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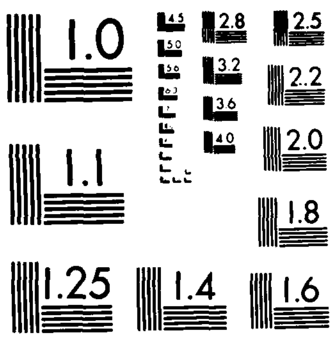

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A REVIEW OF THE PACKING PROBLEM

BY

HERBERT SOLOMON and HOWARD WEINER

TECHNICAL REPORT NO. 383
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Key Words and Phrases: random packing, parking problem, Palasti conjecture, applications.

ABSTRACT

The random packing problem has been of interest to investigators in several disciplines. Physical chemists have investigated such models in two and three dimensions. Because of analytical difficulties, one-dimensional analogues have been explored and these are referred to as the parking problem. A number of results are explored and attempts are made to tie them together. Applications are also highlighted.

INTRODUCTION

This paper is a selected yet comprehensive review of asymptotic methods and results for sequential random packing in one and higher dimensions. In one dimension, it is known as the parking problem. Section I contains theoretical methods for the parking problem developed by Renyi (1958) and another approach by Dvoretzky and Robbins (1964). This is followed by a generalization by Solomon (1967). We also consider simulation and computational methods for obtaining the asymptotic parking constant (mean packing density in one dimension) $C = .74759$ (to five decimal places) developed by some authors and demonstrate its value to 15 decimal places. Higher moments and a central limit theorem, and one-dimensional extensions with non-uniform length cars, are discussed. Section II considers the two-dimensional Renyi case, for which most simulations indicate that the Palasti conjecture, namely the asymptotic parking constant is C^2, is false. The Palasti conjecture in general asserts that mean packing density in n dimensional sequential random space filling is the n^{th} power of mean packing density in one dimension. Higher dimensional simulations indicate a further departure than in two dimensions from the corresponding Palasti conjecture. Section III gives a critical review of two published theoretical approaches to the two-dimensional situation. Comments on possible directions for research on the two-dimensional case are given. Some applications are discussed throughout the paper and featured along with miscellaneous remarks in Section IV.

I. One Dimension

In his pioneering paper, Renyi (1958) mentions a three-dimensional random sphere packing problem posed by the physical chemist, J. D. Bernal. It was thought that the mean random packing density of molecules (spheres) for the inert gases could serve as a classification index for these elements. Since three-dimensional mean packing was mathematically intractable, Renyi decided to explore a simpler situation; namely the random packing (parking) mean in one dimension. The Renyi one-dimensional parking problem is as follows. On the (parking) interval $[0,x]$, $x > 1$, a unit-length segment (car) is placed (parked) with its left end uniformly distributed on $(0,x-1)$. Independent of the first car, a second unit length car is placed uniformly on the interval $(0,x)$. If the second car does not overlap the parked car, then it is considered parked. If it does overlap the first car, then it is discarded. The process continues until no new car may be parked, that is, a unit length is not vacant.

In the mid 1950's, the first author while at Columbia University heard this same problem posed by C. Derman and M. Klein who were colleagues there. They also provided some initial development. The motivation was some consternation by Derman and Klein over the car parking problem in the streets of New York City. Thus on two sides of the Atlantic Ocean at approximately the same time but without any knowledge of each other's efforts and from two different motivations, we have work beginning on the parking problem. Derman and Klein who did not publish on this topic told Robbins, also at Columbia, of their model and he with Dvoretzky, who frequently visited Columbia in the 1950's, developed an approach that appeared later, Dvoretzky and Robbins (1964).

We now present the Renyi (1958) and Dvoretzky and Robbins approaches and results.

Let

(1.1) $N(x)$ = total number of cars of unit length that may be parked on a parking interval on length x.

Let

$$M(x) = E(N(X)). \tag{1.2}$$

The Renyi parking problem in one dimension is to show that

$$\lim_{x\to\infty} M(x)/x = C \tag{1.3}$$

and to evaluate C explicitly. To do this, the original approach of Renyi is now given. The first car is parked at $(t,t+1)$ on the parking interval $(0,x+1)$. The mean number of cars parked to the left of the first car is $M(t)$, and the mean number to the right of $(t,t+1)$ is $M(x-t)$. Since the coordinate t is drawn from a random variable which is uniform on $(0,x)$, it follows that

$$M(x+1) = \frac{1}{x}\int_0^x (M(t) + M(x-t)dt + 1 \tag{1.4}$$

or

$$(1.4a) \qquad M(x+1) = 1 + \frac{2}{x}\int_0^x M(u)du$$

with

$$M(x) = 0, \quad 0 < x < 1$$

$$M(x) = 1, \quad 1 \le x \le 2.$$

Renyi proceeds as follows. Multiplying (1.4a) by x, and taking derivatives, we get

$$(1.5) \qquad xM'(x+1) + M(x+1) = 2M(x) + 1.$$

Let the Laplace transform for $M(x)$ be

$$(1.6) \qquad \phi(x) = \int_0^\infty e^{-sx}M(x)dx,$$

$$s = \sigma + it, \ \sigma > 0$$

That the Laplace transform exists follows from $0 \le M(x) \le x$. Multiplying both sides of (1.5) by e^{-sx} and integrating with respect to x, one obtains, using $M(x) = 0, \ 0 \le x \le 1$,

$$(1.7a) \qquad \int_0^\infty M(x+1)e^{-st}dx = e^s\int_1^\infty M(u)e^{-us}du = e^s\phi(x)$$

$$(1.7b) \qquad \int_0^\infty xM'(x+1)e^{-sx}dx = -\frac{d}{ds}\left[e^s\int_1^\infty M'(u)e^{-us}du\right],$$

and an integration by parts yields, using $0 \le M(x) \le x$,

$$(1.7c) \qquad \int_1^\infty M'(u)e^{-us}du = s\phi(s) .$$

Hence (1.5), (1.7a-c) yield

$$(1.8) \qquad -(se^s\phi(s))' + e^s\phi(s) = 2\phi(s) + \frac{1}{s} .$$

To solve (1.8) for $\phi(s)$, let $w(s) = e^s\phi(s)$, then $w(s)$ satisfies

$$sw'(s) = -2w(s)e^{-s} - \frac{1}{s} . \tag{1.9}$$

Since, for $s > 0$,

$$0 \le w(s) \le e^s \int_1^\infty xe^{-sx}dx , \tag{1.10}$$

it follows that

$$\lim_{s\to\infty} w(s) = 0 , \tag{1.11}$$

hence the first-order differential equation (1.9) may be solved by the method of variation of parameters to yield

$$w(s) = \frac{1}{s^2}\int_s^\infty \exp[-s \int_s^t \frac{1-e^{-u}}{u} du]dt , \tag{1.12}$$

so that

$$\phi(s) = \frac{e^{-s}}{s^2}\int_s^\infty \exp[-s \int_s^t \frac{1-e^{-u}}{u} du]dt . \tag{1.13}$$

Hence

$$\lim_{s\downarrow 0} s^2\phi(s) = \int_0^\infty \exp[-2 \int_0^t (\frac{1-e^{-u}}{u})du]dt = C \simeq .748 . \tag{1.14}$$

Since $\alpha(x) = \int_0^t M(u)du$ is monotone increasing and since

$$s^2 \int_0^\infty e^{-sx}d\alpha(x) = s^2\phi(s) , \tag{1.14a}$$

by a Tauberian theorem, and (1.14),

$$\lim_{x\to\infty} \frac{1}{x^2}\int_0^x M(u)du = \frac{C}{2} . \tag{1.14b}$$

Dividing (1.4) by $x+1$ and letting $x \to \infty$, one obtains (1.3).

Renyi next observes that the linear function

$$\ell(\alpha,x) = \alpha x + \alpha - 1 \tag{1.15}$$

satisfies (1.4) identically for any value of the constant α. It is then shown that

$$\lim_{x\to\infty} (M(x)-Cx) = C-1 . \tag{1.16}$$

The second moment, $M_2(x) = E(N^2(x))$ is then approximated. If $0 \leq t \leq x$ is a value of a uniform random variable on $(0,x)$, then by conditioning on t, one may write

$$N(x+1) = N_a(t) + N_b(x-t) + 1 , \tag{1.17}$$

where $N_a(t)$, $N_b(x-t)$ are independent and $N_a(t)$ is distributed as $N(t)$ and $N_b(x-t)$ is distributed as $N(x-t)$, respectively. From this we obtain the equation

$$M_2(x+1) = \frac{1}{x}\int_0^x (1+M_2(t) + M_2(x-t) + 2M(t) + 2M(x-t) + 2M(t)M(x-t))dt , \tag{1.18}$$

so that

$$M_2(x+1) = 1 + \frac{4}{x}\int_0^x M(t)dt + \frac{2}{x}\int_0^t M(t)M(x-t)dt + \frac{2}{x}\int_0^t M_2(t)dt . \tag{1.19}$$

Using a strengthened version of (1.16), namely

$$M(x) - Cx + C-1 = 0(x^{-n}) \quad \text{all} \quad n \geq 1 , \tag{1.20}$$

it is then inferred that

$$(1.21) \qquad \mathrm{Var}(N(x)) = O(x) .$$

The result of (1.21) is sharpened by Dvoretsky and Robbins (1964) and they obtain a central limit theorem for the packing random variable in two ways. This is described next. For $k = 1,2,3,\ldots$, let

$$(1.22) \qquad \phi_k(x) = E(N(x) - \ell(C,x))^k .$$

Using the conditional independence argument of Renyi, and the fact that $\ell(C,x+1) = \ell(C,t) + \ell(C,x-t) + 1$ for $0 \le t \le x$, it follows that

$$(1.23) \qquad \phi_k(x+1) = \frac{1}{x} \sum_{\ell=0}^{k} \binom{k}{\ell} \int_0^x \phi_\ell(t)\phi_{k-\ell}(x-t)dt .$$

From (1.23) and some asymptotic results for integral equations, they are able to show that for all $x > 0$,

$$(1.24) \qquad \inf_{x \le t \le x+1} \frac{M(t)+1}{t+1} \le C \le \sup_{x \le t \le x+1} \frac{M(t)+1}{t+1} ,$$

a result used later by Blaisdell and Solomon (1970) to obtain estimates of C to many decimal places. Dvoretsky and Robbins also show by induction from (1.23) that for all $0 < \varepsilon < 1$,

$$(1.25) \qquad \phi_k(x) = C_k x^{[k/2]} + O(x^{[k/2]-1+\varepsilon})$$

where

$$(1.26) \qquad C_{2k} = \frac{(2k)!}{2^k k!} \lambda_2^k$$

where

$$(1.27) \qquad 0 < \lambda_2 = \lim_{x\to\infty} \frac{\mathrm{Var}(N(x))}{x} < \infty .$$

From this, it is shown that, if $\varepsilon = \frac{1}{2}$, that

$$(1.28)\qquad E\left(\frac{N(x)-M(x)}{\sqrt{Var(N(x))}}\right)^k = \frac{C_k x^{[k/2]}+o(x^{[k/2]})}{(\lambda_2 x+o(x))^{k/2}} .$$

hence that

$$(1.29)\qquad \lim_{x\to\infty} E\left(\frac{N(x)-M(x)}{\sqrt{Var(N(x))}}\right)^k = \begin{cases} \frac{k!}{2^{k/2}(\frac{k}{2})!}, & k \text{ even} \\ 0 & , k \text{ odd} . \end{cases}$$

Since these are the moments of the normal distribution, which is uniquely determined by its moments, the central limit theorem now follows from the convergence of moments.

<u>Theorem</u>.

$$(1.30)\qquad \frac{N(x)-M(x)}{\sqrt{Var(N(x))}} \xrightarrow{L} N(0,1) .$$

Dvoretzky and Robbins also prove the central limit theorem by a Lyapounov argument using second moments and the conditional independence of $N(x_1),N(x_2),\ldots,N(x_n)$, the numbers of cars parked in disjoint open intervals of $(0,x)$ given that $n-1$ cars have already been parked.

Mannion (1964) has also obtained the asymptotic behavior of $M(x)$ and $Var(N(x))$ by methods similar to Renyi and calculated the approximation

$$(1.31)\qquad \lim_{x\to\infty} \frac{Var(N(x))}{x} = C_2 \doteq .035672 .$$

In (2.9) we compute $C_2 \doteq .038156$ and thus differ slightly.

Ney (1962) and Mullooly (1968) have used a more complex version of Renyi's ideas to extend the asymptotic moment results to cars of random lengths which follow a probability distribution bounded below by a positive constant. See also Weiner (1978).

Goldman, Lewis and Visscher (1974) have also obtained related one dimensional results for random car size where car length follows a given distribution, bounded above and below, and there is a termination probability, such that after a geometric number of attempts to park a car are unsuccessful, the attempts terminate with probability p. The equation for $EN(t) = M(t)$, the mean total number of parked cars, with length given by density $g(x)$, distribution function $G(x)$, is obtained as

$$M(x) = \frac{G(x)}{1-(1-G(x))(1-p)} [1 + \frac{2}{G(x)} \int_0^x \frac{g(u)du}{x-u} \int_0^{x-u} M(s)ds] .$$

If $M_k(x) = EN^k(x)$,

$$M_k(x) = Y(x)[E(L^k|L < x) + \frac{2}{G(x)} \int_0^x \frac{g(u)du}{x-u} \int_0^{x-u} M_k(y)dy]$$

where

$$E(L^k|L < x) = \frac{\int_0^x u^k g(u)du}{G(x)} , \quad k = 1,2,\ldots .$$

Some numerical results indicate that the empirical frequencies of car lengths do not correspond to the theoretical frequency $g(x)$, since, e.g., short cars pre-empt spaces so that longer cars may not park. A conjecture of Lewis and Goldman in Goldman, Lewis and Visscher (1974) that the mean fraction of parking interval covered is an increasing function of the coefficient of variation of the underlying length distribution is somewhat tenable by experiment. See also Weiner (1980b) for a central limit theorem obtained from asymptotic moments similar to that in Dvoretzky and Robbins (1964).

A one-dimensional model by Solomon (1967) for sequential parking of unit length cars on a curb $(0,x)$, $x > 1$ is as follows: The left end of each unit length car is uniformly distributed on $(-\alpha, x-1+\beta)$ where $\alpha+\beta = \eta \leq 2$, $\alpha \leq 1$, $\beta \leq 1$. If the first car lands such that it overlaps $(0,x)$, it is shifted to the right (if it covers the origin) or to the left (if it covers the point x) until it is at $(0,1)$ or $(x-1,x)$, respectively. Otherwise it is parked as in the Renyi model. Successive cars are parked as follows: If a car lands in a space within $(0,x)$ which can accommodate it, then it is parked there. If it overlaps either end of $(0,x)$, it is shifted until it just fits into $(0,x)$ and parked there if and only if there is an empty space at $(0,1)$ or $(x-1,x)$. Otherwise it is discarded, since it then overlaps an already parked car. If the new car overlaps a parked car on the left by a length less than or equal to α, then the new car is shifted to the right until its left endpoint is at the right endpoint of the parked car that overlapped with it. It is then parked if there is space for it, otherwise it is discarded. Similarly, for overlap to the right by a length less than or equal to β, the new car is shifted to the left and parked if there is space. The process continues until no further cars may be parked. To summarize, cars are parked as in the Renyi model, except that cars which overlap are not immediately discarded, and are parked at the next adjacent space contiguous to the already parked car, if available. Let

$$K(s) = \text{mean total number of cars} \tag{1.32}$$

which may be parked on $(0,x)$ in accord with a Solomon model. Then, conditioning on the placement of the first car, one obtains the equation

$$K(x) = 1 + \frac{\eta}{x+\eta-1} K(x-1) + \frac{2}{x+\eta-1} \int_0^{x-1} K(u)du \ . \tag{1.33}$$

For $\eta = 0$, this is the Renyi parking scheme. The asymptotic packing scheme for $\eta = 2$ is indicated as follows, using a systematic method which yields asymptotic packing constants in a form which differs from that of Renyi. From (1.33) with $\eta = 2$,

$$(1.34)\qquad (x+1)K(x) = x+1 + 2K(x-1) + 2\int_0^{x-1} K(u)du \ .$$

Since $K(x) = 0$, $0 < x < 1$, $K(x) = 1$, $1 \leq x \leq 2$, upon multiplying (1.34) by e^{-sx}, $s > 0$, and integrating from 1 to ∞, one obtains, with $\phi(x) = \int_1^\infty e^{-sx}K(x)dx$,

$$(1.35)\qquad \phi'(x) + \phi(s)(2e^{-s} + \frac{2}{s}e^{-s} - 1) = -(\frac{2}{s} + \frac{1}{s^2})e^{-s} \ .$$

We now provide more detail because the methods of Renyi may not apply to the extended model. This model is a one-dimensional analogue of the three-dimensional model where shaking is permitted to allow unit spheres to settle and provide more space for additional spheres, thus increasing the mean packing constant.

A straightforward solution of (1.35) may be obtained using the method of integrating factors, as follows. Let

$$(1.36)\qquad Q(s) = \phi(s)\exp[-s-2e^{-s}-s\int_s^\infty \frac{e^{-u}}{u}du] \ .$$

Since $K(x) \leq x$, for $s > 0$,

$$(1.36a)\qquad \phi(s) \leq e^{-s}/s \ .$$

Then from (1.35), (1.36),

$$(1.37)\qquad Q'(s) = -(\frac{2}{s} + \frac{1}{s^2})e^{-s}(\exp[-s-2e^{-s}-2\int_s^\infty \frac{e^{-u}}{u}]) \ .$$

From (1.36a), (1.37), $Q(\infty) = 0$, and if one integrates $Q'(s)$ from t to ∞ one obtains

$$(1.38)\qquad Q(t) = \int_t^\infty (\frac{2}{s} + \frac{1}{s^2})\exp(-2[s+e^{-s} + \int_s^\infty \frac{e^{-u}}{u}du])ds \ .$$

An interesting representation for Euler's constant γ is

$$(1.39) \qquad \gamma = \int_0^1 \frac{1-e^{-u}}{u}\,du - \int_1^\infty \frac{e^{-u}}{u}\,du .$$

This representation is important in the development of (1.47) - (1.51) that follow. The desired representation (1.39) is obtained as follows:

$$(1.40) \qquad \gamma = -\int_0^\infty e^{-t} \log t \, dt ,$$

and so

$$(1.41) \qquad \gamma = -\int_0^1 e^{-t} \log t \, dt - \int_1^\infty e^{-t} \log t \, dt .$$

Then (1.41) may be written

$$(1.42) \qquad \gamma = -\int_0^1 \log t \, d(1-e^{-t}) + \int_1^\infty \log t \, d(e^{-t})$$

and the integration by parts indicated in each term of (1.42) yields (1.39). The following expression for γ will be needed.

$$(1.43) \qquad \gamma = \int_0^t \frac{1-e^{-u}}{u}\,du + \int_t^1 \frac{1}{u}\,du - \int_t^\infty \frac{e^{-u}}{u}\,du$$

where $0 < t < 1$. This is obtained by writing (1.39) as

$$(1.44) \qquad \gamma = \int_0^t \frac{1-e^{-u}}{u}\,du + \int_t^1 \frac{1-e^{-u}}{u}\,du - \int_1^\infty \frac{e^{-u}}{u}\,du$$

and one may split the middle integral of (1.44) to obtain

$$(1.45) \qquad \gamma = \int_0^t \frac{1-e^{-u}}{u}\,du + \int_t^1 \frac{du}{u} - \int_t^1 \frac{e^{-u}}{u}\,du - \int_1^\infty \frac{e^{-u}}{u}\,du .$$

Combining the last two integrals of (1.45) yields (1.39).

From (1.36), (1.38) one obtains

$$(1.46) \qquad \phi(t) = [\exp(t+2e^{-t}-2+2\int_t^\infty \frac{e^{-u}}{u}du + 2\gamma] \cdot$$

$$\int_t^\infty (\frac{2}{s}+\frac{1}{s^2})\exp(-2[s+e^{-s}-1+\gamma+\int_s^\infty \frac{e^{-u}}{u}du])ds \ .$$

Using a Taylor expansion on the first integrand on the right of (1.44), and integrating term by term yields, for $0 < t < 1$,

$$(1.47) \qquad \gamma = -\log t - \int_t^\infty \frac{e^{-u}}{u}du + \sum_{\ell=1}^\infty \frac{(-1)^{\ell+1}t^\ell}{\ell\ell!} \ .$$

Denote $A(t)$, $0 < t < \infty$ by

$$(1.48) \qquad A(t) = (\frac{2}{t} + \frac{1}{t^2})\exp(-2[t+e^{-t}-1+\gamma+\int_t^\infty \frac{e^{-u}}{u}du]) \ .$$

Upon substituting the expression (1.47) for γ in (1.48), one may let $t \downarrow 0$ to obtain

$$(1.49) \qquad \lim_{t\downarrow 0} A(t) = 1 \ .$$

Hence, from (1.49), for $0 < t \ll 1$,

$$(1.50) \qquad \int_t^\infty A(x)dx \simeq \int_0^\infty A(x)dx - t \ .$$

From (1.46), (1.50), if $0 < t \ll 1$, one may conclude that

$$(1.51) \qquad \phi(t) \simeq \frac{1}{t^2}\int_0^\infty A(x)dx - \frac{1}{t} \ ,$$

(1.52) <u>Claim</u>: $K(x)$ is monotone non-decreasing for $x \geq 1$.

<u>Proof of Claim</u>. $K(x) = 0$, $0 \leq x < 1$ and $K(x) = 1$, $1 \leq x \leq 2$, hence assume the claim true for $0 \leq x \leq y$. Then it suffices to show that $K(y+1) \geq K(y)$. From (1.34)

$$(1.53)\qquad (y+1)(K(y+1)-K(y)) = 1+2(K(y)-K(y-1)) + 2\int_{y-1}^{y} K(u)du - R(y+1)\ .$$

Clearly, for $x \geq 1$,

$$(1.54)\qquad K(x) \geq x/2\ .$$

Hence (1.54) applied to the right side of (1.53) yields

$$(1.55)\qquad (y+1)(K(y+1)-K(y)) \geq y+1-R(y+1) > 0\ ,$$

which completes the proof.

From (1.51), one may apply a Tauberian theorem since (1.52) holds, and conclude that as $x \to \infty$,

$$(1.56)\qquad K(x)/x \sim (\int_0^{\infty} A(u)du)x - 1)\ .$$

Computer quadratures by Solomon (1967) yield

$$(1.57)\qquad 1.80865 < \int_0^{\infty} A(u)du < 1.80866\ .$$

He also achieves the result by simulation.

This method (1.33)-(1.57), while somewhat tedious, is a straightforward way to obtain asymptotic results which may be used for general one-dimensional packing equations, and of course, in particular, to the Renyi model.

An early simulation of the one-dimensional parking problem of Renyi was due to Solomon (1967). This model produced values that agreed with the Renyi mean packing constant and obtained $C = .7500 \pm .02$, where $.02$ is the experimental standard deviation, using a line of 100 units in length. (See Solomon (1967), Table VII, p. 130). A similar simulation on the Solomon model in one-dimension for $\gamma = 2$, and a line length of 100 yielded a Solomon constant of $.8060 \pm .02702$. (See Solomon (1967), Table VI, p. 130).

Comments on selected papers in random packing are given in the four review papers Moran (1966), (1969), Little (1974) and Baddeley (1977).

II. Two Dimensions.

In 1960, shortly after Renyi's one-dimensional solution of the parking problem was published, I. Palasti (1960), an associate of Renyi, considered a two-dimensional analog of Renyi's model, that is, the sequential random packing of unit squares uniformly at random on an $a \times b$ parking rectangle, with $a > 1$, $b > 1$, where a unit square car which overlaps another already parked is discarded. The sides of each unit square car are always parallel to the sides of the $a \times b$ parking rectangle. The process continues until no further unit square cars may be so parked.

If $M(a,b)$ denotes the mean total number of unit square cars which may be so parked, then Palasti conjectured that (1960), (1976)

$$\lim_{a,b\to\infty} M(a,b)/ab = C^2 = (.74759)^2 \doteq .56 . \tag{2.1}$$

See also Solomon (1967).

A careful simulation method used by Blaisdell and Solomon (1970), to assess the Palasti conjecture for the two-dimensional Renyi model is now described. Analytical attempts at results for the two-dimensional case have not met with any success.

A simulation to first verify the value of the Renyi constant in the one-dimensional case is based on a lattice model developed by Mackenzie (1962). The paper of Blaisdell and Solomon (1970) will be quoted in parts. A line with integral length n is filled sequentially at random with non-overlapping intervals of length a, their endpoints having integer coordinates. Hence there is a linear lattice of n points in which a-tuplets of neighboring points are occupied at each trial until the maximum number of neighboring points left vacant is less than or equal to $a-1$. The ratio of the number of points left vacant after the sequential filling of the lattice to the total number of points available is labeled $1-f$. Mackenzie (1962) demonstrated that, as n tends to infinity, $E(1-f) \to 1-\rho$. Thus ρ is analogous to the packing density.

A rearrangement of Mackenzie's equation for ρ yields, as a tends to infinity, and an^{-1} goes to zero,

$$(2.2) \qquad \rho = 1 - (1+an^{-1})(1-C - \tfrac{1}{2}(C-e^{-2\gamma})a^{-1} + O(a^{-2})]$$

where in (2.2), γ is Euler's constant, and hence that, from (2.2), for large a, and an^{-1} approaching zero, that

$$(2.3) \quad \rho \simeq C + (.2162)(a^{-1}) - (.2524)(an^{-1}) + (.2162)(n^{-1}) \ .$$

Enough experiments were done at each pair of values of a and n so that the estimated standard deviation of the average density was close to .0010. Pseudo-random numbers, r, were generated by a multiplicative congruential method, namely

$$(2.4) \qquad r_{i+1} = mr_i \ (\mathrm{mod}(10)^5) \ .$$

The simulation was carried out in this way. The index of a site was $i = (n-a+1)r \ (\mathrm{mod}(10)^9)$ for the line or $i + (n-a+1)^2 r(\mathrm{mod}(10)^9)$ for the plane, with $i = 1,2,\ldots,n-a+1$ for the line, and $i = 1,2,\ldots,(n-a+1)^2$ for the plane, respectively. If the site was not occupied, a new random number was generated. If the site was occupiable, a site of a or a^2 lattice points was covered and the number of sites rendered unoccupiable, $u = 1,2,\ldots,2a-1$ for the line, or $u = 1,2,\ldots,(2a-1)^2$ for the plane, respectively, was subtracted from the preceding number of occupiable sites, denoted x. Random numbers were generated until x was reduced to zero. A stopping rule used was to end the simulation if 5,000 random numbers in a row did not lead to the filling of an occupiable site. A small systematic error was thus made. The times taken to put down the last few sets of points were so long that a program was written to keep track of the number, x, of remaining occupiable sites, and to compute a random occupiable site index $i_x = xr(\mathrm{mod}(10)^9)$ and to put down the sets of points at i_x. A sequence of least-squares line fittings for the one-dimensional case using successively more non-border simulation points yielded good agreement with C to four decimal places, using extrapolation $a^{-1} \doteq 0$, with an estimated standard deviation smaller than .001.

Following Blaisdell and Solomon (1970), the square of equation (2.2) is used to view the Palasti conjecture, ignoring negligible terms, as $\frac{1}{a} \to 0$, $\frac{a}{n} \to 0$. This yields

$$(2.5) \qquad \rho^2 = a_{00} + a_{10}\left(\frac{1}{a}\right) + a_{01}\left(\frac{a}{n}\right) + a_{11}\left(\frac{1}{a}\right)\left(\frac{a}{n}\right)$$

where the a_{ij}, $0 \leq i \leq 1$ may be obtained directly from (2.2).

From (2.5a) and the linearity of the data, a least squares fit (Blaisdell and Solomon (1970), p. 678, eq. 4 and Figure 3, p. 680)

$$(2.5a) \qquad \hat{\rho}_2 = \hat{\rho}^2 = b_{00} + b_{10}\left(\frac{1}{a}\right) + b_{01}\left(\frac{a}{n}\right) + b_{11}\left(\frac{1}{a}\right)\left(\frac{a}{n}\right)$$

yields observed values slightly greater than the values in (2.5). Using non-border points, Blaisdell and Solomon (1970), p. 684, eq. 8, obtain

$$(2.6) \qquad \sqrt{\hat{\rho}} - C = \left(1+\frac{a}{n}\right)\left(.0025 - .0109\left(\frac{1}{a}\right)^2\right) ,$$

with an estimated standard deviation of less than .001. Thus, (2.6) indicates that the Palasti conjecture is false but that the difference is small (about .0025) in two dimensions.

Blaisdell and Solomon (1970) conclude the section on the Palasti conjecture with Renyi sphere-packing simulations which suggest that the absolute accuracy of the Palasti conjecture should decrease with increasing dimension, and indicate the need for appropriate Renyi cubic packings in higher dimensions, which Blaisdell and Solomon (1982) later carried out, and is described below. A condensed summary of the work of Blaisdell and Solomon (1970) is contained in Sharp (1974).

Blaisdell and Solomon (1970) use a result of Dvoretzky and Robbins (1964) previously alluded to in one-dimension, namely

$$(2.7) \qquad \inf_{x \le t \le x+1} \frac{M(t)+1}{t+1} \le C \le \sup_{x \le t \le x+1} \frac{M(t)+1}{t+1}$$

to obtain C to 15 decimal places. By a similar result for variances given by Dvoretzky and Robbins (1964), namely that

$$(2.8) \qquad \lim_{x \to \infty} \mathrm{Var}(N(x))/x = D$$

and

$$(2.9) \qquad \inf_{x \le t \le x+1} \frac{\mathrm{Var}(N(t))}{t+1} \le D \le \sup_{x \le t \le x+1} \frac{\mathrm{Var}(N(t))}{t+1} ,$$

Blaisdell and Solomon (1970) were able to compute D to seven decimals, $D \approx .038156$. Monte Carlo simulations using results of Mackenzie by Blaisdell and Solomon (1970) to estimate C give good agreement with $D \approx .0382$ with an estimated standard deviation of $.0003$. We have already commented that this varies slightly from Mannion's result (1.31), namely $.035672$.

Akeda and Hori (1975) published results of computer experiments which supported the two-dimensional Palasti conjecture. This was based on two-dimensional squares with $n/a = 100$, where a is the value of the floating point mantissa. After notification of the Blaisdell-Solomon (1970) results which did not support the Palasti conjecture, Akeda and Hori carried out further experiments (1976), performed the necessary extrapolation $a/n \to 0$ and obtained a discrepancy from the Palasti conjecture which was small, but significant, namely

$$\sqrt{\hat{\rho}_2} - C \to .0027 \pm .0002 \;, \tag{2.10}$$

where .0002 is the experimental standard deviation. This essentially reproduces the Blaisdell-Solomon result. Akeda and Hori (1976) also studied the three-dimensional Renyi model by a different and undescribed method. For various integral values of a/n, and on extrapolation of $a/n \to 0$, for $\hat{\rho}_3$ the experimental packing constant, they obtained

$$(\hat{\rho}_3)^{1/2} - C = -\,.0018 \pm .0008 \;, \tag{2.11}$$

where .0008 is the experimental standard deviation. Blaisdell and Solomon (1982) comment that the discrepancy in three-dimensions (2.11), is of opposite sign from that in two-dimensions, (2.10), and that the line through the individual points in the three-dimensional case crosses the expected extrapolation line, both results intuitively unappealing.

Finegold and Donnell (1970) published a two-dimensional Renyi simulation by a fine-mesh method. In this method, the parking lot is covered by a fine mesh of squares much smaller than the unit square car. A disc-shaped car is centered on a square, and the car is then represented by the minimum set of fine squares which will completely cover the area of the car. The square parking lot was divided into a mesh of 1024 x 1024 fine squares: the availability status (i.e. whether or not a fine square was available for a car center) was stored as a bit in a single megabit array. By an appropriate counting, the problem of making many trials to avoid overlaps is eliminated, which is unavoidable in the large-mesh method. As the center location for each car is randomly selected directly from those fine meshes available, then the neighboring region thereby excluded to another car is also determined and recorded in the megabit array. To identify thd next vacant fine squares on which a car center may be randomly parked, two additional integers for each row of the 1,024 x 1,024 array are used for bookkeeping purposes. An integer array is used to record the coordinates of the car centers. The method has the advantage that the computer time required to find a location for each car is the same for each car parked, even though the time to park a car is long, including all bookkeeping. The authors claim a packing density of .5538 $\pm$ .0035, with "... simulations with more realistic periodic boundary conditions which do not require extrapolation," and an undescribed "course mesh" method, presumably using a unit square grid which gives lower and upper bounds for the packing density of .5629 $\pm$ .0016 and .5649 $\pm$ .0016, respectively. The authors then claim that as the course and fine mesh results bridge the value $C^2 = .5589$, that the Palasti conjecture should not be rejected.

Blaisdell and Solomon comment ((1982), pp. 383-384) that the assertion quoted above, that extrapolation to $a/n \to 0$ is unnecessary, is incorrect. It is also to be noted that in an unextrapolated coarse mesh method with periodic boundaries (as mentioned in Finegold and Donnell (1979)) cars with centers in the parking square are counted even though a portion of the car may overlap the parking square, hence these edge effects in a finite parking square, prior to extrapolation, can yield too high a value for the packing density $\hat{\rho}_2$.

Jodrey and Tory (1980) gave results on the Palasti conjecture for dimensions 1, 2, 3, 4 based on computer experiments with periodic boundaries and obtained extrapolation values as follows ($C = .74759$):

$$\begin{aligned}(2.12)\qquad \hat{\rho}_1 - C &= -\ .0004 \pm .0005\\ (\hat{\rho}_2)^{1/2} - C &= +\ .0021 \pm .0002\\ (\hat{\rho}_3)^{1/3} - C &= .0029 \pm .0002\\ (\hat{\rho}_4)^{1/4} - C &= .0003 \pm .0006\ .\end{aligned}$$

The values of $(\hat{\rho}_2)^{1/2} - C$ obtained by three of the sets of authors; Akeda and Hori (1976), Blaisdell and Solomon (1970), (1982), and Jodrey and Tory (1980) are in good agreement, .0027, .0025, .0021, respectively, and show a small but significant departure from the two-dimensional Palasti conjecture in a direction that is intuitively appealing. The values of $(\hat{\rho}_3)^{1/3} - C$ obtained by Akeda and Hori (1976) and Jodrey and Tory (1980) are in poor agreement, -.0018 and +.0029 respectively.

The variance of the limiting packing density has also been investigated. The results of Blaisdell and Solomon (1982) are described as follows. Let

$$(2.13)\qquad y_1 = s^2[(\tfrac{a}{n})(1 + \tfrac{a}{n})] \to A_2(a)$$

as $a/n \to 0$, $1/a \to 0$, and $A_2(a)$ is an undetermined power series in $1/a$. A least squares treatment of one-dimensional data in Blaisdell and Solomon (1970) yielded, as noted earlier,

$$(2.14)\qquad \begin{aligned} y_1 = {} & .0381 \pm (.0003) + (.0161 \pm .0065)1/a\\ & + (-.0422 \pm .0238)(\tfrac{1}{a})^2\ ,\end{aligned}$$

where 0.0381 is in good agreement with the result 0.038156 obtained by Blaisdell and Solomon (1970) by using the theoretical bounds for the variance developed by Dvoretzky and Robbins (1964) in (2.9).

A plausible generalization of (2.13) for d dimensions (i.e. choosing d coordinates for a potential parking site as noted earlier, generalizing the development after (2.4) is (Blaisdell and Solomon (1982))

$$y_d = s_d^2[(\frac{a}{n})(1+\frac{a}{n})]^d \quad . \tag{2.15}$$

This was found to be in good agreement with the two-dimensional data (Blaisdell and Solomon (1970), namely,

$$y_2 = .0508 \pm .0010 + (1.397 \pm .3905)(\frac{1}{a})^3 + (-3.6660 \pm 1.1400)(\frac{1}{a})^4 \quad . \tag{2.16}$$

Average values for y_d are given for various authors as follows:

Akeda and Hori (1975)	$y_2 = .0402 \pm .0213$
Akeda and Hori (1976)	$y_2 = .0482 \pm .0135$, $y_3 = .0413 \pm .0206$
Blaisdell and Solomon (1970)	$y_1 = .0393 \pm .0014$, $y_2 = .0526 \pm .0027$
Jodrey and Tory (1980)	$y_1 = .0400 \pm .0058$, $y_2 = .0468 \pm .0069$, $y_3 = .0451 \pm .0101$, $y_4 = .0324 \pm .0176$
Blaisdell and Solomon (1982)	$y_3 = .0519 \pm .0062$, $y_4 = .0518 \pm .0158$.

The experiments in three and four dimensions on the Palasti conjecture for the Renyi model by Blaisdell and Solomon (1982), will now be described. Finite lattices with rigid boundaries are used, (i.e. no overlap of parked cars with boundaries) since this is easier to program and allows an exact accounting of every lattice site as occupied or unoccupiable, although as Jodrey and Tory (1980) have pointed out, this method results in a loss of accuracy when extrapolating to $n \to \infty$. Authors using floating point arithmetic have not agreed with each other nor with the results of Blaisdell and Solomon (1983) in three-dimensions. One possible explanation for this is the occurrence of holes which present a very tight fit in at least one dimension, and may have been missed due to round-off error. The likelihood of these tight fits will increase with increasing dimension and may account for the fact that the packing densities found by Jodrey and Tory (1980) are increasingly lower than those of Blaisdell and Solomon (1970), (1982) in dimensions 2, 3, and 4.

Let, for $d = 1,2,3,4$,

$$(2.17) \qquad x_d = (1-(\hat{\rho}_d)^{1/d})/(1+\frac{a}{n}) - .2524 + .2162a^{-1} .$$

A least squares fit of the data for $d = 3,4$ Blaisdell and Solomon (1982) shows that the intercept (a measure of departure from the Palasti conjecture) increases with dimension. The fitted coefficient of $1/a^2$ (a measure of departure from the limiting equation (2.17)) decreases with dimension. For each dimension 1, 2, 3, 4 respectively, the fit yields almost constant estimates of the parameters, indicating that the model fit is satisfactory. A search was made for further terms which might significantly improve the fit by an all possible subsets regression computer program. The results are that a significant improvement in the standard error of estimate is not made upon addition of the extra terms if $n/2 > 5$. Further, there is no trend in the residuals, tending to indicate that the model is satisfactory. The conclusions, as indicated earlier, are that for these computer experiments by Blaisdell and Solomon (1970), (1982) on random sequential packing of finite lattices with rigid boundaries in dimensions 1, 2, 3, and 4 a discrepancy in the Palasti conjecture in the limit as $\frac{1}{a} \to 0$, $\frac{a}{n} \to 0$ is as follows

$$(2.18) \qquad \begin{aligned} (\hat{\rho}_2)^{1/2} - C &\to .0037 \\ (\hat{\rho}_3)^{1/3} - C &\to .0084 \\ (\hat{\rho}_4)^{1/4} - C &\to .0127 . \end{aligned}$$

The following is a table summarizing the experimental results commented on in Section II, from Blaisdell and Solomon (1982).

TABLE

Average Normalized Variances for Sequential Random Paking Densities

References	Dimension	y_d	Notes
Akeda and Hori (1975)	2	0.0402 ± 0.0213	(a)
Akeda and Hori (1976)	2	0.0482 ± 0.0135	(a)
Akeda and Hori (1976)	3	0.0413 ± 0.0206	
Jodrey and Tory (1980)	1	0.0400 ± 0.0058	
Jodrey and Tory (1980)	2	0.0468 ± 0.0069	
Jodrey and Tory (1980)	3	0.0451 ± 0.0101	
Jodrey and Tory (1980)	4	0.0324 ± 0.0176	
Blaisdell and Solomon (1970)	1	0.0393 ± 0.0014	(b)
Blaisdell and Solomon (1982)	2	0.0526 ± 0.0027	(b)
Blaisdell and Solomon (1982)	3	0.0519 ± 0.0062	
Blaisdell and Solomon (1982)	4	0.0518 ± 0.0158	

Notes. (a) Only the values for squares of edge ≥ 40 have been used.

(b) Only a subset of the values obtained in Solomon and Blaisdell (1970) have been used, to allow about the same number of values for each of the four dimensions.

III. Heuristic Attempts at the Palasti Conjecture.

Palasti (1960) and Weiner (1978) have attempted to prove the Palasti conjecture in the plane. The unproved assumptions on which the purported proofs rely will be indicated.

Palasti (1960) based her proof on her

HYPOTHESIS A:

Let $M(x,y)$ be the mean total number of unit squares which are parked in an $x X y$ parking rectangle in accord with the Renyi parking model. Then there exists a constant A such that

$$(3.1) \qquad |M(x_1+x_2,y) - M(x_1,y) - M(x_2,y)| \leq Ay$$
$$|M(x,y_1+y_2) - M(x,y_1) - M(x,y_2)| \leq Ax.$$

Assuming Hypothesis A, Palasti then attempted to show that

$$\lim_{x,y\to\infty} M(x,y)/xy = \alpha \ , \tag{3.2}$$

where α is an undetermined constant.

However, Palasti (1976) gives the following quoted interpretation to Hypothesis A, (3.1), "Let us consider the illustrative meaning of the hypothesis. We can imagine that the (parking) rectangle has already been filled by unit squares placed randomly in the above mentioned way. Then the hypothesis means that if the (parking) rectangle is divided into two parts by a straight line (or by a band having unit width parallel to the axis X), then at most x unit squares would be intersected by this line (or at most $2x$ unit squares will be partly covered by the unit wide band). Naturally the same is true for the straight lines parallel to the axis, Y, that is at most y unit squares can be touched by these lines." This justification draws the following false conclusion, namely that the mean total number of unit squares which are parked on an x_1Xy rectangle is the same as the mean total number of unit squares parked on an x_1Xy subset of an $(x_1+x_2)Xy$ parking rectangle. This latter assumption has not been proved. That it is most likely false is substantiated by, e.g., simulations of Blaisdell and Solomon (1970), p. 680, where the parking density is found to vary with the size of the parking square. Hence Hypothesis A seems vitiated.

Palasti (1976) considers a different two-dimensional parking model closer to the one-dimensional Renyi model, and shows that her conjecture holds for this model, described as follows: A unit square is dropped uniformly at random on an xXy parking rectangle. The edges of the unit square are prolonged until they intersect the edges of the parking rectangle. The four unit-width columns so formed are each packed with tight-fitting unit squares in accord with a one dimensional Renyi model, with each column treated separately. Each of the four new rectangular regions formed are separately considered as new parking rectangles, a unit square parked on each uniformly at random, and the process continues as before until no new unit square cars may be parked. Let $R(x,y)$ denote the mean total number of unit square cars which may be parked in accord with this Palasti scheme. It is clear that, conditioning on the location of the first parked square, and if $M(x)$ denotes the mean total number of unit length cars parked in accord with a Renyi scheme on a parking segment of length x, with $M(x) = 0$, $0 < x < 1$. $M(x) = 1$, $1 \leq x < 2$, then (this equation differs from that in Palasti (1976) in notational use)

$$R(x+1,y+1) = \frac{4}{xy}\int_0^x\int_0^y R(u,v)\,du\,dv + \frac{2}{x}\int_0^x M(u)\,du + \frac{2}{y}\int_0^x M(u)\,du + 1\,. \tag{3.3}$$

By a tedious argument involving partial differential equations, and invoking unicity of solutions, Palasti is able to show that

$$R(x,y) = M(x)\,M(y) \tag{3.4}$$

is the solution. It is to be noted that (3.3) could have been more simply obtained by direct substitution of (3.4) into (3.3), using the previously given equation

$$M(x+1) = 1 + \frac{2}{x}\int_0^x M(u)\,du, \tag{1.4a}$$

$$M(x) = 0,\ 0 < x < 1,\ M(x) = 1,\ 1 \leq x \leq 2\ ,$$

and from (3.3) it is clear that the Palasti conjecture holds, since, as given earlier,

$$\lim_{x\to\infty} \frac{M(x)}{x} = C\ . \tag{1.3}$$

However, this model was not related to the two-dimensional Renyi model, hence the result (3.5) cannot be applied to prove the Palasti conjecture, which had been proposed in connection with the Renyi model.

Weiner (1978) published a purported proof of the two dimensional Palasti conjecture for both the Renyi and Solomon models. The two dimensional Solomon model involves shifting a unit square dropped uniformly at random on the parking rectangle, which overlaps a prior, parked square either horizontally and or vertically the shortest distance to an empty parking space (if one exists) immediately adjacent to the overlapped parked unit square, and discarding the unit square if no such parking space exists. The process continues until no further unit square cars may be parked. Let $M(x,y)$ and $K(x,y)$ denote the mean total numbers of unit square cars which may be parked on an $x \times y$ parking rectangle. The Palasti conjecture is that

$$(3.6) \qquad \lim_{x,y\to\infty} M(x,y)/xy = C^2 = (.74759)^2$$

and

$$\lim_{x,y\to\infty} K(x,y)/xy = d^2 = (.806)^2 .$$

Weiner's argument rested on some unproven, untuitively argued propositions (Weiner (1978), Lemma 3, p. 806; Lemma 7, p. 809), namely that for x,y sufficiently large,

$$(3.7)(a) \qquad \begin{aligned} M(x,y+1) &\geq M(x,y) \\ M(x+1,y) &\geq M(x,y) \\ M(x+1,y+1) &\geq M(x,y) \end{aligned}$$

$$(3.7)(b) \qquad \frac{M(x+1,y)}{(x+1)y} \leq \frac{M(x,y)}{xy}$$

$$(3.7)(c) \qquad M(x,y) + M(x) \geq M(x,y+1)$$

$$M(x,y) + M(y) \geq M(x+1,y) \ .$$

The intuitive arguments to support (3.7a-c) are based on the fact, as may be straightforwardly shown, using (3.5), (3.6), that for x sufficiently large,

(3.8) $\qquad M(x+1)/x$ is monotone increasing.

If the $(x+1)X(y+1)$ parking rectangle is placed on the positive quadrant with two perpendicular edges along the x and y axes respectively, then the mean total number of unit square parked cars which intersect the line segment from $(1,0)$ to $(1,y+1)$ is clearly $M(y+1)$. Each such parked car was considered to be the initial car in a series of "staggered rows" which reached across the parking rectangle along the x direction to the right side boundary given by the segment $(x+1,0)$ to $(x+1,y+1)$. Considering the x-component of each staggered row alone, the mean total number of unit square cars in each staggered row was deemed to be $M(x+1)$. As pointed out by various critics, however, Hori (1979), (1980); Tanemura (1979), (1980); Tory and Pickard (1979), (1980); Weiner (1979), (1980), did not account for all unit square cars, and there may be non-uniqueness in the definition in certain configurations.

A later argument published by Weiner (1979) attempted again to demonstrate (3.7a-c). For example, that $M(x+1,y) \geq M(x,y)$, a given configuration of parked cars filling the xXy rectangle, the resultant parked cars shrunk in a unique way to unit square size, and the (possibly) resultant opened up parking spaces filled in accord with the Renyi scheme for an $(x+1)Xy$ rectangle. It was then claimed that since every filled parking configuration on xXy can be mapped into $(x+1)Xy$, with (possibly) some empty parking spaces, that it therefore followed that $M(x+1,y) \geq M(x,y)$. However, as pointed out, Hori (1980); Tanemura (1980); Tory and Pickard (1980), such a mapping may not (and probably does not) map the probability distribution of the final configuration on xXy to a probability distribution on $(x+1)Xy$ in any tractable manner. Hence the alternative proof is incomplete. In fact, Weiner (1978) did acknowledge the heuristic, incomplete nature of the arguments. Based on the unproved (3.7a-c), Weiner (1978) attempted to bound $M(x,y)$ above and below by parking models for which integral inequalities could be written, and which would be hopefully related to $M(x)M(y)$.

The purported proofs that the presumed tractable means for these new models, denoted $M_1(x,y)$ and $M_2(x,y)$ indeed satisfy inequalities of the form

$$M_1(x,y) \leq M(x,y) \leq M_2(x,y) \tag{3.9}$$

are based on the unproved assumptions (3.7a,b,c).

IV. Remarks.

1. As a first step in a new theoretical consideration of the two dimensional Palasti conjecture for either of the Renyi or Solomon models, the $(k+1)Xx$ abacus model, with $k \geq 1$ an integer, and $x > 1$ may be easier to consider mathematically. In this model, a $(k+1)Xx$ parking rectangle, oriented with the coordinates of its edges at $(0,0)$, $(x,0)$, $(0,k+1)$, $(x,k+1)$, has k horizontal lines evenly spaced with respective endpoints at $(0,\ell)$, (x,ℓ) for $1 \leq \ell \leq k$. A line is chosen uniformly at random with probability $1/k$ each. A unit square car, oriented as usual, is centered on the chosen line, and dropped horizontally along the line in accord with a one dimensional Renyi (resp. Solomon) model, and parked on that line if and only if it is not tangent to a prior parked car on an adjacent line. The process continues until no further unit square cars may be parked on the abacus. If $M(x,k+1)$, $K(x,k+1)$ denote the mean total numbers of unit square cars parked in accord

with a Renyi or Solomon model, respectively, then it may be possible to obtain, if existent,

$$(3.10) \qquad \lim_{k,x\to\infty} M(x,k)/xk$$

$$\lim_{k,x\to\infty} K(x,k)/xk \; .$$

2. For the ordinary two dimensional Renyi or Solomon models, it is of interest to determine which portions of the inequalities of (3.7a-c) do exist. Assumption (3.7b), that the parking density decreases with size of the parking square area for unit-square cars appears vitiated by the negative slopes of the lines in Figure 3, p. 680, of Blaisdell and Solomon (1970).

3. A similarity between random sphere packing density in n dimensions, $1 \le n \le 6$, and relative frequency of one-syllable English words of length n, and in proportion of matriculates with bachelors, masters, doctoral degrees, is empirically noted by Dolby and Solomon (1975). A plausibility argument relates these phenomena by closeness to spherical base regions in an appropriate space.

4. A recent book by Ambartzumian (1982), on pp. 188-189 indicates that the sequential random parking of identical two-dimensional objects on a large parking area is one of the most difficult problems in combinatorial integral geometry.

5. Random packing models for elections in Japan are introduced in Itoh (1978), (1980); Itoh and Ueda (1978), (1979) to explain percentage gains among candidates of the Liberal Democratic Party, using computer-generated experiments. Consider a stick of length $x \ge 2d$. The stick is divided into two sticks with lengths x_1 and x_2 such that $x_1 \ge d$ and $x_2 \ge d$. Each possible division is continued until all sticks are shorter than $2d$. The sticks obtained by such a procedure correspond to gaps generated by one-dimensional random packing. Consider an election in a certain constituency. The length x corresponds to the total votes obtained

by candidates of a certain political party. The party nominates a candidate if he can obtain at least d votes. The length of each stick, which results from the above procedure, corresponds to the votes obtained by a candidate nominated by the party. Evidence for this model is given in Itoh (1978); Itoh and Ueda (1978), (1979). In Itoh (1980), the asymptotic behavior of the following quantity is studied.

Let $L(x)$ denote the minimum of lengths of gaps generated by a one-dimensional Renyi model of parking unit-length cars on a segment of length $x > 1$. Then

$$P(L(x+1) \geq h) = \frac{1}{x} \int_0^x P(L(y) \geq h)\ P(L(x-y) \geq h) dy \tag{3.11}$$

with

$$P(L(x) \geq h) = \begin{cases} 0, & 0 \leq x < h \\ 1, & h \leq x < 1 \\ 0, & x = 1 . \end{cases}$$

It is shown by Laplace transforms methods that for $0 < h < 1$, there exists an $\alpha(h) > 0$ for which

$$\lim_{x \to \infty} \frac{1}{x} \int_0^x e^{\alpha(h)(x+1)} P(L(x) \geq h) dx = 1 , \tag{3.12}$$

and an algorithm is given to obtain upper and lower bounds for $\alpha(h)$.

6. Higher dimensional $(n > 4)$ sequential random packing schemes have been simulated by Itoh and Solomon (1986) using a Hamming distance criterion for acceptance or rejection of new points. This ongoing work suggests that the Palasti conjecture for this model is also false and that the discrepancy is such that the n^{th} power of the one dimensional packing density is smaller than the n-dimensional density.

7. An application of the work of Itoh and Solomon (1986) is to be study of the amino acid code, in which $4^3 \equiv 64$ words are theoretically possible in the triplet coding system by four species of nucleotides. The actual number of amino acids plus chain terminator in the code is only 20. In the Itoh and Solomon scheme, consider a set of $2^6 \equiv 64$ points in 6-space with coordinates 0 or 1. The distance between the points, called the Hamming distance, is the square of the Euclidean distance between the points. One point is chosen uniformly at random and recorded. Another is chosen uniformly from the remainder and recorded if its Hamming distance is at least 2, otherwise it is discarded. The next point is chosen at random from the remainder and recorded if and only if its Hamming distance from each of the recorded points is at least 2, otherwise it is discarded and the process continued until no further point may be recorded. It was found by simulation that the expected proportion of recorded points based on 10,000 simulations of this procedure was 0.3263, close to $0.328125 \equiv 21/64$.

8. An application to the construction of codes seems to be emerging. Consider binary sequences of length n (i.e. n-vectors with entries either 0 or 1). Let the distance between two such n-sequences be the number of locations with different entries. Define the weight of a sequence to be the number of 1's (i.e. the sum of its entries). An (n,M,d) code is a maximal set of M binary n-sequences such that the minimum of the distances between sequences is d. An alternative way of viewing an (n,M,d) code (MacWilliams and Sloane, 1977, p. 41) is to consider each of the M n-vectors as vertices of an n-cube, so that finding an (n,M,d) code is equivalent to finding the maximal number of non-overlapping spheres (denoted M) centered at the vertices of an n-cube, each with radius $\sqrt{d}/2$. A Golay code G_{24} is a code of $n = 24$ dimensional binary sequences, with $d = 8$, containing one of weight 0, and such that there are certain linear restrictions

on the last 12 entries of each 24-sequence. One consequence of the Golay code construction is that there are 4096 = M code words with these weights

weight	0	8	12	16	24
number of 24-length words	1	759	2576	759	1

Itoh and Solomon (1986 - to appear) have found a clever method of obtaining the Golay code G_{24} which involves a sequential random packing simulation as a key component. Starting with the 0-vector (of weight 0), they choose another 24-vector at random and record it if its Hamming distance is either 8, 12, 16, or 24 from 0, otherwise it is discarded. Continuing until a second point is recorded, a third point, chosen at random, is recorded if and only if its distance from the prior two points is either 8, 12, 16, or 24. The process is continued until a total of 12 vectors are obtained in this manner. If the 12 vectors are linearly independent, then all possible binary sums mod 2 of the 12 vectors are obtained. It was found by simulation that when this procedure produced 4096 sequences, that their weights were distributed precisely as in the G_{24} code given in the previous paragraph. This may suggest a way of producing high order (large n) codes by a sequential random packing mechanism.

9. Other papers which consider aspects of sequential random packing schemes but are not cited in the text of this review are listed under Other References.

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The random packing problem has been of interest to investigators in several disciplines. Physical chemists have investigated such models in two and three dimensions. Because of analytical difficulties, one-dimensional analogues have been explored and these are referred to as the parking problem. A number of results are explored and attempts are made to tie them together. Applications are also highlighted. Keywords:

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