective stress vs. soil strain. Change in fluid pressure was assumed to be related to volume changes of the soil. Fluid inertia was ignored and fluid velocity relative to the soil was part of the process of 'dissipation' of 'excess' fluid pressure built up due to soil volume changes. The procedure did not consider simultaneous coupled motion of the soil and the pore fluid. The theory apparently was successful in predicting liquefaction in layered soil systems subjected to shear excitations applied to the base as in seismic event. However, the approach cannot predict post-liquefaction distribution of pore-water pressures, does not give reliable acceleration and displacement response histories, cannot be extended to two- and three-dimensional cases and its use requires considerable experience and 'judgment' in addition to extensive laboratory testing. For these reasons, the theory is not likely to be useful for application to study of soil systems under blast loading.

Biot's formulation of the coupled problem was the basis of theoretical solutions by Biot [21,22], Deresiewicz [52], Garg [69] and Chakraborty [38] among others. Ghaboussi [73], Prevost [147] and Sandhu [157] implemented this theory in finite element computer programs. In this theory, in addition to the ad hoc assumptions on which Biot had set up the constitutive relationships, he assumed the kinetic energy to be a quadratic function in the velocities of the fluid and the solid including a coupling term rather than simply the

sum of the kinetic energies of the constituents. A dissipation function, quadratic in relative velocity was introduced. Garg's formulation of the dynamic problem is identical to Biot's. However, he assumed the kinetic energies of the constituents to be additive and related the coefficients appearing in the equations of the motion to be properties of the constituents and the volume fractions. While developing finite element solution procedures to Biot's equations, Ghaboussi [73] introduced an arbitrary Rayleigh's type viscous damping term to reflect the intrinsic damping in the saturated soil in addition to the dissipation of relative motion based on permeability and porosity.

Theories of multicomponent mixtures and their applications to dynamics of saturated soils were reviewed with a view to evaluate the 'reasonableness' of the various assumptions by reference to fundamental principles of mechanics. Several approaches have been used to describe the motion of constituents in a multicomponent mixture. One is to refer the motion of each component to its reference configuration. Another is to refer the motion of all constituents to the reference configuration of one of them. Yet another is to refer all motion to the current configuration which is same to all constituents. Superposition of relative diffusive motion of the constituents upon the mean motion of the mixture as a whole is also used. For a binary mixture, most investigators, following Biot, describe the motion of the solid with respect of its reference configuration but the motion of fluid is described as relative to the solid. Another procedure, used by Gibson, is to refer to a material region in the reference configuration consisting of a certain set of particles of one of the constituents so that the bounding surface of this set varies with time. This essentially represents the use of convected coordinate description for the solid in a binary fluid-solid mixture. In most theories the deformation of each constituent is referred to an initial reference configuration but the motion is referred to the place coordinates i.e. the current configuration. For granular porous media, some investigators regard the total deformation as made up of two parts; one related to deformation of the solid particles and the other to their rearrangement i.e. change in pore

geometry. These approaches are of assistance in understanding the relationships between the 'bulk' and 'intrinsic' behavior of constituents. Volume fraction theories for compressible materials introduce the volume fractions as additional variables.

The notion of a mixture as a whole is often introduced. For the case of no relative motion between the constituents, the mixture will have motion and deformation as a single body. However, if relative motion is present, the mixture does not satisfy the axioms of continuity and its corollary, the principle of impenetrability. Therefore, the mixture cannot be regarded as a continuum in motion, the particles of which are assumed to be constructed by superposition of constituent particles, except in the case of no relative motion between the constituents.

Atkin [8] pointed out that the mixture having a mass density calculated as the weighted average of the constituents cannot be associated with a material in the physical sense. The mixture so defined consists only of centers of mass and not a set of particles. It should be kept in mind that the mixture density is purely mathematical entity having no physical interpretation except in the special case of no relative motion. Many investigators introduce a material rate for the mixture. This rate again has no physical meaning being executed not on a particle but a center of mass having only instantaneous existence.

Truesdell postulated equations of balance of mass, linear momentum, moment of momentum and energy for each constituent and required that the form of the equations be the same for each constituent and for the mixture. Apparently, the existence of the mixture as a continuum in motion was implied in this line of thought. To accomplish this identity of form, the total stress tensor, the total heat flux vector and the specific energy supply had to be specially defined and did not equal the sum of the corresponding quantities for the constituents as shown in (III-57), (III-59) and (III-61). The energy of the mixture, however, equalled the sum of the energies of the constituents (III-55). The additivity of stresses, heat fluxes and specific energy supply could be restated using special

definitions for these quantities (III-58), (III-60), (III-62). For component velocities relative to the mean velocity of the mixture approaching zero, the stresses, heat fluxes and the specific energy supply approach additivity. On this basis, Bowen [29] regarded Green's theory to be a special case of Truesdell's. This is incorrect. As the mixture cannot be regarded as a continuum in motion, Truesdell's definitions of the total stress, heat flux and specific energy supply are artificial and unnecessary.

Green [83] postulated the additive property of stress, heat flux and the energy supply. Morland [130] supported the additivity of partial stresses on the ground that tractions are additive and Cauchy's stress principle should hold for total stress and total traction as well as for the constituents. Williams [197] presented similar arguments. The sum of internal energy densities of the constituents did not equal the internal energy density of the mixture in this theory (III-121).

Green derived the balance laws from the invariance of a rate of energy equality. The form of the balance equation is the same as in Truesdell's theory but the terms appearing in these equations have quite different meaning.

Some investigators, considering the special problem of the flow through deformable porous solids, have attempted to write the balance equations in terms of relative motion and porosity, which is essentially is a measure of relative deformation. It would appear that a theory based upon balance equations written for a reference set of particles of the porous soil but in the current configuration would be the most appropriate to the problem. For the one-dimensional quasi-static case this would reduce to Gibson's theory [77].

Fukuo [68] used the equations of mass balance to set up equations in terms of volume fractions of the constituents and specialized these for incompressible solid skeleton and also incompressible fluid. This, based on use of d'Arcy's law in conjunction with reference to a fixed set of particles in reference configuration, may be regarded as a generalization of Gibson's equations. Hsieh and Yew [95] set up a relation between rate of porosity change, rate of volumetric strain

and the relative velocity for a porous solid saturated with an incompressible fluid and undergoing small deformations in absence of thermal and chemical effects. Then these quantities, in the case considered, cannot be treated as independently variable. Goodman admitted the volume fraction as an additional quantity in the equation of energy balance for a porous material. Rate of work terms were associated with the rate of change of porosity over volumes and surfaces using generalized forces.

In setting up constitutive relationships, in addition to the components of total stress on saturated soil, the existence of fluid pressure and the interaction force between components has been long recognized. Skempton [179] eliminated the need for a constitutive equation for the fluid pressure by assuming it to be a function of the stress components or the invariants. Later an explicit dependence upon soil strains was empirically introduced. Biot assumed the existence of an energy function in terms of the soil strains and fluid content/change in density. In theories of mixtures, the quantities for which constitutive relations are required are determined on the basis of *number of variables* vs. the number of equations of kinematics and balance or the quantities appearing in the rate of energy equality or entropy inequality which are not defined by kinematics. The equation of mass continuity for soils with compressible grains involve porosity. This has been the motivation for setting up constitutive equations for porosity. The existence of Helmholtz free energy function has been often assumed. However, if the energies of the constituents be considered additive, there would be no coupling between their constitutive relations. This is contrary to Biot's assumptions and is in line with Adkins' earlier view [1,2] and with Morland [130]. However, if Green's theory be followed, (III-121) would indicate the type of coupling. Crochet and Naghdi's theory [45] would have to be interpreted in that light. This does appear to lead to Biot's equations. Morland's theory [130] admitting coupling between constitutive equations for partial stresses in the solid and the fluid without assuming the existence of energy function for the mixture appears to be worth further examination.

SECTION VI
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