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QUANTITATIVE FEEDBACK THEORY (QFT)
Technique for Designing Multivariable Control Systems

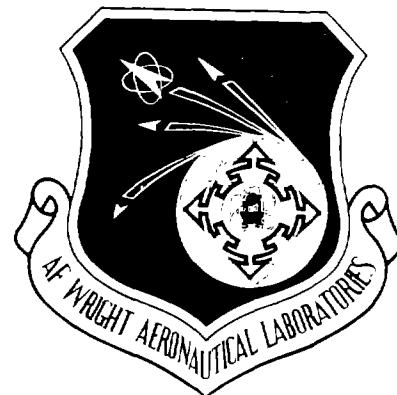
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January 1987

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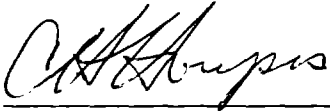
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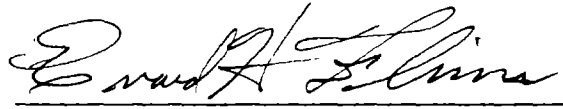
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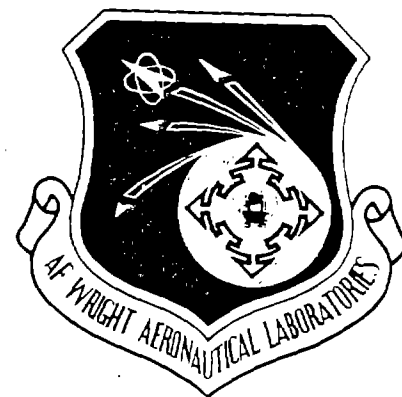
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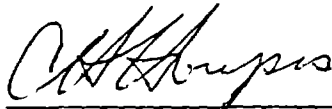
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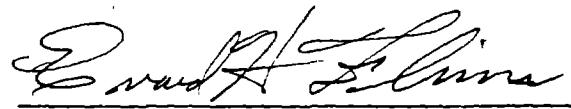
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PREFACE

This report is based upon the technical articles and the unpublished lecture notes of Professor Isaac M. Horowitz and the thesis of AFIT MS students. The report aims to satisfy the desires of practicing engineers and students to have one document that present the Quantitative Feedback Theory (QFT) technique. This is done in this document in a unified and logical manner from the "student" viewpoint. The author has attempted to fulfill this desire by developing this report through various stages of class testing and revision. Any suggestions for further improvement will be welcomed by the author. The author wishes to express his deepest appreciation to his colleague, Professor J.J. D'Azzo, for his editing, suggestions and support for this report. Without the support and encouragement of Mr Evard Flinn, Branch Chief, AFWAL/FIGL and his colleagues, Mr James Morris and Mr Duane Rubertus, this report would not have been a reality.



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TABLE OF CONTENTS

	<u>Page</u>
I. Introduction	1-1
1-1 General Comment of Theory	1-1
1-2 QFT Using The Frequency Response Method	1-2
1-3 Minimum-Phase System Performance Specification	1-5
1-3A Tracking Models	1-5
1-3B Disturbance Rejection Models	1-9
1-4 Summary	1-13
1-5 Problems	1-14
2. SISO LIT Systems	
2-1 Introduction	2-1
2-2 Plant Templates of $P(s)$	2-3
2-3 U-Contour	2-6
2-4 Tracking Bounds $LmB_R(j\omega_1)$ on the N.C.....	2-9
2-5 Disturbance Bounds $B(j\omega_1)$	2-15
2-5A Disturbance Case 1 [$d_2(t) = D_0 u_{-1}(t)$, $d_1(t) = 0$]	2-15
2-5B Disturbance Case 2 [$d_2(t) = D_0 u_{-1}(t)$, $d_2(t) = 0$].....	2-22
2-6 The Composite Boundary, $B_0(j\omega_1)$	2-25
2-7 Shaping of the $L_0(j\omega)$	2-26
2-8 The Design of the Prefilter $F(s)$	2-36
2-9 Design Example	2-39
2-10 m.p. Unstable Plant Template Generation.....	2-55
2-11 Summary	2-57
2-12 Problems	2-58
3. Multiple Input-Multiple Output (MIMO) Uncertain Plants The Frequency Domain	
3-1 Introduction.....	3-1
3-2 The MIMO Plant.....	3-3
3-3 Introduction to MIMO Compensator.....	3-4

3-4	MIMO Compensation.....	3-6
3-5	Justification of the MISO Equivalent Method.....	3-9
3-6	Effective MISO Loops of the MIMO System.....	3-13
3-7	Constraints on the Plant Matrix.....	3-15
3-8	Basically Non-Interacting (BNIC) Loops.....	3-19
3-9	Summary.....	3-20
3-10	Problems.....	3-21
4.	Design Method One-The Single-Loop Equivalents	
4-1	Introduction.....	4-1
4-2	Design Example.....	4-2
4-2A	Performance Tolerances.....	4-3
4-2B	Simplification of the Single-loop Structures.....	4-4
4-3	Equilibrium and Trade-Offs.....	4-6
4-3A	Trade-Off In High Frequency Range.....	4-11
4-3B	Some Universal Design Features.....	4-12
4-4	Problems.....	4-15
5.	MIMO System Design Method Two-Modified Single-Loop Equivalents	
5-1	Introduction.....	5-1
5-2	Design Equations for the 2x2 System.....	5-2
5-3	Reduced Overdesign.....	5-4
5-4	3x3 Design Equations.....	5-4
5-5	m x m System, $m > 3$	5-6
5-6	Conditions for Existence of a Solution.....	5-8
5-6A	Conditions for "a.s. Sensitivity" on a Single-Loop Design.....	5-9
5-6B	Application of Section 5-6A to Design Method 2.....	5-11
5-6C	Inherent Constraints.....	5-13

5-7	Nondiagonal G.....	5-16
5-8	Achievability of a m.p. Effective Plant $P(s)$	5-17
5-9	Conclusions.....	5-24
Appendix A	MISO Design Problem.....	A-1
Appendix B	MIMO Design Problem.....	B-1
Appendix C	Quantitative Feedback Design for Sampled-Data Systems.....	C-1
Appendix D	Longitudinal Handling Qualities Approximations and Bandwidth Minimization For Robust Automatic Flight Control Systems Design Using Quantitative Feedback Theory.....	D-1
Appendix E	Important Observations Regarding The Form Of The Plant Data (Most Important For A Practical Control System Design).....	E-1
References		R-1

LIST OF SYMBOLS

$\underline{P}$	----- Plant with uncertainty
$\mathcal{P}$	----- Script cap pee to denote $\mathcal{P} = \{P\}$
$\mathcal{D}$	----- Script cap dee to denote $\mathcal{D} = \{D\}$
$\mathcal{Q}$	----- Script cap que to denote $\mathcal{Q} = \{Q\}$
$\underline{D}$	----- Roman cap dee to denote the disturbance control ratio matrix
$\underline{T}$	----- Roman cap tee to denote the tracking control ratio matrix
T	----- Script cap tee to denote $T = \{T_{ij}(s)\} = \{T\}$
$\delta_R(j\omega_i)$	---- The difference between B_U and B_L
B_U	----- is the upper bound, LmT_{RU}
B_L	----- is the lower bound, LmT_{RU}
L_o	----- optimal loop transmission function
δ_{hf}	----- is the difference between the augmented bounds of B_U and B_L
γ	----- phase margin angle
ω_b	----- bandwidth frequency of the models for $\underline{T}_R$
ω_ϕ	----- phase-margin frequency
α_p	----- peak value of the disturbance response
$\delta_D(j\omega_i)$	----- the upper value, $LmT_{DU}(j\omega_i)$
$L_o(s), \underline{L}_o(j\omega)$	----- the optimal loop transfer functions
$\underline{B}_R(j\omega_i), \underline{B}_D(j\omega_i), \underline{B}_o(j\omega_i)$	--- bounds on $LmL(j\omega_i)$ for tracking and disturbance, respectively, and the optimal bounds.
λ	----- excess of poles over zeros
y_{r_i}	----- is that portion of the i^{th} output due to the i^{th} input
$y_{d_{ij}}$	----- is that portion of the i^{th} output due to the cross-coupling of the other outputs

b_{ij} ----- upper bound for t_{ij}
 a_{ij} ----- lower bound for t_{ij}
a.s. ----- arbitrarily small
a.l. ----- arbitrarily large
r.h.p. ----- right half-plane
l.h.p. ----- left half-plane
m.p. ----- minimum phase
N.C. ----- Nichols Chart
 B_h ----- Ultra High Frequency Boundary (UHFB)
BW ----- bandwidth

CHAPTER 1 INTRODUCTION

1-1 General Comment of Theory¹⁻³

Quantitative feedback theory (QFT) is a unified theory using the available measureable states that is applied to the design of multiple-input multiple-output (MIMO) systems, and incorporates

- a. multivariable nature of control systems
- b. plant uncertainties
- c. wide variations vs time of plant parameters
- d. robustness performance requirements
- e. disturbance attenuation requirements
- f. nonlinearities in the plant model
- g. requirements for decoupled outputs

The representation of a MIMO system with m inputs and ℓ outputs is shown in Fig. 1-1.

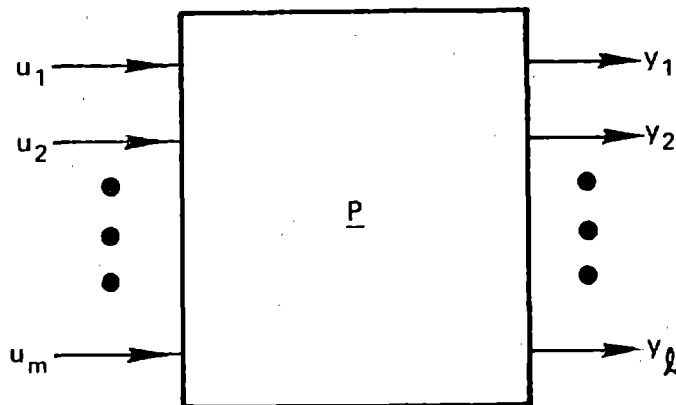


Fig. 1-1 A MIMO uncertain plant

This design technique is applicable to the following problem classes:

1. SISO linear time invariant (LTI) systems
2. SISO nonlinear systems. These are rigorously converted to equivalent class 1 systems whose solutions are guaranteed to work for a large problem class.
3. MIMO LTI systems. The performance specifications on each individual closed-loop system transfer function and on all the disturbance functions must be specified.
4. MIMO nonlinear systems. They are rigorously converted to equivalent class 3 systems whose solutions are guaranteed to work for a large class.

The QFT frequency domain technique is based upon its application to SISO systems. Thus problem classes 3 and 4 are converted into equivalent sets of SISO systems to which the QFT design technique is applied. The objective of this technique is to find compensation functions which guarantee that the performance tolerances are satisfied over the full range of the plant uncertainty. In quantitative design the amount of feedback designed into the system is such as to obtain the desired tolerances, over the given uncertainty range.

1-2 QFT Using The Frequency Response Method

Figure 1-2 represents the feedback control system in which P is the transfer function of the plant which may contain variable parameters, G is the cascade compensator, and F is an input filter transfer function. The output $c(t)$ is required to track the command input $r(t)$ and to reject the disturbances D_1 and D_2 . Figure 1-2 is to be independently designed so that the sensitivity of $c(t)$ to the uncertainty in the plant P is acceptable and to attenuate the effects of the disturbances $d_1(t)$. Also, the filter properties of $F(s)$ must be

designed to achieve the desired tracking response $c(t)$ to the input $r(t)$.

Since the control system in Fig. 1-2 has two measurable quantities, R and C , it is referred to as a canonical two degree-of-freedom (DOF) feedback structure.

If the two disturbances inputs could be measured, then it would be a four DOF structure.

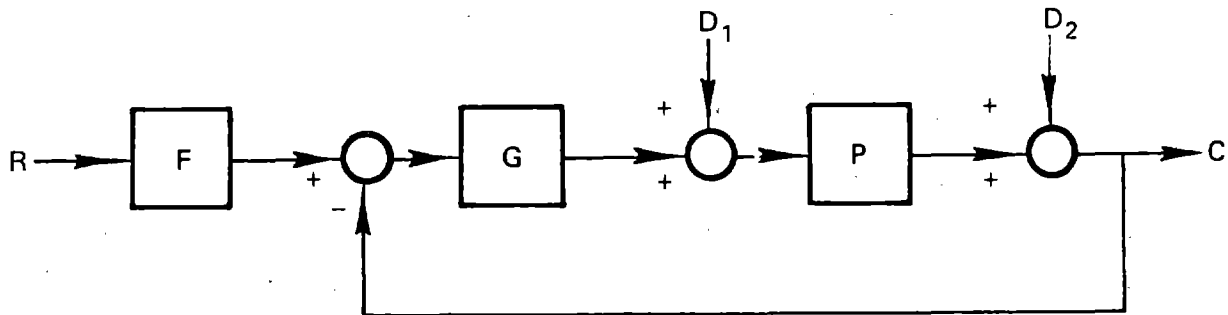


Fig. 1-2. A canonic two degree-of-freedom feedback structure.

The actual design and the cost of feedback are closely related to the extent of the uncertainty and to the narrowness of the performance tolerances. The uncertainty of the plant transfer function is denoted by the set

$$P \equiv \{\underline{P}\} \quad (1-1)$$

and as illustrated by Example 1-1.

Example 1-1 The plant transfer function is

$$P(s) = \frac{K}{s(s+a)} \quad (1-2)$$

where the values of K , a are fixed (no variation) but are uncertain with K in

[1, 10] and a in $[-2, 2]$. The uncertainty of the plant is illustrated by Fig. 1-3.

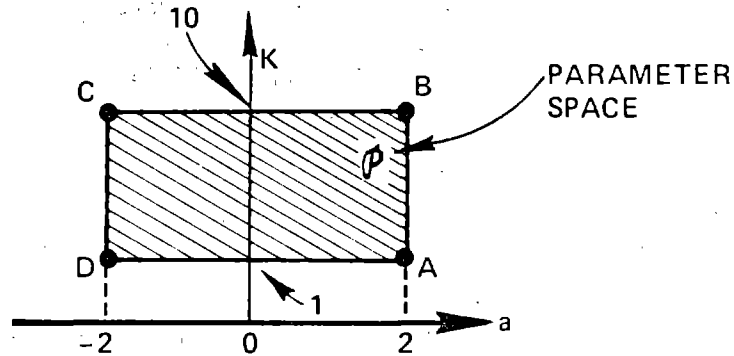


Fig 1-3. Plant uncertainty.

The uncertainty of the disturbance is denoted by

$$\mathcal{D} \equiv \{ \underline{D} \} \quad (1-3)$$

and the acceptable closed-loop transmittances (control ratios) are denoted by

$$\mathcal{T} \equiv \{ \underline{T} \} \quad (1-4)$$

The design objective is to guarantee that each of the actual $T_R(s) = C(s)/R(s)$ or $T_D(s) = C(s)/D(s)$ are a member of the set of acceptable $\underline{T}_R$ and $\underline{T}_D$, respectively, for all $\underline{P} \in \mathcal{P}$. Horowitz has shown that time-varying and nonlinear uncertain plant sets can be converted into equivalent LTI plant sets to which frequency domain techniques can be readily applied and where the fundamental tradeoffs are highly visible. In a feedback system, the principal challenge in control system design is to relate the specifications to the requirements on the loop transmission function $L(s) = G(s)P(s)$ in order to achieve the desired benefits of feedback, i.e., the desired reduction in sensitivity to plant uncertainty and desired disturbance attenuation. The advantage of the frequency domain is that $L(s) = G(s)P(s)$ is simply the multiplication of complex numbers. In the ω -domain it is possible to isolate and pinpoint $\underline{L}(j\omega)$ at every ω_i separately and thus, at each ω_i the optimal bounds on $\underline{L}(j\omega_i)$ can be determined

by the design technique.

1-3 Minimum-Phase System Performance Specifications^{4,11}

In order to apply the QFT technique it is necessary to synthesize the desired or model control ratio based upon the system's performance specifications in the time domain. For the structure of Fig. 1-2 the control ratios are

$$\text{Tracking: } (D_1 = D_2 = 0) \quad (1-5)$$

$$T_R(s) = \frac{F(s)G(s)P(s)}{1 + G(s)P(s)} = \frac{FL}{1 + L}$$

$$\text{Disturbance: } (R = 0) \quad (1-6)$$

$$T_{D_1}(s) = \frac{P(s)}{1 + G(s)P(s)} = \frac{P}{1 + L} \text{ with } D_2 = 0$$

$$T_{D_2}(s) = \frac{1}{1 + G(s)P(s)} = \frac{1}{1 + L} \text{ with } D_1 = 0 \quad (1-7)$$

1-3A Tracking Models

The tracking thumb print specifications, based upon satisfying some or all of the step forcing function figures of merit for underdamped

$(M_p, t_p, t_s, t_r, K_m)$ and overdamped (t_s, t_r, K_m) responses, respectively, for a simple second - order system, is depicted in Fig. 1-4. The time responses $c(t)_U$ and $c(t)_L$ in this figure represent the upper and lower bounds, respectively, of the thumb print specifications, that is, an acceptable response $c(t)$ must lie between these bounds. The Bode plots of $LmT_R(j\omega)$ vs ω for these upper B_U and

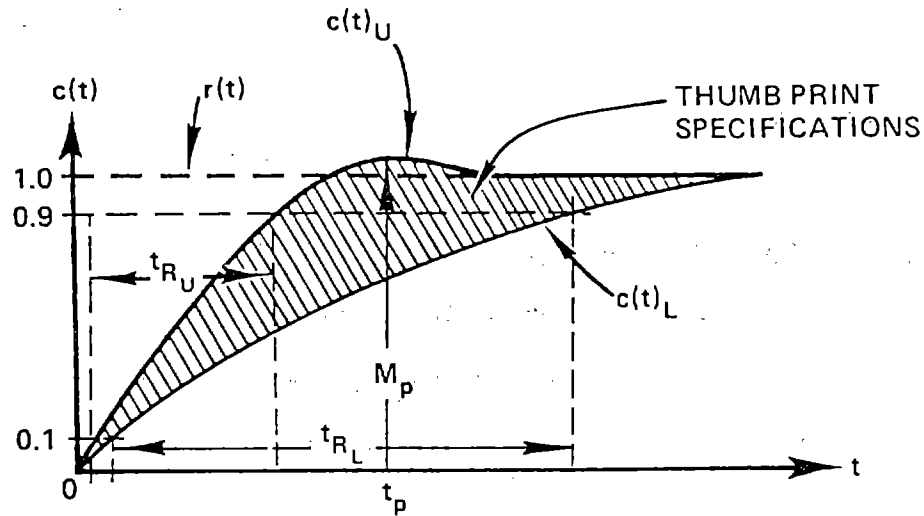


Fig. 1-4. The tracking thumb print specifications.

lower B_L bounds are shown in Fig. 1-5. Note that, for minimum-phase plants, only the tolerance on $|T_R(j\omega)|$ must be satisfied for a satisfactory design. For nonminimum-phase plants, tolerances on $|T_R(j\omega)|$ must also be specified and satisfied in the design process. It is desirable to synthesize T_{R_U} and T_{R_L} so $\delta_R(j\omega_i)$ increases as ω_i increases from the 0-dB crossing frequency of T_{R_U} . This characteristic of $\delta_R(j\omega_i)$ may simplify synthesizing an optimal loop transmission $L_o(s) = G(s)P_o(s)$ as discussed in Sec. 2-7. To synthesize $L_o(s)$ it is necessary to determine the bounds $B_R(j\omega_i)$ which are obtained based upon $\delta_R(j\omega_i)$. Thus this characteristic of $\delta_R(j\omega_i)$ ensures that the bounds $|B_R(j\omega_i)|$ decrease as ω_i increases (see Sec. 2-4).

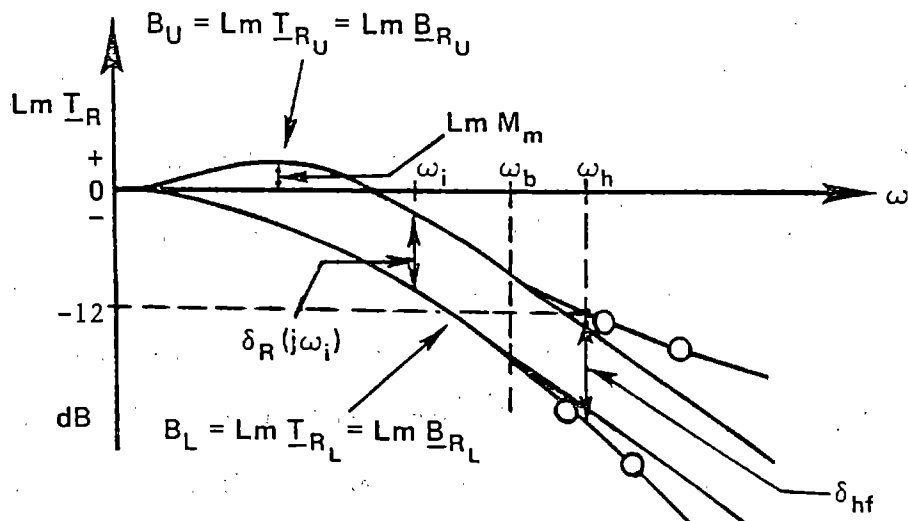


Fig. 1-5 Bode Plots of $T_R(j\omega)$ and $T_{R_U}(j\omega)$

The QFT technique requires that the desired control ratios be modeled to satisfy the required gain K_m and the desired performance specifications for a step input. Thus, M_p , t_p , t_s , and t_r must be satisfied for T_{R_U} and T_s (or t_s) and t_r must be satisfied for T_{R_L} . The simplest model of the control ratio for T_{R_U} is a simple second-order system which is of the form

$$T_{R_U}(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s - p_1)(s - p_2)} \quad (1-8)$$

where $\omega_n^2 = p_1 p_2$. The control ratio $T_{R_U}(s)$ of Eq. (1-8) can also be represented by an equivalent unity feedback system so

$$T_{R_U}(s) = \frac{G_{eq}(s)}{1 + G_{eq}(s)} \quad (1-9)$$

where

$$G_{eq}(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)} \quad (1-10)$$

for which the gain constant for this type 1 system is

$$K_1 = \lim_{s \rightarrow 0} [sG_{eq}(s)] = \omega_n / 2\zeta$$

Equation (1-8) satisfies the requirement that $c(t)_{ss} = R_0$ for $r(t) = R_0 u_{-1}(t)$.

The four figures of merit for a simple second-order underdamped stable system are

$$M_p = 1 + e^{-\zeta\pi / \sqrt{1 - \zeta^2}} \quad (1-11)$$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} \quad (1-12)$$

$$T_s = \frac{4}{|\sigma|} = \frac{4}{\zeta \omega_n} \quad (\text{for } \pm 2\% \text{ criterion}) \quad (1-13)$$

$$K_1 = \frac{\omega_n}{2\zeta} \quad (\text{the ramp error coefficient}) \quad (1-14)$$

Note that there are only two adjustable parameters, ζ and ω_n . Thus, only two of the specifications defined by Eqs. (1-11) through (1-14) can be used to locate the two dominant poles, $p_{1,2} = \sigma \pm j\omega_d$ of Eq. (1-8). Therefore, it may not be possible to satisfy all four specifications. The frequency ω_j for which $\left| \frac{T_R}{T_U}(j\omega) \right| = 0.7071$ is defined as the system frequency bandwidth ω_{b_U} .

The simplest overdamped model for T_{RL} is of the form

$$T_{RL}(s) = \frac{C(s)}{R(s)} = \frac{K}{(s - \sigma_1)(s - \sigma_2)} = \frac{G_{eq}(s)}{1 + G_{eq}(s)} \quad (1-15)$$

where
$$G_{eq}(s) = \frac{\sigma_1 \sigma_2}{s[s - (\sigma_1 + \sigma_2)]}$$

and
$$K_1 = -\sigma_1 \sigma_2 / (\sigma_1 + \sigma_2)$$

For this system $c(t)_{ss} = R_o$ for $r(t) = R_o u_{-1}(t)$. Selection of the parameters σ_1 and σ_2 is used to meet the specifications for t_s and K_1 . This selection also determines the frequency bandwidth ω_{b_L} .

Once the models $T_{RU}(j\omega)$ and $T_{RL}(j\omega)$ are determined, then the plots of Fig. 1-5 can be drawn. Because the models for T_{RU} and T_{RL} are both second order, the high frequency asymptotes in Fig. 1-5 have the same slope. The high frequency range in Fig. 1-5 is defined as $\omega \geq \omega_b$ where ω_b is the model bandwidth frequency of B_U . In the high frequency (hf) range the QFT method may require a larger spread between B_U and B_L , that is,

$$\delta_{hf} = B_U - B_L \quad (1-17)$$

as explained below. This must be achieved by changing B_U and B_L without violating the desired time response characteristics associated with B_U and B_L . This spread can be achieved by augmenting T_{RL} with a negative real axis pole as close to the origin as possible but far enough away not to significantly affect the time response. This additional pole must lower B_L in the high frequency range. The spread can be further increased by augmenting T_{RU} with a zero as close to the origin as possible without significantly affecting the time response. This additional zero must raise the curve B_U in the high frequency range. This is done because at high frequencies δ_{hf} must be larger than the actual variation in the plant, ΔP . This comes about because of the Bode derived theorem which states that $\int_0^\infty \text{Lm} S_P^T d\omega = 0$, i.e. the reduction in sensitivity S_P^T at the lower frequencies must be compensated by an increase in sensitivity at the higher frequencies.

At some high frequency $\omega_i > \omega_h$ (see Fig. 1-5) since $\left| \frac{T_{RL}}{T_{RU}} \right| \approx 0$ then let

$$\delta_R(j\omega_i) \approx B_U - (-\infty) = \infty \text{ dB} \quad (1-17a)$$

1-3B. Disturbance Rejection Models⁶ ----- The requirements for disturbance

rejection for the control system of Fig. 1-2, with a constant disturbance

$d_2(t) = D_o u_{-1}(t)$, and $r(t) = d_1(t) = 0$, is

$$|c(t)| \leq \alpha_p \quad \text{for} \quad t \geq t_x \quad (1-18)$$

as shown in Fig. 1-6. Note that for this situation $c(0) = D_o$. When

$d_1(t) = D_o u_{-1}(t)$ and $r(t) = d_2(t) = 0$, then the requirement is

$$|c(t_p)| \leq \alpha_p \quad (1-19)$$

as shown in Fig 1-7, where $c(0) = 0$.

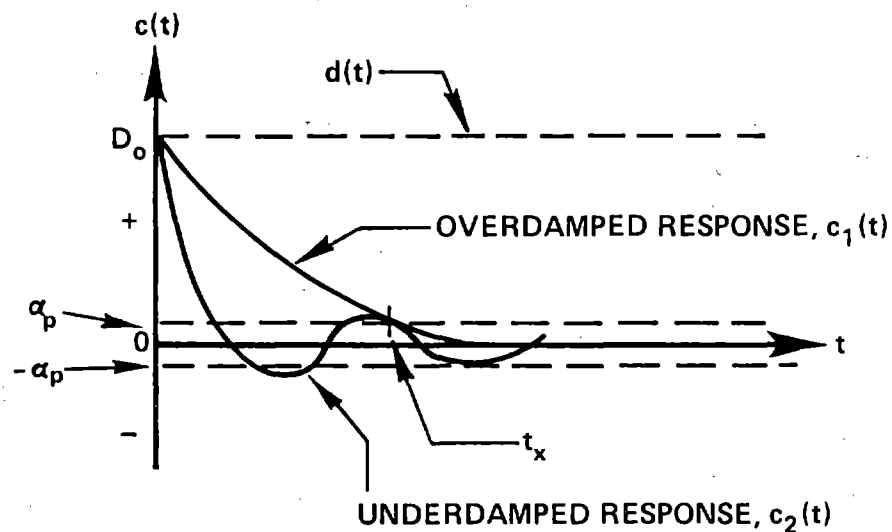


Fig. 1-6 Disturbance rejection time response for Fig. 1-2 with

$$r(t) = d_1(t) = 0.$$

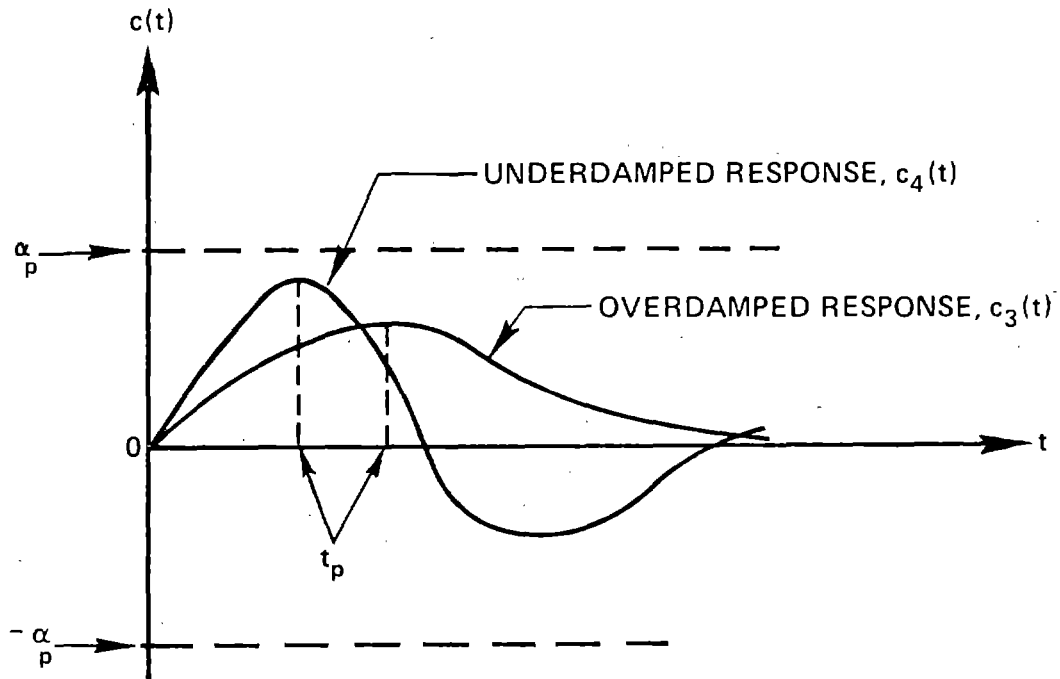


Fig. 1-7 Disturbance rejection time response for Fig. 1-2 with $r(t) = d_2(t) = 0$ and $d_1(t) = D_o u_{-1}(t)$.

The simplest control ratio models for Figs. 1-6 and 1-7, respectively, are shown below:

$r(t) = d_1(t) = 0$ and $d_2(t) = D_o u_{-1}(t)$		
$T_D = \frac{C_1(s)}{D_2(s)} = \frac{s}{s + a} \quad (1-20)$	$C_1(s) = \frac{D_o}{s + a}$	$c_1(t) = D_o e^{-at} \quad (1-21)$
$T_D = \frac{C_2(s)}{D_2(s)} = \frac{s(s + a)}{(s + a)^2 + b^2} \quad (1-22)$	$C_2(s) = \frac{D_o(s + a)}{(s + a)^2 + b^2}$	$c_2(t) = D_o e^{-at} \cos bt \quad (1-23)$
See Fig. 1-6		

$r(t) = d_2(t) = 0$ and $d_1(t) = D_o u_{-1}(t)$		
$T_D = \frac{C_3(s)}{D(s)} = \frac{K_D s}{(s+a)(s+b)} \quad (1-24)$	$C_3(s) = \frac{K_D D_o}{(s+a)(s+b)}$	$c_3(t) = \frac{K_D D_o}{b-a} (e^{-at} - e^{-bt}) \quad (1-25)$
$T_D = \frac{C_4(s)}{D_1(s)} = \frac{K_D s}{s^2 + 2as + a^2 + b^2} \quad (1-26)$	$C_4(s) = \frac{K_D D_o}{(s+a)^2 + b^2}$	$c_4(t) = \frac{K_D D_o}{b} e^{-at} \sin bt \quad (1-27)$
See Fig. 1-7		

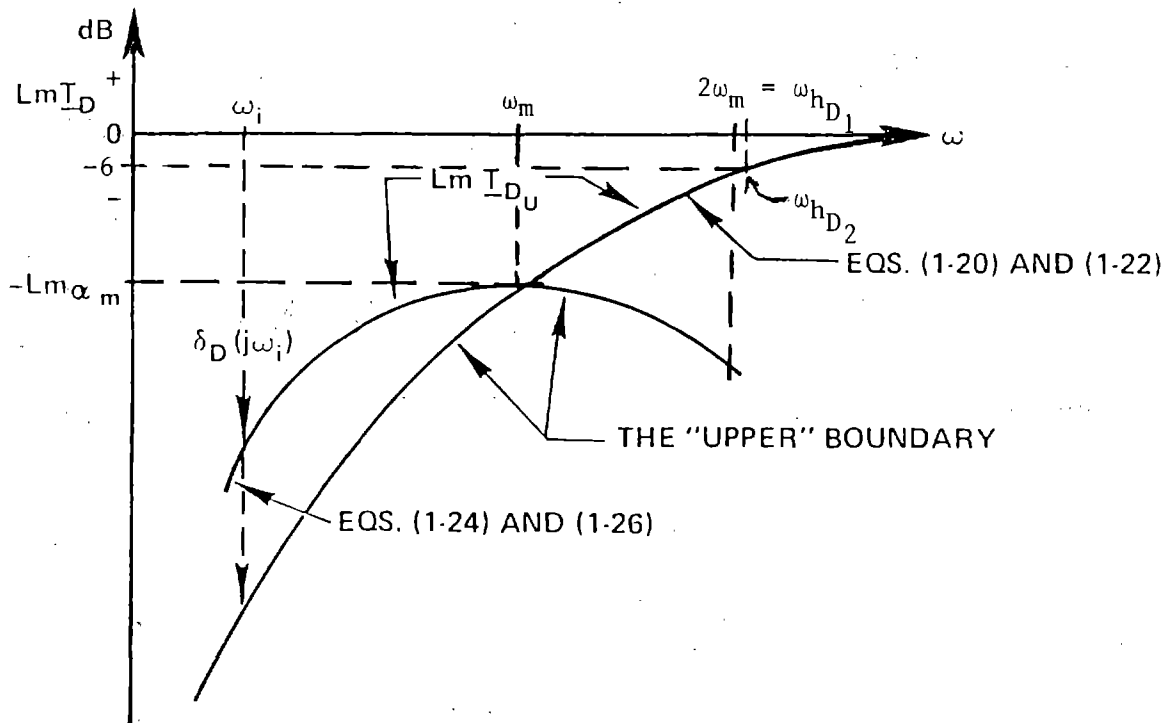


Fig. 1-8 Bode Plots of $T_D(j\omega)$

The Bode plots of Eqs (1-20), (1-22), (1-24), and (1-26) are shown in Fig. 1-8. As seen from this figure, there is only an upper boundary for $L_m \underline{T}_D$.

1-4 Summary

This chapter presents a general introduction to the QFT technique. It is essentially a frequency response design method applied to the design of a control system with an uncertain plant. This design is based upon trying to achieve the desired system performance specifications.

PROBLEMS^{3,4}

Problem 1-1

Given: $P(s) = \frac{10}{s(s+5)}$

Desire $K_v = K_1 = 25s^{-1}$ and $\gamma = \theta_m = 40^\circ$

Do by use of a lag compensator

Note: $T = \frac{L}{1+L}$ where $L = G_c(s)P(s)$

Problem 1-2 Repeat above for a lead compensator

Problem 1-3 Add to the above the requirement of a cross-over frequency

$$\omega_c = 15\text{rps at } L_m L = 0\text{dB}$$

Problem 1-4 Add to the above the desire that the

Gain Margin be 12dB

$$\theta_m = 40^\circ$$

$$\omega_c = 15\text{rps}$$

$$K_1 = 25s^{-1}$$

But also want $\angle \underline{L} \geq -140^\circ$ for $\omega < \omega_c$ and be optimized as described in Sec's 5.8 and 5.9 of reference 3.

2-1 Introduction

The basic concepts of the QFT technique can best be explained by means of a design example. The system configuration is shown in Fig. 1-2 whose uncertainty of the plant is denoted by Eq. (1-1). The system contains three inputs. The first objective is to track a step input $r(t) = u_{-1}(t)$ and satisfy the thumb print specifications of Fig. 1-4 with no steady-state error. The second objective is to attenuate the system response due to step disturbance inputs $d_1(t)$ and $d_2(t)$, as described in Figs. 1-6 and 1-7. An outline of the basic design procedure for the QFT technique, as applied to a minimum phase plant, is as follows:

1. Synthesize the tracking model control ratios $T_R(s)$ in the manner described in Sec. 1-3A, based upon the desired tracking specifications (see Fig. 1-5).
2. Synthesize the disturbance rejection model control ratios $T_D(s)$ in the manner described in Sec. 1-3B, based upon the disturbance rejection specifications.
3. Obtain templates of $\underline{P}(j\omega_i)$ that pictorially describe the range of the plant uncertainty on the Nichols chart (NC).
4. Select a nominal plant from the set of Eq. (1-1) denoted as P_o .
5. In conjunction with Step 6 determine the U-contour based upon the specified values of $\delta_R(j\omega_i)$ for tracking, M_L for disturbance rejection, and V for the universal high-frequency boundary (UHFB) B_h .
6. Steps 2 and 3 are used along with the values of $\delta_D(j\omega_i)$ (see Fig. 1-8) to determine the disturbance bound $\underline{B}_D(j\omega_i)$ on the loop transmission $\underline{L}_D(j\omega_i) = \underline{G}(j\omega_i)\underline{P}(j\omega_i)$. For minimum-phase systems this requires that the synthesized loop transmission must satisfy the requirement that $L_m\underline{L}_D(j\omega_i)$ is on or above the curve for $L_m\underline{B}_D(j\omega_i)$ on the Nichols diagram.

7. The tracking model (Step 1), the templates $\underline{P}(j\omega_i)$ (Step 3), and the values of $\delta_R(j\omega_i)$ (see Fig. 1-5) and M_L [see Eq. (2-11)] are used to determine the tracking bound $\underline{B}_R(j\omega_i)$ on the nominal loop transmission $\underline{L}_{R0}(j\omega_i) = \underline{G}(j\omega_i)\underline{P}(j\omega_i)$. For minimum-phase systems this requires that the synthesized loop transmission must satisfy the requirement that $\text{Lm}\underline{L}_R(j\omega_i)$ is on or above the curve for $\text{Lm}\underline{B}_R(j\omega_i)$ on the Nichols diagram.

8. Plot curves of $\text{Lm}\underline{B}_R(j\omega_i)$ versus $\phi_R = \angle \underline{B}_R(j\omega_i)$ and $\text{Lm}\underline{B}_D(j\omega_i)$ versus $\phi_D = \angle \underline{B}_D(j\omega_i)$ on the same Nichols chart. For a given value of ω_i at various values of the angle ϕ , select the value of $\text{Lm}\underline{B}_D(j\omega_i)$ or $\text{Lm}\underline{B}_R(j\omega_i)$, whichever is the largest value (termed the "worst" or "most severe" boundary). Draw a curve through these points. The resulting plot defines the overall boundary $\text{Lm}\underline{B}_0(j\omega_i)$ vs ϕ . Repeat this procedure for sufficient values of ω_i .

9. In order to achieve the lowest possible value of the loop transmission frequency (phase margin frequency) it is necessary to design $\underline{L}_0(j\omega_i)$ to be as close as possible to the boundary value $\underline{B}_0(j\omega_i)$. Synthesize an $\underline{L}_0(s) = \underline{G}(s)\underline{P}(s)$ utilizing the $\text{Lm}\underline{B}_0(j\omega_i)$ boundaries and the U-contour so that $\text{Lm}\underline{L}_0(j\omega_i)$ is on or above the curve for $\text{Lm}\underline{B}_0(j\omega_i)$ on the Nichols diagram. Note that $\underline{L}_0(j\omega_i) \geq \underline{B}_0(j\omega_i)$ represents the loop transfer function that satisfies the most severe boundary $\underline{B}_R$ and $\underline{B}_D$.

10. Based upon the information available from steps 1 and 9, synthesize an $\underline{F}(s)$ that results in a $\text{Lm}\underline{T}_R$ [Eq. (1-5)] vs ω that lies between B_U and B_L of Fig. 1-5.

11. Obtain the time response data for $c(t)$: (a) with $d(t) = u_{-1}(t)$ and $r(t) = 0$ and (b) with $r(t) = u_{-1}(t)$ and $d(t) = 0$ for sufficient points around the parameter space describing the plant uncertainty (e.g., see Fig. 1-3).

A simple plant is used in the following sections to illustrate the details in applying this QFT design procedure.

2-2 Plant Templates of $P(s)$, $\underline{TP}(j\omega_1)$

The tracking control ratio for Fig. 1-2 with $D_1 = D_2 = 0$ is

$$T_R(s) = F \left[\frac{L}{1 + L} \right] \quad (2-1)$$

where

$$L = GP \quad (2-2)$$

From Eq. (2-1)

$$\text{Lm} \underline{T_R} = \text{Lm} \underline{F} + \text{Lm} \left[\frac{\underline{L}}{1 + \underline{L}} \right] \quad (2-3)$$

Since F is LTI, then

$$\Delta(\text{Lm} \underline{T_R}) = \text{Lm} \underline{T_R} - \text{Lm} \underline{F} = \text{Lm} \left[\frac{\underline{L}}{1 + \underline{L}} \right] \quad (2-4)$$

is the change in T_R due to the uncertainty in P . By the proper design of $I = L_0$ and F , this change in T_R is restricted so that the actual value of $\text{Lm} \underline{T_R}$ will always lie between B_U and B_L . The first step in synthesizing an L_0 is to make templates which characterize the variation of the plant uncertainty for various values of ω_1 over a frequency range of $\omega_x \leq \omega_1 \leq \omega_h^*$. The guidelines for selecting the frequency range for the templates are: (a) for a type-0 plant select $\omega_x = 0$, (b) select three frequency values up to the -3dB point of the B_U plot in Fig. 1-5, (c) select frequency values no less than an octave apart, and (d) for type-1 or higher plants select $\omega_x \neq 0$ which satisfies guidelines (b) and (c).

Obtaining the templates for a plant $\underline{P}(j\omega_1)$ with uncertain parameters may be involved. For example, an aircraft operates over a wide range of speed, altitude, and weight. For each flight condition FC there is a different transfer function. However, a simple plant is used here as an example to

*Select the largest value of ω_h obtained from Figs. 1-5 and 1-8.

illustrate how the templates are obtained for a plant with variable parameters.

Consider the simple plant

$$P(s) = \frac{Ka}{s(s+a)} \quad (2-5)$$

where

$$K \in \{1, 10\}, \quad a \in \{1, 10\} \quad (2-6)$$

The region of plant uncertainty, as prescribed by Eq. (2-6), for the plant of Eq. (2-5) is depicted in Fig. 2-1. The boundary of the plant template can usually be obtained by mapping the boundary of the plant parameter region.

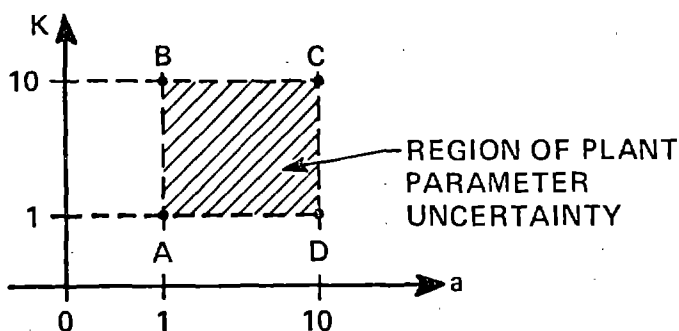


Fig. 2-1 Region of plant uncertainty for Eqs. (2-5) and (2-6).

A number of points on the perimeter of ABCD of Fig. 2-1 are selected, and values of $\text{Lm} \underline{P}(j\omega_i)$ and $\angle \underline{P}(j\omega_i)$ are obtained at each point. This data, for $\omega = \omega_i$, is plotted on a Nichols diagram. A curve is drawn through these points and becomes the contour at the frequency ω_i for the template $\underline{T}_P(j\omega_i)$. A sufficient number points must be selected so that the contour of $\underline{T}_P(j\omega_i)$ accurately reflects the region of uncertainty shown in Fig. 2-1. For example, in addition to the points A, B, C and D, it may also be necessary to include points between B and C and between A and D in Fig. 2-1. The data for the points A, B, C and D at the frequency $\omega = 1$ rps, from Eq. (2-5), is:

$$\begin{aligned} \underline{P}_A &= \frac{1}{j1(1+j1)} = \frac{\sqrt{2}}{2} \angle -135^\circ & \underline{P}_B &= 5\sqrt{2} \angle -135^\circ \\ \underline{P}_C &= \frac{100}{j1(10+j1)} = \frac{100}{\sqrt{101}} \angle -95.7^\circ & \underline{P}_D &= \frac{10}{\sqrt{101}} \angle -95.7^\circ \end{aligned} \quad (2-7)$$

The data of Eq. (2-6) is plotted on the Nichols chart shown in Fig. 2-2. A curve is drawn through the points A, B, C and D as shown in this figure, where the shaded area is labeled $\mathcal{T}_P(j1)$. The contour A, B, C, and D in Fig. 2-2 may be drawn on a plastic sheet (preferably colored) so that a plastic template for $\mathcal{T}_P(j1)$ can be cut out and so labeled. The templates for other values of ω_i are obtained in a similar manner. For the aircraft example, each point A, B, C, D, etc., in Fig. 2-2 represents a given F.C. at $\omega = \omega_i$, i.e., $[\underline{P}_{A/C}(j\omega_i)]_{FC_j}$.

From an analysis of the templates it is often easy to identify a plant P , where $P \in \mathcal{P}$ as the nominal P_0 . For the plant of Eq. (2-5) the values $K = a = 1$ represent the lowest point of each of the templates $\mathcal{T}_P(j\omega_i)$. Although for this example it is convenient to select the nominal plant having the values $K = a = 1$ for all frequencies it should be emphasized that any P in $\mathcal{P}$ can be chosen as the nominal plant.¹³ The significance of the templates can best be illustrated as

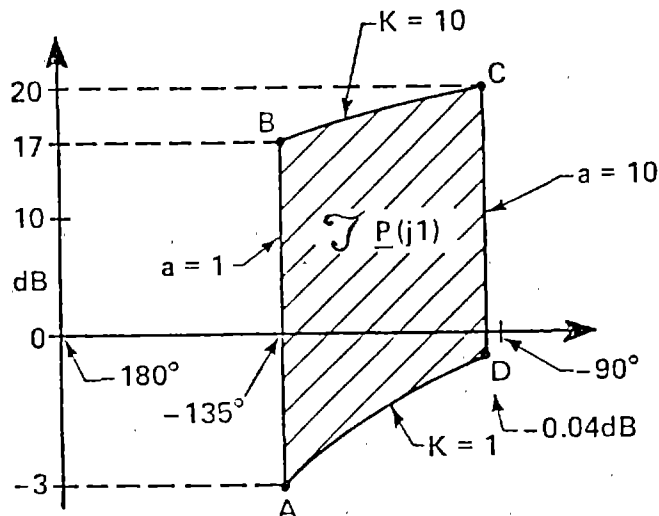


Fig. 2-2 The template $\mathcal{T}_P(j1)$ characterizing Eq. (2-5) over the region of uncertainty of Eq. (2-6).

follows: let $\underline{L} = \underline{G}\underline{P}$ for any $\underline{P} \in \underline{P}$ and $\underline{L}_0 = \underline{G}\underline{P}_0$. Since $\underline{G}$ is LTI, then

$$\delta_P(j\omega_i) = \text{Lm}\underline{L} - \text{Lm}\underline{L}_0 = (\text{Lm}\underline{G} + \text{Lm}\underline{P}) - (\text{Lm}\underline{G} + \text{Lm}\underline{P}_0) = \text{Lm}\underline{P} - \text{Lm}\underline{P}_0 \text{ dB}$$

and

$$\angle \Delta \underline{P} = \angle \underline{P} - \angle \underline{P}_0 \text{ Degrees}$$

Thus if point A in Fig. 2-2: (a) represents $\text{Lm}\underline{P}_0$ then a variation in $\underline{P}$ results in a horizontal translation in the angle of $\underline{P}$ and a vertical translation in the log magnitude value of $\underline{P}$; (b) represents $\text{Lm}\underline{P}$ then the template $\underline{T}\underline{P}(j\omega_i)$ (pattern ABCD) may be translated but not rotated on the Nichols chart by the amount $\text{Lm}\underline{G}(j\omega_i)$ to form $\underline{L}(j\omega)$. That is, $\underline{T}\underline{P}(j\omega_i)$ can be translated vertically by the amount $\text{Lm}\underline{G}(j\omega_i)$ and horizontally by the amount $\angle \underline{G}$. (See the example in Reference 1, pg. 13); (c) represents $\text{Lm}\underline{L}_0$ then any point within or on the $\underline{T}\underline{P}(j1)$ represents the point $\text{Lm}\underline{L}(j1)$ whose corresponding value of $\text{Lm}\underline{T} = \text{Lm}[\underline{L}/(1 + \underline{L})]$ is given by the corresponding M-contours that goes through this point.

2-3 U-Contour^{1,2}

In order to ensure that the resulting synthesized optimum loop transfer function $\underline{L}_0$ does not yield roots of the characteristic equation close to the imaginary-axis, it is necessary to impose an additional constraint of the form $|1 + \underline{L}(j\omega)| \geq$ some constant for all ω and $\underline{P}$. More conveniently this constraint is expressed as $|\underline{L}/(1 + \underline{L})| \leq M_L$ a constant. This constraint establishes a region on the Nichols chart (N.C.) which $\text{Lm}\underline{L}_0(j\omega)$ must not penetrate. The boundary of this region is referred to as the universal high-frequency boundary (UHFB), the U-contour, because this becomes the dominating constraint on $\underline{L}$ at large ω values, and results in a single fixed boundary (see later).

For the two cases of disturbance rejection depicted in Figs. 1-6 and 1-7 the control ratios are, respectively,

$$T_D = \frac{1}{1 + \underline{L}} \quad \text{and} \quad T_D = \frac{\underline{P}}{1 + \underline{L}}$$

Thus, it is necessary to synthesize the optimum $L_o(s)$ so that the disturbances are properly attenuated. For the present, only one aspect of this disturbance response problem is considered, namely, that there will generally be a constraint on the damping factor ζ of the dominant complex pole pair of T_D nearest the $j\omega$ -axis⁶. This damping factor is related to the peaking in

$$\left| \underline{T}(j\omega) \right| = \left| \frac{\underline{L}(j\omega)}{1 + \underline{L}(j\omega)} \right| \quad (2-8)$$

For example consider that the dominant complex-pole pair of Eq. (2-8) results in a peak of 8dB(= LmM_m) for $\zeta = 0.2$, and 2.7dB for $\zeta = 0.4$, etc. Although this large peaking does not appear in T_R due to the design of the filter $F(s)$ it does affect the response for T_D . Although if $d(t)$ is very small a peak of T_D , due to $\zeta = 0.2$, can be "very large" and be such that the requirement [see Eqs (1-18) and (1-19)]

$$\left| c(t) \right| \leq \alpha p$$

may be difficult to achieve. Therefore it is reasonable to add the requirement

$$\left| \underline{T} \right| = \left| \frac{\underline{L}}{1 + \underline{L}} \right| \leq M_L \quad (2-9)$$

where M_L be a constant for all ω and over the whole range of P parameter values. This results on a constraint on ζ of the dominant complex-pole pair of T_D . This constraint can therefore be translated into a constraint on the M_m value of Eq. (2-8). For example, let $LmM_m = 2\text{dB}$ then the oval, agefa, in Fig. 2-3 should correspond to the 2dB M-contour on the N.C.. This results in limiting the peaking of the disturbance response. A value of M_L can be that corresponding to the M_m of T_R . Therefore the top portion, efa, of the M-contour on the Nichols chart which corresponds to the value of the selected value of M_L becomes part of

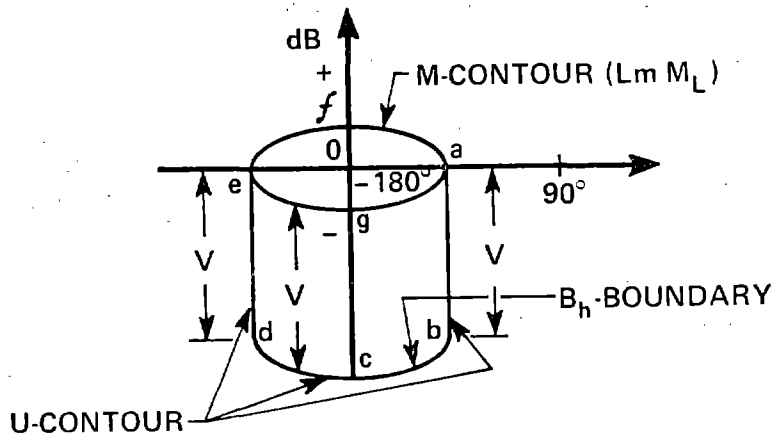


Fig 2-3 U-Contour Construction

the U-contour.

For a large problem class, as $s \rightarrow \infty$ the plant approaches the form of

$$\lim_{s \rightarrow \infty} [P(s)] = \frac{K}{s^\lambda}$$

where λ represents the excess of poles over zeros of $P(s)$. The plant template, for this problem class, approaches a vertical line of length

$$\Delta LmP \cong \lim_{\omega \rightarrow \infty} [LmP_{\max} - LmP_{\min}] = LmK_{\max} - LmK_{\min} = VdB \quad (2-10)$$

If the nominal plant is chosen at $K = K_{\min}$ then the constraint M_L gives a boundary which approaches the U-contour abcdefa of Fig. 2-3.

For the plant of Eqs. (2-5) and (2-6), applying the limiting condition, UHF, to Eq.(2-10) yields

$$\begin{aligned} V &= (\Delta LmP)_{UHF} = \lim_{\omega \rightarrow \infty} [\{Lm(Ka)_{\max} - Lm(j\omega)^2\} - \{Lm(Ka)_{\min} - Lm(j\omega)^2\}] \\ &= Lm(Ka)_{\max} - Lm(Ka)_{\min} = Lm100 - Lm1 = 40dB \end{aligned} \quad (2-11)$$

For the minimum phase plant of Eq. (2-5), where poles are real the plant templates are of the shape of Fig. 2-2. It should be noted that Eq. (2-10) applies for all types of plants.

The high-frequency boundary B_h , the bcd portion of the U-contour in Fig. 2-3, is obtained by measuring down VdB (= 40dB for the example of this Chapter) from the ega portion of the M-contour as illustrated in this figure. The remaining portions of the U-contour, portions ab and de which are not necessarily straight lines, are determined by satisfying the requirements of Eq. (2-9) and $\delta_R(j\omega_1)$. That is, in utilizing the $\underline{TP}(j\omega_1)$ to determine the corresponding $\underline{B}_R(j\omega_1)$ on the N.C., in the manner described in Sec. 2-4, it must not penetrate the M_L -contour.

The characteristics described in this section do not apply if the plant has an unending number of poles or zeros, e.g., bending modes of the form of

$$\prod (s^2 + 2\zeta\omega_n s + \omega_n^2).$$

2-4 Tracking Bounds $\underline{LmB}_R(j\omega)$ on the N.C.

As an introduction to this section the procedure for adjusting the gain of a LTI unity feedback system to achieve a desired value of M_m by use of the Nichols chart is reviewed. Consider a plant whose plot of $\underline{LmP}$ vs $\underline{P}$ is shown in Fig. 2-4 (the solid curve). With $G(s) = A = 1$ and $F(s) = 1$ in Fig. 1-2, then $\underline{L} = \underline{P}$ and the plot of $\underline{LmL}(j\omega) = \underline{LmP}(j\omega)$ is tangent to the $M = 1$ dB curve with a resonant frequency $\omega_{m1} = 1.1$ rps. It is specified that the system gain A be adjusted to produce $\underline{LmM}_m = 2$ dB. The dashed curve in Fig. 2-4 is obtained by raising $\underline{LmL}(j\omega)$ until it is tangent to the 2dB curve. For this example the curve has been raised by the amount $\underline{LmA} = \underline{LmG} = 4.5$ dB ($G = A = 1.679$) and the resonant frequency is $\omega_m = 2.09$ rps.

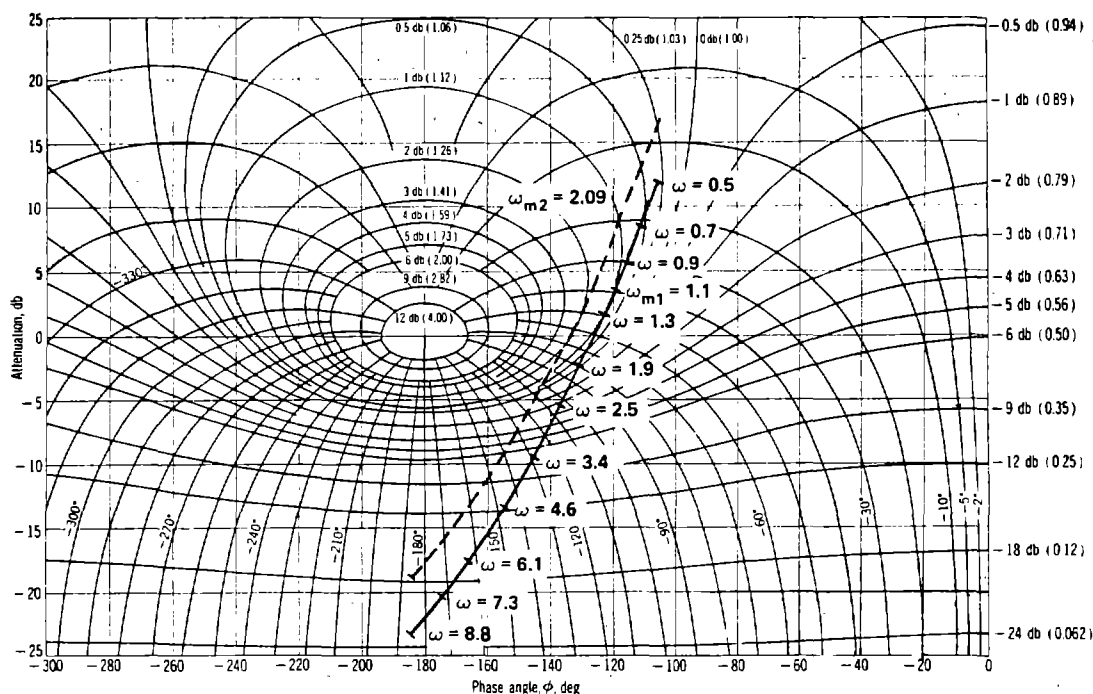


Fig. 2-4 Log magnitude-angle diagram.

Now consider that the plant uncertainty involves only the variation in gain A between 1 and 1.679. It is desired to find a compensator $\underline{G}(j\omega)$ such that the specification $1\text{ dB} \leq \text{LmM}_m \leq 2\text{ dB}$ is always maintained under this plant gain variation while the resonant frequency ω_m remains constant. That is, determine an $\underline{L}(j\omega) = \underline{G}(j\omega)\underline{P}(j\omega)$ such that it is always tangent to an M-contour in the range of $1 \leq \text{LmM} \leq 2\text{ dB}$ with a gain variation of $1 \leq A \leq 1.679$ at the same value of $\omega_m = 2.09$. The manner of achieving this and other time response specifications is the subject of the remaining portion of this chapter.

It is assumed for Eq. (2-4) that the compensators F and G are LTI, that is, they have negligible uncertainty. Thus, only the uncertainty in P contributes to the change in T_R given by Eq. (2-4)*. The specifications require that the actual $\Delta \text{LmT}_R(j\omega_i) \leq \delta_R(j\omega_i)\text{ dB}$ in Fig. 1-5. Thus, it is necessary to determine the resulting constraint or bound $\underline{B}_R(j\omega_i)$ on $\underline{L}(j\omega_i)$. It is convenient to pick a

*If one wishes to allow for some uncertainty in G he can do so by enlarging, in an appropriate manner, the plant template.

nominal plant $P_o(s)$ and derive the bounds on the resulting nominal loop transfer function $L_o(s) = P_o(s) G(s)$.

As an illustration, consider the plot of $Lm\underline{P}(j2)$ vs $\angle\underline{P}(j2)$ for the plant of Eqs. (2-5) and (2-6). As shown in Fig. 2-5, the plant's region of uncertainty $\mathcal{T}\underline{P}(j2)$ is given by the contour ABCD, i.e., $Lm\underline{P}(j2)$ lies on or within the boundary of this contour. The nominal plant transfer function, with $K_a = 1$ and $a_o = 1$, is

$$P_o = 1/s(s + 1) \quad (2-12)$$

and is represented by point A for $\omega = 2$ $[-13.0\text{dB}, -153.4^\circ]$ in Fig 2-5. Note, once a nominal plant is chosen it must be used for determining all the bounds, $\underline{B}_R(j\omega_1)$. Since $Lm\underline{L}(j2) = Lm\underline{G}(j2) + Lm\underline{P}(j2)$, then $\mathcal{T}\underline{P}(j2)$ is translated on the

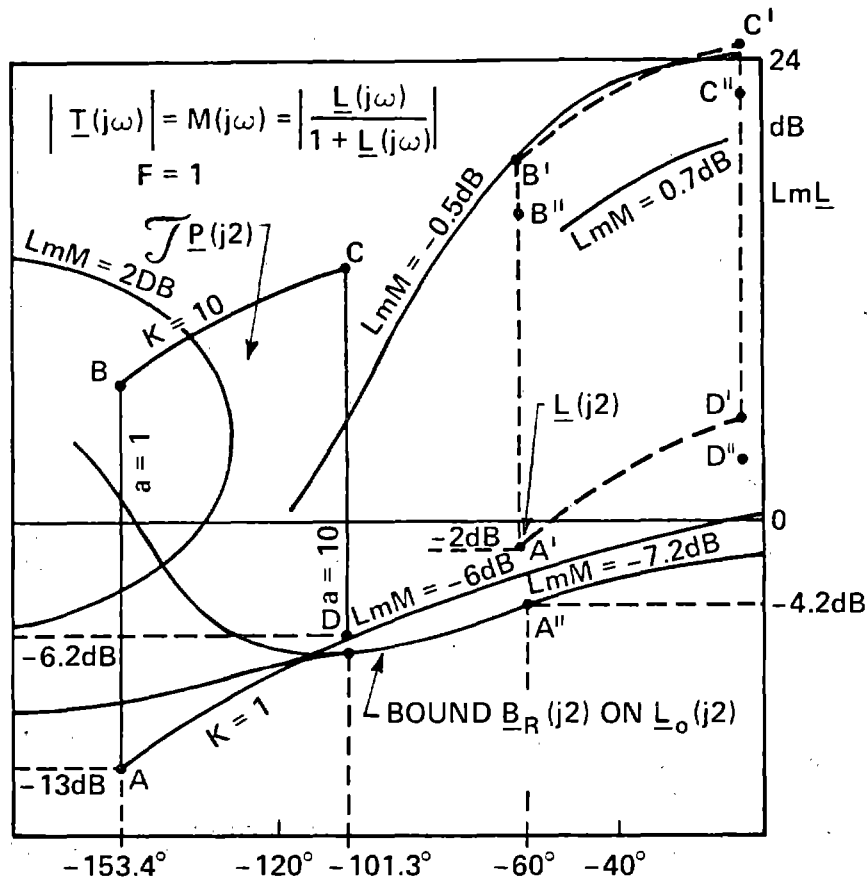


Fig. 2-5 Derivation of bounds $\underline{B}_R(j\omega)$ on $\underline{L}_o(j\omega)$ for $\omega=2$.

Nichols chart by the value of $Lm\underline{G}(j2)$. For example, if a trial design of $\underline{L}(j2)$ requires sliding $\underline{TP}(j2)$ to the position A'B'C'D' in Fig. 2-5, then

$$\left| Lm\underline{G}(j2) \right| = \left| Lm\underline{L}(j2) \right|_{A'} - \left| Lm\underline{P}(j2) \right|_A = \left| -2 \right| - \left| -13 \right| = 11\text{dB} \quad (2-13)$$

$$\angle \underline{G}(j2) = \angle \underline{L}(j2)_{A'} - \angle \underline{P}(j2)_A = -60^\circ - (-153.4^\circ) = 93.4^\circ \quad (2-14)$$

The templates $\underline{TP}(j\omega_i)$ are manipulated to find the position of $\underline{L}_O(j\omega)$ which satisfies the specifications in Fig. 1-5 of $\delta_R(j\omega_i)$. For example, the template for $\omega = 2$ is positioned in Fig. 2-5, at A'B'C'D'. Using the contours of constant $LmM = Lm[\underline{L}/(1 + \underline{L})]$ on the Nichols chart in Fig. 2-5, the maximum occurs at point C' ($M = -0.49\text{dB}$) and the minimum at point A' ($M = -5.7\text{dB}$) so that the maximum change in LmT is, in this case, very closely $(-0.49) - (-5.7) = 5.2\text{dB}$. Suppose that the specifications tolerate a change of 6.5 dB at $\omega = 2$, so the above trial position of $Lm\underline{L}_O(j2)$ is in this case more than satisfactory. The template is lowered on the Nichols chart to A''B''C''D'', where the extreme values of $Lm[\underline{L}/(1 + \underline{L})]$ are at C'' (-0.7dB) and A'' (-7.2dB), i.e.,

$$\left| Lm\underline{L}(j2) \right|_{C''} - \left| Lm\underline{L}(j2) \right|_{A''} = \left| 0.7 - 7.2 \right| = 6.5\text{dB} = \delta_R(j2).$$

Thus, if $\angle \underline{L}_O(j2) = -60^\circ$, then -4.2dB is the smallest or minimum value of $Lm\underline{L}_O(j2)$ which satisfies the 6.5dB specification for $\delta_R(j\omega_i)$. Any smaller magnitude is satisfactory but represents over-design at that frequency. The manipulation of the $\omega = 2$ template is repeated along a new vertical line, and a corresponding new minimum of $\left| \underline{L}_O(j2) \right|$ is found. Sufficient points are obtained in this manner to permit drawing a continuous curve of the bound $\underline{B}_R(j2)$ on $\underline{L}_O(j2)$, as shown in Fig 2-5. The above procedure is repeated at other frequencies, resulting in a family of boundaries $\underline{B}_R(j\omega_i)$ of permissible $\underline{L}_O(j\omega)$. The procedure for determining the boundaries $\underline{B}_R(j\omega_i)$ is summarized as follows:

1. From Fig. 1-5 obtain values of $\delta_R(j\omega_i)$ for sufficient values of ω_i ($\omega_1, \omega_2, \dots, \omega_i, \dots, \omega_h$), preferably an octave apart over the specified bandwidth. (The selection of ω_h results in the bound $\underline{B}_R(j\omega_h)$ passing under the U-contour.)

2. With $\underline{TP}(j\omega_i)$ go to the Nichols chart containing the U-contour to determine the bound $\underline{B}_R(j\omega_i)$ as follows:

a. Use major angle divisions of the Nichols chart for lining up the $\underline{TP}(j\omega_i)$.

b. Select P_o to represent, in general, the lowest point of $\underline{TP}(j\omega_i)$. For the design example shown in this chapter:

(1) Select point A in Fig. 2-5 as P_o .

(2) Line up side A-B of $\underline{TP}(j\omega_i)$ on the -90° line, as shown in Fig. 2-6.

Move the template up or down until the difference $\Delta \text{LmT}_R(j\omega_i)$ between the

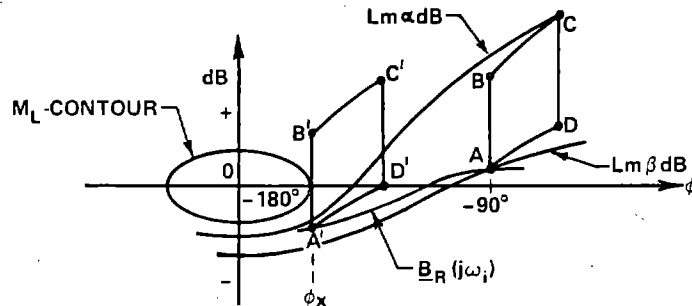


Fig. 2-6 Graphical determination of $\underline{B}_R(j\omega_i)$.

values of two adjacent M-contours ($\text{Lm}\alpha$ and $\text{Lm}\beta$, respectively) is equal to $\delta_R(j\omega_i)$ of Fig. 1-5. That is, in Fig. 2-6, determine the locations of the template on the -90° line where

$$\Delta \text{LmT}_R(j\omega_i) = \text{Lm}\alpha - \text{Lm}\beta = \delta_R(j\omega_i) \quad (2-15)$$

When Eq. (2-15) is satisfied, then point A on the M-contour = $\text{Lm}\beta$ lies on the bound $\underline{B}_R(j\omega_i)$. Mark this point on the Nichols chart. For other plants the shape of the template may be such that if point A represents $\underline{P}_o$, another point of the

template may be the lowest point that satisfies Eq. (2-15). When this equation is satisfied point A still yields points on $\underline{B}_R(j\omega_i)$.

(3) Repeat 2 on the lines -100° , -110° , etc, up to -180° or up until a point of the template becomes tangent to the M_L -contour. No intersection of the M_L -contour by a template is permissible. For example, in moving the template from the -90° line to the left the template may eventually become tangent to the M_L -contour at some angle ϕ_x as illustrated in Fig. 2-6. If the template is moved further to the left it will intersect the M_L contour and will permit a peaking of T greater than M_L . In order to satisfy the requirement of Eq. (2-11) point A' on the $\phi = \phi_x$ line becomes the left boundary or terminating point for the $\underline{B}_R(j\omega_i)$ contour and a point on the U-contour (a point on ab of Fig. 2-3)². Draw a curve through all the points to obtain the contour for $\underline{B}_R(j\omega_i)$. For the plant of this example the U-contour is symmetrical about the -180° axis. Note that in obtaining the bounds $\underline{B}_R(j\omega_i)$ in the manner described in step 2-b(2) only guarantees that the difference between the upper bound $\text{Lm}\underline{T}_U$ and the lower bound $\text{Lm}\underline{T}_L$ for $\text{Lm}\underline{T} = \text{Lm}[\underline{L}/(1 + \underline{L})]$ will be

$$\delta_R(j\omega_i) = \text{Lm}\underline{T}_R(j\omega_i) - \text{Lm}\underline{T}_L(j\omega_i) = \text{Lm}\underline{T}_U(j\omega_i) - \text{Lm}\underline{T}_L(j\omega_i) \quad (2-16)$$

(4) Repeat 2 and 3, generally octaves apart over the range of $\omega_x \leq \omega_i \leq \omega_h$, (i.e., $\omega_x = \omega_1$, $\omega_2 = 2\omega_1$, $\omega_3 = 2\omega_2$, ..., ω_h) until the highest bound $\underline{B}_R(j\omega_i)$ and the lowest bound $\underline{B}_R(j\omega_h)$ on the Nichols chart clears the U-contour. Usually for reasonably damped plants ($\zeta > 0.6$), over the entire region of plant uncertainty, the magnitude of the bounds, $\underline{B}_R(j\omega_i)$, decrease as ω increases. Thus for this type of plant it is desirable to have $\delta_R(j\omega_i)$ increasing as ω_i increases, as discussed in Sec. 1-3A. When this characteristic of $\delta_R(j\omega_i)$ is not observed it is possible to have $|\underline{B}_R(j\omega_j)| > |\underline{B}_R(j\omega_i)|$ for $\omega_j > \omega_k$. For a plant that is highly

underdamped ($\zeta \leq 0.6$), over some portion of the region of plant uncertainty, avoid using an underdamped nominal plant $P_o(s)$. For this latter situation it is desirable, but not necessary, to synthesize T_{RU} and T_{RL} based upon an overdamped $P_o(s)$ model.

2-5 Disturbance Bounds $B_D(j\omega_1)$

In Fig. 1-2 are shown two disturbance inputs. It is assumed that only one disturbance input exists at a time. Both cases are analyzed.

2-5A Disturbances Case 1 [$d_2(t) = D_o u_{-1}(t)$, $d_1(t) = 0$]

Control Ratio - From Fig. 1-2 the control ratio disturbance for the input $d_2(t)$ is

$$T_D(s) = \frac{1}{1 + L} \quad (2-17)$$

Substituting $L = 1/\ell$ into Eq.(2-17) yields

$$T_D(s) = \frac{\ell}{1 + \ell} \quad (2-18)$$

which has the form

$$T(s) = L(s)/[1 + L(s)]$$

Over the specified bandwidth it is desired that $|T_D(j\omega)| \ll 1$ which results in the requirement, from Eq (2-17), that $|L(j\omega)| \gg 1$ (or $|\ell(j\omega)| \ll 1$), i.e.,

$$|T_D(j\omega)| \cong \frac{1}{|L(j\omega)|} = |\ell(j\omega)| \quad (2-19)$$

Disturbance Response Characteristic ---- A time domain tracking response characteristic based upon $r(t) = u_{-1}(t)$, often specifies a maximum allowable peak overshoot M_p . In the frequency domain this specification may be approximated by

$$\left| \frac{M}{R}(j\omega) \right| = |T_R(j\omega)| = \left| \frac{C(j\omega)}{R(j\omega)} \right| \leq M_m \cong M_p \quad (2-20)$$

The corresponding time and frequency domain response characteristics, based upon a step disturbance forcing function are respectively,

$$\left| \underline{M}_D(t) \right| = \left| \frac{c(t)}{d(t)} \right| \leq \alpha_p \text{ for } t \geq t_x \quad (2-21)$$

and

$$\left| \underline{M}_D(j\omega) \right| = \left| \underline{T}_D(j\omega) \right| = \left| \frac{\underline{C}(j\omega)}{\underline{D}(j\omega)} \right| \leq \alpha_m \approx \alpha_p \quad (2-22)$$

Application ---- Let $L = KL'$ and the tracking control ratio is $T = L/(1 + L)$, where K is an unspecified gain, and the specification on system performance is $M_m = 1.12(1dB)$. By means of a N.C. determine the value of K required and obtain the data to plot $\left| \underline{M}(j\omega) \right|$ vs ω . The plot of $Lm\underline{L}'$ vs $\underline{L}'$ on the N.C. has been adjusted until it is tangent to the 1dB contour, resulting in the plot of $Lm\underline{L}(j\omega)$ vs $\underline{L}(j\omega)$ in Fig. 2-7. Intersections of $Lm\underline{L}(j\omega)$ with the M contours provide the data to plot the tracking control ratio $\left| \underline{M}(j\omega) \right|$ vs ω .

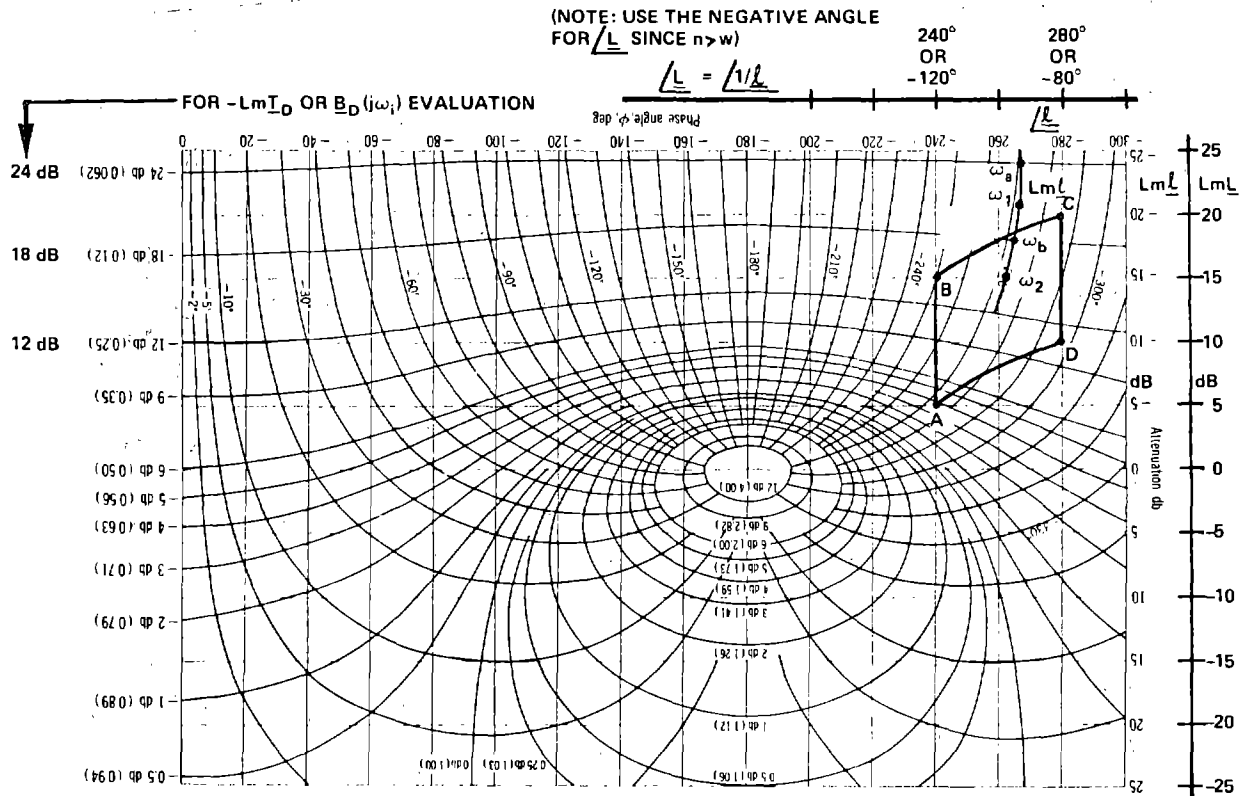
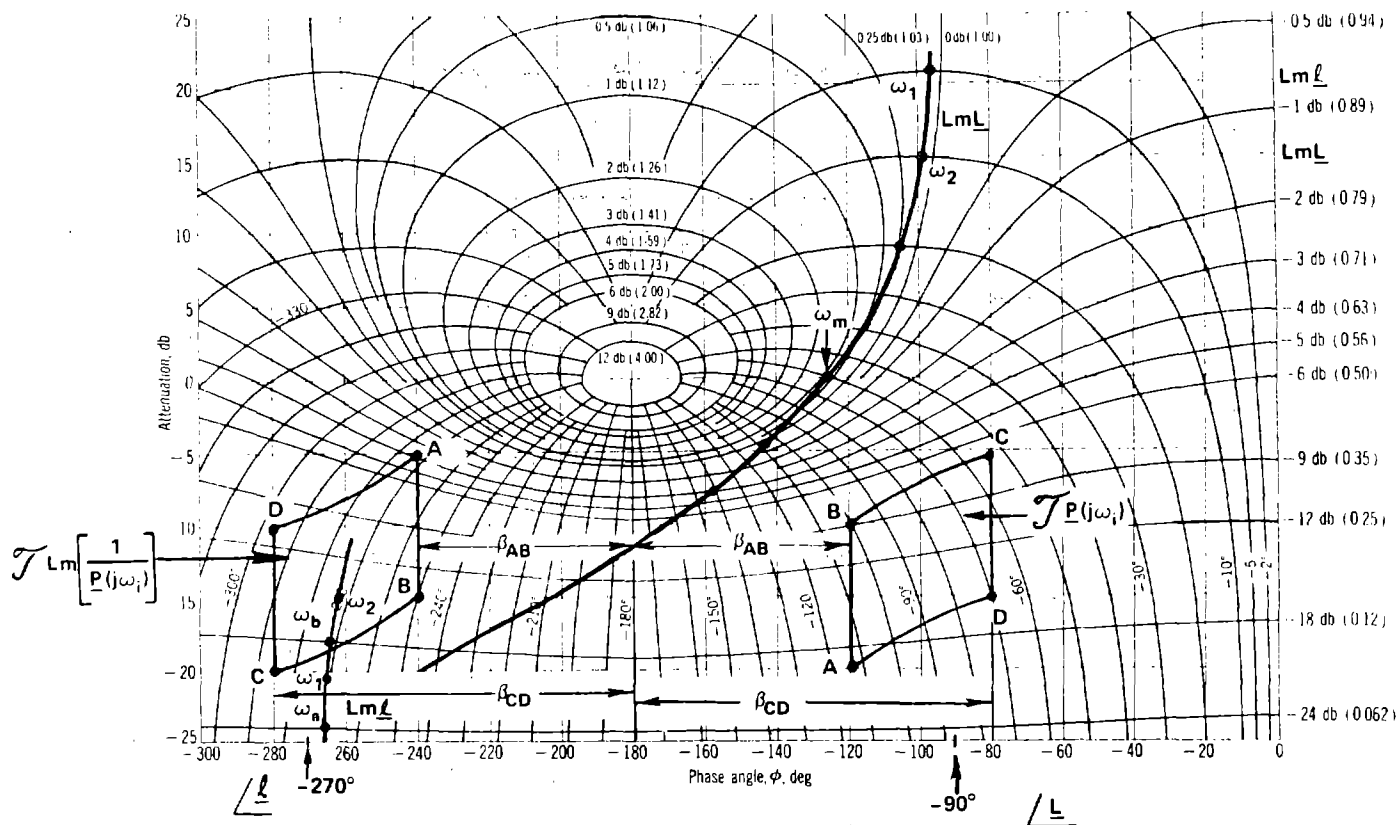
Now consider the corresponding disturbance control ratio for the same control system. The disturbance transfer functions is $T_D = 1/(1 + L) = \underline{\ell}/(1 + \underline{\ell})$, having the desired B.W. $0 \leq \omega \leq \omega_2$ for which $\left| \underline{L} \right| \gg 1$ and $\left| \underline{\ell} \right| \ll 1$. Thus Eq. (2-19) applies within the BW region. Table 2-1 contains data for two points on the Nichols plot of Fig 2-7. The plot of $Lm\underline{\ell}$ vs $\underline{\ell}$ for these two points is also shown in this figure. The N.C. of Fig. 2-7 can be rotated 180° and is redrawn in Fig. 2-8. Since $Lm\underline{\ell}(j\omega) = Lm(1/\underline{L}(j\omega)) = -Lm\underline{L}(j\omega)$, then a negative value of $Lm\underline{\ell}$ yields a + value for $Lm\underline{L}$ as shown in Fig. 2-8. Since $L = KL' = 1/\underline{\ell}$, then

Table 2-1 Data points for a Nichols plot

ω	$\text{Lm } \underline{L}$	$\angle \underline{L}$	$\text{Lm } \underline{\ell}$	$\angle \underline{\ell}$
ω_1	21	-96°	-21	96° (or -264°)
ω_2	15	-98°	-15	98° (or -262°)

$$\underline{\ell}(j\omega) = K^{-1} \underline{\ell}'(j\omega) \quad (2-23)$$

If $\underline{\ell}'(j\omega)$ is given and it is required to determine K^{-1} to satisfy Eq. (2-17), then the plot $\text{Lm } \underline{\ell}'(j\omega)$ vs $\angle \underline{\ell}'(j\omega)$ must be raised or lowered until it is tangent to the $\text{Lm } \alpha_p$ - contour ($|T_D|_{\max} = \alpha_m$). The amount Δ by which this plot is raised or lowered yields the value of K , i.e., $\text{Lm } K^{-1} = \Delta$. Note that this is the same procedure used for the tracking example of Fig. 2-4, except that the adjustment in $\text{Lm } \underline{\ell}(j\omega)$ is K^{-1} .



Templates ---- For a given plant $\underline{P}$ having uncertain parameters, consider that its template $\underline{TP}(j\omega_i)$ for a given ω_i has equal dB differences along its A-B and C-D boundaries, i.e., for a given $\underline{P}(j\omega_i)$, as follows:

$$\Delta(\text{Lm}\underline{P}_B - \text{Lm}\underline{P}_A) = \Delta(\text{Lm}\underline{P}_C - \text{Lm}\underline{P}_D) = 10\text{dB}$$

This template is arbitrarily set on the N.C. as shown in Fig. 2-7. Data corresponding to the template location shown in Fig. 2-7 is given in Table 2-2.

Points	$\angle \underline{P}$	$\angle 1/\underline{P}$
A,B	-120°	120° (or -240°)
C,D	-80°	80° (or -280°)

Table 2-2 Data points for the templates of Fig. 2-7.

The template of the reciprocal, $\text{Lm}[1/\underline{P}(j\omega_i)]$, is arbitrarily set on the N.C. as shown in Fig. 2-7 for the same frequency as for the template of $\text{Lm}\underline{P}(j\omega_i)$ and for the angles of Table 2-2. Note that the template of $\text{Lm}[1/\underline{P}(j\omega_i)]$ is the same as the template of $\text{Lm}\underline{P}(j\omega_i)$ but is rotated by 180° . Thus, it is located by first reflecting the template of $\text{Lm}\underline{P}(j\omega_i)$ about the -180° axis "flipping it over" vertically, and then moving it up or down so that it lies between -5 and -20dB. For the arbitrary location of the template of $\text{Lm}[1/\underline{P}(j\omega_i)]$, note that:

$$(a) \quad \beta_{AB} = 180^\circ + \angle \underline{P} = 180^\circ - 120^\circ = 60^\circ$$

$$\beta_{CD} = 180^\circ + \angle \underline{P} = 180^\circ - 80^\circ = 100^\circ$$

For $1/\underline{P}$, the corresponding angles are

$$\begin{aligned}\angle 1/\underline{P}_{AB} &= -180^\circ - \beta_{AB} = -180^\circ - 60^\circ = -240^\circ \\ \angle 1/\underline{P}_{CD} &= -180^\circ - \beta_{CD} = -180^\circ - 100^\circ = -280^\circ\end{aligned}$$

(b) The templates of $\text{Lm}\underline{P}(j\omega_1)$ are used for the tracker case $T = L/(1 + L)$ and the templates of $\text{Lm}[1/\underline{P}(j\omega_1)]$ are used for the disturbance case of

$$T_D = 1/(1 + L) = \ell/(1 + \ell)$$

Since it is desired, for the disturbance rejection problem, Eq. (2-17), to obtain $\text{Lm}\underline{L}$, then the disturbance boundary $= \underline{B}_D(j\omega_1)$ for $\underline{L}(j\omega_1) = 1/\ell(j\omega)$ are best determined on the rotated N.C. of Fig. 2-8. Thus, the N.C. of Fig 2-7 is rotated clockwise by 180° , where the rotation of the $\text{Lm}[1/\underline{P}(j\omega_1)]$ templates ABCD is reflected in Fig. 2-8. The rotated N.C. is used to determine directly the boundaries, $\underline{B}_D(j\omega_1)$, for $\underline{L}_D(j\omega_1)$. Point A for the simple plant of this design example corresponds to the nominal plant parameters and is the lowest point of the template $\underline{T}\underline{P}(j\omega_1)$. This point is again used to determine the disturbance bounds $\underline{B}_D(j\omega_1)$. The lowest point of the template must be used to determine the bounds and, in general, may or may not be the point corresponding to the nominal plant parameters.

Based upon Eqs. (2-17) and (2-18) and Fig. 1-8

$$\underline{T}_D = \frac{1}{1 + \underline{L}} = \frac{\underline{\ell}}{1 + \underline{\ell}} \leq \alpha(j\omega_1) \quad (2-24)$$

thus

$$\text{Lm}\underline{T}_D = -\text{Lm}[1 + \underline{L}] \leq \text{Lm}\alpha(j\omega_1) = \delta_D(j\omega_1) < 0 \quad (2-25)$$

which is rearranged to

$$-\text{Lm}\underline{T}_D = \text{Lm}[1 + \underline{L}] \geq -\text{Lm}\alpha(j\omega_1) > 0 \text{ dB} \quad (2-26)$$

Since in the BW $|\underline{L}| \gg 1$, then

$$-Lm\underline{T}_D \approx Lm\underline{L} \geq -Lm\alpha(j\omega_1) = -\delta_D(j\omega_1) \quad (2-27)$$

In terms of $\underline{L}(j\omega_1)$, the constant M contours of the N.C. can be used to obtain the disturbance performance $\underline{T}_D$. This requires the change of sign, as shown in Fig. 2-8, of the vertical axis in dB and the M contours.

Bounds $\underline{B}_D(j\omega_1)$ --- The procedure for determining the boundaries $\underline{B}_D(j\omega_1)$ is

1. From Fig. 1-8 obtain values of $\delta_D(j\omega_1)$ from the curve representing Eqs. (1-20) and (1-22) for the same values of frequencies as for the tracking boundary $\underline{B}_R(j\omega_1)$.
2. Determine the $\underline{B}_D(j\omega_1)$ bound as follows:
 - a. Use major angle divisions of the N.C. for lining up the $\underline{T}_P(j\omega_1)$.
 - b. Select the lowest point of $\underline{T}_P(j\omega_1)$ to represent the nominal plant $\underline{P}_0$. For the design example used in this chapter:
 - (1) Select point A in Fig. 2-8 as $\underline{P}_0$.
 - (2) Line up side A-B of $\underline{T}_P(j\omega_1)$, for example, on the -280° line for $\underline{\ell}$ (or the -80° line for $\underline{L}$). Move the template up or down until point A lies on the M-contour that represents $\delta_D(j\omega_1)$. Mark this point on the N.C.
 - (3) Repeat step (2) on the -100° , -120° , etc., lines up to the -180° line or the U-contour. Draw a curve through all the points to obtain the contour for $\underline{B}_D(j\omega_1)$.
 - (4) Repeat steps (2) and (3) over the desired frequency range

$$\omega_x \leq \omega_1 \leq \omega_h.$$

Note that when $|\underline{L}| \gg 1$, then from Eq. (2-24) $|\underline{L}(j\omega_1)| \gg \alpha(j\omega_1)$, thus the M-contour corresponding to $\underline{T}_D(j\omega_1)$ becomes the boundary $\underline{B}_D(j\omega_1) = -LmM$ for $\underline{L}(j\omega_1)$. For example, if $\alpha(j\omega_1) = 0.12$, then $|\underline{L}(j\omega_1)| = 8.33$ and thus $Lm\alpha(j\omega_1) = -18.4\text{dB}$ and $Lm\underline{L}(j\omega_1) = 18.4\text{dB}$

3. Transcribe these $\underline{B}_D(j\omega_1)$ onto the N.C. that contains the bounds $\underline{B}_R(j\omega_1)$.

2-5B Disturbance Case 2 [$d_1(t) = D_o u_{-1}(t)$, $d_2(t) = 0$]

Control Ratio -- From Fig. 1-2 the disturbance control ratio is

$$\underline{T}_D = \frac{\underline{P}}{1 + \underline{G}\underline{P}} \quad (2-28)$$

Assuming point A of the template represents the nominal plant, then Eq. (2-28) is rearranged to

$$\underline{T}_D = \frac{\underline{P}_o}{\underline{P}_o} \left[\frac{\underline{1}}{\underline{1} + \underline{G}} \right] = \frac{\underline{P}_o}{\frac{\underline{P}_o}{\underline{P}} + \underline{G}\underline{P}_o} = \frac{\underline{P}_o}{\frac{\underline{P}_o}{\underline{P}} + \underline{L}} = \frac{\underline{P}_o}{\underline{W}} \quad (2-29)$$

where

$$\underline{W} = (\underline{P}_o/\underline{P}) + \underline{L} \quad (2-30)$$

Thus, from Eq. (2-29) with $\text{Lm}\underline{T}_D = \delta_D$,

$$\text{Lm}\underline{T}_D = \text{Lm}\underline{P}_o - \text{Lm}\underline{W} = \delta_D$$

which yields

$$\text{Lm}\underline{W} = \text{Lm}\underline{P}_o - \delta_D \quad (2-31)$$

Disturbance Response Characteristics -- Based upon Eq. (2-20) the time and frequency domain response characteristics, based upon a unit step disturbance forcing function, are empirically given by, respectively,

$$\left| M_D(t) \right| = \left| \frac{c(t_p)}{d(t)} \right| = \left| c(t_p) \right| \leq \alpha_p \quad (2-32)$$

and

$$\left| \underline{M}_D(j\omega) \right| = \left| \underline{T}_D(j\omega) \right| = \left| \frac{\underline{C}(j\omega)}{\underline{D}(j\omega)} \right| \leq \alpha_m \cong \alpha_p \quad (2-33)$$

where t_p is the peak time.

Bounds $B_D(j\omega_i)$ -- The procedure for determining the boundaries $B_D(j\omega_i)$ is

1. From Fig. 1-8 obtain values of $\delta_D(j\omega_i)$ from the plot representing Eqs. (1-24) and (1-26) for the same values of frequencies as for the tracker boundaries $B_R(j\omega_i)$.

2. Evaluate in tabular form for each value of ω_i the following items in the order given:

$$\text{Lm } \underline{P}_O(j\omega_i), \quad \delta_D(j\omega_i), \quad \text{Lm } \underline{W}(j\omega_i), \quad \left| \underline{W}(j\omega_i) \right|, \quad \frac{\underline{P}_O(j\omega_i)}{\underline{P}(j\omega_i)}$$

The ratio $\underline{P}_O/\underline{P}$ is evaluated at each of the four points of Fig. 2-1 for each value of ω_i . It may be necessary to evaluate this ratio at additional points around the perimeter of the ABCD contour.

3. In Eq. (2-29) replace $\underline{L}$ by its bound B_D . To explain the graphical evaluation of $B_D(j\omega_i)$ the phasor relationship for Eq. (2-30) is shown in Fig. 2-9 for a possible configuration.

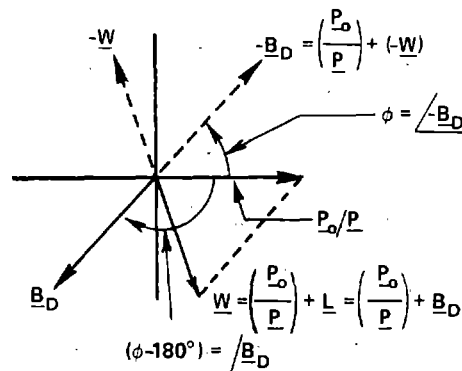


Fig. 2-9 Phasor relationship of Eq. (2-30) with $\underline{L} = B_D$.

4. On polar graph paper draw $T \left[\frac{P}{P_0} \right]$ for each ω_i as shown in Fig. 2-10, where point A is the nominal point.

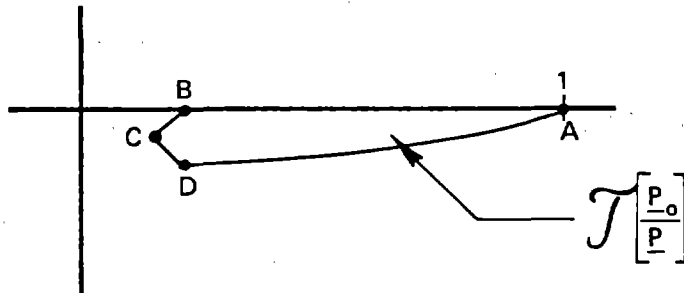


Fig. 2-10 Template in polar coordinates

5. Based upon Fig. 2-9 and the location of the phasor $-\underline{W}$, the solution for $-\underline{B}_D$ is obtained from

$$\frac{-B_D}{-D} = T \left[\frac{P}{P_0} \right] - \frac{W}{D} \quad (2-34)$$

For one value of $\frac{P}{P_0}$ shown in Fig. 2-9, the value $\frac{-W}{P_0}$ is plotted and

$-\underline{B}_D = |-\underline{B}_D|/\angle\phi$ is obtained. This graphical evaluation of $\underline{B}_D = |\underline{B}_D|/\angle\phi - 180^\circ$ is performed for various points around the perimeter of $T[P_O/P_-]$ in Fig. 2-10. A simple graphical evaluation yields a more restrictive bound (the worst case).

By use of a compass mark off arcs with a radius equal to the distance $\left| \frac{W(j\omega_1)}{P(j\omega_1)} \right|$ at a number of points on the perimeter of $T \left[\frac{P_o(j\omega_1)}{P(j\omega_1)} \right]$. Draw a curve which is tangent to these arcs to form a portion of the Q-contour shown in Fig. 2-11. It may be necessary to extend this contour into the second and fourth quadrants.

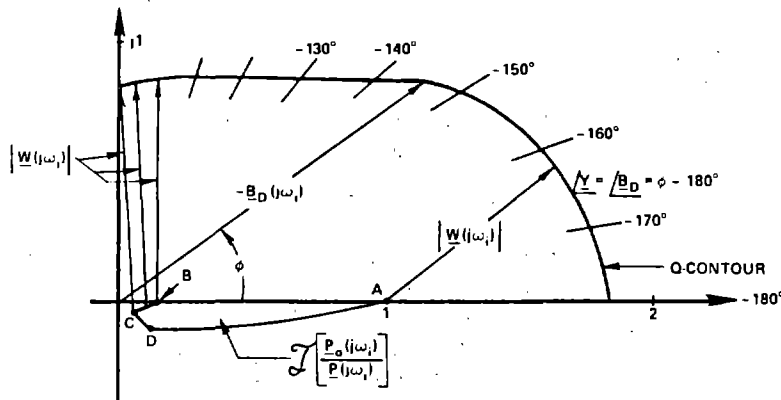


Fig. 2-11 Graphical evaluation of $B_n(j\omega_i)$.

6. Thus based upon Eq. (2-34) and Fig. 2-9, the phasor from the origin of Fig. 2-11 to the Q-contour represents $\underline{B}_D(j\omega_1)$. This contour takes into account the plant uncertainty as represented by $T \left[\frac{\underline{P}_O(j\omega_1)}{\underline{P}(j\omega_1)} \right]$. Depending on the plant type desired for $\underline{L}$, it may be necessary to obtain more of the Q-contour around

the template. In the frequency range $\omega_x \leq \omega_1 \leq \omega_h$, if $\left| \underline{W}(j\omega_1) \right| \gg \left| \frac{\underline{P}_O(j\omega_1)}{\underline{P}(j\omega_1)} \right|$, then the Q-contour is essentially a circle about the origin with radius $\left| \underline{W}(j\omega_1) \right|$ and

$$\left| \underline{W}(j\omega_1) \right| \cong \left| \underline{B}_D(j\omega_1) \right|$$

7. Assuming the partial Q-contour of Fig. 2-11 is sufficient, measure from the graph the length $\underline{B}_D(j\omega_1)$ for every 10° of $\underline{B}_D(j\omega_1)$ and create the following table, for each value of ω_1 :

$\angle \underline{B}_D(j\omega_1)$	$\left \underline{B}_D(j\omega_1) \right $	$\text{Lm } \underline{B}_D(j\omega_1)$
-180°		
-170°		
-160°		
$\vdots$		

8. Plot the values from the table of Step 7, for each value of ω_1 , onto the same N.C. as $\underline{B}_R(j\omega_1)$.

2-6 The Composite Boundary, $\underline{B}_O(j\omega_1)$

The composite bound $\underline{B}_O(j\omega_1)$ that is used to synthesize the desired loop transmission transfer function $\underline{L}_O(s)$ is obtained in the manner shown in Fig. 2-12. That is, the bound $\underline{B}_O(j\omega_1)$, for each value of ω_1 , is composed of that portion of each respective bound $\underline{B}_R(j\omega_1)$ and $\underline{B}_D(j\omega_1)$ that is the most onerous. For the case shown in Fig. 2-12(a) the bound $\underline{B}_O(j\omega_1)$, for each value

of ω_i , is composed of that portion of each respective bound $\underline{B}_R(j\omega_i)$ and $\underline{B}_D(j\omega_i)$ that has the largest value. For the situation of Fig. 2-12(b), the outermost boundary of the two boundaries $\underline{B}_R(j\omega_i)$ and $\underline{B}_D(j\omega_i)$ becomes the perimeter of $\underline{B}_O(j\omega_i)$. The situation of Fig. 2-12 occurs when the two bounds have one or more intersections. If there are no intersection then the bound with the largest value or with the outermost boundary dominates. The synthesized $\underline{L}_O(j\omega_i)$, for the situation of Fig. 2-12(a), must lie on or just above the bound $\underline{B}_O(j\omega_i)$. For the situation of Fig. 2-12(b) the synthesized $\underline{L}_O(j\omega_i)$ must not lie in the interior of the $\underline{B}_O(j\omega_i)$ contour.

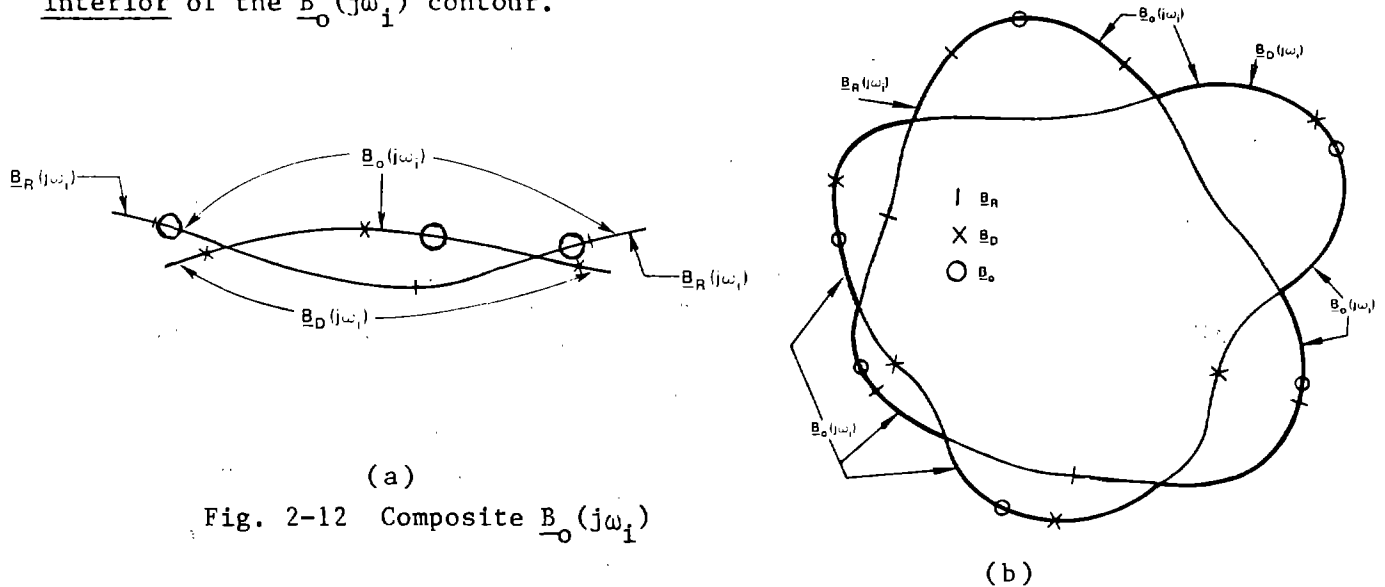
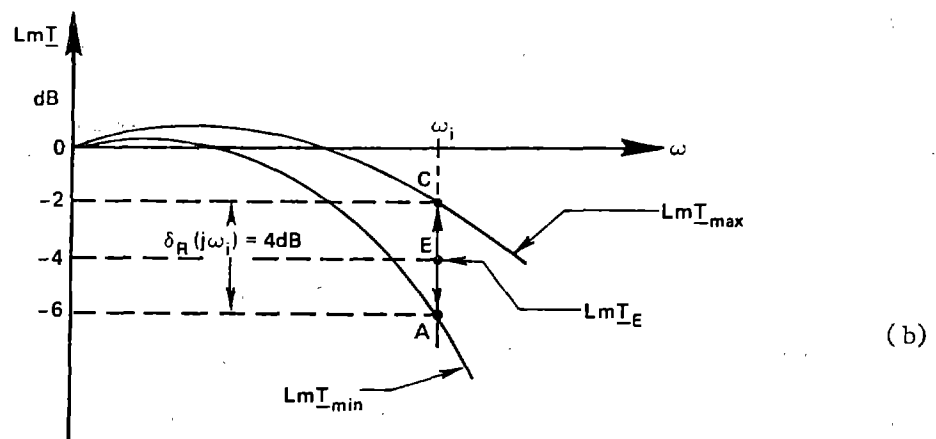


Fig. 2-12 Composite $\underline{B}_O(j\omega_i)$

2-7 Shaping of the $\underline{L}_O(j\omega)$ ^{1,2}

It has been shown^{9,10} that a realistic definition of optimum in an LTI system is the minimization of the high-frequency loop gain K . This gain affects the high frequency response since $\lim_{\omega \rightarrow \infty} [\underline{L}(j\omega)] = K(j\omega)^{-\lambda}$ where λ is the excess of poles over zeros assigned to $\underline{L}(j\omega)$. Thus, only the gain K has a significant effect on the high frequency response and the effect of the other parameter uncertainty is negligible. It has been proven^{9,10} that the optimum $\underline{L}(j\omega)$ lies on the boundary $\underline{B}_O(j\omega)$ at all ω_i , and that such an optimum exists and is unique.

Figure 10.10 is a Bode plot showing the magnitude and phase of a system. The plot is on a log-log scale. The horizontal axis represents the phase angle ϕ in degrees, ranging from -180 to 0. The vertical axis represents the magnitude in decibels (db), ranging from -24 to 0.5. The plot includes a magnitude curve (labeled $L_m \alpha$) and a phase curve (labeled $L_m \beta$). The magnitude curve is labeled $L_m \alpha$ and the phase curve is labeled $L_m \beta$. The plot also shows the asymptotic approximation (labeled $L_o(j\omega)$) and the error (labeled $B_o(j\omega)$). The plot is labeled (a).



(b) Plots of LmT_{max} and LmT_{min} vs ω .

$$\underline{L}_O(j\omega_i) = \underline{G}(j\omega_i)\underline{P}_O(j\omega_i)_A \quad (2-35)$$

$$\underline{P}_O(j\omega_i)_A = \left| \underline{P}_O(j\omega_i) \right|_A \underline{P}_O(j\omega_i)_A \quad (2-36)$$

$$\left[\underline{Lm}\underline{\alpha} - \underline{Lm}\underline{\beta} \right]_{\omega_i} = \delta_R(j\omega_i) = 4\text{dB} \quad (2-37)$$

$$\underline{\alpha}(j\omega_i) = \underline{T}_{\max} = \frac{\underline{L}_O\alpha}{1 + \underline{L}_O\alpha} \quad (2-38)$$

$$\underline{\beta}(j\omega_i) = \underline{T}_{\min} = \frac{\underline{L}_O\beta}{1 + \underline{L}_O\beta} \quad (2-39)$$

For various values of $\delta_R(j\omega_i)$, obtained from Fig. 1-5, data can be obtained from Fig. 2-13(a) to plot $\underline{LmT}_{\max}$ and $\underline{LmT}_{\min}$ vs ω as shown in Fig. 2-13(b).

Assume that the plant parameters vary such that

$$\underline{P}(j\omega_i)_E = \left| \underline{P}(j\omega_i)_E \right| \underline{P}(j\omega_i)_E \quad (2-40)$$

which corresponds to point E of $\underline{TP}(j\omega_i)$ where

$$\underline{L}_E = \underline{GP}_E \quad (2-41)$$

and

$$\underline{T}_E = \frac{\underline{L}_E}{1 + \underline{L}_E} \quad (2-42)$$

and $\underline{LmT}_E = -4\text{dB}$. Thus for any value of $\underline{P}$ on $\underline{TP}(j\omega_i)$ yields a value of $\underline{LmT}$ between points A and C in Fig 2-13(b). Therefore, as seen from the graphical plant parameter variation representation in Fig 2-13(a) any value of $\underline{P}$ that lies in the region of uncertainty (see Fig. 2-1 for the example of this chapter) yields a maximim variation in $\underline{T}(j\omega_i)$ that satisfies the requirement

$$LmT_{\max} - LmT_{\min} \leq \delta_R(j\omega_1) \quad (2-43)$$

Thus, proper design of the prefilter F (see Sec. 2-8) yields an actual T_R that lies between LmT_U and LmT_L in Fig 1-5.

For the plant of Eqs. (2-8) and (2-9), the shaping of $\underline{L}_O(j\omega)$ is shown in Fig. 2-14. A point such as $Lm\underline{L}_O(j2)$ must be on or above the curve labelled $\underline{B}_O(j2)$. Further, in order to satisfy the specifications, $\underline{L}_O(j\omega)$ cannot violate the U-contour. In this example a reasonable $\underline{L}_O(j\omega)$ closely follows the U-contour up to $\omega = 40$ and must then stay below it as shown in Fig. 2-14. Additional specifications are: $\lambda = 4$ and $L_O(s)$ must be Type 1 (one pole at the origin). A reasonable procedure for choosing a rational function $L_O(s)$ which satisfies the above specifications is now described.

The first step is to find the $\underline{B}_O(j\omega)$ which "dominates" $\underline{L}_O(j\omega)$. For example, suppose $\underline{L}_{O1}(j4) = KP'(j4) = 0dB \angle -135^\circ$ (point A in Fig. 2-14) and that at $\omega = 1$ the required $Lm\underline{L}_O(j1)$ is $\approx 27dB$. In order to decrease $Lm\underline{L}_O(j\omega)$ from 27dB to about 0dB in two octaves, the slope of $\underline{L}_O(j\omega)$ must be about -14dB/octave, with $\angle \underline{L}_O < -180$ (see Fig 2-15). Since "absolute" stability is required, $\underline{L}_O(j\omega)$ must have a phase margin of 40° over the entire frequency range for which $\underline{L}_O(j\omega)$ follows the vertical right-hand side of the U-contour and not just at crossover ("crossover" is defined as the frequency ω_ϕ at which $|\underline{L}_O(j\omega)| = 1$). Hence $\underline{B}_O(j1)$ dominates $\underline{L}_O(j\omega)$ more than does $\underline{B}_O(j4)$. In the same way it is seen that $\underline{B}_O(j1)$ dominates over all other $\underline{B}_O(j\omega)$ in Fig. 2-14.

The $\underline{B}_O(j\omega)$ for $\omega < 1$ are not shown in Fig. 2-14. It is assumed that for $\omega < 1$ a slope of -6dB octave suffices (with 27dB at $\omega = 1$, additional values are 33dB at $\omega = 0.5$, 39dB at $\omega = 0.25$ etc.). To maintain -140° for $\omega \geq 1$, the first denominator factor of $\underline{L}(j\omega)$ has a lag corner frequency (lacf) at $\omega = 1$ (i.e. a pole at -1 is chosen), and the value $Lm\underline{L}_O(j1)$ on the straight line approximation is set at 30dB (to allow for the -3dB correction). The function $L_O(s)$ is, so

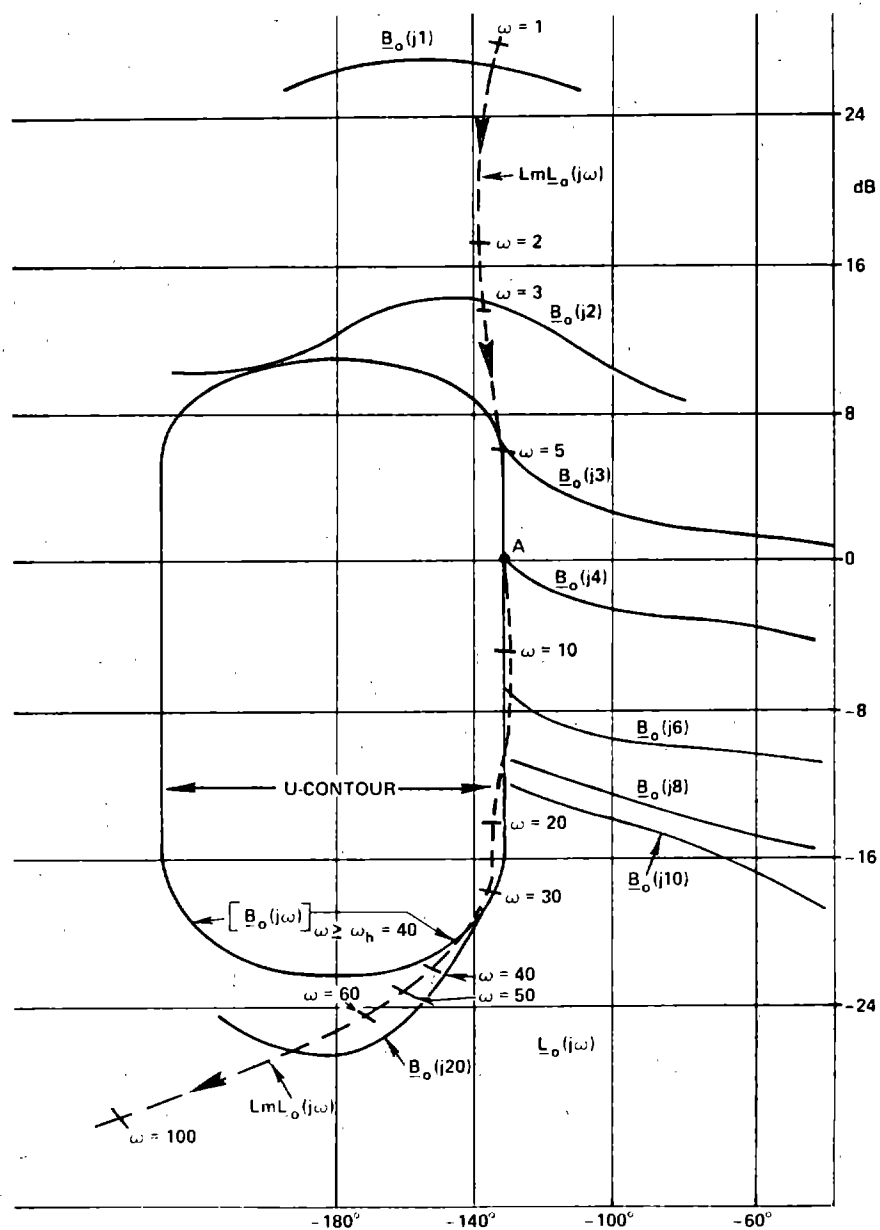


Fig 2-14 Shaping of $\underline{L}_O(j\omega)$ on Nichols chart for the plant of Eqs. (2-5) and (2-6).

far, $L_{o1} = 31.6/s(s + 1)$ whose angle $\angle L_{o1}(j\omega)$ is sketched in Fig. 2-15.

$L_{o1}(j\omega)$ violates the -140° bound at $\omega \geq 1.2$, so a lead corner frequency (lecf) is needed. Where should it be located? At $\omega = 5$, $\angle L_{o1}(j5) = -169^\circ$ (see Fig. 2-15) so a lead angle of 29° is needed. Since a second lacf will be needed, allow an additional 15° for this second lacf, giving a total of $15^\circ + 29^\circ = 45^\circ$ lead angle required at $\omega = 5$. This is achieved by a lecf at $\omega = 5$, i.e., a zero at -5 . This results in

$$L_{o2}(s) = \frac{31.6(1 + \frac{s}{5})}{s(1 + s)}$$

whose phase angle $\angle L_{o2}(j\omega)$ is sketched in Fig. 2-15. In the N.C. of Fig. 2-14, $\omega \geq 5$ or so is the region where the maximum phase lag allowed is -135° (i.e., $\angle L_o(j\omega)$ must be $\geq -135^\circ$). At $\omega = 10$, since $\angle L_{o2}(j10) = -112^\circ$, then $135^\circ - 112^\circ = 23^\circ$ more lag is allowed. But this lacf is to be followed by a lecf, so allow say 10° for it, giving $23^\circ + 10^\circ = 33^\circ$ more lag allowable. This locates the lacf at $\omega = 15.4$ ($\tan 33^\circ = 0.65$, and $10/0.65 = 15.4$), so the lacf is set at $\omega = 15$, giving

$$L_{o3}(s) = \frac{31.6(1 + 0.2s)}{s(1 + s)(s + \frac{s}{15})}$$

A sketch of $\angle L_o(j\omega)$ is shown in Fig. 2-15.

Looking ahead at $\omega = 40$, $Lm L_{o3}(j40) = -20\text{dB}$, so soon $L_o(j\omega)$ can make its asymptotic left turn under the U-contour. The plan is to add two more lecfs, and finally two complex-pole pairs, in order to have an excess λ of poles over zeros of 4 and to minimize the B.W. An lecf is set at $\omega = 40$, giving

$$L_{o4}(s) = \frac{31.6(1 + 0.2s)(1 + \frac{s}{40})}{s(1 + s)(1 + \frac{s}{15})}$$

A sketch of $\underline{L}_{-o4}(j\omega)$ is shown in Fig. 2-15.

In order to achieve (an asymptotic) horizontal segment for $\underline{LmL}_o(j\omega)$, before the final -24dB/octave slope is achieved, a final lecf is needed. Where should this horizontal segment be located? The bottom of the U-contour (see Fig. 2-14) is at -22.5dB. Allow a 2dB safety margin, 3dB correction due to the last lecf, and 1.5dB for the effect of the lecf at $\omega = 40$, giving a total of $-(22.5 + 2 + 3 + 1.5) = -29$ dB. A damping ratio of $\zeta = 0.5$ for the 2 complex pole pairs is used so no correction is needed for them. Thus the final corner frequency for the straight line curve of $\underline{LmL}_o(j\omega)$ is to be at -29dB. Since $\underline{LmL}_{-o4}(j\omega)$ achieves this at $\omega = 60$, the last lecf is at $\omega = 60$. The resulting phase due to L_{o4} , and the lecf at $\omega = 60$, is $-66^\circ - 180^\circ$ could have been selected at this point, but an additional 15° margin is allowed (this is a matter of judgement which depends on the problem--presence of higher order modes, etc.). This means that 100° phase lag is permitted: 50° due to each complex-pole pair ($180^\circ - 66^\circ - 15^\circ \approx 100^\circ$). For $\zeta = 0.6$, this locates the corner frequency at $\omega = 100$. Thus

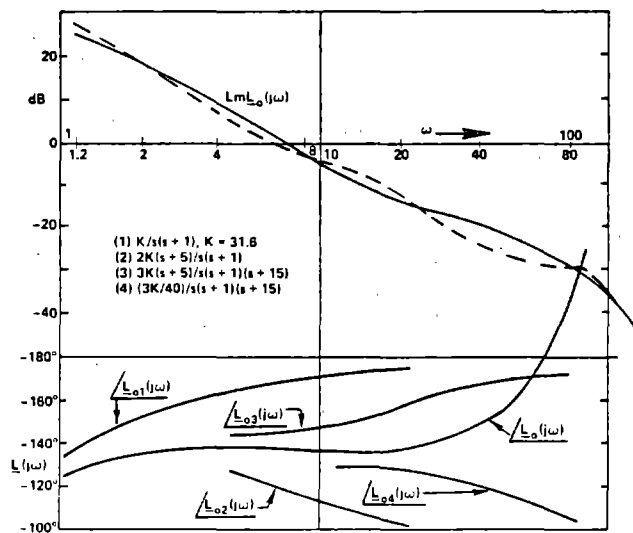


Fig. 2-15 Shaping of $\underline{L}_o(j\omega)$ on Bode plot.

$$L_O(s) = \frac{31.6(1 + 0.2s)(1 + \frac{s}{40})(1 + \frac{s}{60})}{s(1 + s)(1 + \frac{s}{15}) \left[1 + \frac{(1.2)}{100}s + \frac{s^2}{10^4} \right]^2} \quad (2-44)$$

The optimal loop transfer function $\underline{L}_O(j\omega)$ is sketched in Fig. 2-14 and is the solid curve in Fig. 2-15. A well designed, i.e., "economical" $\underline{L}_O(j\omega)$ is close to the $\underline{B}_O(j\omega)$ boundary at each ω_i . The vertical line -140° is the dominating vertical boundary for $\underline{L}_O(j\omega)$ for $\omega < 5$, and the right side of the U-contour line (-135°) is the vertical boundary effectively for $5 \leq \omega \leq 30$. The final $\underline{L}_O(j\omega)$ is pretty good in this respect since it is close to these boundaries. There is a tradeoff between complexity of $\underline{L}_O(s)$ (the number of its poles and zeros) and its final cut-off frequency, now at $\omega = 100$ and ω_ϕ , now at $\omega_\phi = 7$. There is some phase to spare between $\underline{L}_O(j\omega)$ and the boundaries. So use of more poles and zeros in $\underline{L}_O(s)$ permits this cut-off frequency to be reduced a bit below 100, but not by much. On the other hand, if it is desired to reduce the number of poles and zeros of $\underline{L}_O(s)$, then the price in achieving this is a larger cut-off frequency. It is possible to economize significantly by allowing more phase lag in the low frequency range. If -180° is permitted at $\omega = 1$, then a decrease of $\text{Lm}\underline{L}_O(j\omega)$ at a rate of 12dB/octave can be achieved, e.g., with $\text{Lm}\underline{L}_O(j\omega) = 25\text{dB}$ at $\omega = 1$, then it will be 13dB at $\omega = 2$ (instead of the present 18dB). Even with no more saving, this 5dB difference allows a cut-off frequency at about $\omega_\phi = 70$ instead of 100.

Also, Fig. 2-14 reveals [immediately, without any shaping of $\underline{L}_O(j\omega)$ required] that reduction (i.e., easing) of the specifications at $\omega = 1$ to about 21dB (instead of about 26dB) has the same effect as the above. How badly the specifications are compromised by such easing can easily be checked. The design technique is thus highly "transparent" in revealing the trade-offs between

performance tolerances, complexity of the compensation, stability margins, and the "cost of feedback" in bandwidth.

Some general guidelines for the shaping of $\underline{L}_0(j\omega)$ are:

- a. Do not use a CAD program at the onset of your design problem. Use the straight-line approximations on the Bode diagram for the log magnitude.
- b. On the graph paper for the Bode diagram plot the points representing $\text{Lm}\underline{B}_0(j\omega_i)$ and the angles corresponding to the right side of the U-contour (the desired phase margin angle γ) for the frequency range of $\omega_x \leq \omega_y \leq \omega \leq \omega_h$. For the example illustrated in Fig. 2-14 the phase margin angle $\gamma = -45^\circ$ must be maintained for the frequency range of $\omega_y = 5 \leq \omega \leq \omega_h = 40$.
- c. Do the shaping of $\text{Lm}\underline{L}_0(j\omega)$ on the Bode plot using straight line approximations for $\text{Lm}\underline{L}_0(j\omega)$, with the plotted information of Step b, and utilizing the shaping discussion for the frequency range $\omega < \omega_y$ in achieving Eq. (2-44) as guidelines.
- d. Use the characteristics an octave above and below and a decade above and below a corner frequency and at a corner frequency for first order and second order terms^{3,6} along with maintaining the phase margin angle corresponding to the right side of the U-contour in shaping $\underline{L}_0(j\omega)$.
- e. The last two poles that are added to $\underline{L}_0(j\omega)$ are generally a complex pair (the nominal range is $0.5 < \zeta < 0.7$) which tends to minimize the B.W. for the range $\omega \geq \omega_h$ when $K = K_{\max}$.
- f. Once the $\underline{L}_0$ has been shaped, determine $F(s)$, (see Sec. 2-8) and then verify that $\underline{L}_0$ does meet the design objectives by use of a CAD program.

If the synthesized $\underline{L}_0(s)$ yields the desired performance then the compensator is given by

$$G(s) = \frac{\underline{L}_0(s)}{\underline{P}_0(s)} \quad (2-45)$$

Specific guidelines for the shaping of $\underline{L}_O(j\omega)$ are

- a. An optimum design of $\underline{L}_O(j\omega)$ requires that $\underline{L}_O(j\omega_i)$ be on the corresponding bound, $\underline{G}_O(j\omega_i)$ -in practice, have it as close as possible but above it in order to keep the BW of $1 + \underline{L}_O(j\omega)$ to a minimum.
- b. Since exact cancellation of a pole by a zero is rarely possible, any RH-plane poles and/or zeros of $P_O(s)$ should be included in $\underline{L}_O(s)$. A good starting $\underline{L}_O(s)$ is $\underline{L}_{O1}(s) = KP_O(s)$. Further, if it is desired that $c(\infty) = 0$ for $d(t) = u_{-1}(t)$, then in order to insure that $T_D(s)$ have a zero at the origin select $\underline{L}_{O1}(s) = P_O(s)/s$.
- c. If $P(s)$ has an excess of poles over zeros, i.e., $\#p's - \#z's = \lambda$, which is denoted by e^λ , then, in general, the final form of $\underline{L}_O(s)$ must have an excess of poles over zeros of at least $e^{\lambda+i}$ where $i \geq 1$. If the BW is too large, then increase the value of i . Experience shows that a value of $\lambda + i$ of three or more for $L(s)$ yields satisfactory results.^{2,10}
- d. Generally the BW of $\underline{L}_O(s)/[1 + \underline{L}_O(s)]$ is larger than required for an acceptable rise time t_R for $c(t)$ responding to $r(t)$. An acceptable rise time can be achieved by the proper design of the prefilter $F(s)$ (see Sec. 2-8).
- e. Generally it is desirable to first find the bounds $\underline{B}_D(j\omega_i)$ and then the bounds $\underline{B}_R(j\omega_i)$. It may be evident, after finding the first $\underline{B}_R(j\omega_i)$ that all or some of the $\underline{B}_D(j\omega_i)$ are completely dominant compared to $\underline{B}_R(j\omega_i)$. Thus the $\underline{B}_D(j\omega_i)$ boundaries are the optimal boundaries, i.e., $\underline{B}_D(j\omega_i) = \underline{B}_O(j\omega_i)$.
- f. If $\delta_R(j\omega_i)$ is not continuously increasing as ω_i increases then it will be necessary to utilize complex poles and or zeros in $G(s)$ in order to achieve an optimal $\underline{L}_O(s)$. This assumes that the tracking model is not designed to yield this characteristic of $\delta_R(j\omega_i)$.
- g. The ability to shape the nominal loop transmission, $\underline{L}_O(s)$, is an art developed only after much practice and patience. This ability is as important to the control engineer as the use of a hammer to a carpenter.

2-8 The Design of the Prefilter $F(s)$ ^{1,12}

Design of a proper $L_o(s)$ guarantees only that the variation in $|T_R(j\omega)|$ is less than or equal to that allowed. The purpose of the prefilter is to position $Lm[T(j\omega)]$ within the frequency domain specifications. For the example of this chapter the magnitude of the frequency response must lie within the bounds B_U and B_L shown in Fig. 1-5 which are redrawn in Fig. 2-16. One method for determining the bounds on the prefilter $F(s)$ is as follows: Place the nominal point of the ω_i template on the Nichols chart where the $L_o(j\omega_i)$ point occurs (see Fig. 2-17). Record the maximum $LmT_{\max}$ and minimum $LmT_{\min}$ values of

$$LmT(j\omega_i) = \frac{L(j\omega_i)}{1 + L(j\omega_i)} \quad (2-46)$$

obtained from the M-contours. These values are plotted as shown in Fig. 2-16.

Since $T_R = FL/[1+L]$ and

$$LmT_R(j\omega_i) = LmF(j\omega_i) + LmT(j\omega_i) \quad (2-47)$$

then the variations in Eqs. (2-46) and (2-47) are both due to the variation in P thus

$$LmT_{\max} - LmT_{\min} = B_U - B_L = \delta_R(j\omega_i) \quad (2-48)$$

Therefore, based upon Eq. (2-47), it is necessary to determine the range in dB that $LmT(j\omega_i)$ must be raised or lowered to fit within the bounds of the specifications by use of the prefilter $F(j\omega_i)$. The process is repeated for each frequency corresponding to the templates used in the design of L_o . Therefore, in Fig. 2-18 the difference between the $LmT_R - LmT_{\max}$ and the $LmT_R - LmT_{\min}$ curves yields the requirement for $LmF(j\omega)$, i.e., from Eq (2-47)

$$LmF(j\omega) = LmT_R(j\omega) - LmT(j\omega) \quad (2-49)$$

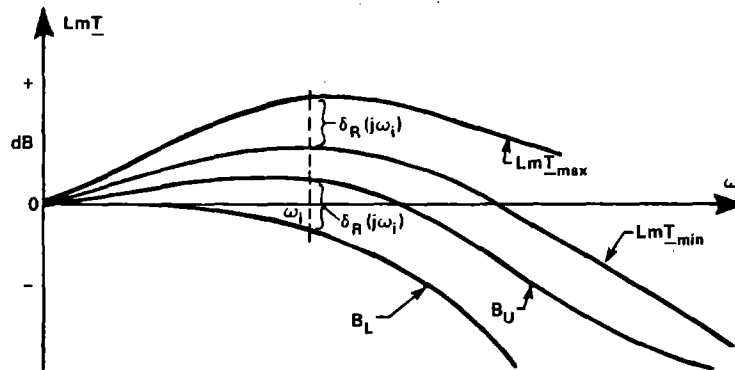


Fig. 2-16 Requirements on $F(s)$

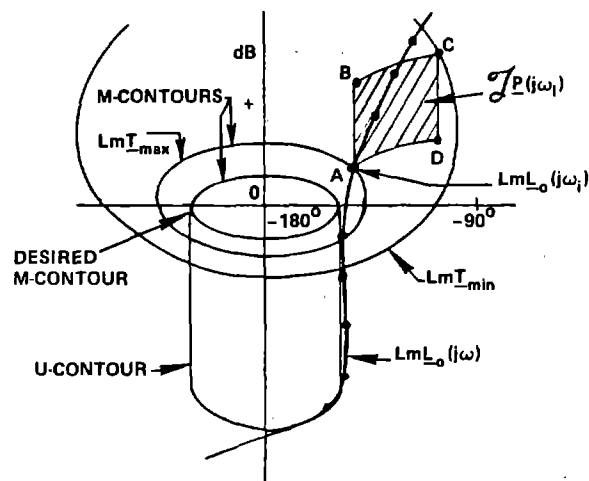


Fig. 2-17 Prefilter Determination.

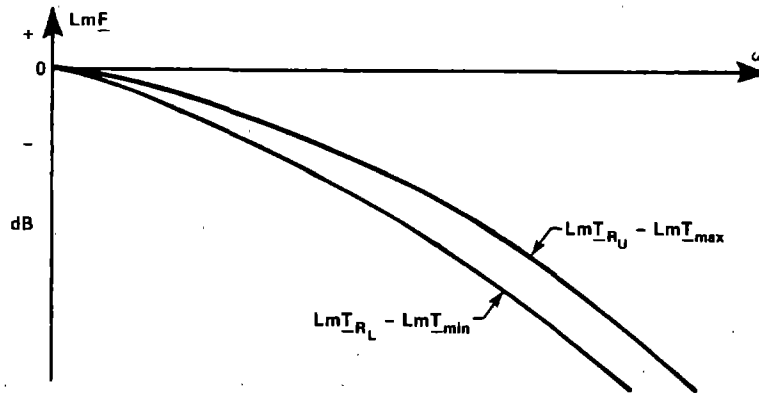


Fig. 2-18 Frequency Bounds on the Prefilter, $F(s)$.

The procedure for designing $F(s)$ is summarized as follows:

a. Use templates to determine $T_{\max}$ and $T_{\min}$ for each ω_i , of $L_0(j\omega)$ on the N.C. This is done by placing $T_P(j\omega_i)$ with its nominal point on the point $LmL_0(j\omega_i)$. Then use the M-contours to determine $T_{\max}(j\omega_i)$ and $T_{\min}(j\omega_i)$ (see Fig. 2-17).

b. Obtain the values of LmT_{RU} and LmT_{RL} for various values of ω_i , from Fig. 1-5.

c. From the values obtained in steps a and b plot

$$[LmT_{RU} - LmT_{\max}] \text{ and } [LmT_{RL} - LmT_{\min}] \text{ vs } \omega$$

as shown in Fig. 2-18.

d. By use of straight line approximations synthesize an $F(s)$ so that its $LmF(j\omega)$ lies within the plots of step c. For step forcing functions the resulting $F(s)$ must satisfy

$$\lim_{s \rightarrow 0} [F(s)] = 1 \quad (2-50)$$

Thus for the $\underline{L}(j\omega)$ obtained, the plot of $\text{Lm}(\underline{L}(j\omega)/[1 + \underline{L}(j\omega)])$ may be larger or smaller than LmT_{R_U} or LmT_{R_L} of Fig. 1-5, but $\delta_R(j\omega_1)$ is satisfied. By the proper design of the input filter $F(s)$ $\text{LmT}_R = \text{LmFL}/[1 + \underline{L}]$ will lie within the bounds of LmT_{R_U} and LmT_{R_L} . In problems with very large uncertainty and in disturbance rejection requiring a very large $|\underline{L}(j\omega)|$ over a "large" B.W. then

$$\frac{\underline{L}}{1 + \underline{L}} \approx 1 \rightarrow T \approx F \quad (2-51)$$

For these situations design $F(s)$ in the manner of Sec. 1-3A. In those situations where the bounds are chosen as $\underline{B}_O = \underline{B}_D$ (ignoring tracking) it is possible that the plots of step c of the filter design procedure may intersect at one or more points. This can occur in the design of sampled-data control systems (see Appendix C). When this situation occurs draw a straight line approximation for $\text{LmF}(j\omega)$ through the intersection point(s) that lies between the two curves. This may result in some degradation in the satisfaction of the tracking specifications.

2-9 Design Example

This design example is for the control system of Fig. 1-2 with $r(t) = d_2(t) = u_{-1}(t)$ and $d_1(t) = 0$. The plant

$$P = \frac{Ka}{s(s+a)} \quad 1 \leq K \leq 10, \quad 1 \leq a \leq 10 \quad (2-52)$$

has the nominal values: $a = K = 1$.

Step 1 Modeling the tracking control ratio $T_R = C/R$.

$$T_{R_U} : M_p = 1.2, t_s \leq 2s$$

$$T_{R_L} : \text{Overdamped}, t_s \leq 2s$$

$$(a) \quad (T_{R_U})_T = \frac{19.753}{(s + 2 \pm j3.969)} \quad (2-53)$$

A zero is inserted in Eq. (2-53) which does not affect the desired performance specifications but widens δ_{hf} between T_{RU} and T_{RL} in the high frequency range. Thus

$$T_{RU} = \frac{0.6584(s + 30)}{(s + 2 \pm j3.969)} \quad (2-54)$$

whose figures of merit are

$$M_p = 1.2078, \quad t_R = 0.342s., \quad t_p = 0.766s., \quad t_s = 1.84s.,$$

$$LmM_m = 1.95dB, \quad \omega_m = 3.4 \text{ rps.}$$

$$(b) \quad (T_{RL})_T = \frac{120}{(s + 3)(s + 4)(s + 10)} \quad (2-55)$$

which is modified to yield

$$T_{RU} = \frac{8400}{(s + 3)(s + 4)(s + 10)(s + 70)} \quad (2-56)$$

The pole at -70 is inserted in Eq. (2-55) for the same reason that the zero is inserted in Eq. (2-53). The figures of merit for Eq. (2-56) are:

$$M_p = 1, \quad t_R = 1.02s., \quad t_s = 1.849s.$$

(c) Determination of $\delta_R(j\omega_1)$

From the data for the Lm plots of Eqs. (2-54) and (2-56), the following values are obtained for $\delta_R(j\omega_1)$

ω	$\delta_R(j\omega_1), \text{dB}$
0.5	0.25
1	1.06
2	3.728
5	10.57
10	9.851
15	11.522
100	40

Due to the disturbance rejection specifications (see Step 6) the above values are sufficient. Normally, values of ω an octave apart up to ω_h should be obtained.

Step 2 Modeling the disturbance control ratio $T_{DU} = C/D_2$

Since $c(0) = 1$, it is desired that $c(t)$ decay "as fast as possible," i.e.,
 $|c(t)| \leq 0.01$ for $t \geq t_x = 60\text{ms}$.

Let the model disturbance control ratio be of the form

$$T_{DU} = \frac{C}{D_2} = \frac{s(s+g)}{(s+g)^2 + h^2} \quad (2-57)$$

where for $d_2(t) = u_{-1}(t)$

$$c(t) = e^{-gt} \cos ht \quad (2-58)$$

The desired specifications (see Fig. 1-6) are satisfied by choosing

$$T_{DU} = \frac{s(s+70)}{s+70 \pm j18} \frac{N}{D} = \frac{1}{1+L_D} \quad (2-59)$$

$$\text{where } L_D = \frac{D - N}{N} = \frac{70(s + 74.6286)}{s(s + 70)} \quad (2-60)$$

For Eq. (2-59)

$$c(t) \leq 0.01 \text{ at } t_x \approx 0.0565s$$

Steps 3 and 4 Forming the templates

Analysis of P is based upon the nominal values $K = a = 1$ (Point A of Fig. 2-19).

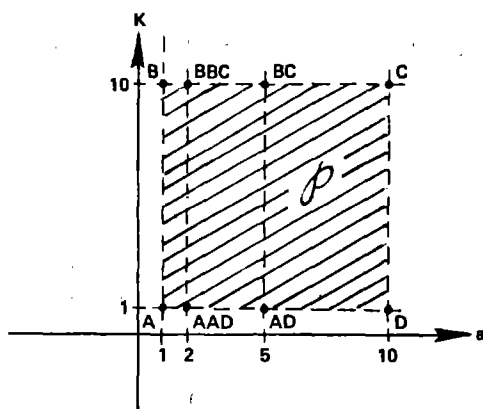


Fig. 2-19 Template Points.

$$\omega = 0.5$$

$$\text{Point A: } P(s) = \frac{1}{s(s + 1)} \longrightarrow \text{Lm} \left[\frac{1}{j0.5(1 + j0.5)} \right] = 5\text{dB}, -116.6^\circ$$

$$\text{Point B: } P(s) = \frac{10}{s(s + 1)} \longrightarrow \text{Lm} \left[\frac{10}{j0.5(1 + j0.5)} \right] = 25\text{dB}, -116.6^\circ$$

$$\text{Point BC: } P(s) = \frac{50}{s(s + 5)} \longrightarrow \text{Lm} \left[\frac{50}{10.5(5 + j0.5)} \right] = 26\text{dB}, -95.7^\circ$$

$$\text{Point C: } P(s) = \frac{100}{s(s + 10)} \longrightarrow \text{Lm} \left[\frac{100}{j0.5(10 + j0.5)} \right] = 26\text{dB}, -93^\circ$$

$$\text{Point D: } P(s) = \frac{10}{s(s + 10)} \longrightarrow \text{Lm} \left[\frac{10}{j0.5(10 + j0.5)} \right] = 6\text{dB}, -93^\circ$$

$$\text{Point AD: } P(s) = \frac{5}{s(s + 5)} \longrightarrow \text{Lm} \left[\frac{5}{j0.5(5 + j0.5)} \right] = 6\text{dB}, -95.7^\circ$$

The evaluation of $\underline{P}(j\omega_i)$ and $\text{Lm}\underline{P}(j\omega_i)$ at the various points for the remaining values of ω_i , is done in a similar manner as above. Table 2-3 presents the values of $\text{Lm}\underline{P}(j\omega_i)$ and $\underline{P}(j\omega_i)$ for various values of ω_i . The values in Table 2-3 are used to draw the templates $\underline{TP}(j\omega_i)$ shown in Fig. 2-20.

Step 5 Determination of the U-Contour

(a) Determination of V [see Eq. (2-10)]: for the plant of Eq. (2-52)

$$\lim_{s \rightarrow \infty} \left[\frac{P}{s} \right] = \left[\frac{ka}{s^2} \right] \quad \Delta \text{Lm} \left[\frac{P}{s} \right]_{\max} = \text{Lm} \left[\frac{Ka}{s} \right]_{\max} = 40\text{dB} \quad (2-61)$$

To determine the value of ω_h where $\Delta \text{Lm}\underline{P} \approx V = 40\text{dB}$, select various values of ω_i as shown below. For $\omega = 100$

$$\text{Point A } \text{Lm}\underline{P} = \text{Lm} \left[\frac{1}{j100(1 + j100)} \right] = -80\text{dB}$$

$$\text{Point B } \text{Lm}\underline{P} = \text{Lm} \left[\frac{10}{j100(1 + j100)} \right] = -60\text{dB}$$

$$\text{Point C } \text{Lm}\underline{P} = \text{Lm} \left[\frac{100}{j100(1 + j100)} \right] = -40\text{dB}$$

$$\text{Point D } \text{Lm}\underline{P} = \text{Lm} \left[\frac{10}{j100(10 + j100)} \right] = -60\text{dB}$$

From this data it is seen that the maximum value of $\Delta \text{Lm}\underline{P}$ occurs between points A and C. For other values of ω_i the maximum values of $\Delta \text{Lm}\underline{P}$ are:

$$\omega = 15: \quad \left| -47.06 - (-8.64) \right| = +38.4\text{dB}$$

$$\omega = 16: \quad \left| -48.20 - (-9.6) \right| = +38.6\text{dB}$$

$$\omega = 20: \quad \left| -52 - (-13) \right| = +39\text{dB}$$

$$\omega = 40: \quad \left| -64 - (-24.35) \right| = 39.65\text{dB}$$

Therefore it can be seen, for this example, that V has been achieved essentially at $\omega_h \cong 40$.

Table 2-3 Data Points for $TP(\omega_1)$

Point ↓	$\omega_1 \rightarrow$	0.5	1	2	5	10	15
A	LmP	5	-3	-13	-28.1	-40	-47
	$\angle P$	-116.6°	-135°	-153.5°	-168.7°	-174.3°	-176.2°
B	LmP	25	17	7	-8.1	-20	-27
	$\angle P$	-116.6°	-135°	-153.5°	-168.7°	-174.3°	-176.2°
BBC	LmP	25.8	19	11	-2.6	-14.2	-21.1
	$\angle P$	-104.1°	-116.6°	-135°	-158.2°	-168.7°	-172.4°
BC	LmP	26	19.8	13.3	3	-7	-13.5
	$\angle P$	-95.7°	-101.3°	-111.8°	-135°	-153.5°	-161.6°
C	LmP	26	20	13.8	5	-3	-8.64
	$\angle P$	-93	-95.7°	-101.3°	-116.6°	-135°	-146.4°
D	LmP	6	-0.04	-6.2	-14.95	-23	-28.6
	$\angle P$	-93°	-95.7°	-101.3°	-116.6°	-135°	-146.4°
AD	LmP	6	-0.17	6.67	-17	-27	-33.5
	$\angle P$	-95.7	-101.3°	-111.8°	-135°	-153.5°	-116.6°
AAD	LmP	5.65	0.97	-9	-22.6	-34.2	-41.1
	$\angle P$	-104.1°	-116.6°	-135°	-158.2	-168.7°	-172.4°

Note: Values of Lm are in dB

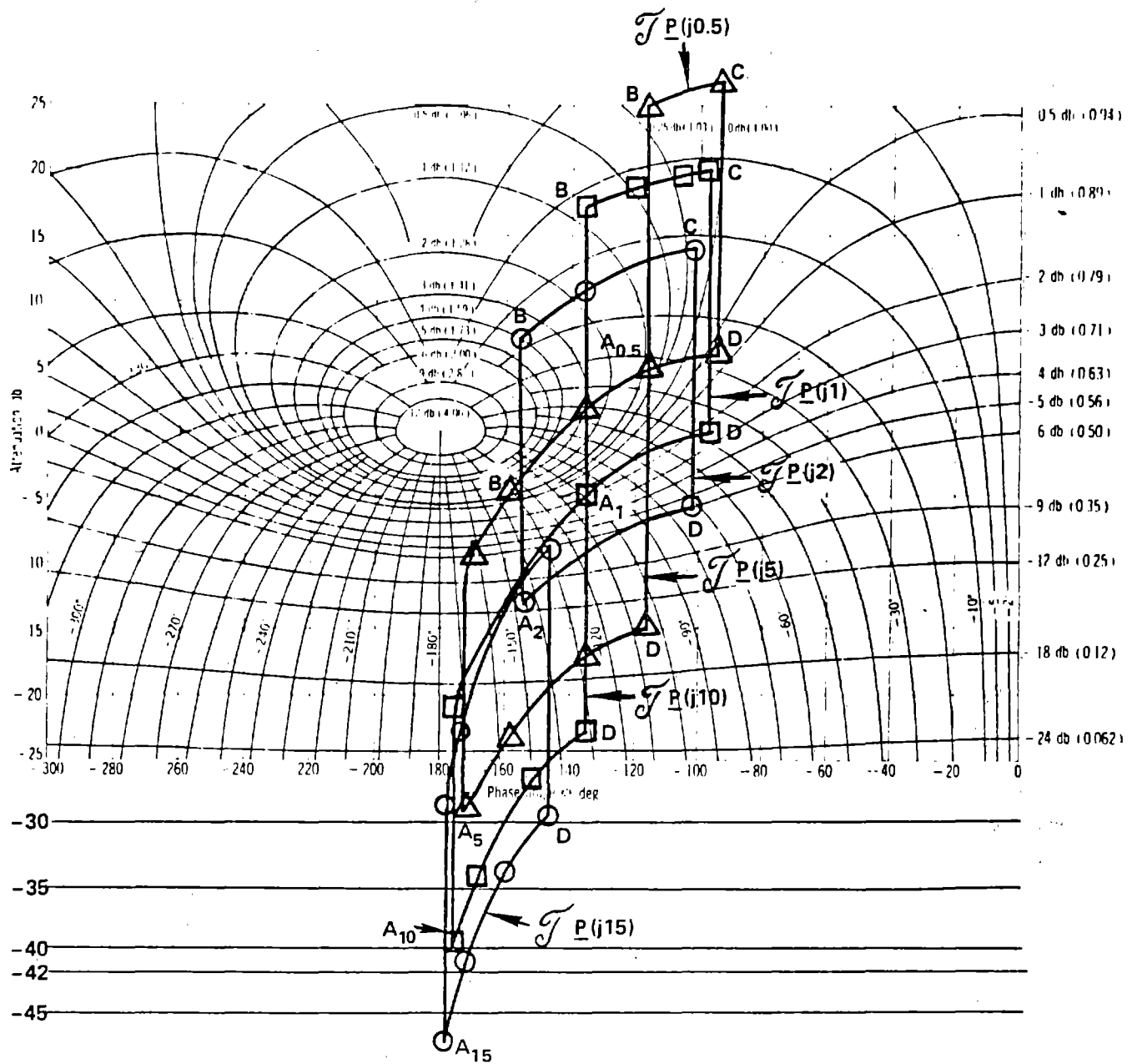


Fig. 2-20 Construction of the plant templates.

(b) Determination of the B_h boundary

Select the M-contour that represents the desired value of M_L for T (see Sec. 2-3). At various points around the lower half of this contour measure down 40dB and draw the B_h boundary in the manner shown in Fig. 2-3.

Step 6 Determination of bounds $B_D(j\omega_1)$ for T_{D_U} Eq. (2-57)

$$T_{D_U} = \frac{C}{D} = \frac{1}{1+L} = \frac{\ell}{1+\ell} \quad (2-62)$$

where $\ell = 1/L = 1/PG$ and $C = DT_{D_U}$.

For the disturbance rejection case:

$$\Delta LmC = \Delta Lm \left[\frac{\ell}{1+\ell} \right] = \Delta Lm T_{D_U} \quad (2-63)$$

Note that there are no lower bounds on the response $c(t)$. Thus, choose

$$\underline{L} \geq \underline{B}_D(j\omega_1)$$

to satisfy

$$Lm \left[\frac{C}{\underline{D}} \right] = Lm \left[\frac{\underline{\ell}}{1+\underline{\ell}} \right] \leq Lm \left[\frac{T_{D_U}}{\underline{D}} \right] \quad (2-64)$$

for each frequency ω_1 of $T_D(j\omega_1)$ over the frequency range $0 \leq \omega \leq 15$ rps.

Equation (2-59) yields the data of Table 2-4 for various values of ω .

Table 2-4 Values for Eq. (2-59)

ω	$\underline{\text{LmT}}_{\underline{\text{D}}}$, dB	$\angle \underline{\text{T}}_{\underline{\text{D}}}$, degrees
0.5	-43.5	89.65
1	-37.5	89.3
2	-31.44	88.6
5	-23.5	86.4
10	-17.52	82.85
15	-14.1	79.31

Point A of the templates of Step 3 yields the maximum value $\underline{\ell} = 1/\underline{L}$. For $0 \leq \omega \leq 15$ the $[\underline{\text{T}}_{\underline{\text{D}}}(\underline{j}\omega_1)]$ contours (see Fig. 2-21) become the contours for $\underline{\text{B}}_{\underline{\text{D}}}$.

Step 7 Determination of the bounds $\underline{\text{B}}_{\underline{\text{R}}}(\underline{j}\omega_1)$

By use of the templates, the values of $\delta_{\underline{\text{R}}}(\underline{j}\omega_1)$ given in Step 2, the $\underline{\text{M}}_{\underline{\text{L}}}$ contour and the $\underline{\text{B}}_{\underline{\text{h}}}$ boundary the bounds $\underline{\text{B}}_{\underline{\text{R}}}(\underline{j}\omega_1)$ and the U-contour are determined and are drawn in Fig. 2-22.

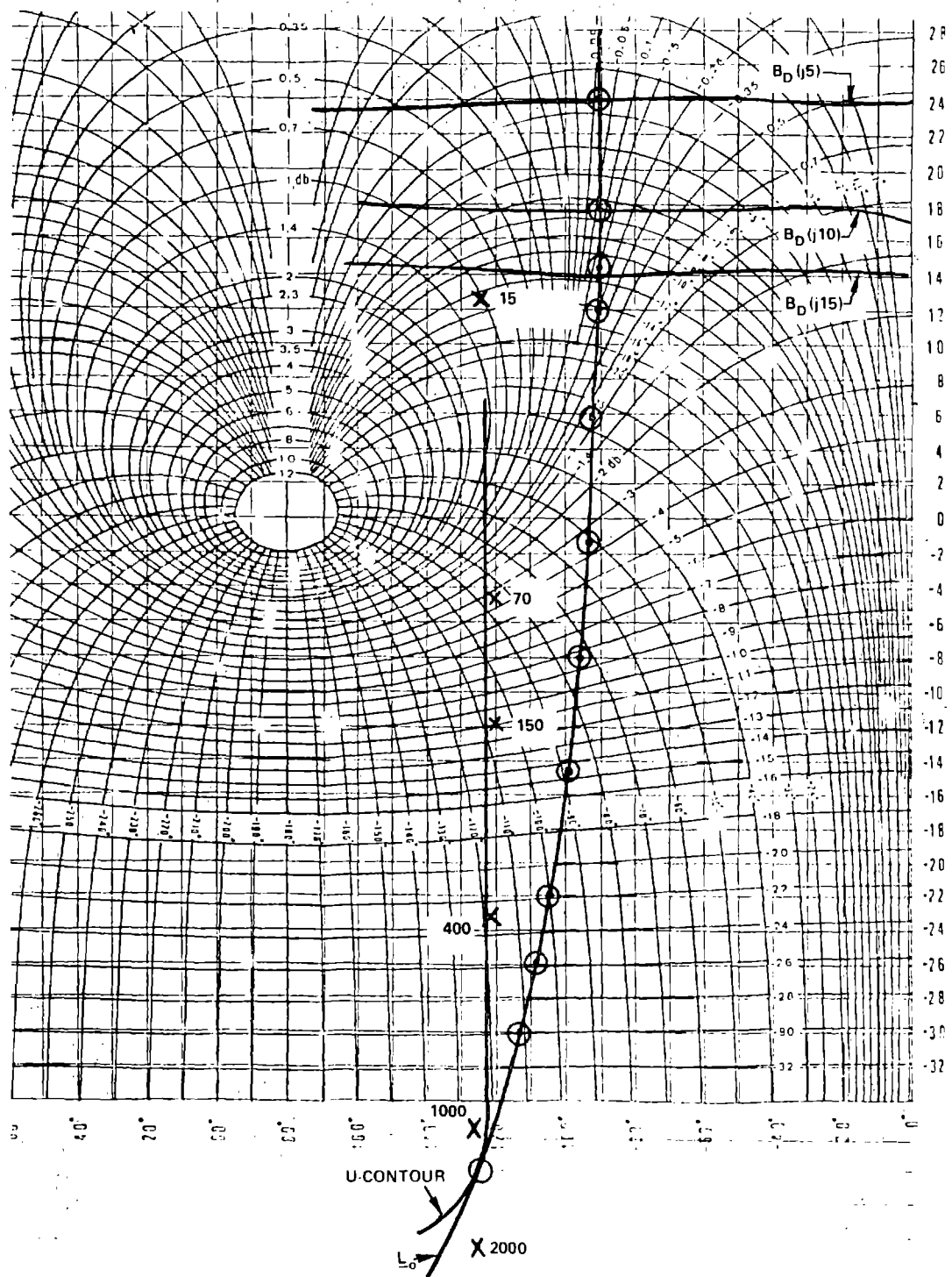


Fig. 2-21 Construction of the $B_D(j\omega_1)$ contours.

Step 8 Determination of the optimal (composite) bounds $\underline{B}_O(j\omega_1)$

An analysis of Figs. 2-21 and 2-22 reveal that the bounds $\underline{B}_D(j\omega_1)$, for this example all lie above the tracking contours $\underline{B}_R(j\omega_1)$. Thus the $\underline{B}_D$ contours of Fig. 2-21 are more severe and become the $\underline{B}_O(j\omega_1)$ contours for the overall system of Fig. 2-23.

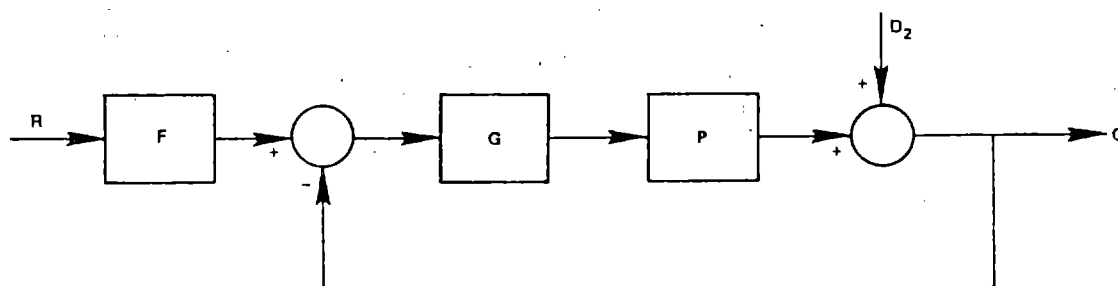


Fig. 2-23 The MISO System

Step 9 Synthesizing or Shaping of $L_O(s)$

A. Minimum structure -- Since $P(s)$ is Type-1, then $L_O(s)$ must be at least Type-1. Equation (2-60), obtained from the model $T_D(s)$ is used as the starting approximation $L_{O1}(s)$ for $L_O(s)$ since it yields the upper bound T_{D_U} . Normally, this is not advisable since it can yield, as in this example, a very large value for the phase margin frequency ω_ϕ . Equation (2-60) yields the data of Table 2-5 for various values of ω_1 . Since, for this example, $\underline{B}_O = \underline{B}_D$ then it is necessary to extend Table 2-5 by use of Eq. (2-60) to include higher values of ω . These additional values will provide sufficient disturbance bounds which will yield an $\underline{L}_O$ that does not intersect the U-contour and results in an actual $\underline{T}_D(j\omega)$ which has essentially the same frequency response characteristics as Eq. (2-59).

Table 2-5 Values of Eq. (2-60) for various values of ω_i .

ω	$Lm\underline{L}_D$, dB	$\angle \underline{L}_D$, degrees
0.5	43.5	-90.025
1	37.5	-90.05
2	31.44	-90.1
5	23.48	-90.25
10	17.45	-90.5
15	13.91	-90.73

B. Structure of $\underline{L}_o(s)$

(1) Start with $\underline{L}_{o1}(j\omega)$ and build up to $\underline{L}_o(j\omega)$ so that it follows the U-contour. For the range $0 \leq \omega \leq 15$ it is desired that both magnitude and angle satisfy the condition

$$|\underline{L}_o| \angle \underline{L}_o \cong |\underline{L}_D| \angle \underline{L}_D \cong |\underline{B}_o| \angle \underline{B}_o \quad (2-65)$$

i.e., $\underline{L}_o$ be as close as possible to the model $\underline{L}_D$ obtained in the manner described in Chapter 1.

(2) Using four cycle semi-log graph paper and the table below, construct or shape $\underline{L}_o(s)$ to satisfy Eq. (2-60) for $0 \leq \omega \leq 15$. Then for $\omega \geq 15$ shape $\underline{L}_o(s)$ to be as close as possible to the U-contour. To satisfy Eq. (2-65) cannot bring $\underline{L}_o$ too close to the U-contour too soon. This is done by assuming the format of the optimal transfer function as

$$\underline{L}_o(j\omega) = \underline{L}_{o1}(j\omega) \left[\frac{(j\omega - z_1)(j\omega - z_2) \dots}{(j\omega - p_3)(j\omega - p_4) \dots \left[\left(1 - \frac{\omega^2}{\omega_n^2}\right) + j2\zeta \frac{\omega}{\omega_n} \right]} \right] \quad (2-66)$$

which has w real zeros, $n = w + \lambda - 2$ real poles, and a pair of complex conjugate poles. For this example a table having the format shown in Table 2-6 is used to assist in obtaining the desired $L_o(s)$.

Table 2-6 Data for loop shaping

		Angle contribution of pole(p) and zero(z)					
w	$\angle L_{o1}$	P_3 -800	z_1 -1000	P_4 -2000	z_2 -3000	$P_{5,6}$ -9000 + j10000	$\angle L_o$
1	-90.1°	-	-	-	-	-	-90.1°
10	-90.5°	-0.75°	0.6°	-0.25°	0.18°	-	-90.7°
15	-90.7°	-1.05°	0.9°	-0.45°	0.28°	-	-91°
50	-91.85°	-3.6°	1.05°	-1.4°	1°	-	-94.8°
100	-91.5°	-7.1°	5.7°	-2.9°	1.9°	-1°	-94.9°
200	-91.1°	-14°	11.4°	-5.8°	3.75°	-1°	-96.8°

The resulting optimal loop transfer function is

$$L_o(s) = \frac{80(1 + \frac{s}{74.63})(1 + \frac{s}{1000})(1 + \frac{s}{3000})}{s(1 + \frac{s}{70})(1 + \frac{s}{800})(1 + \frac{s}{2000})(1 + \frac{2\zeta}{\omega_n}s + \frac{s^2}{\omega_n^2})} \quad (2-67)$$

$$= \frac{7.224 \times 10^9 (s + 74.63)(s + 1000)(s + 3000)}{s(s + 70)(s + 800)(s + 2000)(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

$$\omega_n^2 = 1.81 \times 10^8 \quad \zeta = 0.67 \quad K = 7.224 \times 10^9$$

$L_o(j\omega)$ is drawn on the Bode plot of Fig. 2-24. It is also plotted on the N.C. of Fig. 2-21.

C. Determination of $G(s)$ Use $L_o(s)$ and $P_o(s)$, where $P_o(s) = 1/s(s+1)$ is the nominal plant, to obtain the transfer function $G(s) = L_o(s)/P_o(s)$.

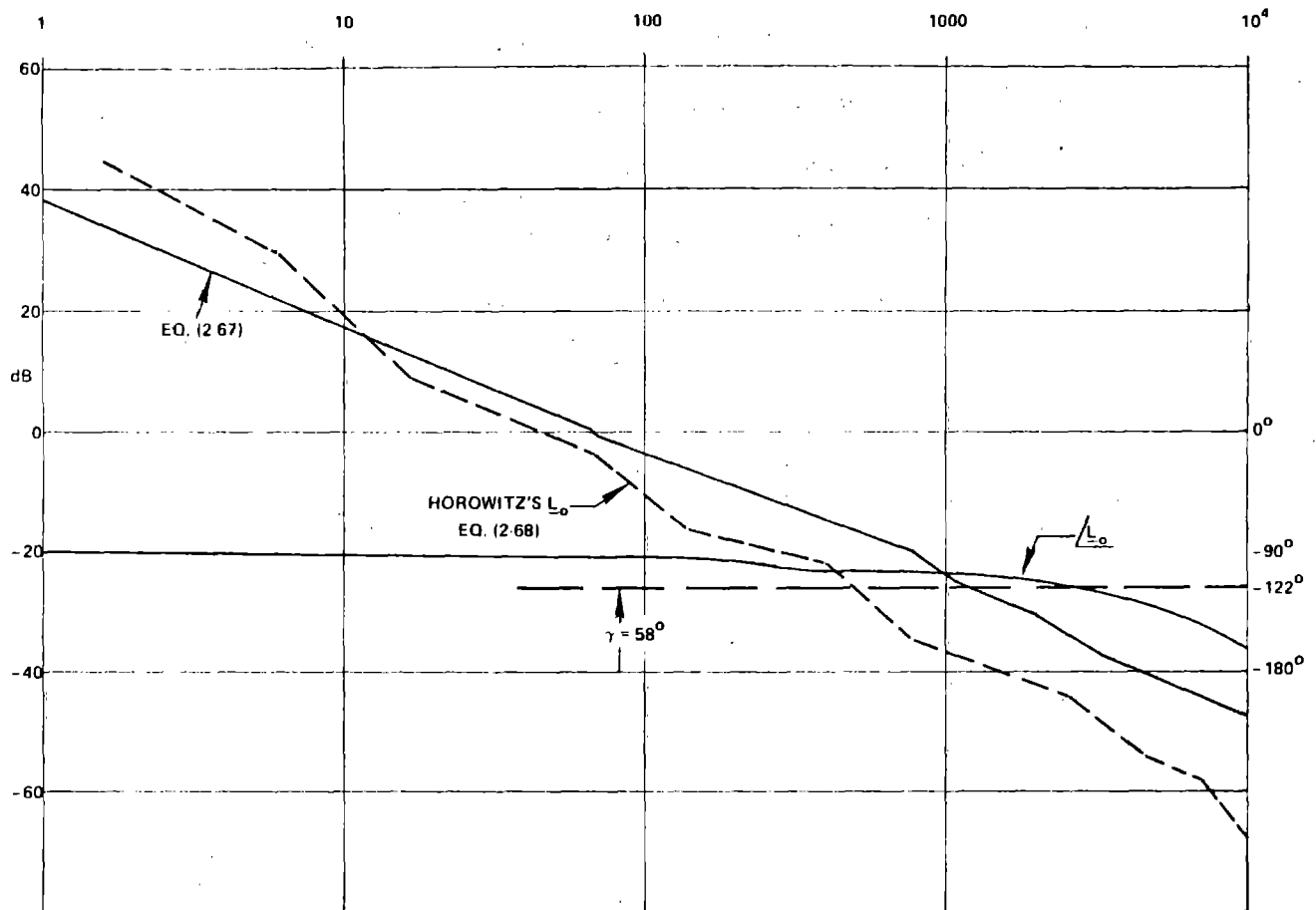


Fig. 2-24 Bode of $\underline{L}_O(j\omega)$ of Eq. (2-67)

Step 10 The filter $F(s)$ needs now to be synthesized to yield the desired tracking performance of $c(t)$ due to the input $r(t)$ in the manner described in Sec. 2-8. This is left as an exercise for the reader (see Prob. 2-6).

Note that Professor Horowitz obtained the following alternate $\underline{L}_O(s)$

$$\underline{L}_O(s) = \frac{K_{\text{New}} \left(1 + \frac{s}{18}\right) \left(1 + \frac{s}{150}\right) \left(1 + \frac{s}{750}\right) \left(1 + \frac{s}{4500}\right)}{s \left(1 + \frac{s}{7}\right) \left(1 + \frac{s}{80}\right) \left(1 + \frac{s}{400}\right) \left(1 + \frac{s}{2500}\right) \left[1 + \frac{s}{8000} + \left(\frac{s}{8000}\right)^2\right]} \quad (2-68)$$

where $K_{\text{New}} \cong 2K_{\text{Old}} = 159.57$. The Bode plot of Eq. (2-69) is shown in Fig. 2-24. Note that at $\omega = 10^4$ Eq. (2-68) is less than Eq. (2-67) by about 18dB. Also, the ω_n of Eq. (2-68) is 8000 vs 13,500 for Eq. (2-67). Note that

Eq. (2-68) yields a phase margin frequency ω_ϕ of 300rps versus 760rps for Eq. (2-67). The plot of Eq. (2-68) on the N.C. can be drawn through the "X's" shown on Fig. 2-21 which follow the boundaries more closely. Using Eq. (2-60) for $L_{O1}(s)$ is not recommended as it usually yields a larger value for ω_ϕ . It is best to start with $L_{O1} = K/s^m$ representing the desired Type m transfer function. With this L_{O1} insert zeros and poles to try to achieve an optimal $L_O(s)$.

Step 11 Time responses for a disturbance input: $d(t) = u_{-1}(t)$

For each point of Fig. 2-19 substitute the corresponding data into

$$C_D(s) = \frac{1}{1 + P(s)G(s)} D(s) \quad (2-69)$$

to determine $c_D(t)$ for each point. The response characteristics for points A, B, C, and D are:

Point	$C_D(t_x)$	Specified t_x, s	Actual t_x for $C_D(t) = 0.01$
A	0.00297	0.06	0.05
B	-8.3×10^{-5}	↓	0.0078
C	0.00106		0.00145
D	0.01049		0.11

At point D the desired condition $c(t) \leq 0.01$ at 60ms is not quite met. Raising the gain from 7.2×10^9 to 7.73×10^9 , an increase of 0.6dB, results in meeting the specifications at all four points.

2-10 m.p. Unstable Plant Template Generation

In the generation of templates for m.p. unstable plants, proper care must be exercised in analyzing the angular variation of $\underline{P}(j\omega)$, as the frequency is varied from zero to infinity. As an example, consider the plant

$$\underline{P}(j\omega) = \frac{K(j\omega - z_1)}{j\omega(j\omega - p_1)(j\omega - p_2)(j\omega - p_3)} \quad (2-70)$$

where p_1 and p_2 are a complex-conjugate pair and

$$\angle \lim_{\omega \rightarrow \infty} \underline{P}(j\omega) = \angle \frac{K}{(j\omega)^3} = -270^\circ \quad (2-71)$$

In Eq (2-70) the convention is assumed in the limit that each $j\omega$ term contributes an angle of $+90^\circ$. Two cases are analyzed for Eq.(2-70) as follows:

Case 1--A m.p. stable plant is assumed where the poles and zeros of Eq (2-70) are plotted in Fig 2-25(a). In order to satisfy Eq (2-70), p_1 , for $\omega_1 \leq \omega_{d1} = 1$, is taken clw and for $\omega_1 \geq \omega_{d1}$ it is taken as cclw where all other angles are taken cclw as illustrated in Fig 2-25(a) and in Table 2-6.

Case 2--The poles and zeros of Eq (2-70), for a m.p. unstable plant are plotted in Fig 2-25(b). To satisfy Eq (2-70) all angles are taken cclw as illustrated in Fig 2-25(b) and in Table 2-6.

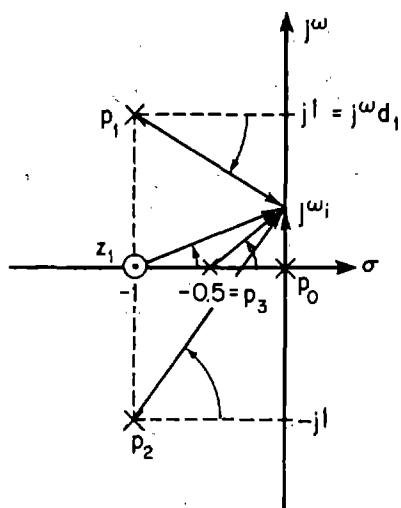
Note: (a) that the difference in the angle of $\underline{P}(j0)$ between the two cases is 360° and (b) that the angle of $\underline{P}(j\omega)$ varies continuously in a given direction as the frequency is varied between zero and infinity. This feature is very important in obtaining a template where P contains both m.p. stable and unstable plants.

The following guidelines should be used in the angular determination of $\angle \underline{P}(j\omega_1)$:

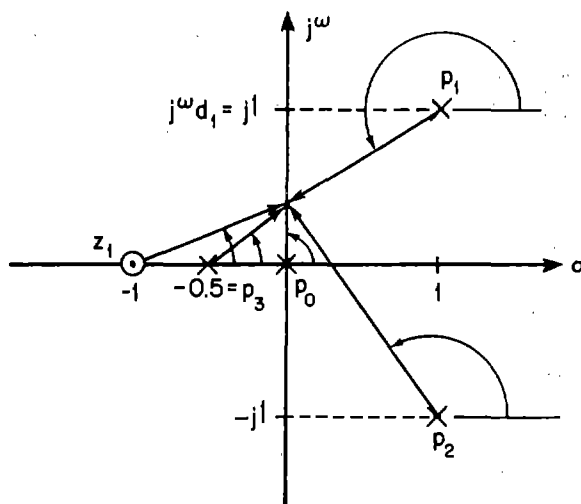
1. For m.p. stable plants all angular directions are taken so that their values always lie within the range of -90° to $+90^\circ$.
2. For m.p. unstable plants (a) the angular directions for the l.h.p. poles and zeros of $P(s)$ are taken in the same manner as for the m.p. stable plant and (b) the angular directions of all r.h.p. poles and zeros are all taken cclw.

Table 2-6 Angular variations of the poles and zeros of the m.p. stable and unstable plants of Fig 2-25.

Case	ω_1	Angle					
		$\angle p_0$	$\angle p_1$	$\angle p_2$	$\angle p_3$	$\angle z_1$	$\angle \underline{P}(j\omega_1)$
1	0	$+90^\circ$	-45°	$+45^\circ$	0°	0°	-90°
	1	$\downarrow$	0°	63.5°	63.5°	$+45^\circ$	-172°
	∞	$\downarrow$	$+90^\circ$	$+90^\circ$	$+90^\circ$	$+90^\circ$	-270°
2	0	$+90^\circ$	$+225^\circ$	$+135^\circ$	0°	0°	-450°
	1	$\downarrow$	$+180^\circ$	$+116.5^\circ$	$+63.5^\circ$	$+45^\circ$	-405°
	∞	$\downarrow$	$+90^\circ$	$+90^\circ$	$+90^\circ$	$+90^\circ$	-270°



(a)



(b)

Fig 2-25 Plot of the zero and poles of a m.p. (a) stable plant;
(b) unstable plant

2-11 Summary

This chapter has presented a robust design technique that permits a control system to satisfy the desired performance specifications within the specified range of plant parameter variation. Since the QFT technique is based upon the design of the loop transmission function $L(s)$ of the multiple input-single output (MISO) control system of Fig. 1-2 it is referred to as a SISO design technique. Although this chapter has dealt with minimum phase plants the QFT technique can be applied to nonminimum phase plants.¹³ Some suggestions on how to apply the QFT design technique to nonminimum phase and/or unstable plants are given in Appendix C, and in the Design Checklist on pages E-3 through E-10.

2-12 Problems

(Note: refer to Chapter 5 of Reference 4 for a review of signal flow graphs).

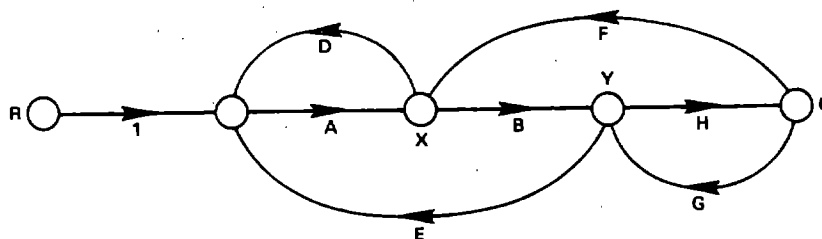
2-1 A system is described by the differential equations

$$\ddot{y} - Az = r \quad Bz + Dy + \dot{r} = 0$$

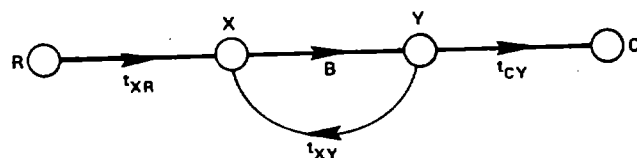
where r is the input and y is the output

- Draw the signal flow graph of the system.
- Determine the loop transmission function.

2-2



- Determine the transfer function C/R .
- Redraw the flow graph into the fundamental feedback (single loop) flow graph in which B is the reference, i.e.,



Identify the transfer functions t_{xr} , t_{ys} and t_{xs} in terms of A , B , C etc.

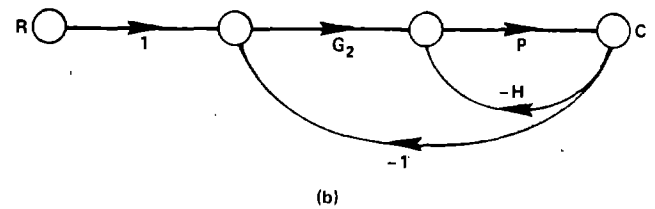
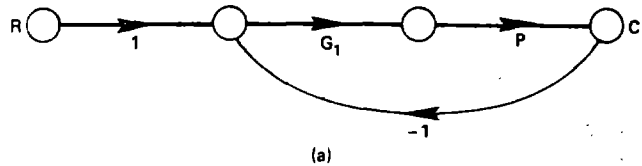
Hint: First determine t_{xr} and t_{ys} by letting $B = 0$.

2-3 In figure (a) $P(s) = 1/s$ and $G_1(s) = 1/(s+2)$, and in figure (b) $P(s) = 1/s$ and $H(s) = s/(s+1)$.

- Find $G_2(s)$ so that $C(s)/R(s)$ is the same for both configurations.
- Find $S_P^T(s)$, system sensitivity to P , for each configuration. If these

are different which one yields smaller values of $|\underline{S}_P^T(j\omega)|$ over a range of finite values of ω ?

(c) Which configuration is the best for minimizing the parameter variation effects on $C(s)$?



2-4 Given $L(s) = G(s)P(s)$ and $P(s) = (10 - s)P_1(s)$ where $P_1(s)$ has no poles or zeros in the right-half plane. It is desired that $|\underline{L}(j\omega)| > 1$ from $\omega = 0$ to ω_ϕ with ω_ϕ as large as possible, that $|\underline{L}(j\omega)| \geq 2$ for $0 \leq \omega \leq 0.5\omega_\phi$, that the phase margin $\gamma \geq 30^\circ$, that the gain margin $L_{ma} \geq 12\text{dB}$, that the excess of poles over zeros for $L(s)$ be two, and that the corner frequency of the final pole to be inserted into L be $\leq 10\omega_\phi$. Design $L(s)$.

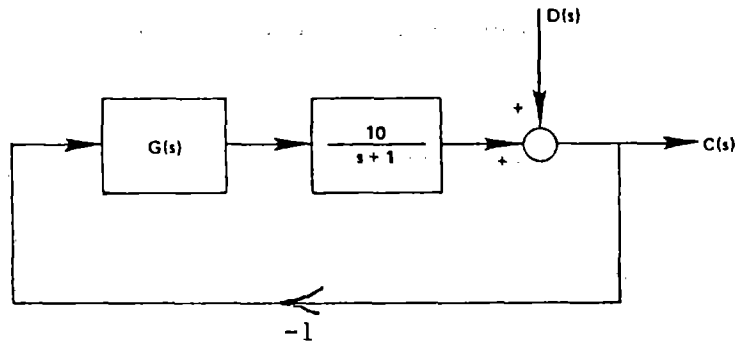
2-5 For the control system shown $P(s) = 10/(s + 1)$, and $d(t) = u_{-1}(t)$. It is desired that

$$\lim_{t \rightarrow \infty} [c(t)] \leq 10^{-3}$$

and that $|c(t)| < 0.35$ within one second and not exceed this value. Design $G(s)$ so that these specifications are satisfied where

$$\lim_{s \rightarrow \infty} G(s) = 0$$

and that $|\underline{G}(j\omega)|$ be as small as possible for all ω . (NOTE: Figure is on the top of the next page.)



2-6 For the design example of Sec. 2-9.

- (a) Obtain the templates.
- (b) Draw $L_m L_o$ vs L_o on a N.C.
- (c) Obtain the data in order to draw the plots of Fig 2-18.
- (d) Determine a suitable $F(s)$.
- (e) Obtain the five response characteristics for $T_R(s)$ for the points A, B, C and D. Are the desired time response specifications achieved at each point?

2-7 For the control system shown in Figure (a) $P(s) = Ka/s(s+a)$ and the

parameter variations shown in figure (b) are $K_x \leq K \leq K_y$ and $a_w \leq a \leq a_z$.

(Note parameter variation values are to be supplied by the instructor, a typical set is $1 \leq K \leq 10$ and $2 \leq a \leq 12$). Design

- (a) Design $G(s)$ and $F(s)$ so that the following specifications are met:

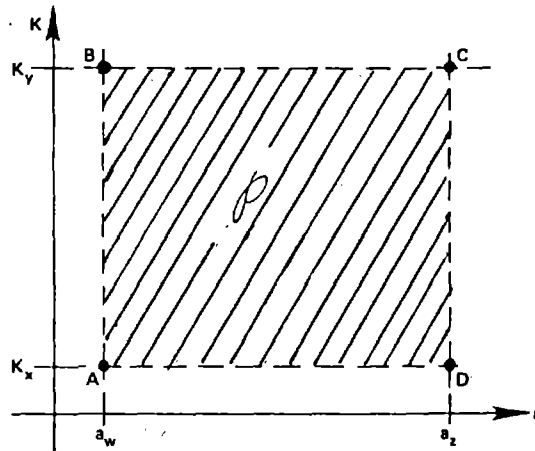
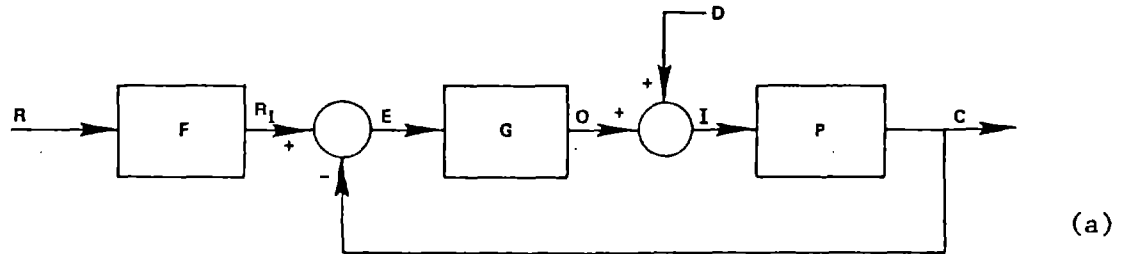
For T_R : $T_U \rightarrow M_p = 1.2$ ($L_m M_m = 1.58\text{dB}$) and $t_s \leq 1.8\text{sec}$.

$T_L \rightarrow$ overdamped response and $t_s \leq 1.8\text{sec}$.

For T_D : $c(t_p) \leq 0.1$ where $t_p \leq 60\text{ ms}$, $c(\infty) = 0$ "and"

$$r(t) = d(t) = u_{-1}(t).$$

- (b) For your design, for each point A, B, C and D in figure (b), obtain $c(t)$ due to $r(t) = u_{-1}(t)$ and $d(t) = 0$. Repeat for $d(t) = u_{-1}(t)$ and $r(t) = 0$.



THE FREQUENCY DOMAIN3-1 Introduction

The quantitative feedback theory (QFT) synthesis technique for highly uncertain linear time-invariant MIMO plants has the following features¹³:

1. It is quantitative in nature. The extent of parameter uncertainty is defined a priori and so are the tolerances on the system responses to command inputs and /or on the system response to disturbance functions.
2. The synthesis problem is converted into a number of single-loop problems, in which parameter uncertainty, external disturbances, and performance tolerances are derived from the original MIMO problem. The solutions to these single-loop problems are guaranteed to work for the MIMO plant. It is not necessary to consider the system characteristic equation. Any technique may be used for the single-loop design problems--state space, frequency response, or even cut and try.
3. The design is tuned to the extent of the uncertainty and the performance tolerances. The design for a MIMO system, as stated previously involves the design of an equivalent set of MISO system feedback loops. The design process for these individual loops is the same as the design of a SISO system described in Chap. 2. In general, an $\ell \times m$ open-loop MIMO plant can be represented in matrix notation as

$$\underline{y}(t) = \underline{P}\underline{u}(t) \quad (3-1)$$

where $\underline{y}(t)$ = ℓ -dimensional plant output vector

$\underline{u}(t)$ = m -dimensional plant input vector

$\underline{P}$ = $\ell \times m$ plant transfer function matrix relating $\underline{u}(t)$ to $\underline{y}(t)$

Consider the MIMO plant of Fig 3-1 for which $\underline{P}$ is a member of $\mathcal{P}$ ($\underline{P} \in \mathcal{P}$), the set of all plant transfer functions relating each input to each output. When a system has variable parameters, which are known or unknown, the transfer function matrix of the plant may be represented by the associated set $\{P_1, P_2, \dots\}$. This set of transfer function matrices is contained in $\mathcal{P}$. The design method requires that the uncertainty in $\underline{P}$ be known or is at least bounded. In any MIMO system with m inputs there are at most m outputs which can be independently controlled³. Therefore, the same dimensions, $l = m$, are used for both the input and output vectors in the design procedure presented here. If the model defines an unequal number of inputs and outputs, the first step is to modify the model so that the dimensions of the input and output are the same. The system is then defined as being of order $m \times m$.

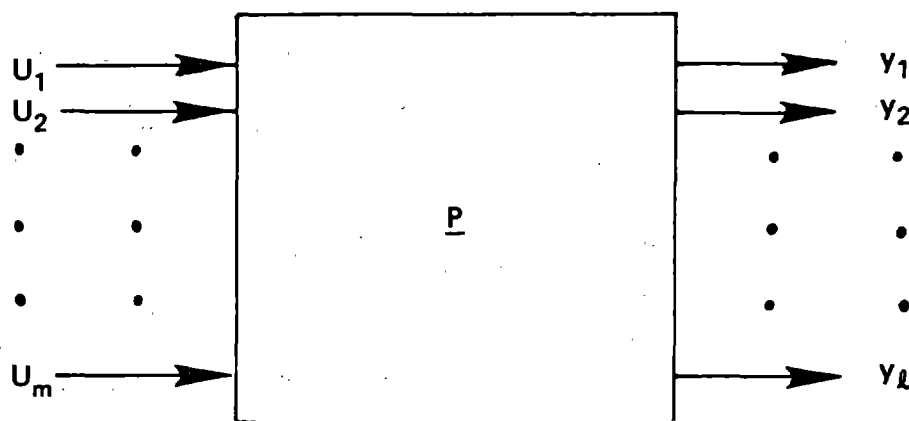


Fig. 3-1 A MIMO plant

Horowitz has shown, (see Sec. 3-6) by using fixed point theory^{14,20} that the MIMO problem for an $m \times m$ system can be separated into m equivalent single-loop MISO systems and m^2 prefilter/disturbance problems, which are each designed as outlined in Chap. 2.¹³

3-2 The MIMO Plant

The P matrix may be formed from either the system state space matrix representation or from the system linear differential equations. The state space representation for a linear-time-invariant MIMO system is:

$$\begin{aligned}\dot{\underline{x}}(t) &= \underline{A}\underline{x}(t) + \underline{B}\underline{u}(t) \\ \underline{y}(t) &= \underline{C}\underline{x}(t)\end{aligned}\tag{3-2}$$

where the A, B and C are constant matrices. The plant transfer function matrix P(s) is evaluated as

$$\underline{P}(s) = \underline{C} [s\underline{I} - \underline{A}]^{-1} \underline{B}\tag{3-3}$$

If the plant model consists of m coupled linear-time-invariant differential equations, the general plant model for a MIMO system with two inputs and two outputs has the form

$$\begin{aligned}a(s)y_1(s) + b(s)y_2(s) &= fu_1(s) + gu_2(s) \\ c(s)y_1(s) + d(s)y_2(s) &= hu_1(s) + iu_2(s)\end{aligned}\tag{3-4}$$

where a(s) through d(s) are polynomials in s, f through i are constant coefficients, y₁(s) and y₂(s) are the outputs, and u₁(s) and u₂(s) are the inputs. In matrix notation the system is represented by

$$\begin{bmatrix} a(s) & b(s) \\ c(s) & d(s) \end{bmatrix} \underline{Y}(s) = \begin{bmatrix} f & g \\ h & i \end{bmatrix} \underline{U}(s)\tag{3-5}$$

This is defined as a 2x2 system. In the general case with m inputs and m outputs, the system is defined as mxm. Let the matrix premultiplying the output vector Y(s) be M(s) and the matrix premultiplying the input vector U(s) be N. Equation 3-5 may then be written as

$$\underline{M}(s)\underline{Y}(s) = \underline{N}\underline{U}(s)\tag{3-6}$$

The solution of Eq. (3-6) for the output Y(s) yields

$$\underline{Y}(s) = \underline{M}^{-1}(s)\underline{N}\underline{U}(s) = \underline{P}(s)\underline{U}(s)\tag{3-7}$$

Thus, the $m \times m$ plant transfer function matrix $\underline{P}(s)$ is

$$\underline{P}(s) = \underline{M}^{-1}(s)\underline{N} \quad (3-8)$$

This plant matrix $\underline{P}(s) = \begin{bmatrix} p_{ij}(s) \end{bmatrix}$ is a member of the set $\mathcal{P} = \{\underline{P}(s)\}$ of possible plant matrices which are functions of the uncertainty in the plant parameters. In practice, only a finite set of $\underline{P}$ matrices are formed, representing the extreme boundaries of the plant uncertainty under varying conditions. Only linear time-invariant systems are considered in this text.

3-3 Introduction to MIMO Compensation

Figure 3-2 represents an $m \times m$ MIMO closed-loop system in which $\underline{F}, \underline{G}, \underline{P}, \underline{T}$ are each $m \times m$ matrices, and $\mathcal{P} = \{\underline{P}\}$ is a set of matrices due to plant uncertainty. There are m^2 closed-loop system transfer functions (transmissions) $t_{ij}(s)$ relating the outputs $y_i(s)$ to the inputs $r_j(s)$, i.e., $y_i(s) = t_{ij}(s)r_j(s)$. In a quantitative problem statement, there are tolerance bounds on each $t_{ij}(s)$, giving m^2 sets of acceptable regions $\tau_{ij}(s)$ which are to be specified in the design, thus $t_{ij}(s) \in \tau_{ij}(s)$ and $\mathcal{T}(s) = \{\tau_{ij}(s)\}$. The application of QFT to 2×2 and 3×3 systems has been developed and is illustrated in later sections of this chapter.

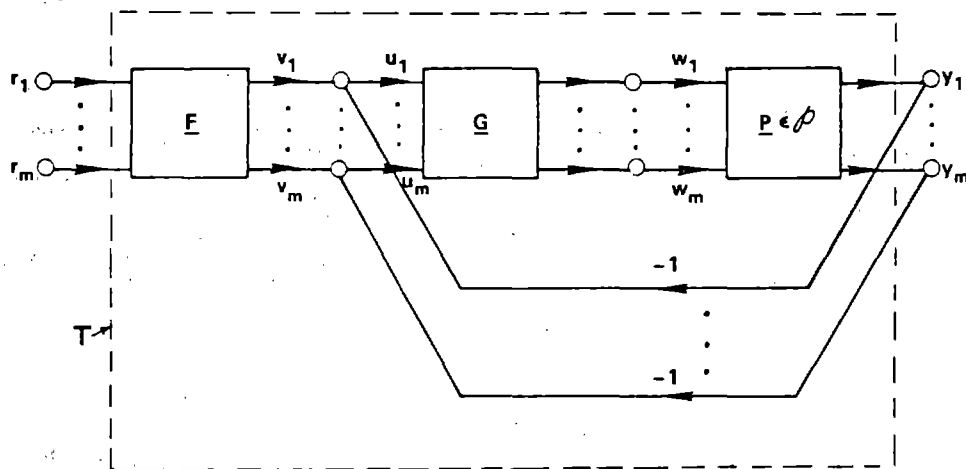


Fig. 3-2 MIMO feedback structure.

From Fig. 3-2 the following equations can be written:

$$\underline{y} = \underline{P}\underline{x} \quad \underline{x} = \underline{G}\underline{u} \quad \underline{u} = \underline{v} - \underline{y} \quad \underline{v} = \underline{F}\underline{r}$$

In these equations $\underline{P}(s) = [p_{ij}(s)]$ is the matrix of plant transfer functions, $\underline{G}(s)$ is the matrix of compensator transfer functions and is often simplified so that it is diagonal, that is, $\underline{G}(s) = \text{diag} \{g_i(s)\}$, and $\underline{F}(s) = [f_{ij}(s)]$ is the matrix of prefilter transfer functions which may also be a diagonal matrix. The first two expressions yield:

$$\underline{y} = \underline{P}\underline{G}\underline{u}$$

which is utilized with the remaining two expressions to obtain

$$\underline{y} = \underline{P}\underline{G}[\underline{v} - \underline{y}] = \underline{P}\underline{G}[\underline{F}\underline{r} - \underline{y}]$$

This equation is rearranged to yield

$$\underline{y} = [\underline{I} + \underline{P}\underline{G}]^{-1} \underline{P}\underline{G}\underline{F}\underline{r} \quad (3-9)$$

where the system control ratio relating $\underline{r}$ to $\underline{y}$ is

$$\underline{T} = [\underline{I} + \underline{P}\underline{G}]^{-1} \underline{P}\underline{G}\underline{F} \quad (3-10)$$

To appreciate the difficulty of the design problem, note the very complex expression for t_{11} below for the case $m = 3$ with a diagonal $\underline{G}$ matrix.

However, the QFT design procedure systematizes and simplifies the manner of achieving a satisfactory system design.

$$\begin{aligned} t_{11} = & \{ [p_{11}f_{11}g_1 + p_{12}f_{21}g_2 + p_{13}f_{31}g_3] \\ & [(1 + p_{22}g_2)(1 + p_{33}g_3) - p_{23}p_{32}g_2g_3] \\ & - [p_{21}f_{11}g_1 + p_{22}f_{21}g_2 + p_{23}f_{31}g_3] \\ & [p_{12}g_2(1 + p_{33}g_3) - p_{32}p_{13}g_2g_3] \\ & + [p_{31}f_{11}g_1 + p_{32}f_{21}g_2 + p_{33}f_{31}g_3] \\ & [p_{23}p_{12}g_2g_3 - (1 + p_{22}g_2)p_{13}g_3] \} / \\ & \{ (1 + p_{11}g_1)[(1 + p_{22}g_2)(1 + p_{33}g_3) \\ & - p_{23}p_{32}g_2g_3] - p_{21}g_1[p_{12}g_2(1 + p_{33}g_3) \\ & - p_{32}p_{13}g_2g_3] + p_{31}g_1[p_{12}p_{23}g_2g_3 \\ & - p_{13}g_3(1 + p_{22}g_2)] \} \end{aligned} \quad (3-11)$$

There are $m^2 = 9$ such $t_{ij}(s)$ expressions (all have the same denominator), and there may be considerable uncertainty in the nine plant transfer functions $p_{ij}(s)$. The design objective is a system which behaves as desired for the entire range of uncertainty. This requires finding nine $f_{ij}(s)$ and three $g_i(s)$ such that each $t_{ij}(s)$ stays within its acceptable region $\tau_{ij}(s)$ no matter how the $p_{ij}(s)$ may vary. Clearly, this is a very difficult problem. Even the stability problem alone, ensuring that the characteristic polynomial [the denominator of Eq. (3-11)] has no right-half plane zeros for all possible $p_{ij}(s)$, is extremely difficult. Most design approaches treat stability for a fixed parameter set, neglecting uncertainty, and attempting to cope with the plant uncertainty by trying to design the system to have conservative stability margins. Two QFT design techniques have been developed for such systems and are presented in this chapter. In both approaches the MIMO system is converted into an equivalent set of single-loop systems, and then the MISO design method of Chap. 2 is used.

3-4 MIMO Compensation

The basic MIMO compensation structure for a two-by-two MIMO system is shown in Fig 3-3. The structure for a three-by-three MIMO system is shown in Fig. 3-4. They consist of the uncertain plant matrix $\underline{P}$, the diagonal compensation matrix $\underline{G}$, and the prefilter matrix $\underline{F}$. This chapter considers only a diagonal $\underline{G}$ matrix, though a non-diagonal $\underline{G}$ matrix allows the designer much more design flexibility.¹⁹ These matrices are defined as follows:

$$\underline{G} = \begin{bmatrix} g_{11} & 0 & \dots & 0 \\ 0 & g_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & g_{mm} \end{bmatrix} \quad \underline{F} = \begin{bmatrix} f_{11} & f_{12} & \dots & f_{1m} \\ f_{21} & f_{22} & \dots & f_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m1} & f_{m2} & \dots & f_{mm} \end{bmatrix} \quad \underline{P} = \begin{bmatrix} p_{11} & p_{12} & \dots & p_{1m} \\ p_{21} & p_{22} & \dots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \dots & p_{mm} \end{bmatrix} \quad (3-12)$$

The dashes in Eq. (3-12) denote the $\underline{G}$, $\underline{F}$, and $\underline{P}$ matrices for a 2x2 system. Substituting these matrices into Eq. (3-10) yields the $t_{ij}(s)$ control ratios relating the i^{th} output to the j^{th} input. From these $t_{ij}(s)$ expressions the signal flow graph of Fig. 3-5 is obtained. The signal flow graph of Fig. 3-6 for a 3x3 system is obtained in a similar manner.

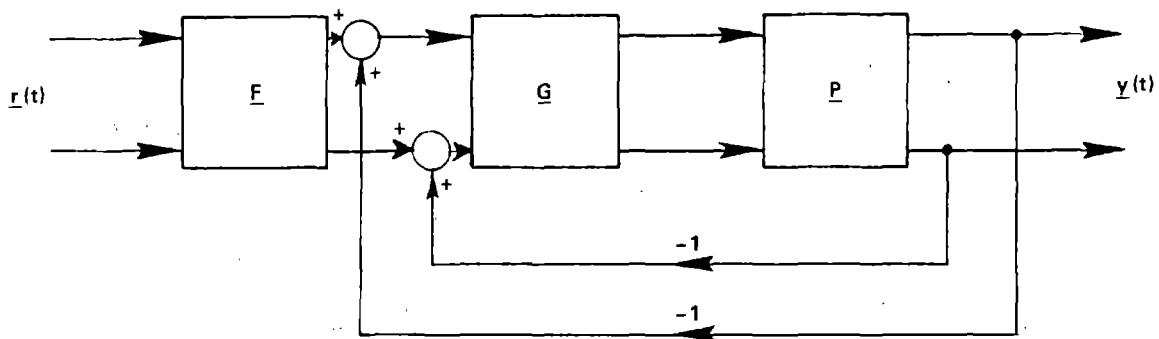


Fig. 3-3 MIMO Control Structure (Two-by-Two System).

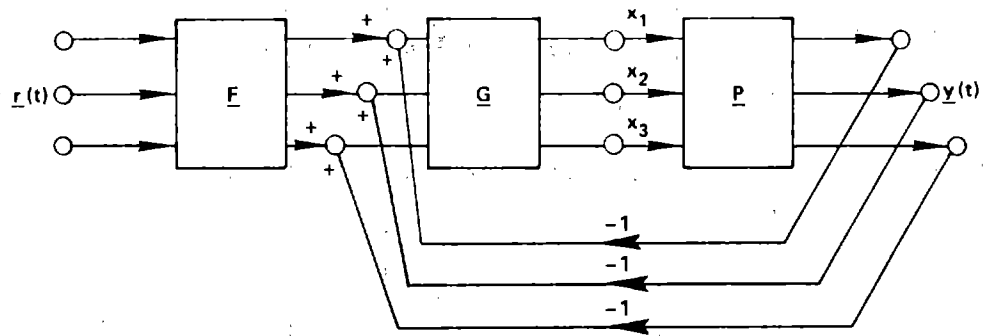


Fig. 3-4 MIMO Control Structure (Three-by-Three System).

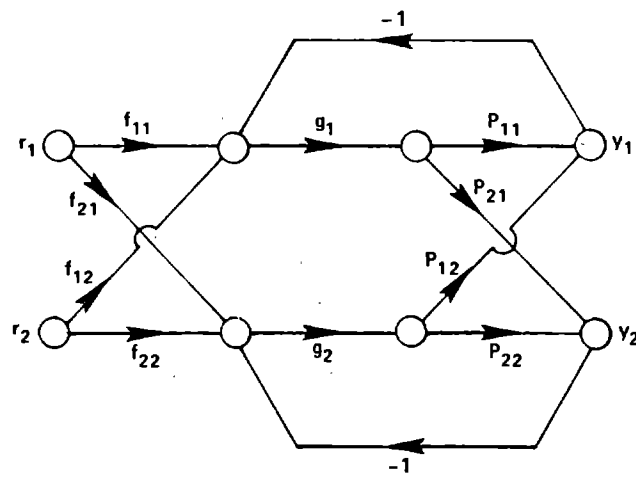


Fig. 3-5 Two-by-Two MIMO System Schematic.

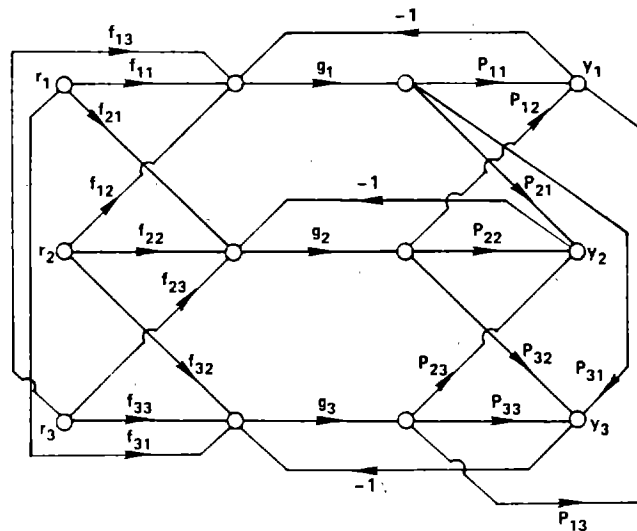


Fig. 3-6 Three-by-Three MIMO System Schematic.

3-5 Justification of the MISO Equivalent Method¹³

The objective of this section is to find a suitable mapping that permits the analysis and synthesis of a MIMO control system by a set of equivalent MISO control systems. This mapping results in m^2 equivalent systems, each with two inputs and one output. One input is designated as a "desired" input and the other as a "disturbance" input. First, Eq. (3-10) is premultiplied by $\underline{I} + \underline{P}\underline{G}$ to obtain

$$[\underline{I} + \underline{P}\underline{G}] \underline{T} = \underline{P}\underline{G}\underline{F} \quad (3-13)$$

When $\underline{P}$ is nonsingular, then premultiplying both sides of this equation by

$$\underline{P}^{-1} \text{ yields } [\underline{P}^{-1} + \underline{G}] \underline{T} = \underline{G}\underline{F} \quad (3-14)$$

Let

$$\underline{P}^{-1} = \begin{bmatrix} p_{11}^* & p_{12}^* & \dots & p_{1m}^* \\ p_{21}^* & p_{22}^* & \dots & p_{2m}^* \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ p_{m1}^* & p_{m2}^* & \dots & p_{mm}^* \end{bmatrix} \quad (3-15)$$

The m^2 effective plant transfer functions are formed as

$$q_{ij} \equiv 1/p_{ij}^* = \det \underline{P} / \text{Adj}_{ij} \underline{P} \quad (3-16)$$

The $\underline{Q}$ matrix is then formed as

$$\underline{Q} = \begin{bmatrix} q_{11} & q_{12} & \dots & q_{1m} \\ q_{21} & q_{22} & \dots & q_{2m} \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ q_{m1} & q_{m2} & \dots & q_{mm} \end{bmatrix} = \begin{bmatrix} 1/p_{11}^* & 1/p_{12}^* & \dots & 1/p_{1m}^* \\ 1/p_{21}^* & 1/p_{22}^* & \dots & 1/p_{2m}^* \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ 1/p_{m1}^* & 1/p_{m2}^* & \dots & 1/p_{mm}^* \end{bmatrix} \quad (3-17)$$

where $\underline{P} = [p_{ij}]$, $\underline{P}^{-1} = [p_{ij}^*] = [1/q_{ij}^*]$, and $\underline{Q} = [q_{ij}] = [1/p_{ij}^*]$.

The matrix $\underline{P}^{-1}$ is partitioned to the form

$$\underline{P}^{-1} = [p_{ij}^*] = [1/q_{ij}] = \underline{A} + \underline{B} \quad (3-18)$$

where $\underline{A}$ is the diagonal part and $\underline{B}$ is the balance of $\underline{P}^{-1}$, thus $\lambda_{ii} = 1/q_{ii} = p_{ii}^*$,

$b_{ii} = 0$, and $b_{ij} = 1/q_{ij} = p_{ij}^*$ for $i \neq j$. Next, rewrite Eq. (3-14) using Eq.

(3-18) and with $\underline{G}$ diagonal. This yields $[\underline{A} + \underline{G}]\underline{T} = \underline{GF} - \underline{BT}$ which produces

$$\underline{T} = [\underline{A} + \underline{G}]^{-1} [\underline{GF} - \underline{BT}] \quad (3-19)$$

This is used to define the desired fixed point mapping where each of the

m^2 matrix elements on the right side of Eq. (3-19) can be interpreted as a MISO

problem. Proof of the fact that design of the individual MISO feedback loops

will yield a satisfactory MIMO design is based on the Schauder fixed point

theorem.¹⁴ This theorem is described by defining a mapping $\underline{Y}(\underline{T})$ by

$$\underline{Y}(\underline{T}) \equiv [\underline{A} + \underline{G}]^{-1} [\underline{GF} - \underline{BT}] \quad (3-20)$$

where each member of $\underline{T}$ is from the acceptable set $\mathcal{T}$. If this mapping has a

fixed point, i.e., $\underline{T} \in \mathcal{T}$ such that $\underline{Y}(\underline{T}) = \underline{T}$, then this $\underline{T}$ is a solution of Eq.

(3-19). Recalling that $\underline{A}$ and $\underline{G}$ in Eq. (3-19) are both diagonal, the 1,1 ele-

ment on the right side of Eq. (3-20) for the 3x3 case, for a unit impulse

input yields the output

$$y_{11} = \frac{q_{11}}{1 + g_{11}q_{11}} \left[g_{11}f_{11} - \left(\frac{t_{21}}{q_{12}} + \frac{t_{31}}{q_{13}} \right) \right] \quad (3-21)$$

This corresponds precisely to the first structure in Fig. 3-7 and is the control

ratio that relates the i^{th} output to the j^{th} input, where $i = j = 1$ in Eq.

(3-21). Similarly each of the nine structures in Fig. 3-7 corresponds to one of

the elements of $\underline{Y}(\underline{T})$ of Eq. (3-20). The general transformation result of

m^2 MISO system loops is shown in Fig. 3-8. Figure 3-7 shows the four effective

MISO loops (in the boxed area) resulting from a two-by-two system and the nine

effective MISO loops resulting from a three-by-three system.¹³ The control

ratios for unit impulse inputs, for the $m \times m$ system of Fig. 3-8 obtained from

Eq. (3-20) have the form

$$y_{ij} = w_{ii}(v_{ij} + d_{ij}) \quad (3-22)$$

where

$$w_{ii} = q_{ii}/(1 + g_i q_{ii}) \quad (3-23)$$

$$v_{ij} = g_i f_{ij} \quad (3-24)$$

and

$$d_{ij} = -\sum_{k \neq i} \left[\frac{t_{kj}}{q_{ik}} \right] \quad k = 1, 2, \dots, m \quad (3-25)$$

Equation (3-25) represents the interaction between the loops.

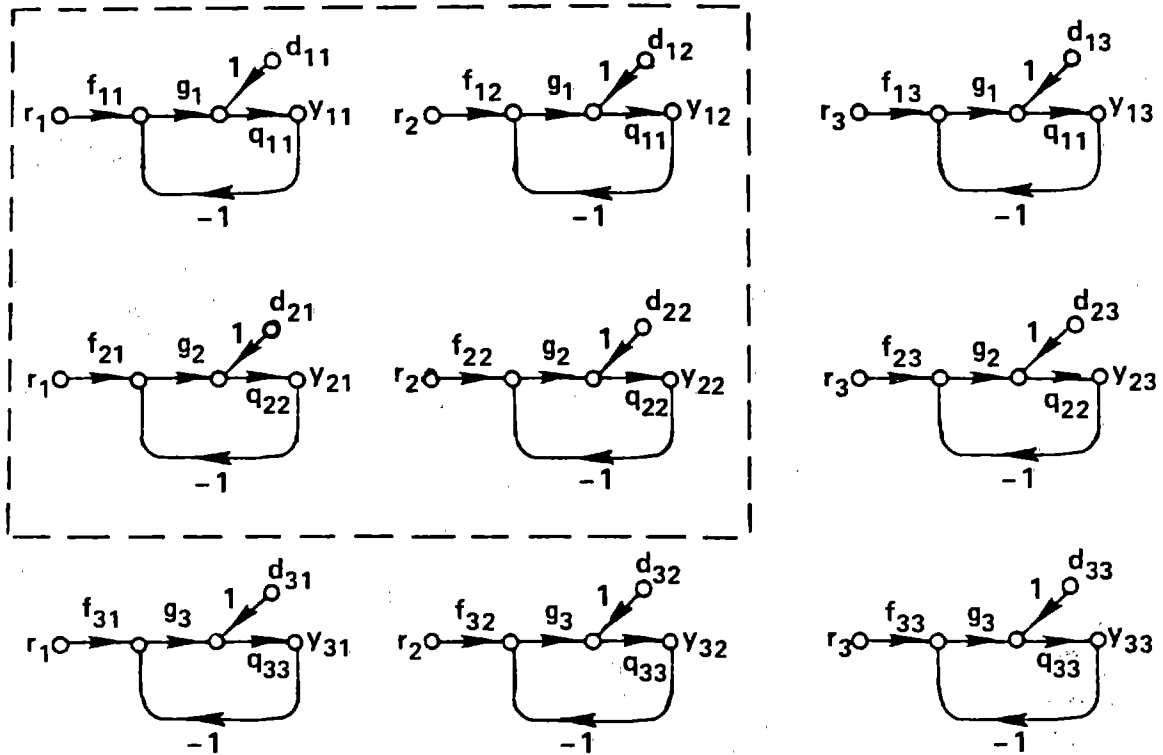


Fig. 3-7 Effective MISO Loops Two-By-Two (Boxed in Loops)
and Three-By-Three (All 9 Loops)

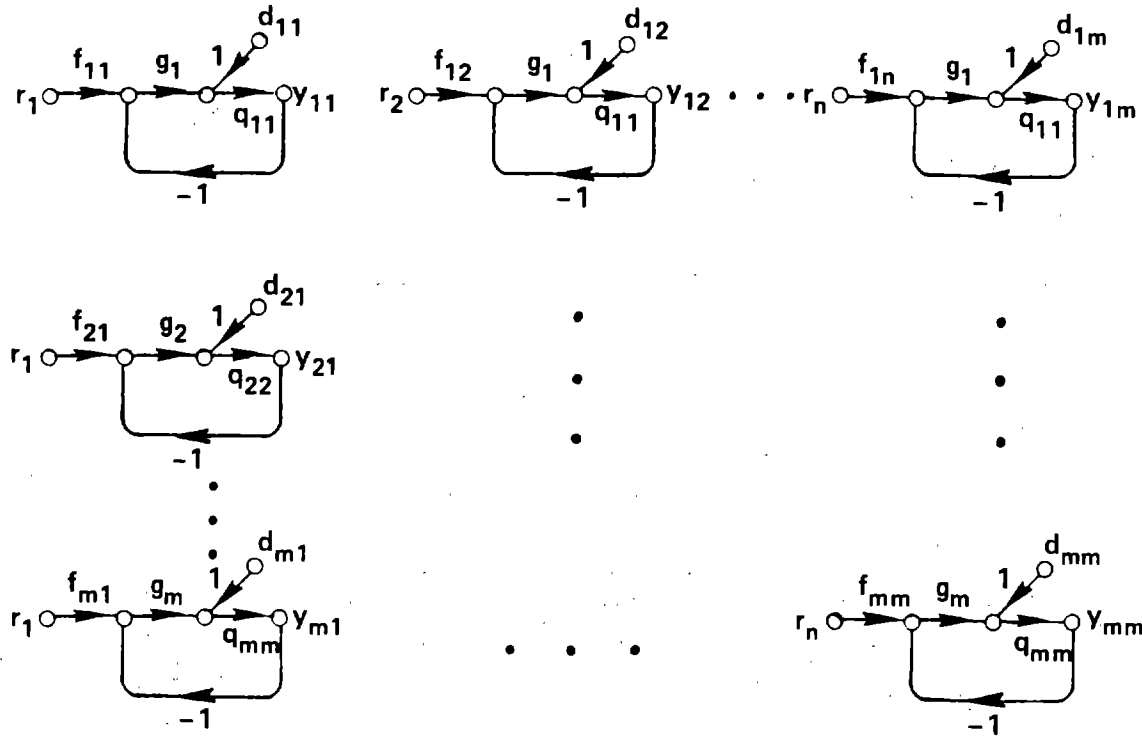


Fig. 3-8 Effective MISO Loops (in general)

Thus Eq. (3-22) represents the control ratio of the i^{th} MISO loop where the transfer function $w_{ii}v_{ij}$ relates the i^{th} output to the i^{th} "desired" input r_i and the transfer function $w_{ii}d_{ij}$ relates the i^{th} output to the j^{th} "disturbance" input d_{ij} . The transfer function of Eq. (3-22) can thus be expressed, with $r_i(s) = 1$, as

$$y_{ij} = (y_{ij})_{r_j} + (y_{ij})_{d_{ij}} = y_{r_j} + y_{d_{ij}} \quad (3-26a)$$

or

$$t_{ij} = t_{r_{ij}} + t_{d_{ij}} \quad (3-26b)$$

where

$$t_{r_j} = y_{r_j} = w_{ii}v_{ij} \quad (3-27a)$$

$$t_{d_{ij}} = y_{d_{ij}} = w_{ii}d_{ij} \quad (3-27b)$$

and where now the upper bound, in the low frequency range is expressed as

$$b_{ij} = b'_{ij} + \tau_{dij} \quad (3-27c)$$

Thus

$$\tau_{dij} = b_{ij} - b'_{ij} \quad (3-27d)$$

represents the maximum portion of b_{ij} allocated towards disturbance rejection in and b'_{ij} represents the upper bound for the tracking portion of t_{ij} .

For any particular loop there is a disturbance input which is a function of all the other loop outputs. The object of the design is to have each loop track its desired input while minimizing the outputs due to the disturbance inputs.

In each of the nine structures of Fig. 3-7 it is necessary that the control ratio $y_{ij}(s)$, with $r(s) = 1$, must be a member of the acceptable $t_{ij} \in \tau_{ij}(s)$. All the $g_i(s)$, $f_{ij}(s)$ must be chosen to ensure that this condition is satisfied, thus constituting nine MISO design problems. If all of these MISO problems are solved, there exists a fixed point, and then $y_{ij}(s)$ on the left side of Eq. (3-20) may be replaced by a t_{ij} and all the elements of T on the right side by t_{kj} . This means that there exist nine t_{ij} and t_{kj} , each in its OK set, which is a solution to Fig. 3-2. If each element is 1:1, then this solution must be unique. A more formal and detailed treatment is given in Reference 14.

3-6 Effective MISO Loops of the MIMO System

There are two design methods for designing MIMO systems. In the first method each MISO loop in Figs. 3-7 and 3-8 is treated as an individual MISO design problem, which is solved using the procedures explained in Chap. 2. The $f_{ij}(s)$ and $g_i(s)$ are the compensator elements of the $\underline{F}(s)$ and $\underline{G}(s)$ matrices described previously.

The disturbances $d_{ij}(s)$ expressed by Eq. (3-25) represent the interaction between the loops

$$d_{ij} = - \sum_{k \neq i} \frac{b_{kj}}{q_{ik}}, \quad k = 1, 2, \dots, m \quad (3-28)$$

where the numerator b_{kj} is the upper response bound, (T_{Rij} or T_D in Figs. 1-5 and 1-8 respectively), for the respective input/output relationship. These are obtained from the design specifications.¹³ The first subscript k refers to the output variable, and the second subscript j refers to the input variable. Therefore, b_{kj} is a function of the response requirements on the output y_k due to the input r_j . The lower bound a_{kj} needs defining only when there is a command input.

For the disturbance rejection problem, the responses must be less than some bound [i.e. $(y_{ij})_{d_{ij}} \leq \tau_{d_{ij}}$]. Thus the loop equations become:

$$\tau_{d_{ij}} \geq (y_{ij})_{d_{ij}} = \frac{d_{ij} |q_{ii}|}{|1 + g_i q_{ii}|} \quad (3-29)$$

where $g_i q_{ii} = L_i$. Reorganizing:

$$|1 + L_i| \geq \frac{d_{ij} |q_{ii}|}{\tau_{d_{ij}}} \quad (3-30)$$

using Eq. (3-28), and rewriting Eq. (3-30):

$$|1 + L_i| \geq \left| - \sum_{k \neq i} \frac{b_{kj}}{q_{ik}} \right| \frac{|q_{ii}|}{\tau_{d_{ij}}} \quad (3-31)$$

For example, in a three-by-three system for the first loop, L_1 , where $i = 1$, $j = 2$, and in the first term $k = 2$ and in the second term $k = 3$, Eq. (3-31) becomes:

$$|1 + L_1| \geq \frac{\left| \frac{b_{22}}{q_{12}} + \frac{b_{32}}{q_{13}} \right| |q_{11}|}{\tau_{d12}} \quad (3-32)$$

Remember to use only the magnitude in the disturbance calculations. This assumes the worst case.

The second design method is an improvement over the first method. It involves using the information on the $g_i(s)$ and $f_i(s)$ from the loops previously designed. This reduces the overdesign inherent in the early part of the design process. It may involve a trade-off in the design parameters. The final loop, is designed using the exact equation.¹⁷ The exact equation represents the last loop and the interactions caused by all the other loops. The order in which the loops are designed may be significant. Any order can be used, but some orders produce less overdesign (less bandwidth) than others. The last loop designed has the least amount of overdesign, therefore do the most constrained loop first, and continue through all the loops.

3-7 Constraints on the Plant Matrix¹⁴

In order to use the QFT technique the following critical condition must be satisfied.

Condition 1: $\underline{P}$ must be nonsingular for any combination of possible plant parameters, thus $\underline{P}^{-1}$ exists.

In the high frequency range $y_{-r_i}(j\omega)$ approaches zero as $\omega \rightarrow \infty$. Therefore, Eq. (3-26) can be approximated by

$$\underline{t}_{ij} \approx \underline{y}_{dij} \quad (3-33)$$

Thus from Eqs. (3-22), (3-23) and (3-28), with impulse input func-

tions, Eq. (3-33) can be expressed as

$$y_{dij} = \frac{d_{ij} q_{ii}}{1 + g_i q_{ii}} \quad (3-34)$$

Consider first the 2x2 plant, i.e., $m = 2$. Specifying that

$|y_{d11}| \leq b_{11}$ (the given upper bound in the high frequency range), Eqs. (3-28) and (3-34) yield, for $i = j = 1$,

$$b_{11} \geq \left| \frac{-b_{21}}{q_{12}} \right| \left| \frac{q_{11}}{1 + L_1} \right| \quad (3-35)$$

For $i = 2, j = 1$, where it is specified that $y_{d21} \leq b_{21}$ (the given upper bound), Eqs. (3-28) and (3-34) yield

$$b_{21} \geq \left| \frac{-b_{11}}{q_{21}} \right| \left| \frac{q_{22}}{1 + L_2} \right| \quad (3-36)$$

Equations (3-35) and (3-36) are rearranged to

$$\left| 1 + L_1 \right| \geq \frac{b_{21}}{b_{11}} \left| \frac{q_{11}}{q_{12}} \right| \quad (3-37)$$

$$\left| 1 + L_2 \right| \geq \frac{b_{11}}{b_{21}} \left| \frac{q_{22}}{q_{21}} \right| \quad (3-38)$$

Multiplying Eqs. (3-37) by Eq. (3-38), where $L_1 = L_2 \approx 0$ in the high frequency range, results in

$$1 \geq \left| \frac{q_{11} q_{22}}{q_{12} q_{21}} \right| \quad (3-39)$$

Since

$$\underline{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}, \quad \underline{P}^{-1} = \begin{bmatrix} p_{ij}^* \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} p_{22} & -p_{12} \\ -p_{21} & p_{11} \end{bmatrix},$$

$$\underline{Q} = \begin{bmatrix} q_{ij} \end{bmatrix} = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$$

and $q_{ij} = 1/p_{ij}^*$ then

$$\begin{aligned}
q_{11} &= \frac{1}{p_{11}^*} = \frac{\Delta}{p_{22}} & q_{12} &= \frac{1}{p_{12}^*} = \frac{-\Delta}{p_{12}} \\
q_{21} &= \frac{1}{p_{21}^*} = \frac{-\Delta}{p_{21}} & q_{22} &= \frac{1}{p_{22}^*} = \frac{\Delta}{p_{11}}
\end{aligned} \tag{3-40}$$

Substituting Eqs. (3-40) into Eq. (3-39) yields

Condition 2 (2x2 plant): As $\omega \rightarrow \infty$

$$|p_{11}p_{22}| > |p_{12}p_{21}| \tag{3-41}$$

Since p_{11} and p_{22} are elements of the principal diagonal of $\underline{P}$, then Eq. (3-41) is the diagonal dominance condition for the 2x2 plant. This condition is obtained considering only the left column of the MISO loops for the 2x2 plant of Fig. 3-7. This condition may also be obtained by using the right column of MISO loops in Fig. 3-7 since the loop transmissions L_1 and L_2 are again involved in the derivation.

Next consider the 3x3 plant, i.e., $m = 3$. For $i = j = 1$ (the left column of Fig. 3-7), where $\underline{L}_1 = \underline{L}_2 = \underline{L}_3 \approx 0$ as $\omega \rightarrow \infty$ and where it is specified that

$$|y_{d11}| \leq b_{11}, |y_{d21}| \leq b_{21}, \text{ and } |y_{d31}| \leq b_{31},$$

Eqs. (3-28) and (3-34) yield, for $i = 1, 2, 3$ respectively,

$$1 \geq \left| \frac{q_{11}}{b_{11}} \right| \left[\left| \frac{b_{21}}{q_{12}} \right| + \left| \frac{b_{31}}{q_{13}} \right| \right] \tag{3-42}$$

$$1 \geq \left| \frac{q_{22}}{b_{21}} \right| \left[\left| \frac{b_{11}}{q_{21}} \right| + \left| \frac{b_{31}}{q_{23}} \right| \right] \tag{3-43}$$

$$1 \geq \left| \frac{a_{33}}{b_{31}} \right| \left[\left| \frac{b_{11}}{a_{31}} \right| + \left| \frac{b_{21}}{a_{32}} \right| \right] \quad (3-44)$$

Letting

$$\lambda_1 = \frac{b_{21}}{b_{11}} \quad (3-45)$$

$$\lambda_2 = \frac{b_{31}}{b_{11}}$$

Equations (3-42) through (3-44) become, respectively,

$$1 \geq |a_{11}| \left[\left| \frac{\lambda_1}{a_{12}} \right| + \left| \frac{\lambda_2}{a_{13}} \right| \right] \quad (3-46)$$

$$\lambda_1 \geq |a_{22}| \left[\left| \frac{1}{a_{21}} \right| + \left| \frac{\lambda_2}{a_{23}} \right| \right] \quad (3-47)$$

$$\lambda_2 \geq |a_{33}| \left[\left| \frac{1}{a_{31}} \right| + \left| \frac{\lambda_1}{a_{32}} \right| \right] \quad (3-48)$$

From Eqs. (3-47) and (3-48) and using

$$\gamma_{ij} = \frac{a_{ii}a_{jj}}{a_{ij}a_{ji}} \bigg|_{\substack{i=2 \\ j=3}} = \frac{a_{22}a_{33}}{a_{23}a_{32}} \quad (3-49)$$

The following expressions for λ_1 and λ_2 are obtained

$$\lambda_1 \geq \frac{\left| \frac{a_{22}}{a_{21}} \right| + \left| \frac{a_{22}a_{33}}{a_{23}a_{31}} \right|}{|1 - \gamma_{23}|} \quad (3-50)$$

$$\lambda_2 \geq \frac{\left| \frac{a_{33}}{a_{31}} \right| + \left| \frac{a_{22}a_{33}}{a_{21}a_{32}} \right|}{|1 - \gamma_{23}|} \quad (3-51)$$

Substitute Eqs. (3-50) and (3-51) into Eq. (3-46) to obtain

$$\left| \frac{q_{11}}{q_{12}} \right| \left[\left| \frac{q_{22}}{q_{21}} \right| + \left| \frac{q_{22}q_{33}}{q_{23}q_{31}} \right| \right] + \left| \frac{q_{11}}{q_{13}} \right| \left[\left| \frac{q_{33}}{q_{31}} \right| + \left| \frac{q_{33}q_{22}}{q_{21}q_{32}} \right| \right] \leq |1 - \gamma_{23}| \quad (3-52)$$

Substituting $q_{ij} = 1/p_{ij}^*$ into Eq. (3-52) yields

$$\left| \frac{p_{12}^*}{p_{11}^*} \right| \left[\left| \frac{p_{21}^*}{p_{22}^*} \right| + \left| \frac{p_{23}^*p_{31}^*}{p_{22}^*p_{33}^*} \right| \right] + \left| \frac{p_{13}^*}{p_{11}^*} \right| \left[\left| \frac{p_{31}^*}{p_{33}^*} \right| + \left| \frac{p_{21}^*p_{32}^*}{p_{22}^*p_{33}^*} \right| \right] \leq \left| 1 - \frac{p_{23}^*p_{32}^*}{p_{22}^*p_{33}^*} \right| \quad (3-53)$$

Multiplying Eq. (3-53) by $p_{11}^* p_{22}^* p_{33}^* = p_{123}^*$ yields:

Condition 3 (3x3 plant; applies only for the method 1 design technique of Chap. 4): As $\omega \rightarrow \infty$

$$|p_{123}^*| \geq |p_{12}^*| \left[|p_{21}^*p_{33}^*| + |p_{23}^*p_{31}^*| \right] + |p_{13}^*| \left[|p_{22}^*p_{31}^*| + |p_{21}^*p_{32}^*| \right] + |p_{11}^*p_{23}^*p_{32}^*| \quad (3-54)$$

See Reference 14 for higher order plants. This condition is necessary only if Eq. (3-8) is used to generate the plant.

The first condition ensures controllability of the plant since the inverse of $\underline{P}$ produces the effective transfer functions used in the design. If the $\underline{P}$ matrix, resulting from the original ordering of the elements of the input and output vectors does not satisfy the second condition, then a reordering of the input and output vectors may result in satisfying these conditions.

3-8 Basically Non-Interacting (BNIC) Loops

A basically non-interacting (BNIC) loop¹³ is one in which the output $y_k(s)$ due to the input $r_j(s)$ is ideally zero. Plant uncertainty and loop interaction makes the ideal response unachievable. Thus, the system performance specifications describe a range of acceptable responses for the commanded output and a maximum tolerable response for the uncommanded outputs. The uncommanded outputs are treated as disturbances.

3-9 Summary

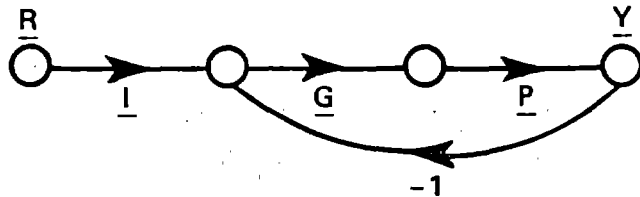
This chapter describes the multiple input-multiple output closed-loop system and the plant matrix. Guidelines for finding the $\underline{P}$ matrix, which relates the input vector to the output vector are given.

The method of representing a MIMO system by an equivalent set of MISO systems is presented using $\underline{P}^{-1}$. Two design approaches, which are discussed in detail in Chaps. 4 and 5 respectively, are available in which the equivalent MISO loops are designed according to the MISO design method of Chap. 2.

3-1. The plant matrix and the desired control ratio matrix for a 2x2 system are

$$\underline{P} = \begin{bmatrix} \frac{1}{s+1} & \frac{0.2}{(s+1)(s+2)} \\ \frac{0.5}{s+1} & \frac{0.5s}{(s+1)(s+2)} \end{bmatrix} \quad \underline{T} = \begin{bmatrix} \frac{2}{s+2} & 0 \\ 0 & \frac{2}{s+2} \end{bmatrix}$$

The one degree of freedom is shown below where $\underline{I}$ is the identity matrix and $\underline{I} = \underline{F}$.



- Determine $\underline{P}^{-1}$
- Determine $\underline{G}$ that results in achieving the desired $\underline{T}$.
- Suppose the actual compensator transfer function is

$$\underline{G} = \begin{bmatrix} \frac{(0.5 + \epsilon_1)s}{s+2} & g_{12} \\ \frac{-(0.5 + \epsilon_2)}{s-0.2} & g_{22} \end{bmatrix} \quad \text{where } \epsilon_1 > 0 \text{ and } \epsilon_2 > 0$$

Determine $t_{11}(s)$ and $t_{21}(s)$. Do these control ratios yield stable responses? Hint: Analyze $\underline{T} = [\underline{I} + \underline{P}\underline{G}]^{-1}\underline{P}\underline{G}$; especially $\underline{P}\underline{G}$.

- For this part assume $t_{11}(s) = t_{22}(s) = (s-2)/[(s+2)(\tau s+1)]$ and $t_{12}(s) = t_{21}(s) = 0$ where $\tau > 0$. Determine $\underline{G}(s)$ and compare it with $\underline{G}$ of part (b).

3-2. In a 2x2 MIMO system, the system poles due to feedback are the zeros of $\Delta = \det [\underline{I} + \underline{P} \underline{G}]$. Assume that $\underline{G}$ is a diagonal matrix. Prove that the zeros of interest are those of $\Delta' = (1 + g_1 q_{11})(1 + g_2 q_{22}) - q_{11} q_{22} / q_{12} q_{21}$. Hint: show that $(\Delta / p_{11}) = (\Delta' / q_{11})$.

3-3. Let Fig. 3-1 represent a 2x2 MIMO plant where the plant is described by

$$\ddot{y}_1 + A\dot{y}_1 + By_1 + Cy_2 = E_1\dot{u}_1 + E_2\dot{u}_1 + E_3u_2$$

$$Jy_2 + Ky_1 + Hy_1 = Qu_2$$

(a). Determine $\underline{P}$.

(b). Determine $\underline{P}^{-1}$ directly from $\underline{P}$. Suppose $\underline{P}$ has the form

$$\underline{P} = \frac{\begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix}}{d} \quad \text{thus} \quad \underline{P}^{-1} = \frac{\begin{bmatrix} n_{22} & -n_{12} \\ -n_{21} & n_{11} \end{bmatrix}}{\Delta}$$

where $\Delta = n_{11}n_{22} - n_{12}n_{21}$. By dividing d into Δ you should obtain the quotient $E_1Qs + E_2Q$ and no remainder. This reveals that for most practical MIMO plants of the above form if $\underline{P}^{-1}$ is derived from $\underline{P}$ then the common poles of the elements of $\underline{P}$ should not appear as common zeros of the elements of $\underline{P}^{-1}$.

(c). Utilizing Eq. (3-6) prove that the remainder is zero.

Hint: $\underline{P} = \underline{M}^{-1}\underline{N}$ and $\underline{U} = \underline{P}^{-1}\underline{Y}$.

DESIGN METHOD ONE - THE SINGLE-LOOP EQUIVALENTS¹

4-1 Introduction

Let $\tau_{ij}(s)$ be the set of acceptable transfer functions $t_{ij}(s) = y_i(s)/r_j(s)$ (output $y_i(s)$ in response to command input $r_j(s)$ in Fig. 3-2). Let the plant inverse matrix $\underline{p}^{-1} = [1/q_{ij}]$, so the plant set $\underline{p}^{-1}$ generates sets of $q_{ij} = \{q_{ij}\}$. Replace the $m \times m$ MIMO problem by m single loops and m^2 prefilters (see Fig. 3-7 for $m = 3$). In Fig. 3-7, $\underline{p}^{-1} = [1/q_{ij}]$, and the uncertainty in $\underline{p}$ generates sets $\mathcal{Q} = \{q_{ij}\}$. The disturbance

$$d_{11} = -\left(\frac{t_{21}}{q_{12}} + \frac{t_{31}}{q_{13}}\right) \quad t_{kl} \in \tau_{kl} \quad (4-1)$$

in Fig. 3-7 is any member of a set d_{11} generated by the t_{k1} in τ_{k1} and the q_{ik} in q_{ik} , $k = 2, 3$.

In general in Fig. 3-7,

$$d_{ij} = \{d_{ij}\}, \quad d_{ij} = - \sum_{k \neq i} \frac{t_{kj}}{q_{ik}}, \quad t_{kj} \in \tau_{kj} \quad (4-2)$$

for the set of acceptable t_{kj} transfer functions. For the top row of MISO loops in Fig. 3-7, the MISO design problem is to find $L_1 = g_1 q_{11}$, f_{11} such that the output is a member of the set τ_{11} for all q_{11} in q_{11} and for all d_{11} in $\mathcal{D}$. Similarly, for the middle MISO loop in the top row of Fig. 3-7 find $L_1 = g_1 q_{11}$, f_{12} so that its output is in τ_{12} for all q_{11} in q_{11} and all d_{12} in d_{12} etc. Note that L_1 is the same for all the MISO structures in the first row of Fig. 3-7 etc. In each of these three structures, the uncertainty problem (due to the sets q_{1i} , d_{1i}) gives bounds on the level of feedback L_1 needed, and so the toughest of these bounds must be satisfied by L_1 .

If the designer designs these MISO systems to satisfy their above stated specifications, then it is guaranteed that these same f_{ij} , g_i satisfy the MIMO uncertainty problem. It is not necessary to consider the highly complex system characteristic equation [denominator of Eq. (3-11)] with its uncertain p_{ij} plant parameters. System stability (and much more than that) for all $\underline{P}$ in $\mathcal{P}$ is automatically guaranteed. It is easier to present the important ideas by means of a design example.

4-2 Design Example

The power of the technique is illustrated by presenting the results of Reference 13, done as a Master's thesis by a typical graduate student who like nearly all control graduates, has had no courses dealing with uncertainty. Here the plant and uncertainties are

$$p_{ij} = \frac{sA_{ij} + B_{ij}}{s^2 + Es + F}$$

$A_{11} \in [2, 4]$	$A_{12} \in [0.5, 1.1]$	$A_{22} \in [5, 101]$	$\left. \vphantom{\begin{matrix} A_{11} \\ A_{12} \\ A_{22} \\ A_{31} \\ B_{11} \\ B_{12} \\ B_{13} \\ B_{21} \\ B_{22} \\ B_{23} \\ B_{31} \\ B_{32} \\ B_{33} \\ E \\ F \end{matrix}} \right\} (4-3)$
$A_{31} \in [-0.8, -1.8]$	all other $A_{ij} = 0$		
$B_{11} \in [-0.15, 1]$	$B_{12} \in [-1, -2]$		
$B_{13} \in [1, 4]$			
$B_{21} \in [1, 2]$	$B_{22} \in [5, 10]$	$B_{23} \in [-1, -4]$	
$B_{31} \in [-1, -2]$	$B_{32} \in [15, 25]$		
$B_{33} \in [10, 20]$			
$E \in [-0.2, 2]$	$F \in [0.5, 2]$		

Note that p_{12} is always nonminimum phase and p_{11} is also nonminimum phase for part of the parameter range. Also, the plant is unstable for part of the

parameter range. At small ω there is no diagonal dominance in any row, nor in columns 1,2. Hence, Rosenbrock's technique¹⁵ cannot be used even if there was no parameter uncertainty.

Since the MISO plants in Fig. 3-7 are $q_{ii} = \det \underline{P} / \text{Adj}_{ii} \underline{P}$, it follows that if $\det \underline{P}(s)$ has right half-plane zeros, the equivalent single-loop transmissions are nonminimum phase.^{**} The technique can still be used, but success can be guaranteed only if the performance tolerances are compatible with the nonminimum phase character, just as in the single-loop system⁹.^{*} Here $\det \underline{P}(s)$ has all its zeros in the interior of the left-half plane, so this problem does not arise.

4-2A Performance Tolerances

These are on the magnitudes of the elements of the 3x3 closed loop system transmission matrix $T = [t_{ij}(s)]$, in the frequency domain $|t_{ij}(j\omega)|$ which suffices when they are minimum phase. In fact, as previously noted¹¹, time-domain specifications on the system output and on as many of its derivatives as desired, may then be achieved by means of bounds on $|t_{ij}(j\omega)|$. They are shown in Fig. 4-1(a) for $|t_{ii}(j\omega)|$, $i = 1,2,3$, and in Fig. 4-1(b) for $|t_{12}|$, $|t_{32}|$. The four remaining specifications are $|t_{ij}(j\omega)| \leq 0.1$ for all ω . The first five are called 'interacting' and the latter four 'basically non-interacting' (BNIC), because ideally zero t_{13} , t_{21} , t_{23} , t_{31} are desired for all ω . But this is impossible because of parameter uncertainty, so upper bounds must be assigned. The matter of minimum or nonminimum phase character of the BNIC elements, is of no concern; t_{12} , t_{32} are deliberately chosen interacting, in order to obtain a design problem with considerable variety. The functions which satisfy the assigned tolerances on t_{ij} constitute a set τ_{ij} .

*Refer to Appendix C (pages C-1 through C-16) and the Design Checklist (pages E-3 through E-10) dealing with r.h.p. zeros and/or poles.

**See Sec. 5-8.

4-2B Simplification Of The Single-loop Structures

For the second row of Fig 3-7 $y_{ij}(s)$ has the two components due to $r_j(s) = 1$ and to $d_{ij}(s)$. Their respective outputs are

$$y_{ri} = \frac{f_{ij}L_i}{1 + L_i}, \quad y_{di} = \frac{d_{ij}q_{ii}}{1 + L_i}, \quad L_i = g_i q_{ii} \quad (4-4a-c)$$

At a fixed $P \in P$ (and hence fixed $m^2 q_{ij}$) $|y_{di}|_{\max}$ occurs at

$$|d_{ij}|_{\max} = \left| \sum_{k \neq i} t_{kj} / q_{ik} \right|_{\max}$$

The relative phases of the t_{kj} / q_{ik} are not known so the extreme,

$$\sum_{k \neq i} |t_{kj}|_{\max} / |q_{ik}| = \sum_{k \neq i} b_{kj} / |q_{ik}| = d_{ik} \quad (4-5)$$

must be used in the design. That is, where b_{ii} and b_{ki} represent the upper bounds of the control ratios t_{ii} and t_{ki} , respectively, in Fig. 4-1. Also shown in this figure are the corresponding lower bounds a_{ii} and a_{ki} , respectively.

There are two kinds of performance tolerances. In the BNIC t_{ki} ($k, i = 1, 3; 2, 1; 2, 3$; etc.) it is necessary that

$$|y_{ri} + y_{di}| = |\tau_{ij}(j\omega) + \tau_{di}(j\omega)|_{\max} \leq |\tau_{ij}|_{\max} + |\tau_{di}|_{\max} = \quad (4-6)$$

$$|\tau_{di}|_{\max} \leq b_{ij}(\omega)$$

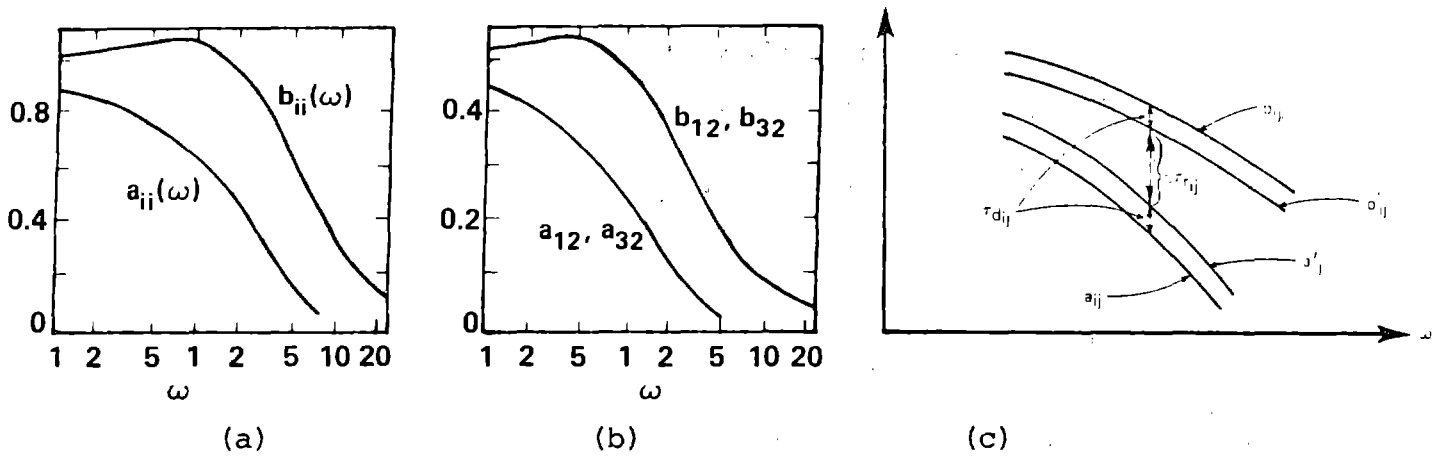


Fig. 4-1 Tolerances (a) on $|\underline{t}_{ii}(j\omega)|$, (b) on $|\underline{t}_{12}|$, $|\underline{t}_{32}(j\omega)|$, and (c) allocation.

because the relative phase of the two terms is not known, so it is clearly best to force $\tau_{ij} = 0$ by setting $f_{ij} = 0$ in Eq. (4-4) for $(1,3;2,1;2,3;3,1)$. Let some fixed $\underline{p}$ be chosen as the nominal plant matrix $\underline{p}_0 = [P_{ij}]$, generating a nominal $\underline{p}_0^{-1} = [1/q_{ij0}]$ and the nominal loop transmissions $L_{i0} = g_i q_{iio}$, so $L_i = L_{i0} q_{iij} / q_{iio}$. In this example, the nominal plant parameters of Eq. (4-3) are chosen as $A_{11} = 2$, $B_{11} = 0.1$, $A_{12} = 1.1$, $B_{12} = -1$, $B_{13} = 4$, $B_{21} = 2$, $A_{22} = 5$, $B_{22} = 5$, $B_{23} = 4$, $A_{31} = 1.8$, $B_{31} = 2$, $B_{32} = 20$, $B_{33} = 10$, $E = 2$, $F = 2$. Using Eqs. (4-4b) and (4-5) condition given by Eq. (4-6] can now be written (for the BNIC $\underline{t}_{ij}$) to yield the design specification.

$$\left| \frac{1}{1 + L_{i0} \frac{q_{iij}}{q_{iio}}} \right| \leq \frac{b_{ij}}{d_{ij} q_{iij}} = \frac{b_{ij}}{\sum_{k \neq i} \left| \frac{q_{iij}}{q_{ik}} \right| b_{ki}} \quad (4-7)$$

Specification type Eq. (4-7) is denoted as a D-type, specifically D_{ij} .

The specifications on the interacting $\underline{t}_{ij}$ terms (all $\underline{t}_{ii}$ and $\underline{t}_{12}, \underline{t}_{32}$) are, using Eq. (4-4),

$$a_{ij}(\omega) \leq |\underline{t}_{ij}(j\omega) + \underline{t}_{dik}(j\omega)| \leq b_{ij}(\omega)$$

with a_{ij} , b_{ij} the designated bounds in Fig. 4-1. Suppose that due to the uncertainty in q_{ii} , $|t_{ij}|$ of Eq. (4-4) is in the interval $[a'_{ij}, b'_{ij}]$ [see Fig. 4-1(c)]. Since the relative phases of t_{ij} , t_{dij} are not a priori known, it is necessary that

$$|b'_{ij} + t_{dij}| \leq b_{ij}, |a'_{ij} - t_{dij}| \geq a_{ij} \quad (4-8)$$

From Eq. (4-4) the above may be summarized by Eq. (4-8) and

$$\Delta \left| \frac{L_i}{1 + L_i} \right| \leq \frac{b'_{ij}}{a'_{ij}}, \left| \frac{1}{1 + L_i} \right| \leq \left| \frac{t_{dij}}{d_{ij}q_{ii}} \right| = t_{dij} / \sum_{k \neq i} \left| \frac{q_{ii}}{q_{ik}} \right| b_{ki} \quad (4-9a,b)$$

Equation (4-9a) is denoted as A_{ij} , Eq. (4-9b) as B_{ij} . Note that Eq. (4-9b) is almost the same as D_{ij} of Eq. (4-7) except that t_{dij} is not known and must be suitably chosen together with a'_{ij}, b'_{ij} subject to Eq. (4-8). This gives us Table 4-1.

4-3 Equilibrium and Trade-Offs

There are interactions between the rows of Fig. 3-7 via the specifications b_{ij} . For example, suppose b_{21} is decreased from 0.1 to δ . (Decrease of b_{21} is permissible because the BNIC $|t_{21}| \leq b_{21}$ is required.) This decreases $|d_{11}|$ and $|d_{31}|$ in Fig. 3-7, which eases the burden on L_1 and L_3 for satisfying $y_{11} \in \tau_{11}$ and $y_{31} \in \tau_{31}$, assuming that y_{11} and y_{31} dominate L_1 and L_3 respectively. But it makes it harder on L_2 , because now $|y_{21}| \leq \delta$ instead of 0.1. This does not matter if y_{22} dominates L_2 . If so, b_{21} can be decreased until say y_{21} imposes the same burden on L_2 as does y_{22} , denoted by $y_{21} \sim y_{22}$. Any further decrease of b_{21} involves 'trade-off', i.e. sacrifice of L_2 for the sake of L_1 and/or L_3 . However, it is con-

Table 4-1. Constraints on the equivalent single-loop structures.

A_{11}, A_{12}		B_{11}	B_{12}	D_{13}
$\Delta \left \frac{L_1}{1 + L_1} \right $	$\leq \frac{b_{11} b_{12}}{a_{11} a_{12}}$	$\frac{1}{1 + L_1} \leq \frac{b_{21} \left \frac{q_{11}}{q_{12}} + b_{31} \left \frac{q_{11}}{q_{13}} \right \right }{\tau d_{11}}$	$\frac{b_{22} \left \frac{q_{11}}{q_{12}} + b_{32} \left \frac{q_{11}}{q_{13}} \right \right }{\tau d_{12}}$	$\frac{b_{23} \left \frac{q_{11}}{q_{12}} + b_{33} \left \frac{q_{11}}{q_{13}} \right \right }{b_{13}}$
D_{21}		A_{22}		
$\frac{1}{1 + L_2} \leq \frac{b_{21} \left \frac{q_{22}}{q_{21}} + b_{31} \left \frac{q_{22}}{q_{23}} \right \right }{b_{11} \left \frac{q_{22}}{q_{21}} + b_{31} \left \frac{q_{22}}{q_{23}} \right \right }$	$\Delta \left \frac{L_2}{1 + L_2} \right $	$\frac{b_{22}}{a_{22}} \leq \frac{b_{22}}{a_{22}}$	$\frac{1}{1 + L_2} \leq \frac{b_{12} \left \frac{q_{22}}{q_{21}} + b_{32} \left \frac{q_{22}}{q_{23}} \right \right }{\tau d_{22}}$	$\frac{b_{23}}{b_{13} \left \frac{q_{22}}{q_{21}} + b_{33} \left \frac{q_{22}}{q_{23}} \right \right }$
D_{31}		A_{32}, A_{33}		
$\frac{1}{1 + L_3} \leq \frac{b_{31} \left \frac{q_{33}}{q_{31}} + b_{21} \left \frac{q_{33}}{q_{32}} \right \right }{b_{11} \left \frac{q_{33}}{q_{31}} + b_{21} \left \frac{q_{33}}{q_{32}} \right \right }$	$\Delta \left \frac{L_3}{1 + L_3} \right $	$\frac{b_{32}}{a_{32}} \leq \frac{b_{32}}{a_{32}}$	$\frac{1}{1 + L_3} \leq \frac{b_{32} \left \frac{q_{33}}{q_{31}} + b_{22} \left \frac{q_{33}}{q_{32}} \right \right }{\tau d_{32}}$	$\frac{1}{b_{13} \left \frac{q_{33}}{q_{31}} + b_{23} \left \frac{q_{33}}{q_{32}} \right \right } \leq \frac{b_{33}}{\tau d_{33}}$

ceivable that before $y_{21} \sim y_{22}$ occurs, either y_{12} (or $y_{13} \sim y_{11}$, or y_{32} (or $y_{33} \sim y_{31}$. There are a bewildering multitude of possibilities and options. Is there a simple rule for determining when 'free' easing of burdens is no longer possible, and only 'trade-off' is available? This question is partly answered by the following definition and theorem.

Definition

Equilibrium exists when it is impossible to reduce the burden on any L_i , without increasing it on some other L_j .

Theorem

A necessary and sufficient condition for equilibrium is when all the L_i are dominated by the members of the same column, i.e. y_{ij} , $i = 1, 2, \dots, m$ dominate the L_i , with j any fixed column. This does not preclude $y_{ij} \sim y_{ik} \sim \dots$ for one or more i, k . See Reference 13 for proof.

Clearly, a number of equilibria can exist, as any column may be chosen. This variety offers the designer very useful design freedom. He may wish to economize on some L_x , according to the circumstances - for example, if the sensor on y_x is noisier than the others, or q_{xx} has elastic modes in a rather low ω range, so it is important to reduce $|L_x(j\omega)|$ rapidly compared with ω . It is also emphasized that all the above discussion is at a fixed ω value. It may be desirable and it is certainly possible to have different columns dominating at different ω values, and the design technique offers this kind of flexibility. The above led to the low and medium frequency bounds on L_{i0} shown in Fig. 4-2.

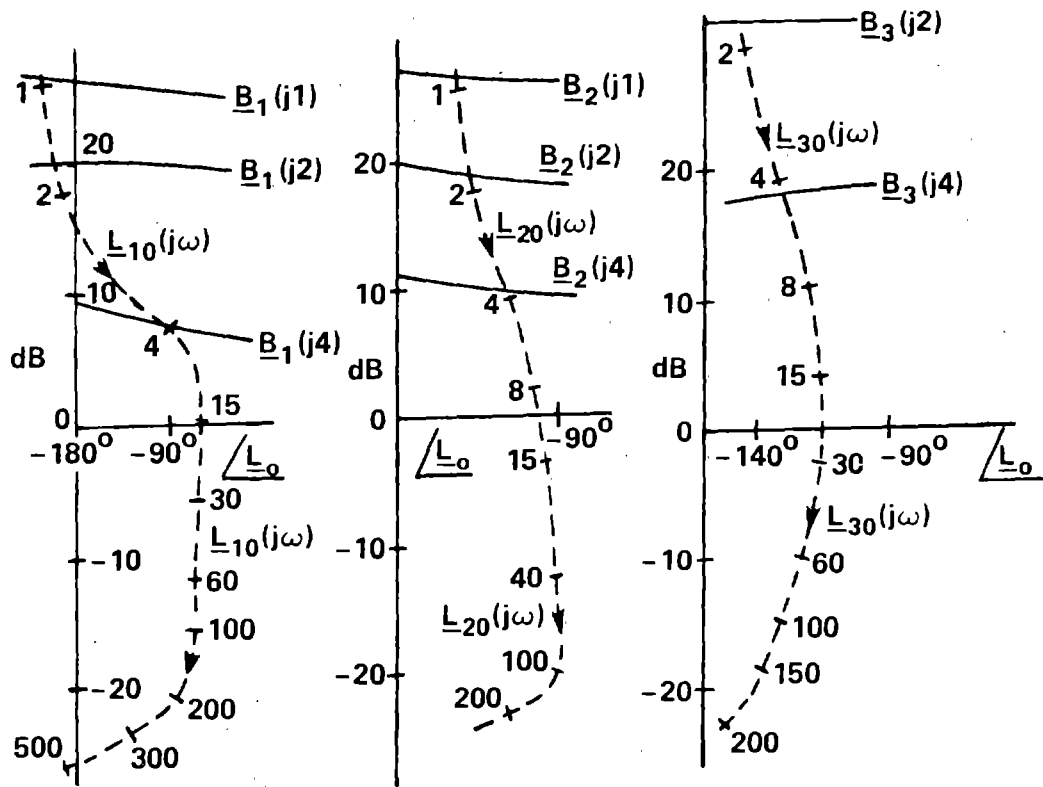


Fig. 4-2 Bounds in Nichols Chart on $\underline{L}_{io}(j\omega)$, $i = 1, 2, 3$, and L_i designed.

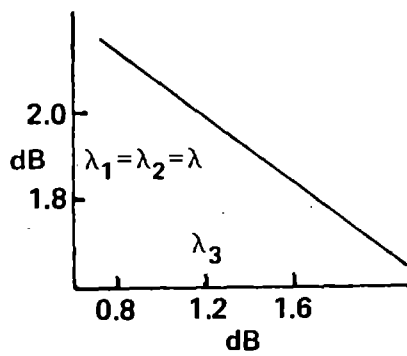


Fig. 4-3 Optimum relation between $\lambda_1 = \lambda_2$ and λ_3 for $\omega > 100$.

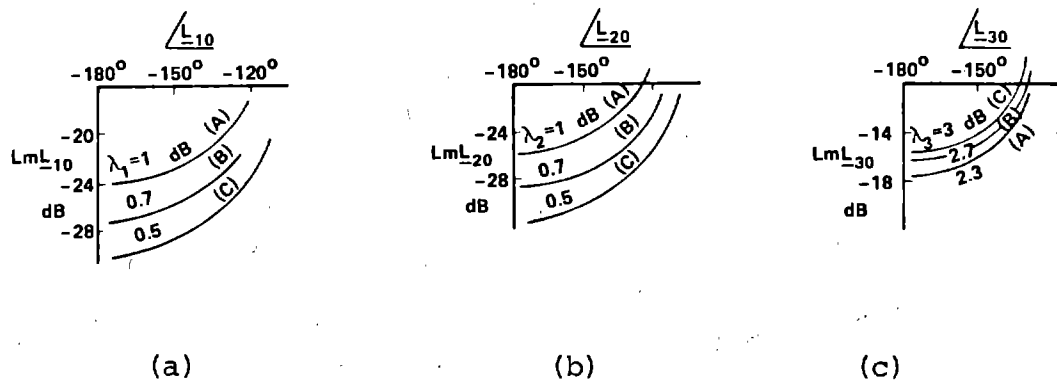


Fig. 4-4 Resulting bounds on L_{10} for various λ , λ_3 values, for $\omega > 100$.

Trade-off

After equilibrium has been reached, it is still possible and may be desirable to 'trade-off' - sacrifice one or more L_i for others, e.g. in the situations noted above. Thus, say column 1 dominates all L_i but it is highly desirable to reduce L_1 . In Fig. 3-7 one or both of b_{21}, b_{31} may be reduced thereby decreasing $|d_{11}|$ but making the specifications harder on y_{21}, y_{31} . Assuming that $y_{11} \approx y_{12}$ then it is necessary also to reduce one or both of b_{22}, b_{32} . At low ω values, the amount of such improvement is quite limited, because $b_{ij} > a_{ij}$ (in the interacting y_{ij}), and the difference between b_{ij} and a_{ij} is small at low ω (see Figs. 4-1a,b). Also, as $b_{ij} \rightarrow a_{ij}$, the sacrifice of $L_i \rightarrow \infty$, for the sake of very small improvement of some other L . In serious situations, the designer can then consider modifying the (a_{ij}, b_{ij}) tolerances. This is another of the valuable insights offered by the design technique. The trade-off situation is radically different in the high frequency range.

4-3 A Trade-off In High Frequency Range

At large ω , the $a_{ij}(\omega)$ of the interacting t_{ij} of Figs. 4-1a,b become so small that they can be made zero. Hence, the A_{ij} in Table 4-1 can be ignored and each $\tau_{d_{ij}} = b_{ij}$, so all the constraints in the table are now of the D type, i.e. replace the B_{ij} by D_{ij} . There is now no limit on the permissible reduction of the b_{ij} and the problems of 'equilibrium' and 'trade-off' become so simplified, as to permit analytical derivation of the bounds $B_i(\omega)$ on $L_{i0}(j\omega)$. First the claim that the b_{ij} are optimally chosen so that $D_{1i} \sim D_{1j}, \dots, D_{mi} \sim D_{mj}$, for all i, j is easily proven. Suppose, (with no loss of generality), that D_{11} dominates L_1 . Reduce b_{12} (recall it has replaced $\tau_{d_{12}}$) until $D_{12} \sim D_{11}$. This certainly doesn't hurt L_1 and helps D_{12} , $i \neq 1$. Hence if any D_{i2} dominates L_i , this reduction of b_{12} helps L_i . If not, a single D_{i2} does so, then there is at least no harm in this reduction of b_{12} . Similarly, reduce b_{13} , until $D_{13} \sim D_{12} \sim D_{11}$. Using exactly the same argument, one can deduce that $D_{23} \sim D_{22} \sim D_{21}$, etc., for each row.

The above result applies even if the $|q_{11}/q_{12}|_{\max}$ are different in the elements of row 1, etc. However if they are the same, as they are in this case then let $v_{ij} \triangleq |q_{ii}/q_{ij}|_{\max}$ which appear in row i of Table 4-1.

The above give

$$\frac{b_{11}}{v_{12}b_{21} + v_{13}b_{31}} = \frac{b_{12}}{v_{12}b_{22} + v_{13}b_{32}} = \frac{b_{13}}{v_{12}b_{23} + v_{13}b_{33}} \triangleq \lambda_1 \quad (4-10)$$

$$\frac{b_{k1}}{\sum_{i \neq k} v_{ki}b_{i1}} = \frac{b_{k2}}{\sum_{i \neq k} v_{ki}b_{i2}} = \dots \triangleq \lambda_k \quad (4-11)$$

Consider the m equations

$$b_{k1} / \sum_{i \neq k} v_{ki} b_{i1} = \lambda_k \quad \text{for } k = 1, \dots, m$$

which constitute a linear homogeneous set in the m b_{k1} . For a solution to exist the determinant of the coefficient matrix must be zero. This matrix, denoted by ϕ has v_{ij} for its off-diagonal and $-1/\lambda_i$ as its diagonal elements. Exactly the same matrix results by taking any α value in the set of m equations

$$b_{k\alpha} / \sum v_{ki} b_{i\alpha} = \lambda_k, \quad k = 1, \dots, m$$

The choice of the λ_k is up to the designer but he can choose only $m - 1$ of them. The condition $\det \phi = 0$, determines the last λ value. In this way, the designer can deliberately sacrifice some loops in order to help others.

For any fixed j , the above set of m equations in b_{kj} ($k = 1, \dots, m$) is homogeneous, so the ratios b_{kj}/b_{1j} , $k = 2, \dots, m$ are determined by the choice of the λ_k . Even if the b_{kj}/b_{1j} emerge very large, one can always make the b_{1j} small enough so that the b_{kj} satisfy the tolerances. Hence, it is not necessary to solve the equations for the b_{kj}/b_{1j} . Of course, this is true only because all $a_{kj} = 0$. If the above approach is used at lower ω , where some or all $a_{kj} \neq 0$, the b_{kj}/b_{1j} should be calculated, in order to be certain that the tolerances are not violated.

4-3B Some universal design features

In this example the bounds $B_i(j\omega)$ on $L_{30}(j\omega)$ tend to be significantly larger (more stringent) than those on L_{10} , L_{20} in the low and middle ω range, where the trade-off opportunities are limited because the interacting

$a_{ij} \neq 0$. In the higher ω range, when Eqs. (4-10) and (4-11) apply, there is much greater scope for such trade-off. This was utilized to help L_3 in the higher ω range by setting $\lambda_1 = \lambda_2 = \lambda$, and solving $\det \underline{\Phi} = 0$ for λ_3 . As the v_{ij} are functions of ω , $\det \underline{\Phi} = 0$ gives a relation between the λ_i which is also a function of ω . The result is shown in Fig. 4-3 for $\omega > 100$, for which the $v_{ij}(j\omega)$ are fairly constant with ω . Given $|1 + \underline{L}_i|^{-1} \leq \lambda_i$, $\underline{L}_i = \underline{L}_{i0} q_{i11}/q_{i10}$, one can find the bounds on $\underline{L}_{i0}$, which of course depend on the set $\underline{Q} = \{q_{i11}\}$. The results are shown in Fig. 4-4, where Fig. 4-4(a) corresponds to $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 2.3$ dB, Fig. 4-4(b) $\lambda_1 = \lambda_2 = 0.7$, $\lambda_3 = 2.7$ dB and Fig. 4-4(c) $\lambda_1 = \lambda_2 = 0.5$, $\lambda_3 = 3$ dB. The $q_{i11}(j\omega)$ are almost constant with ω for $\omega > 100$, so these bounds apply for all $\omega > 100$. As expected, the larger λ , the easier it is on $\underline{L}_{10}, \underline{L}_{20}$ and the harder on $\underline{L}_{30}$. The big difference in the bounds in Fig. 4-4 is due to the small λ compared with the large λ_3 that were deliberately used. For this reason the actual $\underline{L}_{30}(j\omega)$ was able to reach its final asymptotic slope at $\omega \sim 250$ (see Fig. 4-5) compared with $\sim 450, 230$ for $\underline{L}_{10}, \underline{L}_{20}$ even though $|\underline{L}_{30}|$ is much more than $|\underline{L}_{10}|, |\underline{L}_{20}|$ at low and medium ω . The $\underline{L}_{i0}$ chosen to satisfy the bounds are shown in Fig. 4-5. Design simulation results were highly satisfactory and are shown in Fig. 4-6 for extreme plant parameter combinations.

The reader is urged, in performing a QFT design, to constantly refer to the Design Checklist on pages E-3 through E-10. Also, this checklist and Appendix C discuss how the QFT design technique can be applied to plants having r.h.p. zeros and/or poles.

It should be noted that if each p_{ij} of $\underline{P}$ does not have the same value of excess of poles over zeros then as $\omega \rightarrow \infty$ the templates may not be straight lines.

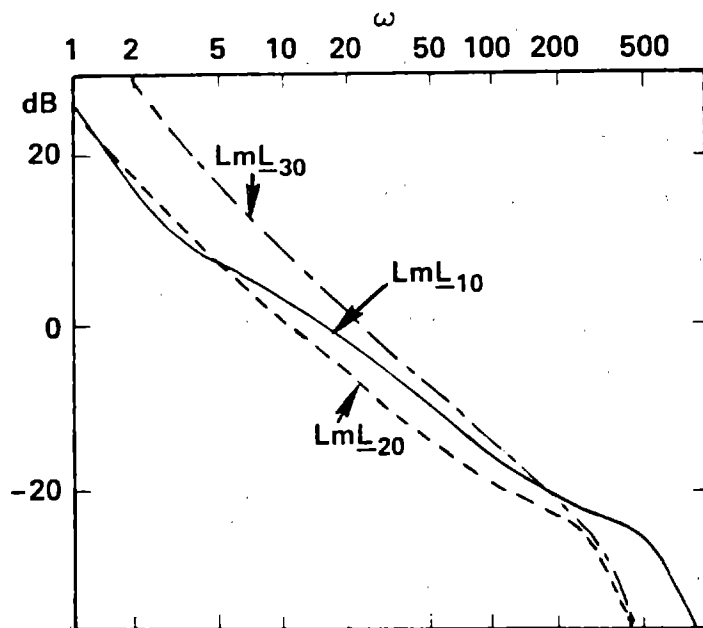


Fig. 4-5 Bode plots of $L_{i0}(j\omega)$

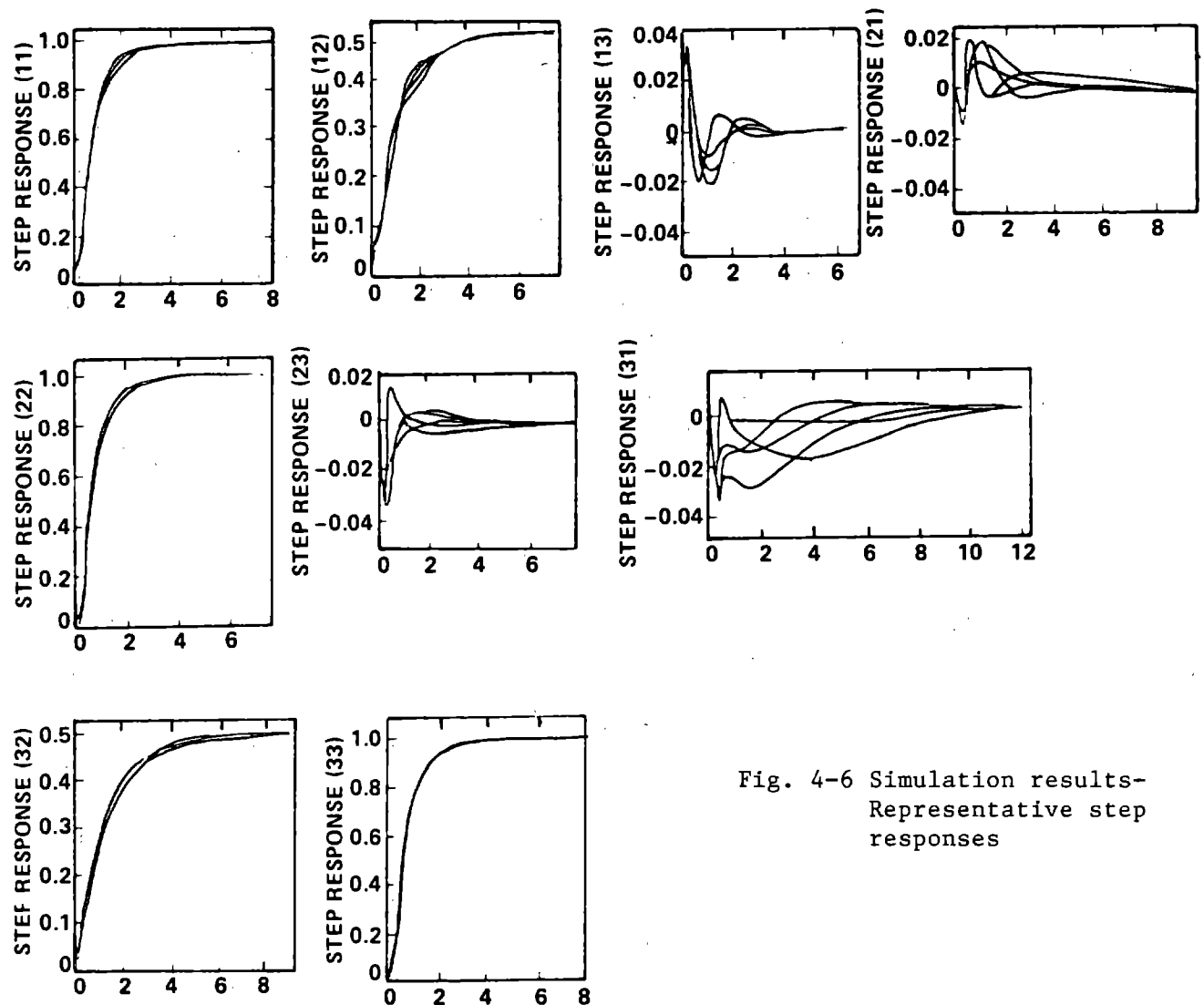


Fig. 4-6 Simulation results-
Representative step
responses

4-4 Problems

4-1 In a 2x2 MIMO system $r_1 \neq 0$ and $r_2 = 0$, thus only t_{11} and t_{21} are involved in the design. The specifications on t_{11} are:

ω	0.1	0.2	0.5	1	2	5	10
b_{11}	1.02	1.07	1.09	0.97	0.65	0.25	0.05
a_{11}	0.95	0.92	0.80	0.55	0.20	0	0

The specification on t_{21} is that $b_{21} = 0.01$ for all values of ω and the specification on each L_i satisfies

$$L_m \left| \frac{\underline{L}_i}{1 + \underline{L}_i} \right| \leq 2.3\text{dB}$$

The plant is described as follows:

$$q_{11} = k_{11}, q_{12} = k_{12}, q_{21} = k_{21} \text{ and } q_{22} = k_{22}$$

$$0.2 \leq k_{11} \leq 1, 0.5 \leq k_{12} \leq 2, 2 \leq k_{21} \leq 5, \text{ and } 0.1 \leq k_{22} \leq 0.5$$

Note: q 's are assumed as real numbers (gain uncertainty only)

thus $\gamma_{\max} = (1)(0.5)/(0.5)(2) = 0.5$. Choose the following nominal values

$$q_{11o} = 0.2 \text{ and } q_{22o} = 0.1$$

(a) Find the bounds on $\underline{L}_{1o}$ at $\omega = 0.1, 0.2, 0.5, 1$ and 2 .

(b) Use

$$\left| \frac{1}{1 + \underline{L}_2} \right| \leq \left| \left(\frac{b_{21}}{b_{11}} \right) \left(\frac{q_{21}}{q_{22}} \right) \right|$$

to find the bounds on $\underline{L}_{2o}$ for the same values of ω as in part (a).

Hint: use the "inverse N.C." for this purpose, i.e., let $\underline{\ell}_2 = 1/\underline{L}_2$

to obtain the form $\underline{\ell}_2/(1 + \underline{\ell}_2)$.

4-2 Given $\underline{T} = [\underline{I} + \underline{P} \underline{G}]^{-1} \underline{P} \underline{G} \underline{F}$ whose poles are given by

$$(1 + P_{11}g_1)(1 + P_{22}g_2) - P_{12}g_2P_{21}g_1$$

Prove that instead, can consider the zeros of

$$(1 + q_{11}g_1)(1 + q_{22}g_2) - \frac{q_{11}q_{22}}{q_{12}q_{21}}$$

4-3 Given a 2x2 plant whose Q matrix is

$$Q = \frac{1}{s} \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

where

$$1 \leq k_{11} \leq 2, 0.5 \leq k_{12} \leq 2, 5 \leq k_{21} \leq 10, 0.5 \leq k_{22} \leq 1$$

$$|q_{11}q_{22}| < |q_{12}q_{21}| \text{ as } s \rightarrow \infty$$

$$r_1(t) = 0$$

The specifications based upon input $r_2(t)$ are:

$$|t_{12}| < 0.1 \text{ for all } \omega$$

and for $Lm \ t_{22}(j\omega)$

ω	0	0.5	1	2	5	10	20
$Lmb(j\omega)$	0	1	1	0	-4	-10	-20
$Lma(j\omega)$	0	-2	-4	-8	-20	$-\infty$	$-\infty$

Additional specifications on L_1 and L_2 , respectively, are

$$Lm \left| \frac{L_1}{1 + L_1} \right| \leq 3dB \quad Lm \left| \frac{L_{21}}{1 + L_{21}} \right| \leq 3dB$$

- Design $L_1(s)$ and $L_{21}(s)$ by designing $L_1(s)$ first and letting $f_{12} = 0$.
- Redesign $L_{21}(s)$ where $0.5 \leq k_{22} \leq 1.2$ and $L_1(s)$ of part (a) is not affected by the change in k_{22} .

5-1 Introduction

The design technique of Chap. 4 inherently involves some overdesign, seen from Eq. (4-1), in which t_{21} , t_{31} can be any members of their acceptable sets τ_{21} , τ_{31} and q_{12} , q_{13} any members of their uncertainty sets. As noted in Sec. 4-2, it is therefore necessary to use the worst cases. Actually, there is a correlation between the t_{21} , t_{31} and the q_{11} , q_{13} , etc, that is, it is possible that q_{12} is large when t_{21} is large etc. Such correlation can only help make d_{11} smaller. But it is not possible to use this correlation, and so must take largest t_{21} , smallest q_{12} , etc. This is the price paid for converting the MIMO problem into the much simpler MISO problems, and avoiding having to work with the horrendous denominator in Eq. (3-11).

Another disadvantage of the above is that there emerges a certain inequality (see Sec. 3-7) which must be satisfied by the plant elements: for $m = 2$, as $\omega \rightarrow \infty$ it is

$$|p_{11}p_{22}(s)| > |p_{12}p_{21}(s)| \quad \text{for all } p \in P \quad (5-1)$$

or vice versa. (Rosenbrock's¹⁵ dominance condition is tougher. It requires this inequality to be satisfied over the entire frequency range, not just as $s \rightarrow \infty$. His method can therefore not be used for the 3x3 example of Chap. 4.) The equivalent of Eq. (5-1) for $m = 3$, as $\omega \rightarrow \infty$, it is

$$\begin{aligned} |p_{123}^*| = |p_{11}^*p_{22}^*p_{33}^*| &> |p_{11}^*p_{23}^*p_{32}^*| + |p_{12}^*p_{21}^*p_{33}^*| \\ &+ |p_{12}^*p_{23}^*p_{31}^*| + |p_{13}^*p_{22}^*p_{31}^*| + |p_{13}^*p_{21}^*p_{32}^*| \end{aligned} \quad (5-2)$$

It turned out that Eq. (5-2) was not satisfied by the FY-16CCV 3x3 lateral system for any $p \in P$. Thus it was necessary to seek a modification of the technique that succeeds in avoiding the condition of Eq. (5-2), and involves less overdesign than the first method. The necessary constraints for method two are discussed in Sec. 5-6 and on page E-10.

5-2 Design equations for the 2x2 system

In method two the same design equations as before are used for t_{11} , t_{12} or alternatively t_{21} , t_{22} (as in Fig 3-7 for $m = 2$). Thus for Fig. 5-1 and for an impulse input, Eqs. (3-22) through (3-25) yield

$$t_{1j} = y_{1j} = \frac{f_{1j}L_1 + d_{1j}q_{11}}{1 + L_1}, \quad L_i = g_i q_{ii} \quad (5-3a-d)$$

$$d_{1j} = \frac{-t_{2j}}{q_{12}}, \quad t_{2j} \in \tau_{2j}, \quad \underline{P}^{-1} = \begin{bmatrix} p_{ji}^* \end{bmatrix} = \begin{bmatrix} 1/q_{ji} \end{bmatrix}$$

The objective is to choose $f_{11}(s)$, $f_{12}(s)$, $g_1(s)$ such that the $y_{1j}(s)$ have no right half-plane (rhp) poles, and satisfy the tolerances on $|t_{1j}(j\omega)|$, $\forall P \in \mathcal{P}$ and $\forall t_{2j} \in \tau_{2j}$ appearing in d_{1j} . These are precisely the MISO design problems of Sec. 3-6 through 3-8.

When t_{1j} is substituted for y_{1j} in Eq. (5-3), the resulting equations are exact. Hence, if indeed $y_{1j} \in \tau_{1j}$ for all t_{2j} (of d_{1j}) $\in \tau_{2j}$,

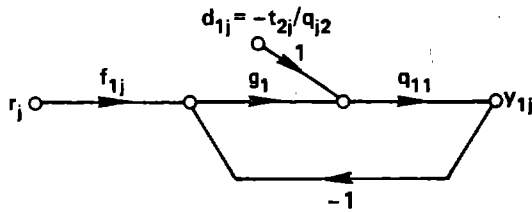


Fig. 5-1 MISO structure for t_{1j}
($j = 1, 2$), Eq. (5-3)

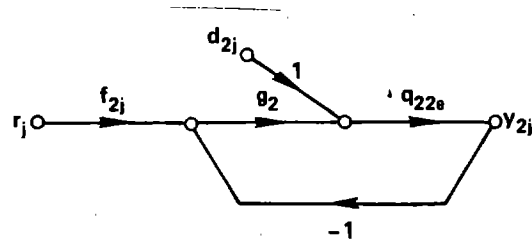


Fig. 5-2 MISO structure for t_{2j}
($j = 1, 2$), Eq. (5-4)

$\underline{P} \in \mathcal{P}$, then the design objectives for t_{1j} have been achieved by the f_{11} , f_{12} , g_1 but only if actually $t_{2j} \in \tau_{2j}$.

The final step, in the 2x2 system, is to choose f_{21} , f_{22} , g_2 to ensure that the $t_{2v}(s)$ have no rhp poles and that $t_{2v} \in \tau_{2v}$, $\underline{p} \in P$. In Sections 3-6 and 3-7 the design equations are again Equation (5-3) with 2 replacing 1 and the d_{2v} containing t_{1v} . Instead, here, there are used equations independent of the t_{1v} , by simply finding t_{2j} from $\underline{T} = (\underline{I} + \underline{PG})^{-1}\underline{PGF}$:

$$\begin{aligned} t_{2j} &= \frac{f_{2j}L_{2e} + d_{2j}}{1 + L_{2e}}, \quad L_{2e} = g_2 q_{22e} = \frac{g_2 q_{22}(1 + L_1)}{1 - \gamma_{12} + L_1} \\ \gamma_{ji} &= \frac{p_{ij}p_{ji}}{p_{ii}p_{jj}}, \\ d_{2j} &= \frac{g_1 f_{11} p_{21} (1 - \gamma_{12})}{1 - \gamma_{12} + L_1}, \quad \underline{p} = \begin{bmatrix} p_{ji} \end{bmatrix} \end{aligned} \quad (5-4a-e)$$

with the MISO structures of Fig. 5-2. This design is done after the design Eq. (4-9a) has been completed by means of Eqs. (5-3a-d), so that L_1 , f_{11} , and f_{12} are known.* It is then necessary to find g_2 , f_{21} , and f_{22} so that in Fig. 5-2 the outputs y_{21} , y_{22} are stable and satisfy the tolerances on $|t_{21}|$ and $|t_{22}|$ respectively. These are single-loop problems similar to Fig. 5-1, except that only the uncertainty $\underline{p} \in P$ need be considered, as the d_{2j} in Eqs. (5-4a-e) are not functions of the elements of τ_{ij} , which they are in Eqs. (5-3a-d). At each step, design execution is that of single input-output single loop systems--which is what makes this design procedure so tractable.

The theoretical justification of the above design procedure is as follows: the design specifications are satisfied for t_{2v} of Fig. 5-2, by proper choice of the f_{2v} and g_2 for the given f_{1v} and g_1 . So the design is satisfactory for t_{21} , t_{22} because Eqs. (5-4a,b) are exactly the expressions for t_{21} and t_{22} (even if the specifications for t_{12} , t_{11} are not satisfied). Now Fig. 5-1 has

*Use L_1 not L_{10} (see page E-3).

been designed (via Eqs. 5-3a-d), so that y_{11} , y_{12} are stable and satisfy the specifications on t_{11} and t_{12} $\forall \underline{P} \in \underline{P}$, if the t_{2j} appearing in d_{1j} are in τ_{2j} (which they are). The equations for y_{11} and y_{12} correspond precisely to those for t_{11} and t_{12} , respectively. Thus, no fixed point theory is needed to rigorously justify this design procedure, although the idea and approach are motivated by the fixed point method in Method one. There, the design equations for t_{21} and t_{22} are of the same form as Eqs. (5-3a-d) with the d_{2j} functions of t_{1j} , so fixed point theory is required to justify Method one.

5-3 Reduced Overdesign¹⁷

Figures 5-1 and 5-2 are the same as Fig. 3-7 (for 2x2 system)¹⁴ in which t_{21} and t_{22} appear in the disturbance d_{11} and d_{12} , respectively. There is inherent overdesign in Fig. 5-1 because in reality there is a correlation between the t_{21} and t_{22} and the q_{ij} of $\underline{P}^{-1}$. This correlation is not being exploited. The uncertainties in $t_{21} \in \tau_{21}$, $t_{22} \in \tau_{22}$ are assumed independent of $\underline{P} \in \underline{P}$ in Figs. 5-1 and 5-2. But such overdesign does not exist in Fig. 5-2 because d_{21} and d_{22} are not functions of the elements of any τ_{ij} . In Fig. (3-7), for the second row of the 2x2 system, the design for t_{21} and t_{22} , by method 1, involved overdesign precisely as in the first row of this figure and in Fig. 5-1. The above procedure can, of course, be reversed with y_{21} and y_{22} using Eq. (5-3) with 1 replaced by 2 and with t_{11} and t_{12} using Eqs. (5-4) by exchanging numbers 1,2.

5-4 3x3 Design Equations

Let y_{1j} , $j = 1,2,3$, be the same as in Secs. 3-6 and 3-7.

$$y_{1j} = \frac{f_{1j}L_1 + d_{1j}q_{11}}{1 + L_1}, \quad L_1 = g_1q_{11}, \quad d_{1j} = -\sum_k t_{kj}/q_{1k} \quad (5-5a-c)$$

giving MISO problems. L_1 and the three f_{1j} are chosen so that y_{1j} are stable and satisfy the tolerances on t_{1j} for all $t_{kj} \in \tau_{kj}$ appearing in d_{1j} in

Eq. (5-5c) and for all $\underline{p} \in P$. The equations for y_{2j} are obtained from the second row of Fig. 3-8 in which t_{1j} and t_{3j} appear, but the t_{1j} are replaced by the y_{1j} of Eq. (5-5), giving for $j = 1, 2, 3$

$$y_{2j} = \frac{f_{2j} L_{2e} + d_{2j}}{1 + L_{2e}}, \quad L_{2e} = \frac{g_{22} q_{22}}{1 - \frac{\gamma_{12}}{1 + L_1}}, \quad p_{21}^e = \frac{L_1 p_{21}^*}{1 + L_1}, \quad L_{2e} = g_{22} q_{22e} \quad (5-6a-d)$$

$$d_{2j} = \frac{L_{2e}}{g_2} \left\{ t_{3j} \left[\frac{p_{21}^* p_{13}^*}{p_{11}^* (1 + L_1)} - p_{23}^* \right] - f_{1j} p_{21}^e \right\}, \quad \underline{p}^{-1} = [p_{ij}^*].$$

again MISO problems. L_1 and the f_{1j} are known from the designs of Eqs. (5-5). In Eqs. (5-6) the f_{2j} and L_{2e} are chosen so that y_{2j} are stable for all $t_{3j} \in \tau_{3j}$ appearing in d_{2j} and, of course, for all $\underline{p} \in P$. Although the forms for d_{2j} and L_{2e} in Eqs. (5-6) are different from those in Eqs. (5-5), they are otherwise identical in form, so the design techniques for both are basically the same, as detailed in Reference 17.

Finally, MISO design equations for t_{3j} are obtained by finding t_{3j} from $\underline{T} = [\underline{I} + \underline{P}\underline{G}]^{-1} \underline{P}\underline{G}\underline{F}$, or from the third row of Fig. 3-7 for $i = 3$ and eliminating t_{1i} , t_{2i} by means of Eqs. (5-5) and (5-6)

$$t_{3j} = \frac{f_{3j} L_{3e} + d_{3j}}{1 + L_{3e}}, \quad L_{3e} = \frac{L_3 \zeta}{\zeta - \Lambda}, \quad L_3 = q_{33} g_3$$

$$\zeta = (1 + L_1)(1 + L_{2e}) - \gamma_{12}$$

$$\Lambda = \gamma_{23}(1 + L_1) + \gamma_{13}(1 + L_{2e}) - (\gamma_{12}\mu_2 + \gamma_{13}\mu_3)$$

$$\mu_2 = \frac{p_{23}^* p_{31}^*}{p_{21}^* p_{33}^*}, \quad \mu_3 = \frac{p_{32}^* p_{21}^*}{p_{31}^* p_{22}^*} \quad (5-7a-j)$$

$$d_{3j} = \frac{f_{1j} L_1 q_{33} \eta_1 + f_{2j} L_{2e} q_{33} \eta_2}{\zeta - \Lambda}$$

$$\eta_1 = q_{22} p_{21}^* p_{32}^* - p_{31}^* (1 + L_{2e})$$

$$\eta_2 = q_{11} p_{12}^* p_{31}^* - p_{32}^* (1 + L_1)$$

Note: See Eq. (3-49) for the expressions for the γ_{ij} terms of Eq. (5-7).

Since L_1 , L_2 f_{1j} , f_{2j} are known, the only unknowns in the above are the f_{3j} and g_3 . Equations (5-7a-j) constitute single-loop uncertainty problems, for which the technique of Chap. 2 applies, i.e. they are chosen so that the t_{3j} are stable and satisfy the tolerances τ_{3j} . Note, again, that at each step, design execution is that of MISO single-loop systems.

The justification of the above design approach is the same as for the 2×2 case. Suppose the nine f_{ij} and three g_i in Eqs. (5-7a-j) are such that the t_{3j} are stable and their tolerances are satisfied, which is so by definition here. Now the design based on Eqs. (5-6a-c) guarantees that the t_{2j} tolerances are satisfied, providing the t_{3j} appearing in d_{2j} are $\in \tau_{3j}$ - which is the case here. Hence, the t_{2j} tolerances are also satisfied. Finally, the design based on Eqs. (5-5a-c) guarantees that the t_{1j} tolerances are satisfied, providing that the t_{2j} and t_{3j} , appearing in d_{1j} are $\in \tau_{2j}$ and τ_{3j} respectively, which has been established. In the above, there is some overdesign in Eqs. (5-5a-c) because the $t_{ij} \in \tau_{kj}, (k = 2, 3), (j = 1, 2, 3)$ appear as disturbances uncorrelated to the plant uncertainty. In Eqs. (5-6a-d) there is less overdesign because only the t_{3j} so appear, while there is no such overdesign in Eqs. (5-7a-j). Of course, the order can be changed and equations of the form of Eqs. (5-5a-c) are used for the second or third channel, etc.

5-5 $m \times m$ System, $m > 3$

The procedure for generating the design equations for $m \times m$ MIMO systems with $m > 3$, should be clear from the above. One uses for any channel (say the first) design equations in which all the $t_{ij} (i \neq 1)$ appear as disturbances. Denote these as Eqs. A. These equations can be derived or obtained from Reference 14, Eqs. (4a,b) with $u = 1$. For the next chosen channel (say the

first) design equations in which all the $t_{ij}(i \neq 1)$ appear as disturbances. Denote these as Eqs. A. These equations can be derived or obtained from Reference 14, Eqs. (4a,b) with $u = 1$. For the next chosen channel (say the second), one starts with Eqs. (4a,b) of Reference 14 with $u = 2$, but eliminates all the t_{1j}, t_{2j} by means of Eqs. A above. Denote the resulting design equations as Eqs. B. For the next chosen channel (say the third), start again with Eqs. (4a,b) of Reference 14 with $u = 3$, but eliminate all the t_{1j}, t_{2j} by means of Eqs. A, B above. The process continues until the end, and the theoretical justification is the same as given above for $m = 2, 3$.

Mixtures of the first and second techniques may also be used. For example, for the 3x3 system, Eqs. (5-5) type equations, the first two rows of Fig. 3-7 may be used for both channels 1, 2 and Eqs. (5-7a-j) for channel 3. The theoretical justification is now as follows: the design for channel 3 is correct by definition then the fixed point theory, precisely as in Reference 14, is used to justify the designs for channels (1,2). This method was used for the 3x3 FY16-CCV lateral design modes¹⁶.

The two sets of design equations for t_{1j}, t_{2j} are taken as the mappings on the acceptable sets τ_{ij} and the third set of mappings is simply $t_{3j} \in \tau_{3j}$. The nine f_{ij} and the three g_i have been chosen so that these mappings map τ_{ij} into themselves, etc., so a fixed point exists, etc., as in (Reference 14). For larger m , it is clear that a larger variety of mixtures is possible, giving the designer useful flexibility. However, he must understand MISO design theory used in the design execution, which reveals the cost of feedback and the available tradeoffs among the loops, in order to be able to exploit this flexibility to its fullest extent.

There are additional important advantages in Method 2. Conditions given by Eqs. (5-1) and (5-2) are no longer necessary. Instead the principal condition to be satisfied (to achieve arbitrary small sensitivity) is that $\det \underline{P}$ has no right half-plane zeros. The reader is referred to Reference 17 for details.

5-6 Conditions for Existence of a Solution

This section considers the conditions required for the applicability of the above design technique. Also, it considers the inherent, irreducible conditions applicable for LTI compensations in general, and compares the two sets of conditions. This is done for "arbitrarily small (a.s.) sensitivity" defined as achievement of a.s. sensitivity of the t_{ij} over arbitrarily large (a.l.) bandwidth. Such "a.s. sensitivity" also achieves a.l. attenuation over a.l. bandwidth of external disturbances acting on the plant. This problem has also been studied in an abstract setting by Zames and Bensoussan¹⁸.

In Fig. 5-1, it is required that $y_{11} \in \tau_{11}$ and $y_{12} \in \tau_{12}$, for all $\underline{P} \in \mathcal{P}$, $t_{21} \in \tau_{21}$, and $t_{22} \in \tau_{22}$. In the general m case, the disturbance component in, for example, y_{11} is

$$y_{d11} = \frac{d_{1j}q_{11}}{1 + g_1q_{11}} = -\sum_{k \neq 1} t_{k1} \left(\frac{(\text{Adj } \underline{P})_{1k} / (\text{Adj } \underline{P})_{11}}{1 + g_1 \frac{\det \underline{P}}{(\text{Adj } \underline{P})_{11}}} \right) \quad (5-8)$$

R.h.p. poles of $(\text{Adj } \underline{P})_{1k}$ are normally cancelled by similar poles in $\det \underline{P}$, since $(\text{Adj } \underline{P})_{1k}$ is a term in the expansion of $\det \underline{P}$. R.h.p. zeros of $(\text{Adj } \underline{P})_{11}$ are, of course, normally cancelled by similar ones in the denominator. There may be exceptional cases when in $\det \underline{P}$, for example, a r.h.p. pole of $(\text{Adj } \underline{P})_{1k}$ is cancelled by an identical zero of p_{1k}^* and does not appear in the other terms of $\det \underline{P}$. Such cases are excluded.

5-6A Conditions For "a.s. Sensitivity" In Single-Loop Design

In Eqs. 5-3, it is seen that a.s. tolerances over a.l. bandwidth (i.e. "a.s. sensitivity") for t_{11} , t_{12} are achievable if $L_1 = g_1 q_{11}$ can be made a.l. over a.l. bandwidth. Indeed this is at least theoretically possible, if q_{11} satisfies certain constraints. These have been detailed in Appendix 1 of Reference 14 so are only qualitatively described here by means of Fig. 5-3.

Figure 5-3 is the extended logarithmic complex plane (N. C.). Since $\underline{p}$ ranges over P , the set $\{\underline{L}_1 = \underline{g}_1 \underline{q}_{11}(j\omega)\}$ is not a single complex number (at any fixed ω) in the N. C. but a region, denoted as $T_p[\underline{L}(j\omega)]$, the template of $\underline{L}_1$, which is the same as $T_p[\underline{q}_{11}(j\omega)]$ but translated vertically by $20\log[\underline{g}_1(j\omega)]$ dB and horizontally by $\angle \underline{g}_1(j\omega)$ degrees, because $L_1 = g_1 q_{11}$. In a design with significant plant uncertainty $T_p[\underline{L}_1(j\omega)]$ must lie relatively high up, above the zero dB line as shown in Fig. 5-3 for ω_1 . This is so over the important ω range of t_{11} and t_{12} (their bandwidths generally), in order to achieve the desired sensitivity reduction. Such large values for $T_p[\underline{L}_1(j\omega)]$ can be maintained theoretically for any finite ω range, if q_{11} is m.p. For those ω for which $T_p[\underline{L}_1(j\omega)]$ is so located, the uncertainty in the magnitude and phase of $\underline{q}_{11}(j\omega)$ (i.e. the area of $T_p[\underline{q}_{11}(j\omega)]$, can be arbitrarily large. [Note, however, that in order to maintain $T_p[\underline{L}_1(j\omega)]$ above the zero db line, any zeros of $\underline{q}_{11}(j\omega)$ on the $j\omega$ axis must be known and finite in number in order for $g_1(s)$ to be assigned poles there. (Obviously, transcendental compensation can be used for special countable cases.) If the range of such $j\omega$ axis zeros is uncertain, then the specifications

$$a_{ij}(j\omega) \leq |t_{ij}(j\omega)| \leq b_{ij}(j\omega), \forall p \in P \quad (5-9a)$$

$$\tau_{ij}(j\omega) = t_{ij}(j\omega) = \text{set of acceptable } t_{ij}(j\omega) \quad (5-9b)$$

must be modified to permit such zeros of t_{1j} .]

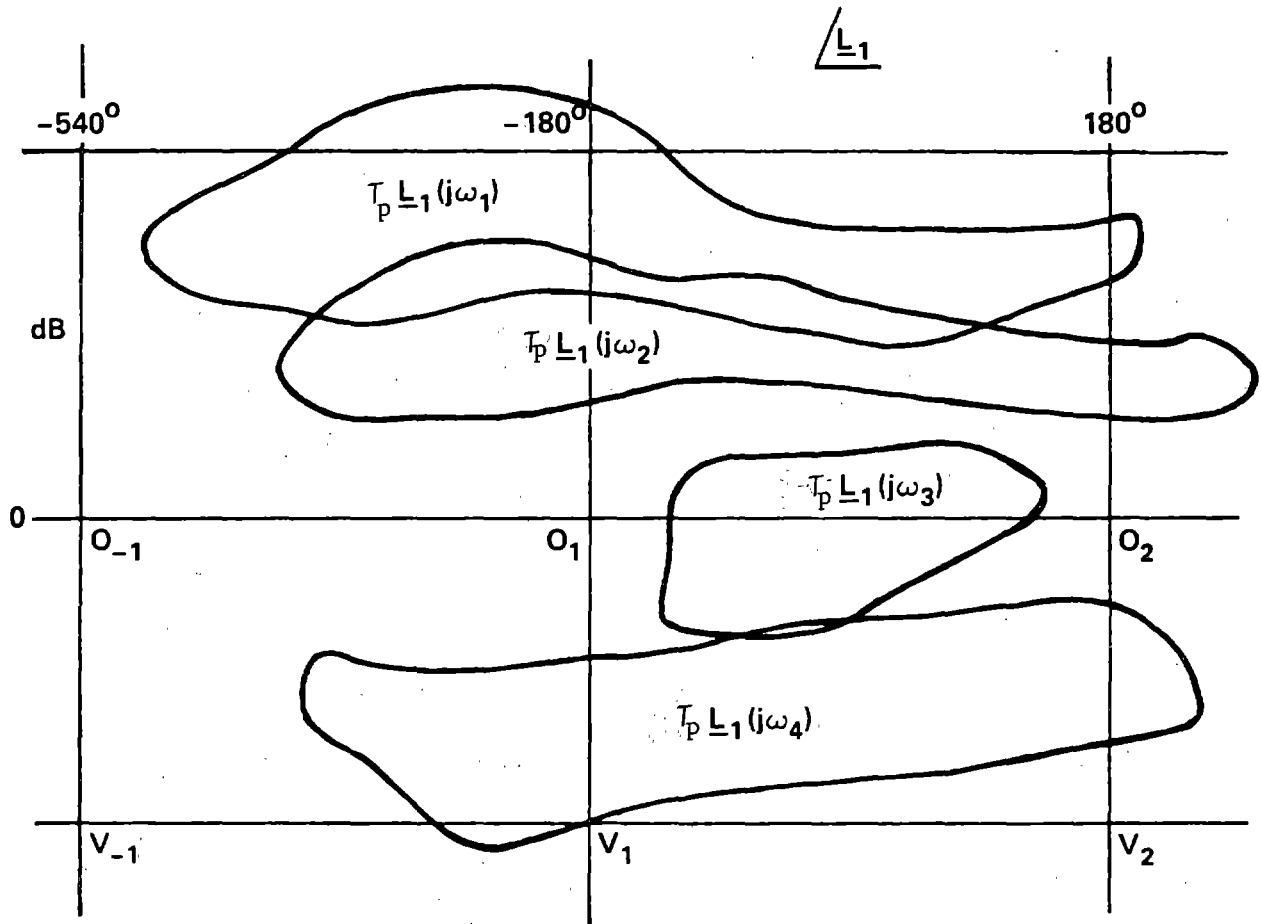


Fig. 5-3 Templates of $\underline{L}_1(j\omega)$ on logarithmic complex plane (Nichols chart)

Sooner or later $\underline{L}_1(j\omega)$ must decrease and $\rightarrow 0$ as $\omega \rightarrow \infty$. Stability over the range of P requires that $T_p \underline{L}_1(j\omega)$ move downward in between the vertical lines V_1 and V_2 without the points $\dots O_{-1} O_1 O_2, \dots$ lying in any of the $T_p \underline{L}_1(j\omega)$. This appears to allow $(360^\circ - 2\theta_m)$ degrees phase width for $T_p \underline{L}_1(j\omega)$, with θ_m phase margin. However $\angle \underline{L}_1(j\omega)$ must be negative on the average, in order for $|\underline{L}_1(j\omega)|$ to decrease. So in practice only $(180^\circ - \theta_m)$

phase width is tolerable in this range. As ω increases and the $T_{pL_1}(j\omega)$ descend lower on the chart below the zero dB line, clearly their width may increase again, but it is essential that the points $\dots 0_1, 0_2, \dots$ never be a part of any $T_{pL_1}(j\omega)$. Unstable q_{11} are included in the above discussion and don't require separate treatment. It follows from the above that the manageable uncertainties depend on the assigned t_{ij} tolerances, but two important constraints, for $m \leq 3$, are stated here for the case of "a.s. sensitivity":

Constraint 1

$q_{11} = \det \underline{P} / (\text{Adj. } \underline{P})_{11}$ must be m.p.

Constraint 2

Suppose $\exists \omega_h, \exists$ for all $\omega > \omega_h$, the width of $T_{pq_{11}}(j\omega)$ exceeds $(180^\circ - \gamma)$, γ a desired phase margin, then it is impossible to achieve a.s. tolerances for $\omega > \omega_h$. This prevents "a.s. sensitivity" if $q_{ii} = k \Pi(s + z_v) / \Pi(s + p_u)$, with the k uncertainty including a sign change which is independent of the signs of the z_v and p_u . Also excluded is a factor $(1 + \tau s)$ in the numerator or denominator of q_{ii} , with the uncertainty in τ including a sign change which is independent of other parameters. See Constraint 1 and 2 application on page E-10.

5-6B Application of Section 5-6A to Design Method 2

Constraints 1 and 2 therefore apply to the $q_{ii} = \det \underline{P} / (\text{Adj } \underline{P})_{ii}$ of the first channel i , used in the design technique of Secs. 5-2 and 5-4. So from Constraint 1, q_{ii} must be m.p. $\forall \underline{P} \in \mathcal{P}$ (rhp poles are tolerable), for "a.s. sensitivity" design. Suppose $\underline{P} = [p_{ij}]$ has each $p_{ij} \rightarrow k_{ij}/s^\lambda$ as $s \rightarrow \infty$. For $m = 2$

$$q_{ii} = \frac{p_{11}p_{22} - p_{12}p_{21}}{p_{jj}} \rightarrow \frac{k_{11}k_{22} - k_{12}k_{21}}{k_{jj}s^\lambda} \quad (5-10)$$

Let $\underline{K} = [k_{ij}]$, so Constraint 2 states that there may be no change in the sign of $\det \underline{K}/k_{jj}$ as $\underline{P}$ ranges over P^* . In this chapter it is assumed that all $k_{ij} > 0$ for all $\underline{P} \in P$, so Constraint 2 gives $k_{11}k_{22} > k_{12}k_{21} \forall \underline{P} \in P$, or vice versa. To remove the ambiguity it is also assumed that the plant terminals are numbered so that for at least one $\underline{P} \in P$, $k_{11}k_{22} > k_{12}k_{21}$, so Constraint 2 yields

$$k_{11}k_{22} > k_{12}k_{21} \quad \forall \underline{P} \in P \quad (5-11)$$

This is a diagonal dominance condition as $s \rightarrow \infty$. Zames and Bensoussan¹⁸ have defined a diagonal dominance condition as $s \rightarrow \infty$, in more abstract form.

The above applies to the first channel, say no. 1, for which Eqs. (5-3) are used. Equations (5-4) are used for the second channel. The minimum phase condition of Constraint 1 therefore applies to $q_{22}(1 + L_1)$, most of which is not new because m.p. $(1 + L_1)$, $\det \underline{P}$ are already required. As for Constraint 2, there are two extremes. "A.s. sensitivity" can be achieved by L_1 bandwidth $\gg$ that of L_2 (denoted by $BW(L_1) \gg BW(L_2)$), and then Eq. (5-4b) implies Constraint 2 applies to q_{22} . It can also be achieved by the opposite strong inequality, and then Constraint 2 applies to p_{22} . For the condition assumed with Eqs. (5-10) and (5-11) with no sign changes in the k_{ij} , the results are the same. It may also be so in the general case but this would require consideration of simultaneous sign changes among the k_{ij} , which is not done here.

For $m = 3$ the application of the constraints to the first channel makes it applicable to q_{11} . Application to the second, Eqs. (5-6), gives the same results as for the $m = 2$ case, because L_{2e} has the same form in both cases

*See Constraint 2 amplification for method 2 on page E-10.

[compare to Eqs. (5-4b) and (5-6b)]. If "a.s. sensitivity" is achieved (as it may be) by $BW(L_1) \gg BW(L_2) \gg BW(L_3)$, the result is that the constraints of Sec. 5-6A apply to q_{11} , q_{22} , q_{33} . If the opposite is done, in the notation of Eq. (5-7), they apply to q_{11} , $\frac{q_{22}}{1 - \gamma_{12}}$, and

$$\frac{q_{33}(1 - \gamma_{12})}{1 - (\gamma_{12} + \gamma_{23} + \gamma_{13}) + (\gamma_{12}\mu_2 + \gamma_{13}\mu_3)}$$

It has not been ascertained whether these two sets of constraints are identical. However Sec. 5-6C shows that the constraints of Sec. 5-6A must always apply to each q_{ii} , $i = 1$ to m . Constraints for $m > 3$ may be similarly developed.

5-6C Inherent Constraints

It is important to determine whether the above constraints are due to the specific design technique or are inherent in the problem itself. For this purpose examine Eq. (5-4a) for t_{22} . How can "a.s. sensitivity" of t_{22} be achieved despite large uncertainty in $\underline{P}$? Clearly by large L_{2e} , the usual feedback method. Large L_{2e} is achieved by large g_2q_{22} because large L_1 (needed likewise for small t_{11} sensitivity) gives $L_{2e} = g_2q_{22} = L_2$. The latter also attenuates d_{22} , which may not be small because of g_1 in its numerator. This same principle applies to all t_{ij} , and is basically the same as that derived from examination of Eq. (5-3), i.e., there is need for arbitrarily large L_2 and L_1 over arbitrarily large bandwidth in order to achieve "a.s. sensitivity". But do the constraints of Sec. 5-6.1 apply to q_{22} and q_{11} ?

This is indeed so, and proven by Eq. (5-3a) and its analog for t_{2j} (by interchanging 1,2), by simply asking whether a stable t_{11} is possible if $1 + L_1$ has r.h.p. zeros? For if not, and since a.l. L_1 over a a.l. bandwidth is needed, it follows that L_1 (and L_2) must satisfy the constraints. Suppose

$(1 + L_1)$ has r.h.p. zeros. These are r.h.p. poles of t_{11} , unless in Eq. (5-3a), for $i = 1$, the numerator of t_{11} has these same zeros. Suppose it has them, and there is a small change in f_{21} . Since $\underline{F}$ is outside the feedback loops, system stability is unaffected. The zeros of $1 + g_1 q_{11}$ in Eq. (5-3a) are thereby unaffected, so neither should the zeros of the numerator of Eq. (5-3a), for $i = 1$. The term $f_{11} g_1 q_{11}$ is unaffected, but t_{21} is affected -- see Eqs. (5-4) with $i = 1$. Hence, the hypothesis $(1 + L_1)$ has r.h.p. zeros is untenable, so the constraints apply to q_{11} and q_{22} .

Comparing these results with Sec. 5-6B, the conclusion is that "a.s. sensitivity" may be achieved by design method two, with plant constraints which are inherent and irreducible, i.e., not more severe than inherently necessary. This is achieved by letting $BW(\underline{L}_1) \gg BW(\underline{L}_2) \gg BW(\underline{L}_3)$, wherein Constraints 1 and 2 are the only constraints applicable to q_{11} , q_{22} , q_{33} , which have been shown to be inherent. This is associated with the following design order: first L_1 [Eqs. (5-5)], second L_2 [Eqs. (5-6)], third L_3 [Eqs. (5-7)]. The design procedure is facilitated by such inequalities, because then in Eq. (5-6b), $L_{2e} \rightarrow L_2 = g_2 q_{22}$ over its important design range, and $L_{3e} \rightarrow L_3 = g_3 q_{33}$. If other factors are equal ($T_{pq_{ii}}$) tolerances on t_{ij} , $j = 1, 2, 3$ for each i , there is a natural tendency for this order of the inequalities because of the inherent greater overdemand of L_1 , lesser of L_2 and least of L_3 (recall Sec. 5-3 and the last paragraph in Sec. 5-4).

However, the above is rather of theoretical, academic interest, because it applies only for "a.s. sensitivity", defined at the beginning of Sec. 5-6. Thus it applies if one is given a plant with specific uncertainty range and

is challenged to achieve a.s. performance tolerances over a.l. bandwidths. However, given a MIMO plant set P and sets of tolerances τ_{ij} ($i, j = m$), it is conceivable that the latter are not achievable by method two, but are inherently achievable. The reason (e.g., for $m = 2$) is that in Eqs. (3-22) through (3-25), the demand on L_1 to achieve the desired sensitivity reduction, may be greater than is inherently needed, because of the overdesign, discussed in Sec. 5-3, and is therefore unachievable, because of the nature of P (e.g.) it has some nonminimum phase elements. Thus, a specific problem may be incompatible with the constraints only due to this overdesign. There may exist as yet undiscovered better methods with reduced demand on L_1 , which renders them compatible with the constraints. From Sec. 5-3, it is clear that the best method is achieved with Eqs. (5-3) by maximum use of the correlation which exists between the P uncertainty and the $t_{2j} \in \tau_{2j}$ in Eqs. (5-3). A suggestion for this purpose has been given in Reference 14. One subdivides the plant set into subsets P_j which are correlated with the subsets τ_{ij_u} of τ_{ij} . Equations (5-3), similarly Eqs. (5-5) and (5-6), are now applied to these pairs P_u, τ_{ij_u} separately for each u . This approach has not as yet been attempted in any numerical problem.

It is worth noting that the constraints of the diagonal dominance type as $s \rightarrow \infty$, appear in the design technique of Reference 14. However, they are always present there, even if "a.s. sensitivity" is not attempted. In design method two they are in effect only for "a.s. sensitivity". Hence, it is possible that a specific synthesis problem with given τ_{ij} , P sets may not be solvable by method one, but is solvable by method two. This is the case in Reference 14.

5-7 Nondiagonal G

The constraints on $\underline{P}$ in Sec. 5-6 are deduced on the assumption that $\underline{G}$ is diagonal. Are these constraints eased if nondiagonal $\underline{G}$ is used? To answer this question, let $\underline{H}$ be a fixed LTI pre-compensator matrix inserted ahead of the plant and let $\underline{V} = \underline{P}\underline{H}$ be the new effective plant in the set $\mathcal{V} = \{\underline{P}\underline{H}, \underline{P} \in \mathcal{P}\}$. The design techniques with diagonal $\underline{G}$, are now applied to set $\mathcal{V}$ instead of set $\mathcal{P}$. If $\underline{H}$ is helpful in overcoming some constraint, then it is necessary that the constraint violated by $\underline{P}$, is not violated by $\underline{V}$. The constraint $\det \underline{P}$ is m.p. $\forall \underline{P} \in \mathcal{P}$ is not eased at all, because $\det \underline{V} = (\det \underline{P})(\det \underline{H})$, and obviously cancellation of r.h.p. zeros of $\det \underline{P}$ by $\det \underline{H}$ cannot be done for many reasons. The other important constraint involving diagonal dominance as $s \rightarrow \infty$, is also not eased, because it applies to the sign of $\det \underline{V}$ not changing, as $s \rightarrow \infty$. Thus the constraints on $\mathcal{P}$ for "a.s. sensitivity" are not eased by nondiagonal $\underline{G}$.

However, $\underline{H}$ may be very helpful in reducing the amount of feedback needed to achieve specified tolerance sets given by Eq. (5-9), for a given plant set $\mathcal{P}$, so that a design unachievable by diagonal $\underline{G}$ (say, because of non-m.p. $\underline{P}$ or sensor noise problems) may be achievable via $\underline{H}$. For example in Eq. (5-4a), L_{2e} must handle the uncertainties due to L_{2e} itself and attenuate the effective disturbance set $\{d_{2j}\}$. For basically noninteracting tolerances on $t_{kj}(k \neq i)$, f_{ij} is made zero, so only the latter need exist. It may be possible to considerably reduce $\|d_{2j}\|_{\max}$ by means of $\underline{H}$, by making $\underline{V} = \underline{P}\underline{H}$ quasidiagonal, even though $\underline{P}$ has large nondiagonal components.

Off diagonal plant elements appear in all the design Eqs. (3-26) through (3-28) and Eqs. (5-4) through (5-7) in the 'disturbance' components, so their reduction via $\underline{H}$ is desirable. How is this systematically done in the case of significant $\underline{P}$ uncertainty? For $m = 2$, let the normalized $\underline{H}$ have 1

for its diagonal elements and $h_{12} = \mu$, $h_{21} = \nu$. Then $v_{12} = \mu p_{11} + p_{12}$ and $v_{21} = p_{21} + \nu p_{22}$. The objective is to minimize over P $\max |v_{12}|, |v_{21}|$ at each ω . Sketches of the sets $\{ \frac{p_{12}}{p_{11}}(j\omega) \}$, $\{ \frac{p_{22}}{p_{21}}(j\omega) \}$ in the complex plane, are clearly very helpful in choosing $\mu(j\omega)$, $\nu(j\omega)$. However, one should check the effect on the resulting sets of $v_{11} = p_{11} + \mu p_{12}$, $v_{22} = \mu p_{21} + p_{22}$, because of the obligations on the loop transmissions due to their uncertainties. The final choice depends on the relative importance of the two terms in the numerators of y_{1j}, y_{2j} in Eqs. (5-3) and (5-4). (See Reference 14, Secs. 3.2, 4 for discussion relevant to this topic).

If the elements of $\underline{P}$ have a r.h.p. pole in common, i.e. $\underline{P} = \underline{P}_1/(s - p)$, then one should not try to diagonalize $\underline{P}$ by means of $\underline{P}\underline{H} = \underline{\Lambda}$ diagonal, because in practice $\underline{H} = \underline{P}_a^{-1} \underline{\Lambda}$ with $\underline{P}_a^{-1} \neq \underline{P}^{-1}$ exactly, giving $\underline{P}\underline{H} = \underline{P} \underline{P}_a^{-1} \underline{\Lambda}$ with r.h.p. dipoles. Instead, one tries to diagonalize $\underline{P}_1$ by means of $\underline{P}_1 \underline{H} = \underline{\Lambda}$, giving $\underline{H} = \underline{P}_1^{-1} \underline{\Lambda}$, and $\underline{P}\underline{H}$ and $\underline{P}_1 \underline{P}_a^{-1} \underline{\Lambda}/(s - p)$.

5-8 Achievability of a.m.p. Effective Plant $\det \underline{P}_e$

For some control system (e.g., flight control) there are often available more control inputs (ℓ) than outputs (m). The inputs and outputs of an aircraft may be selected in such a manner as to yield the $m \times m$ aircraft plant matrix $\underline{P}(s)$. If the constraint $\det \underline{P}(s)$ be m.p. is not satisfied then it may be possible to achieve a m.p. $\det \underline{P}_e(s)$ for an effective $m \times m$ plant $\underline{P}_e(s)$ by augmenting the basic $\ell \times m$ plant $\underline{P}_b$ by a $m \times \ell$ gain matrix $\underline{\Delta} = [\delta_{ij}]$ or a frequency sensitivity matrix $\underline{\Delta} = [\delta_{ij}(s)/h_{ij}(s)]$ as shown in Fig. 5-4. The expression for the output $\underline{y}(s)$ may be obtained by two different approaches as

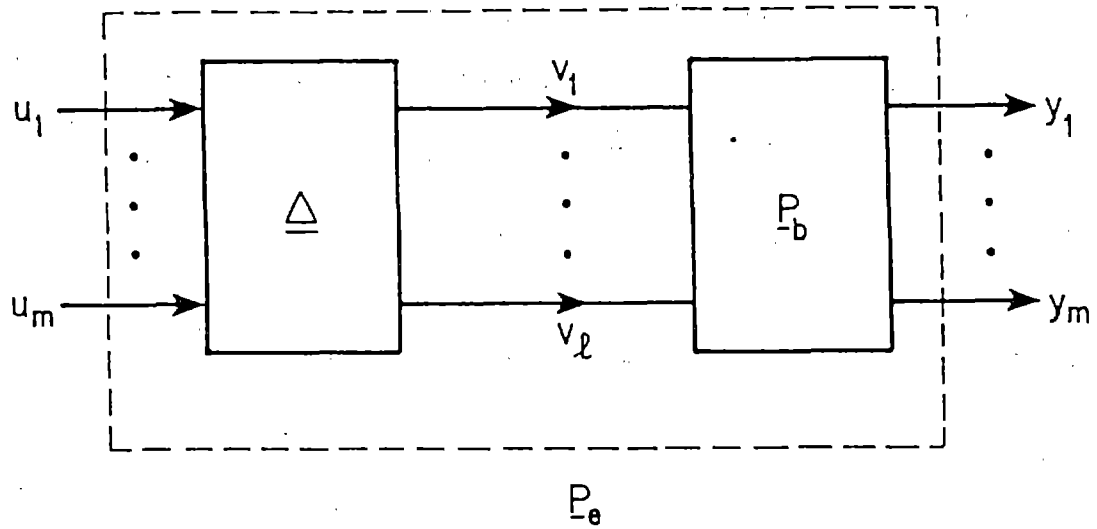


Fig. 5-4 An $m \times m$ Effective Plant $P_e(s)$

Case A--From the differential equations describing the system of Fig. 5-4 obtain

$$\underline{D}(s)\underline{y}(s) = \underline{N}(s)\underline{u}(s) \quad (5-12)$$

where $\underline{D}(s)$ and $\underline{N}(s)$ are $m \times m$ matrices of polynomials and the output $\underline{y}(s)$ and input $\underline{u}(s)$ are $m \times 1$ vectors. Equation (5-12) is manipulated to yield

$$\underline{y}(s) = \underline{D}^{-1}(s)\underline{N}(s)\underline{u}(s) = \underline{P}_e(s)\underline{u}(s) \quad (5-13)$$

where the effective plant matrix is given by

$$\underline{P}_e(s) = \underline{D}^{-1}(s)\underline{N}(s) \quad (5-14)$$

Thus from Eq. (5-14) the following expressions are obtained:

$$\underline{D}(s)\underline{P}_e(s) = \underline{N}(s) \quad (5-15)$$

$$\det \underline{D}(s) \det \underline{P}_e(s) = \det \underline{N}(s) \quad (5-16)$$

$$\det \underline{P}_e(s) = \frac{n_e(s)}{d_e(s)} \quad (5-17)$$

Where $n_e(s) = \det \underline{N}(s)$, $d_e(s) = \det \underline{D}(s)$, and $n_e(s)$ and $d_e(s)$ are polynomials. Thus $n_e(s)$ must not have any r.h.p. zeros in order for Eq. (5-17) to be m.p. and satisfy the constraint that $\det \underline{P}_e(s)$ be m.p.

Case B--From the block diagram of Fig. 5-4, where $\underline{y}(s)$ is a $\ell \times 1$ vector and $\underline{\Delta}$ is only a gain matrix the following expressions are obtained:

$$\underline{y}(s) = \underline{P}_b(s)\underline{v}(s) \quad (5-18)$$

$$\underline{v}(s) = \underline{\Delta} \underline{u}(s) \quad (5-19)$$

$$\underline{y}(s) = \underline{P}_b(s)\underline{\Delta} \underline{u}(s) \quad (5-20)$$

From Eqs. (5-13) and (5-20) it is seen that the effective plant matrix can also be expressed by

$$\underline{P}_e(s) = \underline{P}_b(s)\underline{\Delta} \quad (5-21)$$

where $\underline{P}_b(s) = [p_{ij}(s)]$, $p_{ij}(s) = n_{ij}(s)/d(s)$, $\underline{\Delta} = [\delta_{ij}]$, $n_{ij}(s)$

is a polynomial, $m \leq \ell$, and δ_{ij} is a gain to be determined in order to try to achieve a m.p. det $\underline{P}_e(s)$. Note that although in this section all the $p_{ij}(s)$ are considered to have a common denominator, i.e., $d(s) = d_{ij}(s)$, in general the p_{ij} elements of $\underline{P}_b$ can have different denominators.

By use of the Binet-Cauchy formula⁴⁶ Eq. (5-21) may be expressed as follows:

$$\det \underline{P}_e(s) = \det[\underline{P}_b(s)\underline{\Delta}] = \sum_i p_i(s) \delta_i \quad (5-22)$$

where $p_i(s)$ and δ_i are the determinants of appropriate $m \times m$ submatrices of $\underline{P}_b(s)$ and $\underline{\Delta}(s)$ respectively. A sufficient condition for $\det[\underline{P}_b(s)\underline{\Delta}]$ to be minimum phase is that at least one $p_i(s)$ be minimum phase. A short proof of this fact is as follows: let $p_i(s)$ be the determinant of the $m \times m$ submatrix of $\underline{P}_b(s)$ formed by the columns indexed by $i_1, \dots, i_m$. Assume that this $p_i(s)$ is minimum phase. Choose $\underline{\Delta}$ so that the $m \times m$ submatrix formed by the rows indexed by $i_1, \dots, i_m$, is the identity matrix while all remaining rows are zero. Then, clearly $\det[\underline{P}_b(s)\underline{\Delta}] = p_i(s)$. The number of terms in the summation of Eq. (5-22) is given by

$$\alpha_z = \frac{\ell!}{(\ell - m)!m!} \quad (5-23)$$

Let the submatrices of $\underline{P}_b$ and $\underline{\Delta}$, of Eq. (5-22), be represented, respectively, by $\underline{P}_{b_w}$ and $\underline{\Delta}_{b_w}$ where $w = 1, 2, \dots, \alpha_z$. The $\det \underline{P}_e(s)$ obtained from Eq. (5-22) is

$$\det \underline{P}_e(s) = \det \underline{P}_{b1} \det \underline{\Delta}_{b1} + \det \underline{P}_{b2} \det \underline{\Delta}_{b2} + \dots + \det \underline{P}_{bw} \det \underline{\Delta}_{bw} + \dots + \det \underline{P}_{b\alpha_z} \det \underline{\Delta}_{b\alpha_z} \quad (5-24)$$

where

$$\underline{P}_{bw} = \left[\frac{n_{ij}(s)_w}{d(s)} \right] \quad (5-25)$$

and

$$\underline{\Delta}_{bw} = [\delta_{ijw}] \quad (5-26)$$

Thus each term in Eq. (5-24) can be expressed as

$$\det \underline{P}_{bw} \det \underline{\Delta}_{bw} = \frac{1}{d^m(s)} \det | n_{ij}(s)_w | \det | \delta_{ijw} | \quad (5-27)$$

which permits Eq. (5-24) to be written as follows:

$$\det \underline{P}_e(s) = \frac{1}{d^m(s)} \left\{ \det | n_{ij}(s)_1 | \det | \delta_{ij1} | + \det | n_{ij}(s)_2 | \det | \delta_{ij2} | + \dots + \det | n_{ij}(s)_{\alpha_z} | \det | \delta_{ij\alpha_z} | \right\} = \frac{n'_e(s)}{d^m(s)} \quad (5-28)$$

where $n'_e(s)$ is a polynomial. Since Eq. (5-17) and (5-28) represent the same open-loop system of Fig. 5-4 then

$$\frac{n_e(s)}{d_e(s)} = \frac{n'_e(s)}{d^m(s)} \quad (5-29)$$

which results in $d_e(s) = d(s)$ and

$$n_e(s) = \frac{n'_e(s)}{d^{m-1}(s)} \quad (5-30)$$

Based upon Eqs. (5-28) through (5-30) $d^{m-1}(s)$ must be a factor of $n_{ij}(s)_w$.

The following development illustrates that a sufficient condition for the existence of a m.p. $\det \underline{P}_e(s)$ is that at least one $\det \underline{P}_{bw}(s)$ must be m.p. Assume

$$\det | n_{ij}(s)_1 | \det | \delta_{ij1} | = \det | n_{ij}(s)_1 | k_1 \quad (5-31)$$

in Eq. (5-28) is m.p. where $k_1 = \det | \delta_{ij1} |$ is a scalar. Factor out $d^{m-1}(s)$ from every term in Eq. (5-30) and let

$$k_1 \frac{\det | n_{ij}(s)_1 |}{d^{m-1}(s)} = k_1 N_1(s) \quad (5-32)$$

Also, let the summation of the remaining terms, after factoring out $d^{m-1}(s)$ from each term, be expressed as

$$k_2 N_2(s) = \frac{1}{d^{m-1}(s)} \{ \det | n_{ij}(s)_2 | \det | \delta_{ij2} | + \dots + \det | n_{ij}(s)_{\alpha_z} | \det | \delta_{ij\alpha_z} | \} \quad (5-33)$$

where k_2 is a scalar. Note k_1 , k_2 and $N_2(s)$ are now functions of δ_{ij} which are to be selected in order to try to achieve a m.p. $\underline{P}_e(s)$. Based upon Eqs. (5-32) and (5-33), Eq. (5-28) can be expressed as follows:

$$\det \underline{P}_e(s) = \frac{1}{d(s)} [k_1 N_1(s) + k_2 N_2(s)] \quad (5-34)$$

In order for $\det \underline{P}_e(s)$ to be m.p. then the zeros of

$$k_1 N_1(s) + k_2 N_2(s) = 0 \quad (5-35)$$

must all be in the l.h.p. Equation (5-35) is manipulated to the mathematical format of

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \begin{bmatrix} N_1(s) \\ N_2(s) \end{bmatrix} = -1 \quad (5-36)$$

which permits a root-locus analysis of Eq. (5-34). Since the zeros of Eq.

(5-36) are in the l.h.p. then the weighting factors δ_{ij} are selected in

hopes that all roots of Eq. (5-35) lie in the l.h.p. for all $\underline{P}_b \in \mathcal{P}$. This

assumes that throughout the region of plant parameter uncertainty, the

initially chosen m.p. submatrix in Eq. (5-28) is m.p. for all $\underline{P}_b \in \mathcal{P}$ and be

expressed by Eq. (5-32). To enhance the achievability of m.p. $\det \underline{P}_e(s)$ the

following guidelines may be used:

1. Determine the number α_y of submatrices $\underline{P}_{b_w}$ of Eq. (5-28) that are m.p.
2. Select one of the m.p. submatrices to be identified as Eq. (5-32).
3. The values of δ_{ij} of $\underline{\Delta}_{b_w}$ associated with
 - (a) The remaining $\alpha_y - 1$ m.p. $\underline{P}_{b_w}$ be altered in such a manner as to increase the values of their corresponding $\det \underline{\Delta}_{b_w}$.
 - (b) The $\alpha_z - \alpha_y$ non m.p. $\underline{P}_{b_w}$ be altered in such a manner as to decrease the value of their corresponding $\det \underline{\Delta}_{b_w}$.

By changing the values of the gains δ_{ij} in the manner described will result in the $\alpha_y - 1$ m.p. terms of Eq. (5-33) to dominate in the resulting expression for $N_2(s)$. This dominance of the $\alpha_y - 1$ terms in $N_2(s)$ may result in $N_2(s)$ being m.p. Also, it may enhance the achieveability of a m.p. $\underline{P}_e(s)$ by the analysis of Eq. (5-36).

Once the $n_{ij}(s)$ and δ_{ij} are specified, in order to simplify the root-locus computational effort, it is necessary to factor out $d(s)$ from the numerator polynomials of Eqs. (5-32) and (5-33). Depending upon the CAD package that is utilized in determining $d(s)$ and the numerator polynomials of these equations a perfect factoring may not exist.

Now consider the most general case, i.e.,

$$\det \underline{P}_{b_w} = \det \left| \frac{n_{ij}(s)_w}{d_{ij}(s)_w} \right| = \frac{n_w(s)}{d_w(s)} \quad (5-37)$$

and

$$\det \underline{\Delta}_{b_w} = \det \left| \frac{\delta_{ij}(s)_w}{h_{ij}(s)_w} \right| = \frac{\delta_w(s)}{h_w(s)} \quad (5-39)$$

where the $d_{ij}(s)$ and $h_{ij}(s)$ are not all the same and where $n_w(s)$,

$d_w(s)$, $\delta_w(s)$ and $h_w(s)$ are scalar polynomials. Hence

$$\det \underline{P}_e(s) = \sum_{\alpha_z} \frac{n_w(s)\delta_w(s)}{d_w(s)h_w(s)} \quad (5-40)$$

For each value of w , $n_w(s)/d_w(s)$ has a range of uncertainty which may be correlated with that of the remaining $\alpha_z - 1$ submatrices of $\underline{P}(s)$. Thus for the general case some or all of the $n_w(s)/d_w(s)$ may have r.h.p. zeros and/or poles. The problems now becomes one of trying to choose fixed $\delta_w(s)$ and $h_w(s)$ polynomials so that the $\det \underline{P}_e(s)$ has no r.h.p. zeros over the entire range, or failing that to have them as relatively "far-off" as possible. This problem has been studied by several researchers and the resulting techniques may be used for this purpose^{33,55}.

Thus the Binet-Cauchy formula permits the determination if an m.p. effective plant $\det \underline{P}_e(s)$ is achievable over the region of plant uncertainty.

5-9 Conclusions

This chapter has presented new synthesis techniques for highly uncertain mxm MIMO lti feedback systems with output feedback, with the following features:

- (a) There is detailed control over the m^2 individual system transfer functions.
- (b) The MIMO uncertainty problem is rigorously converted into a number of MISO uncertainty problems. Solutions of the latter are guaranteed to be satisfactory for the former. Relatively simple MISO single loop feedback techniques can be used to solve the MISO problems.

- (c) For "arbitrary small sensitivity" over arbitrary large bandwidth, the technique in Secs. 5-2 through 5-4, 3 give constraints on the plant which are inherent and irreducible, i.e. every lti compensation technique has these constraints.
- (d) Part of the constraints (at infinite s) in (c) were always present in the previously developed MISO equivalent techniques¹⁴, i.e. even if "a.s. sensitivity" was not required. They are present in the new techniques only for a.s. sensitivity. Also, fixed point theory is not required for justification of the new technique.
- (e) The overdesign inherent in the fixed point techniques¹⁴, has been reduced but some overdesign is still present.
- (f) These techniques are applicable to the design of a reconfigurable aircraft with surface failures⁴⁴.

The reader is urged, in performing a QFT design, to constantly refer to the Design Checklist on pages E-3 through E-10. Also, this checklist and Appendix C discuss how the QFT design technique can be applied to plants have r.h.p. zeros and/or poles.

Appendix A

MISO Design Problem

Given: $P(s) = \frac{k}{s + a}$ with plant uncertainty

$$1 \leq k \leq 100 \quad \text{and} \quad -3 \leq a \leq 3$$

Specifications on $|T_R(j\omega)|$

ω	0	0.5	1	2	5	10	20
$Lmb(j\omega)$	0	1	1	0	-4	-10	-20
$Lma(j\omega)$	0	-2	-4	-8	-20	$(-\infty)$	$(-\infty)$

$$M_L \leq 3\text{dB}$$

$$\lambda \geq 3$$

No specifications on T_D

Nominal Plant: $P_o(s) = 1/s$ ($k = 1, a = 0$)

Problem: Design $L_o(s)$ and $F(s)$ to meet the specifications.

Solution:

A. Template Generation -- The set of plant transfer functions $P_i(a_i, k_i)$, where $P_i \in P$, used to obtain the templates are: $P_1(3,1)$, $P_2(0,1)$, $P_3(-3,1)$, $P_4(-3,100)$, $P_5(0,100)$, and $P_6(3,100)$. The templates all have the shape shown in Fig. A-1 where $V = 40\text{dB}$ and as ω increases the templates become narrower. It is left to the reader to obtain the templates in order to verify the solution to this problem.

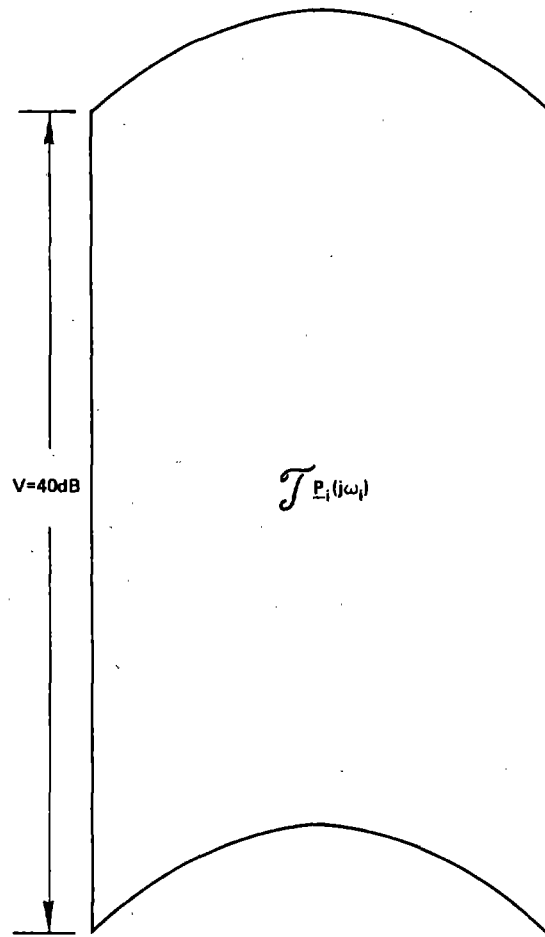


Fig. A-1 Template $\underline{P}_i(j\omega_i)$

B. Determination of Boundaries $\underline{B}_R(j\omega_i)$ --- The templates are used to determine the boundaries $\underline{B}_R(j\omega_i)$ and, along with the values of LmM_L and, V to determine the U-contour as shown in Fig. A-2.

C. Determination of $L_O(s)$ --- The design of an optimum $L_O(s)$ is not unique. Two possible designs of an $L_O(s)$ are shown in Fig. A-2. The designs at the onset should be based upon a qualitative analysis of

$$L(s) = C(s)P(s) = kG(s)/(s + a) \quad (A-1)$$

and

$$L_O(s) = G(s)P_O(s) = kG(s)/s \quad (A-2)$$

Trial 1 -- A type 0 $L_O(s)$ function requires the compensator to be of the format

$$G(s) = \frac{Ks(\quad)\dots\dots}{(\quad)(\quad)\dots\dots} \quad (A-3)$$

which from Eqs. (A-1) and (A-2) results in, respectively,

$$L(s) = \frac{kKs(\quad)\dots\dots}{(s + a)(\quad)(\quad)\dots\dots} \quad (A-4)$$

and

$$L_{O1}(s) = \frac{kK(\quad)\dots\dots}{(\quad)(\quad)\dots\dots} \quad (A-5)$$

When $a \neq 0$, Eq. (A-4) results in

$$T_{R1}(s) = \frac{kKs(\quad)\dots\dots}{(\quad)(\quad)\dots\dots} F(s) \quad (A-6)$$

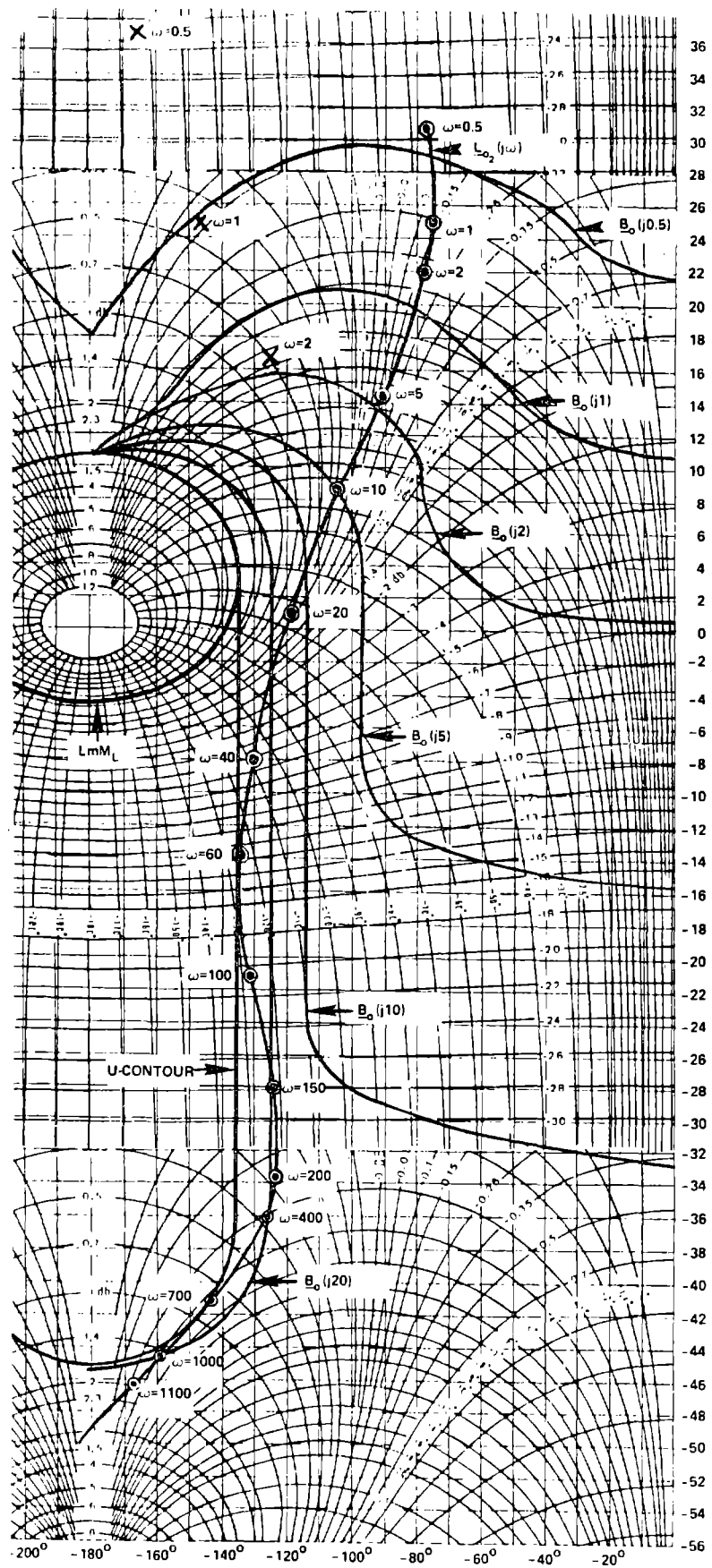


Fig. A-2 Plots of $B_R(jw_1)$ and $L_O(jw)$

and $\lim_{\omega \rightarrow \infty} T_{R1}(j\omega) = -\infty \text{ dB}$ which is unacceptable.

Trial 2 -- A Type 1 $L_o(s)$ requires the compensator to be of the format

$$G(s) = \frac{K(\quad)\dots\dots}{(\quad)(\quad)\dots\dots} \quad (\text{A-7})$$

which from Eqs. (A-1) and (A-2) results in, respectively,

$$I_1(s) = \frac{kK(\quad)\dots\dots}{(s+a)(\quad)(\quad)\dots\dots} \quad (\text{A-8})$$

and

$$L_{o2}(s) = \frac{kK(\quad)\dots\dots\dots}{s(\quad)(\quad)\dots\dots\dots} \quad (\text{A-9})$$

When $a = 0$ $T_R(s)$ corresponds to a Type 1 system and when $a \neq 0$ $T_R(s)$ corresponds to a Type 0 system. Assuming this is acceptable an $L_o(s)$ is synthesized, as shown in Fig. A-2, and results in

$$G(s) = \frac{8.504 \times 10^6 (s+1)(s+12)(s+120)}{(s+a)(s+10)(s+26.5)(s+720 \pm j960)} \quad (\text{A-10})$$

In order to achieve the desired specification $e(\infty) = r(\infty) - c(\infty) = 0$, for $r(t) = u_{-1}(t)$, then the compensator $G(s)$ must be a Type 1 transfer function in order to yield a Type 1 system for $a \neq 0$ and a Type 2 system for $a = 0$. The synthesized loop transmission function is

$$L_{o3} = \frac{kK(\quad)\dots\dots}{s^2(\quad)(\quad)\dots\dots} \quad (\text{A-11})$$

for which three points (denoted by X) are shown in Fig. A-2. It is left to the reader to complete the synthesis of an $L_o(s)$.

D. Design of F(s) --- Based upon the design procedure in Chap. 2, the following nonunique filter is obtained:

$$F(s) = \frac{1.5}{s + 1.5} \quad (A-12)$$

E. Simulation of Design --- A simulation of the system for $L_o(s)$ is made for each $P_i(a_i, k_i)$, where $i = 1, \dots, 6$, based upon Eqs. (A-10) and (A-12). The simulations indicate that all the specifications on $|\underline{T}_R(j\omega)|$ are met except, as expected, when $a \neq 0$ and $\omega = 0$. The reader can simulate the system utilizing his $L_o(s)$.

3

Appendix B

MIMO Design Problem

Given: A 2x2 plant where

$$Q(s) = \frac{1}{s} \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \quad (B-1)$$

has the following independent plant uncertainties:

$$\begin{aligned} 1 \leq k_{11} \leq 2 & \qquad \qquad \qquad 0.5 \leq k_{12} \leq 2 \\ 5 \leq k_{21} \leq 10 & \qquad \qquad \qquad 0.5 \leq k_{22} \leq 1 \end{aligned}$$

The diagonal dominance condition (Sec. 3-7) requires that as $\omega \rightarrow \infty$,

$$|q_{11}q_{22}| < |q_{12}q_{21}| \Rightarrow 2 < 2.5$$

Specifications:

- (a) There is only one command input: $r_2(t)$
- (b) $|t_{12}| < -20\text{dB}$ for all ω
- (c) For $|t_{22}|$:

Table B-1

ω	0	0.5	1	2	5	10	20
$\text{Lm} \underline{b}_{22}(j\omega)$	0	1	1	0	-4	-10	-20
$\text{Lm} \underline{a}_{22}(j\omega)$	0	-2	-4	-8	-20	$-\infty$ (-40)	$-\infty$ (-80)

The values of $\underline{a}_{22}(j\omega)$ are considered to be zero for $\omega > 5$, and thus

$$\text{Lm}[\underline{a}_{22}(j\omega)] \approx -\infty \quad \omega > 5$$

(d) For $\underline{L}_1 = g_1 q_{11}$ and $\underline{L}_2 = g_2 q_{22}$, respectively:

$$\text{LmM}_{\underline{L}_1} \leq \text{Lm} \left[\frac{\underline{L}_1}{1 + \underline{L}_1} \right] \leq 3\text{dB} \quad (\text{B-2})$$

$$\text{LmM}_{\underline{L}_2} \leq \text{Lm} \left[\frac{\underline{L}_2}{1 + \underline{L}_2} \right] \leq 3\text{dB} \quad (\text{B-3})$$

Problem: Design $L_{1_o}(s)$, $L_{2_o}(s)$ and $f_{22}(s)$

A. $L_{1_o}(s)$ Design --- For loop one design, see Fig. B-1, let $f_{12} = 0$ (i.e., decouple output 1 from input 2). With $f_{12} = 0$ then the loop 1 design becomes strictly a disturbance rejection problem (see Fig 3-8).

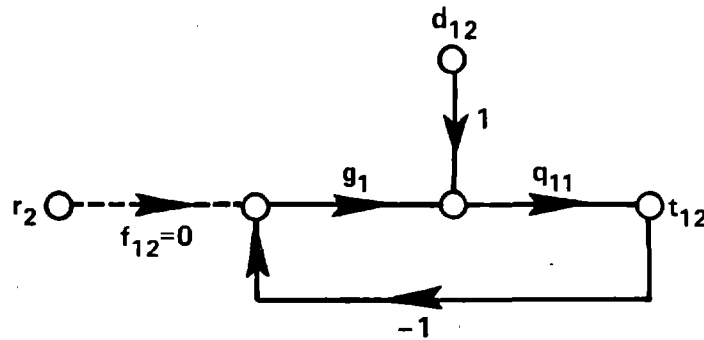


Fig. B-1 Loop One

Thus from Eqs. (3-22) through (3-25), where for an impulse input $t_{1j} = y_{1j}$,

$$t_{12} = \frac{q_{11}(g_1 \overset{0}{\cancel{f_{12}}} + d_{12})}{1 + g_1 q_{11}} = \frac{q_{11} d_{12}}{1 + g_1 q_{11}} \quad (\text{B-4})$$

Based upon the given disturbance rejection specification

$$\left| \underline{t}_{12} \right| = \left| \frac{q_{11} d_{12}}{1 + g_1 q_{11}} \right| \leq 0.1 \quad (B-5)$$

which is rearranged to

$$\left| 1 + g_1 q_{11} \right| \geq \left| \frac{q_{11} d_{12}}{0.1} \right| \quad (B-6)$$

Since

$$d_{12} = -t_{22}/q_{12} \quad (B-7)$$

Equation (B-7) is substituted into Eq. (B-6) to yield

$$\left| 1 + g_1 q_{11} \right| \geq \left| \frac{10 q_{11} t_{22}}{q_{12}} \right| \quad (B-8)$$

Applying the maximum tracking specification value for $\left| \underline{t}_{22} \right|$ yields

$$\left| 1 + g_1 q_{11} \right| \geq \left| \frac{10 b_{22} q_{11}}{q_{12}} \right| \quad (B-9)$$

Thus, this constraint imbeds the disturbance rejections bounds specifications on $y_{12}(t)$. As a consequence the disturbance bounds $\underline{B}_D(j\omega_1)$ become the composite bounds $\underline{B}_O(j\omega_1)$. For the plant uncertainty a template can be generated for q_{11} which is good (for this particular design problem) for all ω . The template is simply a 6dB vertical line. Also, from the specification $Lm_{L1} \leq 3dB$ and the uncertainty in q_{11} at $\omega = \infty$, the universal high frequency boundary can be drawn as shown in Fig. B-2. The nominal plant q_{11_0} is chosen to be the bottom of the 6dB template, thus

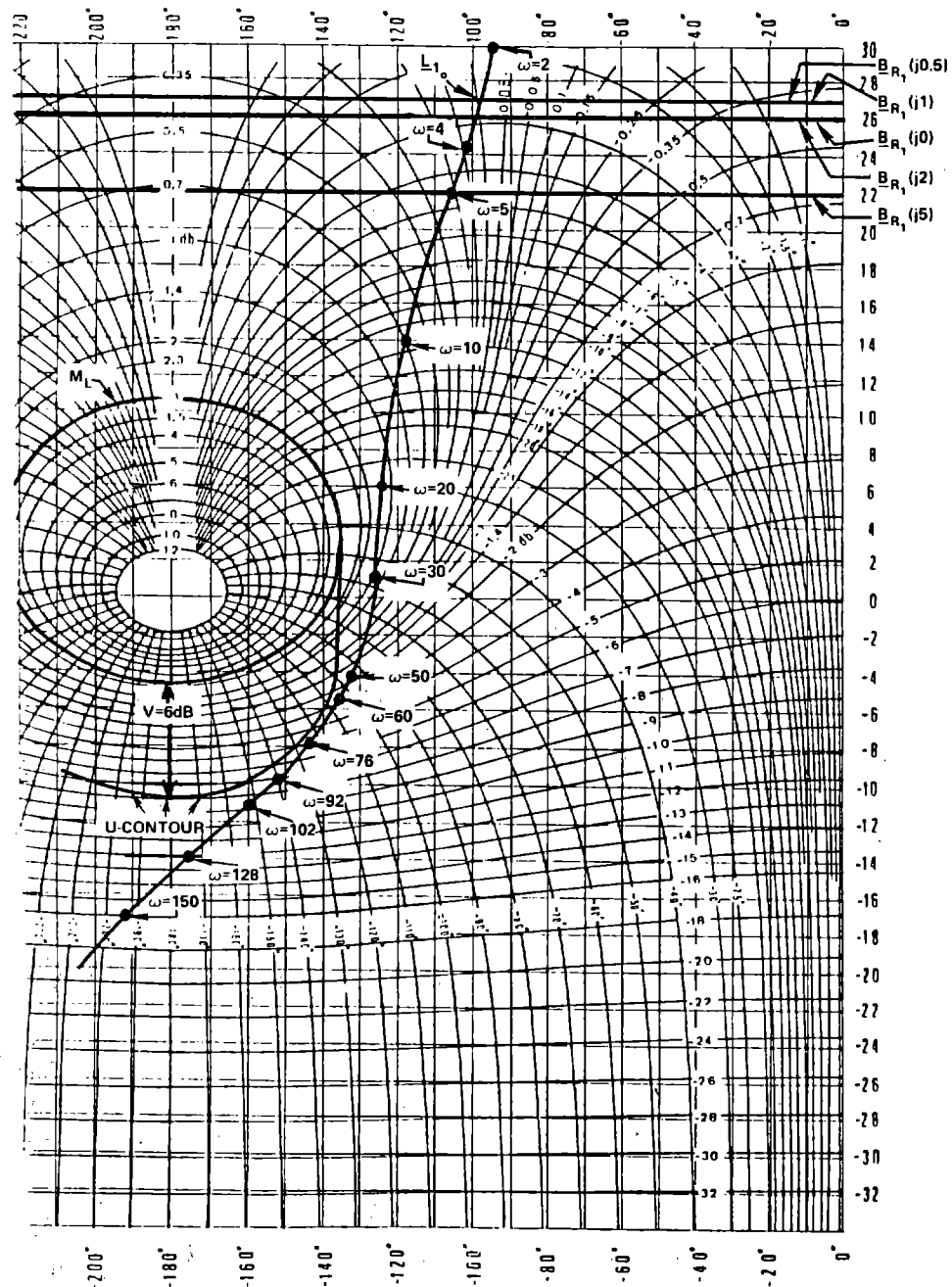


Fig. B-2. $L_1(s)$ design

$$q_{11}(s) = \frac{1}{s} \quad (B-10)$$

For lower ω there generally $|g_1 q_{11}| > 1$ Eq. (B-9) can be simplified to

$$|g_1| \geq |10b_{22}/q_{12}| \quad (B-11)$$

Multiplying both sides of Eq. (B-11) by q_{11_o} yields

$$|L_{1_o}| = |g_1 q_{11_o}| \geq \left| \frac{10b_{22} q_{11_o}}{q_{12}} \right| \quad (B-12)$$

where $L_{1_o} = g_1 q_{11_o}$. The worst case scenario must be chosen from the specifications in order to apply Eq. (B-12). Therefore, for $q_{12_{min}} = 0.5/s$ and $q_{11_o} = 1/s$, Eq. (B-12) becomes

$$|L_{1_o}| \geq |20b_{22}| \quad (B-13)$$

or

$$LmL_{1_o} \geq 26 + Lm b_{22} = B_{o_1}(j\omega_i) \quad (B-14)$$

Thus Eq. (B-14) reveals that for this example the composite bounds $B_{o_1}(j\omega)$ are straight lines whose magnitudes are a function of frequency. Based upon Eq. (B-14) and the given specifications on t_{22} in Table B-1 the bounds are:

Table B-2

ω_i	0	0.5	1	2	5	10	20
$LmB_{o_1}(j\omega_i)$	26	27	27	26	22	N/A	N/A

Since the value of $\underline{b}_{22}(j\omega)$ is very small for $\omega > 5$, the tracking frequency range is effectively $0 \leq \omega \leq 5$. The value $\underline{Lma}_{22}(j\omega) = -\infty$ yields $\delta_R(j\omega) = \underline{Lmb}_{22}(j\omega) - \underline{Lma}_{22}(j\omega) \rightarrow \infty \text{ dB}$ for $\omega > 5$ and the tracking problem will always be satisfied in this frequency range. It is therefore necessary to shape the loop transmission $\underline{L}_{10}(j\omega)$ only for $0 \leq \omega \leq 5$. Only the disturbance rejection problem remains for $\omega > 5$.

Since the p_{ij} elements of $\underline{P}$, and in turn all the elements of $\underline{Q}$, and the nominal q_{110} are type 1 functions then $\underline{L}_{10}(s)$ should be chosen as a type 1 or higher function. A Type 1 function is chosen for $\underline{L}_{10}(s)$. An initial simple $\underline{L}_{10}(s)$ design containing only one pole at the origin, one complex-pole pair and zeros resulted in $\omega_\phi = 65 \text{ rps}$. In order to reduce the value of the phase margin frequency the following $\underline{L}_{10}(s)$ is synthesized:

$$\underline{L}_{10}(s) = \frac{540838(s + 25)}{s(s + 10)(s + 84 \pm j112)} \quad (\text{B-15})$$

where $\omega_n = 140 \text{ rps}$ and is shown in Fig. B-2. Note that the approximation made in Eq. (B-12) is justified because all the values in Table B-2 are at least a magnitude above one.

Since $\underline{L}_{10} = g_1 q_{110}$ then

$$g_1(s) = \frac{\underline{L}_{10}(s)}{q_{110}(s)} = \frac{540838(s + 25)}{(s + 10)(s + 84 \pm j112)} \quad (\text{B-16})$$

B. $\underline{L}_{2eq}$ Design (Improved Method) --- The equations for loop 2, see Fig. B-3 and Eqs. (3-22) through (3-25), are

$$t_{22} = \frac{q_{22}(g_2 f_{22} + d_{22})}{1 + g_2 q_{22}} \quad (\text{B-17})$$

where $d_{22} = -t_{12}/q_{21}$ and

$$t_{12} = \frac{q_{11}d_{12}}{1 + g_1q_{11}} \quad (\text{B-18})$$

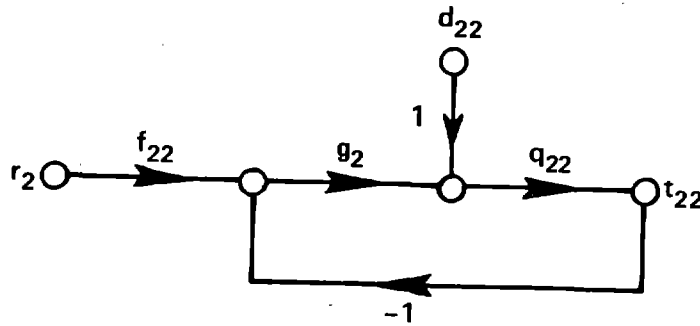


Fig. B-3 Loop 2

Instead of making the necessary substitutions and performing the associated algebra the following generalized loop 2 equations, see Eqs. (5-4a-e), are utilized in the design.

$$\gamma_{12} = \frac{q_{11}q_{22}}{q_{21}q_{12}} \quad (\text{B-19})$$

$$q_{22e} = \frac{q_{22}(1 + L_1)}{1 + L_1 - \gamma_{12}} \quad (\text{B-20})$$

$$L_{2eq} = g_2q_{22e} \quad (\text{B-21})$$

and

$$d_{22e} = \frac{g_1 f_{12} p_{21} (1 - \gamma_{12})}{1 - \gamma_{12} + L_1} = 0 \quad (B-22)$$

since $f_{12} = 0$. Thus loop 2 is only a tracking problem (see Fig. 3-7).

Before generating the templates for q_{22e} it is necessary to ascertain whether all the poles and zeros of q_{22e} lie in the left-half s-plane over the region of uncertainty. The proper design of $L_1(s)$ guarantees that the zeros of $1 + L_1(s)$ are in the l.h.p. Since a m.p. plant is assumed, then $q_{22e}(s)$ will also be m.p. However, it is possible for $1 + L_1(s) - \gamma_{12}(s)$ of Eq. (B-20) to have r.h.p. zeros which become r.h.p. poles of $q_{22e}(s)$ and therefore of $L_{2eq}(s)$. This possibility must be verified before proceeding with the design. The knowledge of the presence of r.h.p. poles is necessary in order to correctly interpret the data for $|q_{22e}(j\omega_1)|$ and $\angle q_{22e}(j\omega_1)$ over the frequency range of interest (see Sec. 2-10).

Plant uncertainty case 10 of Table B-3 represents an unstable plant. At low frequencies the templates are rectangles 6dB in height and essentially 360° wide. That is, the template for $\omega = 0.5$ is essentially 6dB in height and 360° wide, and for $\omega = 20\text{rps}$ it is 6dB in height, 345° wide at the top and 352° wide at the bottom. The templates for $\omega = 130\text{rps}$ and $\omega = 150\text{rps}$ are approximately 180° and 138° wide respectively. It is thus apparent, in order to have a negative phase angle at all frequencies for $L_{2eq}(j\omega)$, templates must be generated for high enough frequencies such that they shrink in width sufficiently in order to allow a gap on the right side of the U-contour for $L_{2eq}(j\omega)$ to "squeeze" through and achieve the desired value of $\gamma = 45^\circ$ (for this example). This analysis of the plant template data reveals: (1) that the width (change in angle) of the templates is the crucial factor in the shaping of L_{2eq} of the template, and

Table B-3

Combinations using plant maximums and minimums					
Case	k_{11}	k_{12}	k_{21}	k_{22}	Comments
1.	1	.5	5	.5	←unstable plant
2.	1	.5	5	1	
3.	1	.5	10	.5	
4.	1	.5	10	1	

5.	1	2	5	.5	
6.	1	2	5	1	
7.	1	2	10	.5	
8.	1	2	10	1	

9.	2	.5	5	.5	
10.	2	.5	5	1	
11.	2	.5	10	.5	
12.	2	.5	10	1	

13.	2	2	5	.5	
14.	2	2	5	1	
15.	2	2	10	.5	
16.	2	2	10	1	

<u>Random Points</u>					
17.	1.5	1	7	1	←nominal plant
18.	2	1.5	9	.75	
19.	1.75	.75	6	.75	
20.	1.25	1.25	8	.5	

(2) that the template shapes go from being "very wide" (essentially rectangular), as shown in Fig. B-4, at "low" frequencies to the shape shown by the shaded area in Fig. B-4 at "high" frequencies. As shown in Fig. B-4, that in going from $\omega = 0.5$ to $\omega = 300$ rps the templates start to "droop" down to the right. This droop becomes more exaggerated if more template points are plotted from other unstable plants within $\mathcal{Q}$. These unstable plants occur for this example, when the condition

$$\gamma_{12} = \frac{k_{11}k_{22}}{k_{12}k_{21}} > 0.5 \quad (\text{B-23})$$

is satisfied. This can be seen by partitioning the characteristic equation of Eq. (B-20) to yield $L_1/(1 - \gamma_{12}) = -1$ for which a root-locus analysis can be made as the value of γ_{12} is varied. The "very wide" characteristics of some of these templates can best be appreciated by the reader by plotting a positive and negative real pole (a and $-a$) in the s -plane. The reader is then urged to analyze the angular contribution of $\left| 1 + j\left(\frac{\omega}{a}\right) \right|$ and $\left| 1 + j\left(\frac{\omega}{-a}\right) \right|$ as the frequency is varied from zero to infinity.

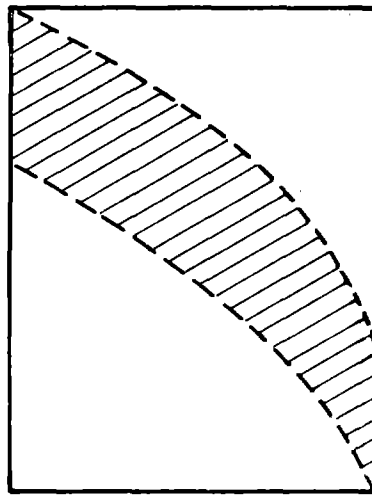


Fig. B-4 Template shapes

In order to simplify the process of generating the bounds rectangular (solid curve) templates are used as shown in Fig. B-4. Although making this assumption yields an overdesign.

The data for the templates (see Fig. B-5) that play the key role in achieving the desired value of γ is given in Table B-4. Note that at $\omega = 500$ the template is 20dB in height which the value of $V(\text{UHFB})$ used to obtain the U-contour.

The templates are used to obtain the $\frac{B_R(j\omega_i)}{e} = \frac{B_O(j\omega_i)}{e}$ bounds of Fig. B-6 for the loop 2 design. Note that for $\omega < 150\text{rps}$ the bounds are considered to be essentially straight lines. That is, these bounds are represented by the straight line tangent to the top of the LmM_L contour since all the templates for $\omega < 150\text{rps}$ are greater than 135° in width. A stable plant is chosen to be the nominal plant, see Table B-3, and is located at the lower right-hand corner of the templates. Thus the plant gain values, k_{ij} , for the nominal plant are: $k_{11} = 2$, $k_{12} = 2$, $k_{21} = 10$, and $k_{22} = 0.5$. For these nominal values, where $\gamma_{12} = 0.05$, let

$$\alpha = 1 - \gamma_{12} = 0.95 \quad (\text{B-24})$$

thus from Eqs. (B-15), (B-19) through (B-22), and (B-24) obtain

$$\begin{aligned} q_{22\text{eq}} &= \frac{k_{22}[s^4 + 178s^3 + 21,280s^2 + (196,000 + 540,838k_{11})s + 13,520,950k_{11}]}{s[\alpha s^4 + 178\alpha s^3 + 21,280\alpha s^2 + (196,000\alpha + 540,838k_{11})s + 13,520.950k_{11}]} \\ &= \frac{0.5263158(s^4 + 178s^3 + 21,280s^2 + 1,277,676s + 27,041,900)}{s^5 + 178s^4 + 21,280s^3 + 667,303.1579s^2 + 14,232,578.955 + 0} \\ &= \frac{0.5263158(s + 49.2349 \pm j6.0882)(s + 39.7651 \pm j96.9859)}{s(s + 17.3915 \pm j24.9674)(s + 71.6085 \pm j101.2169)} \end{aligned} \quad (\text{B-25})$$

Table B-4

ω , rps	Δdb (height)	$\Delta \theta^\circ$ (width)
150	14.5	138
170	15.8	81
180	15	61
200	16.6	37
300	18	10
500	20	~ 0

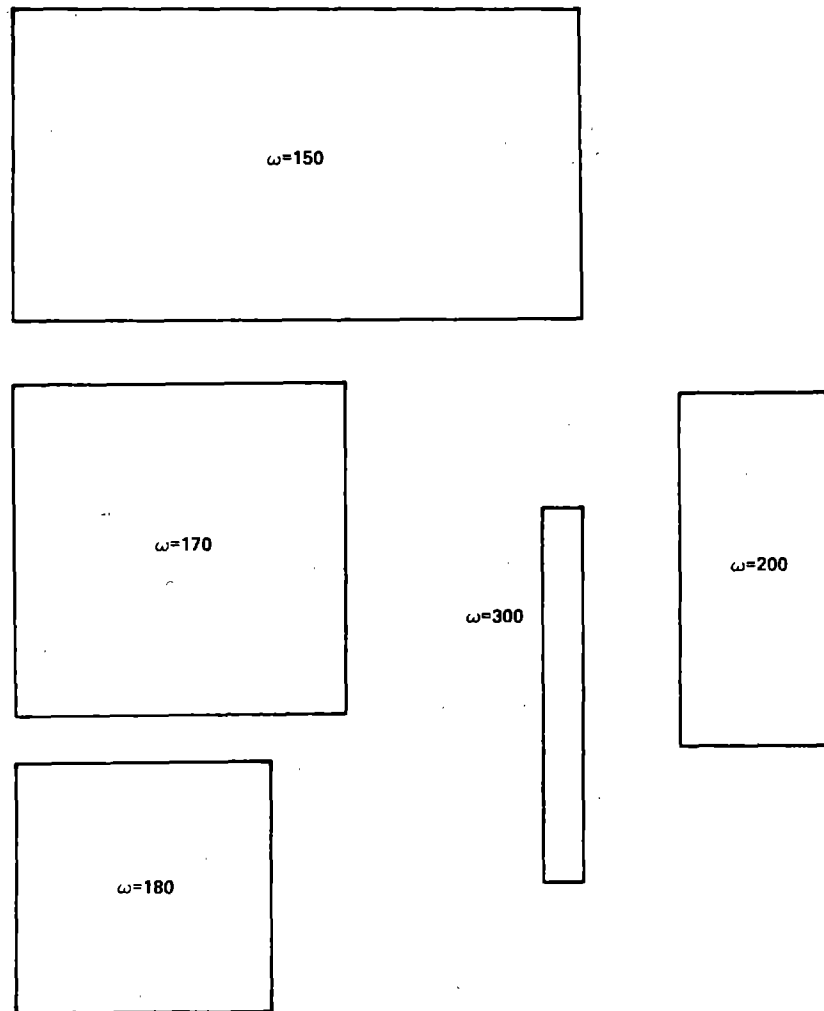


Fig. B-5 Tracking Templates for Loop 2
(using a rectangular approximation)

In order to make $g_2(s)$ "fairly simple" the nominal plant is used in Eq. (B-21) which results in a nonunique g_2 being designed rather than L_{2eq} where

$$L_{2eq}(s) = g_2(s)q_{22e} \quad (B-26)$$

A synthesized (nonunique) $L_{2eq}(s)$ is shown in Fig. B-6. The hump in L_{2eq} at $\omega \approx 150\text{rps}$ is caused by the complex zeros ($\omega_n = 104.8\text{rps}$) of the nominal plant. These zeros have a low damping ratio ($\zeta \approx 0.38$) which causes the "rapid" change in the phase angle. Usually a complicated $g_2(s)$ results when adjusting $L_{2eq}(j\omega)$ so that it crosses the $\underline{B}_e(j150)$ bound near the right side of the N.C. with $L_{2eq}(j\omega)$ being on the corresponding $\underline{B}_e(j\omega_i)$ bound. However, in order to obtain a simpler $g_2(s)$, the approach in this example is to design $g_2(s)$ so that $L_{2eq}(s)$ crosses the first few bounds at a phase angle of approximately -90° . As it turns out, for this example, $g_2(s)$ does not contribute very much to the phase angle at the $\underline{B}_e(j\omega_i)$ crossing for $\omega \leq 150\text{rps}$; i.e., the phase angle is due essentially to the nominal plant. The resultant $g_2(s)$ is

$$g_2(s) = \frac{3.3155 \times 10^{10} (s + 1000)}{(s + 500)(s + 3500 \pm j 6062.1778)} \quad (B-27)$$

where $\omega_n = 7000$ and $\zeta = 0.5$. A second design for $g_2(s)$ can be made by choosing smaller values for the real pole and zero of Eq. (B-27) in order to have $L_{2eq}(s)$ track down the right side of the U-contour more closely. In general a more complicated $g_2(s)$ may be synthesized that approaches the optimal loop transmission function

$$L_{2eq}(j\omega_i)_{opt} = \underline{0} = \underline{0} / \underline{0} \quad (B-28)$$

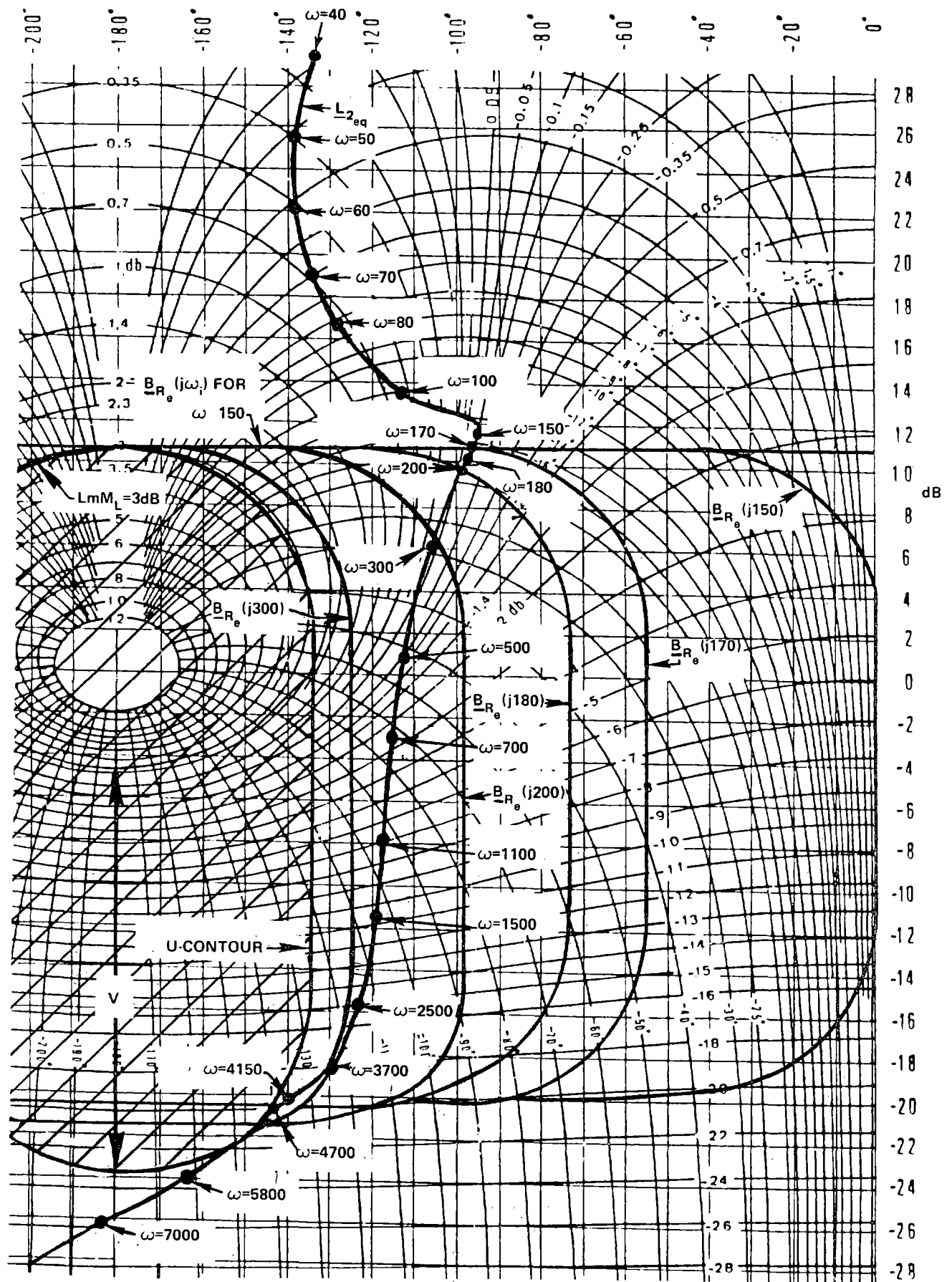


Fig. B-6 $B_{R_e}(j\omega_i)$ and $L_2(j\omega)$ plots.

whose magnitude lies on $\underline{B}_o(j\omega_1)$, for each value of ω_1 , and whose phase angle $\angle \underline{B}_o(j\omega_1)$ lies on the right side of the U-contour.

C. Prefilter $f_{22}(s)$ Design --- The prefilter design procedure of Sec. 2-8

requires the use of the templates to obtain the values of Lmt_{-2max} and Lmt_{-2min} where $\underline{t}_2 = \underline{L}_2/(1 + \underline{L}_2)$ over the frequency range $0 < \omega \leq 20$ rps. In this frequency range, for this example, the templates are between 345° and 360° wide. As a consequence, the location on the N.C. where these templates are placed result in small positive values for $Lmt_{-2} (\leq 0.1\text{dB})$.

Thus small variations exist for Lmt_{-2} , i.e., $\Delta t_2 = Lmt_{-2max} - Lmt_{-2min}$ will be a very small number in the range $0 < \omega \leq 20$ rps. For example, the bottom of the template, for $\omega = 20$, would be placed on the $Lm\underline{L}_2$ curve at the

$Lm\underline{L}_2(j20) = 38.2\text{dB}$ point to yield $Lmt_{-2max} \approx 0.1\text{dB}$ and a value for Lmt_{-2min} something less than 0.1dB . Thus for $\omega < 20$ Δt_2 is a very small number and

$|\underline{t}_2| \approx 1$ for $P = \{\underline{P}\}$. Therefore, the corresponding figures of Fig. 2-18, for this example, are obtained by plotting the data from Table B-1, i.e.,

$LmT_{R_U} - LmT_{-max} = Lmb_{-22}$ and $LmT_{R_L} - LmT_{-min} = Lma_{-22}$. The filter chosen is

$$f_{22}(s) = \frac{1}{s + 1} \quad (B-28)$$

Note: care must be taken in determining the location of the pole of Eq. (B-29) since it may be necessary to lower the value of $|f_{-22}|$ at large values of $\omega (\geq 10)$, where $a_{22} = 0$, since this affects the value of $|\underline{t}_{12}|$, etc. [(see Eqs. (B-7) through (B-9)].

D. Simulation --- The computer data for y_{22} for a number of plants from P resulted in $|\underline{t}_2| \approx 1$ for $0 \leq \omega \leq 20$. Table B-5 presents the time response characteristics of $y_{22}(t)$, for a unit step forcing function, for three

cases from Table B-3. These cases are chosen on the following basis:

Table B-5

Case	M_p	$t_{s,s}$	$t_{s,s}$	Final Value	Figure
10 (unstable) plant	1.000	3.91574	2.19709	1.000	B-8
15 (nominal)	1.000	3.91276	2.19726	1.000	B-7
16 (high gain)	1.000	3.91276	2.19722	1.000	B-9

- (1) Nominal plant --- The loop transmission is obtained based upon the nominal plant.
- (2) Unstable plant --- A "worst" case situation representative of the unstable plant region of P .
- (3) High gain plant --- A high gain plant is chosen because it is representative of the top of the templates on the "nominal plant side."

TIME RESPONSE - T22 - UNIT STEP - NOMINAL PLANT

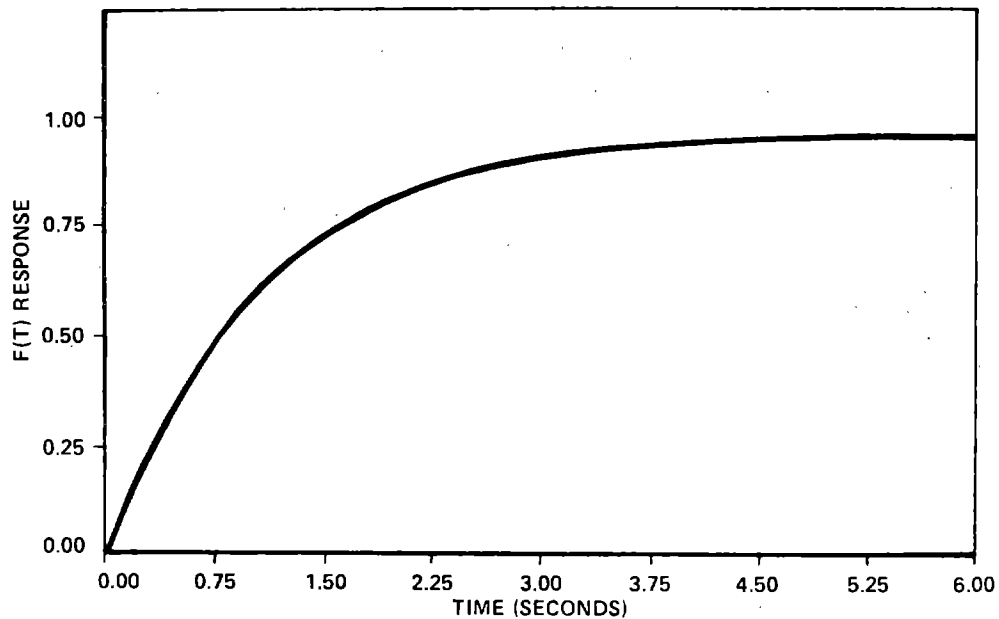


Fig. B-7 Time response $y_{22}(t)$ for $r_2(t) = u_{-1}(t)$:
nominal plant case.

TIME RESPONSE - T22 - UNIT STEP - UNSTABLE PLANT

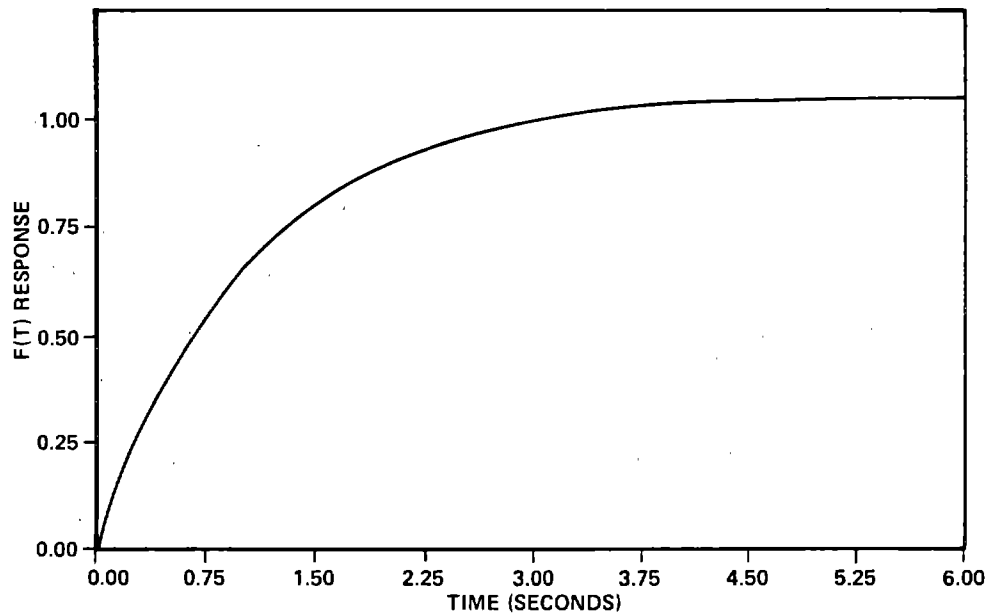


Fig. B-8 Time response $y_{22}(t)$ for $r_2(t) = u_{-1}(t)$:
unstable plant case.

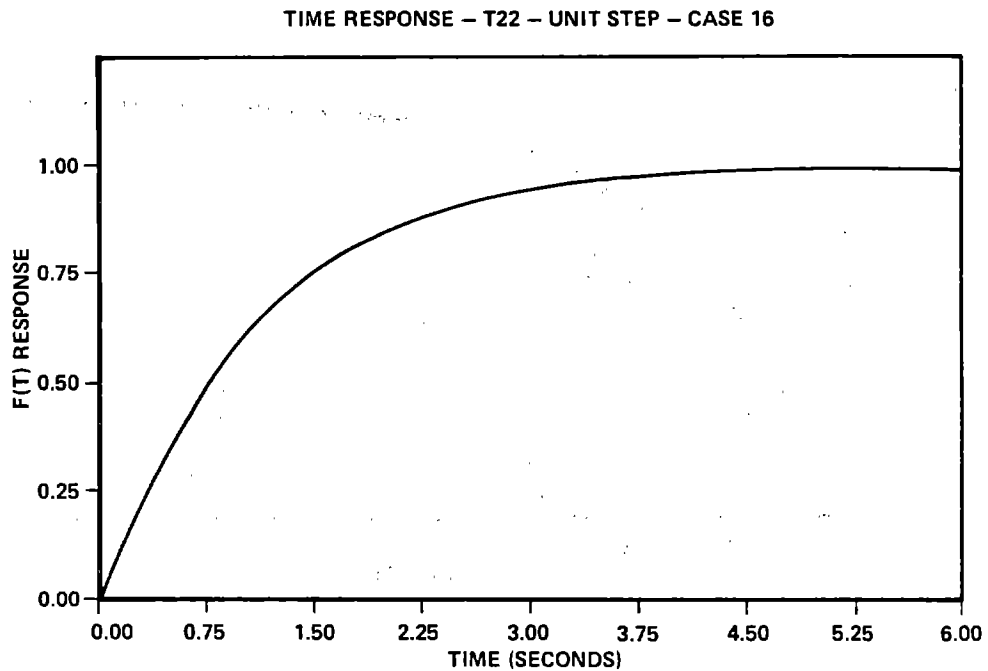


Fig. B-9 Time response $y_{22}(t)$ for $r_2(t) = u_{-1}(t)$:
high gain case.

Since all the other cases of Table B-3 are extremely similar to one of the three cases of Table B-5 they are not simulated.

The frequency domain specifications, as demonstrated by the simulation, are met for loop 2. Although time response specifications are not prescribed $y_{22}(t)$ exhibits respectable rise and settling times. Because of the gross over design any other plant from the plant parameter space P should meet the specifications.

In designing loop one the disturbance rejections bounds are dropped to an arbitrary low value for the shaping of $\underline{L}_1$. The synthesized $\underline{L}_1$ for $\omega = 20$ lies on $\underline{B}_{o1}(j20) = 6\text{dB}$ but for $\omega = 10$ $\text{Lm}\underline{L}_1(j10) \approx 14.5\text{dB} < \underline{B}_{o1}(j10) = 16\text{dB}$. Thus a check on the disturbance rejection $\underline{t}_{12}(j\omega)$ performance for loop one

is made. From Eqs. (B-4) and (B-7) obtain

$$t_{12} = \frac{-t_{22}}{q_{12}} \left[\frac{q_{11}}{1 + L_1} \right] = - \frac{st_{22}}{k_{12}} \left[\frac{q_{11}}{1 + L_1} \right] \quad (B-30)$$

where $q_{12} = k_{12}/s$, $d_{22} = 0$, and

$$t_{22} = \frac{q_{22}(g_{22}f_{22} - d_{22})}{1 + L_{2eq}} = \frac{f_{22}L_{2eq}}{1 + L_{2eq}} \quad (B-31)$$

Substituting Eq. (B-31) into Eq. (B-30) yields

$$t_{12} = - \left[\frac{f_{22}L_{2eq}}{1 + L_{2eq}} \right] \left[\frac{s}{k_{12}} \right] \left[\frac{q_{11}}{1 + L_1} \right] \quad (B-32)$$

Only the plant parameters associated with the unstable q_{22e} plant (case 10) are used in Eq. (B-31) for a simulation. No other plants are simulated to determine if the disturbance rejection specification are met. The frequency response plot for Eq. (B-32) using the parameters for the unstable plant is shown in Fig. B-10. As seen from this figure the -20dB specification is met. The simulation yields $t_{12}(t_p) = -0.0579$ at $t_p \cong 0.026s$, $|t_{12}(t_s)| = 0.002$ (2% of the specified maximum magnitude of 0.1) at $t_s = 2.61s$, and $t_{12}(\infty) = 0$.

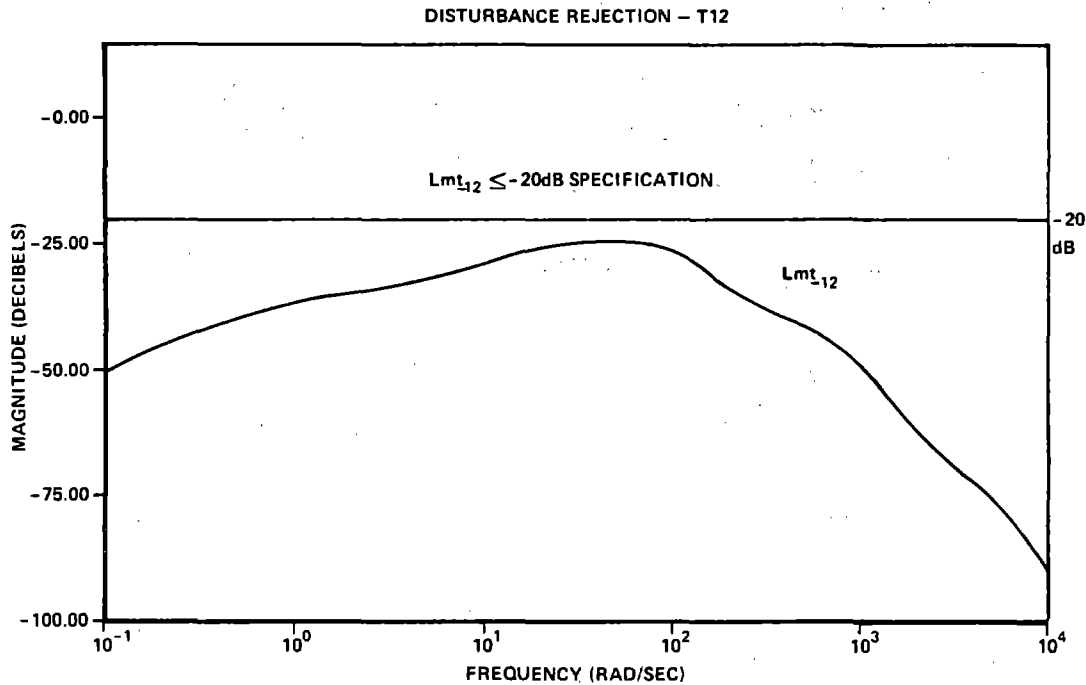


Fig. B-10 Frequency response $\underline{t}_{12}(j\omega)$: unstable case

E. Summary --- In some problems $\gamma_{12} \ll 1$ and therefore $q_{22e} \cong q_{22}$ [see Eq. (B-20)] which simplifies the design. For the example of this appendix $\gamma_{12\max} = 0.8$. Thus for $0.5 < \gamma_{12} \leq 0.8$ there will exist an unstable q_{22e} , i.e., q_{22e} will have r.h.p. poles. An unstable plant with a pole $p_1 > 0$ requires a loop transmission with a crossover frequency $\omega_\phi \cong > 2p_1^{33}$. Therefore L_2 has to have a large crossover frequency. In order for L_{2e} to have a lower value for its ω_ϕ it is necessary to design L_1 so that $1 + L_1 - \gamma_{12}$ does not have r.h.p. zeros. This makes L_1 costlier, the benefit going to L_{2eq} (or g_2). This freedom can be used as a trade-off feature to be used according to the relative sensor noises, etc.

Quantitative Feedback Design for Sampled-Data Systems

I.M. Horowitz* and Y.K. Liao*

Abstract

In quantitative feedback theory (QFT), the plant uncertainty is defined by a set $\mathcal{P} = \{P\}$ of possible plants. The problem is to guarantee that the system response is in a specified acceptable set $\mathcal{A}$, for all P in $\mathcal{P}$. QFT has been developed for large classes of plants imbedded in continuous feedback structures. This paper extends QFT to sampled-data structures. A central problem is to find the minimum sampling frequency $(\omega_s)_{\min}$ needed. The greater the plant uncertainty and the narrower the performance tolerances, the larger $(\omega_s)_{\min}$ must be. The detailed design procedure parallels very closely that for continuous systems, by using the complex variable w , which maps the unit circle in the z domain to the imaginary axis in the w domain.

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C-1. Introduction

A central problem in feedback control design is to achieve desired performance despite plant uncertainty. The problem is made quantitative by defining the set (due to uncertainty) $\mathcal{P} = \{P\}$ of possible plants, and a set $\mathcal{A} = \{A\}$ of acceptable closed-loop transfer functions. Such quantitative feedback techniques (QFT) have been developed for a number of problem classes: single input-single output (siso), single and multiple-loop (Horowitz and Sidi 1972, Horowitz and Wang 1979), multiple input-multiple output (Horowitz 1979a, Betzold et al. 1984); also for nonlinear and time-varying plants in which $\mathcal{P}$ is rigorously converted into a linear time invariant (lti) set $\mathcal{P}_{eq}$ which is equivalent to $\mathcal{P}$ with respect to $\mathcal{A}$, permitting lti design (Horowitz 1976, 1981).

However, in all of these, both the plant and the compensation were assumed continuous. As digital compensators are becoming very popular (Houpis and Lamont 1985), it is important to extend the above techniques to sampled-data systems. The first step is to do so for the single-loop system of Fig. 1, as basically the same design philosophy is used in the others. The design procedure presented here parallels considerably that of the continuous one. The treatment is self-contained, but there is omitted elaboration of those topics available in (Horowitz and Sidi 1972), which are basically the same for both systems. Three complex variables are freely used, as convenient,

$$\left. \begin{aligned} s &= \sigma + j\omega, & z &= e^{sT}, & w &= u + jv = \frac{z-1}{z+1} \\ v &= \tan \frac{\pi\omega}{\omega_s} \text{ on } \sigma = 0, & \omega_s &= 2\pi/T \end{aligned} \right\} \quad (1a-d)$$

T is the sampling period, ω_s the sampling frequency. Also, as is customary in the literature, the notation $F(z)$ is used for the pulse transfer function $F^*(s) = Z F(s)$, $F(w)$ for $F(z)$ under (1c).

Previous work. If ω_s is apriori fixed, then QFT can proceed in the v domain (1c,d) precisely as in the continuous system, described in (Horowitz 1963, Sec. 11.15). It will succeed only if ω_s is large enough for the specific uncertainty problem. Here, ω_s is not apriori known and the derivation of the minimum ω_s needed, is one of the principal design problems.

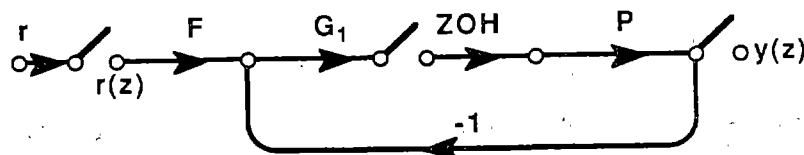


Fig. 1 The siso two-degree-of-freedom sampled data feedback system with lti continuous plant P . (In the chapters the word MISO is used in place of siso.)

c-2. Design Procedure

In Fig. 1, a zero-order-hold (ZOH) is assumed in front of the plant, so

$$\left. \begin{aligned} T(z) &\triangleq \frac{y(z)}{r(z)} = F(z) \frac{L(z)}{1+L(z)}, \\ L(z) &= G_1(z) \frac{(z-1)}{z} P_e(z) \triangleq G(z) P_e(z), \quad P_e(z) = Z\left(\frac{P(s)}{s}\right). \end{aligned} \right\} \quad (2a-c)$$

If a first-order-hold is used, $P_e(z)$ would be $Z(P(s)/s^2)$. $T(\cdot)$ will always be used henceforth for the closed-loop system transfer function, to distinguish it from the sampling period T .

Only the plant uncertainty is assumed significant, although one could exaggerate it to allow for some G uncertainty. Hence, due to plant uncertainty

$$\left. \begin{aligned} \Delta \log T(z) &= \Delta \log \frac{L(z)}{1+L(z)} \\ \Delta \log L(z) &= \Delta \log P_e(z). \end{aligned} \right\} \quad (3a, b)$$

At this point it is convenient to introduce a numerical example:

Example:

$$\left. \begin{aligned} P(s) &= \frac{k}{s(s+a)}, \quad \text{with uncertainties} \\ k &\in [1, 10], \quad a \in [1, 4]. \end{aligned} \right\} \quad (4)$$

c-2.1 Closed-loop system specifications

It has been shown that specifications on $\Delta \log |T(j\omega)|$ suffice to control the time-domain response in stable minimum-phase (mp) systems and those in which the right half-plane (rhp) zeros are known (Horowitz 1976). It will be seen that the latter case applies here, even though the continuous plant $P(s)$ is assumed mp (see Secs. 2.4, 4.1). It is easier to translate time-domain specifications into equivalent ones in the frequency (ω) domain for continuous systems, shown in Fig. 2, with the requirement that

$$a(\omega) \leq |T(j\omega)|_{\text{db}} \leq b(\omega) \quad \text{for all } P \text{ in } \mathcal{P}. \quad (5)$$

It is well known that for $\omega \ll \omega_s$ (a factor of .10 is adequate), any (continuous) $H(j\omega) \approx TH^*(j\omega)$ (sampled). Hence, over a "low enough" part of the spectrum, (5) modified by T , may be used. For QFT purposes it certainly suffices if this can

be done up to where $a(\omega) \sim -16$ db (10 rps in Fig. 2). The reason is that for $\omega > 10$, $a(\omega)$ decreases rapidly, so the allowable $|T(j\omega)|$ uncertainty (i.e. $b(\omega) - a(\omega)$ in db), soon exceeds the plant uncertainty.

The only compelling constraint for larger ω is then of a reasonable stability margin totally independent of $a(\omega)$, $b(\omega)$ (see Sec. 2.4). Is the range $[0, 10]$ rps "low enough"? Now, in general in feedback systems, the price paid for the benefits of feedback is in the bandwidth of the loop transmission L of (2) (Bode 1945), which tends to force ω_s to be significantly larger than what would be required by the sampling theorem. Hence, it is very reasonable to simply use for the sampled model, the same $a(\omega)$, $b(\omega)$ of the continuous model

$$a(\omega) \leq T |T^*(j\omega)| \leq b(\omega) \quad \text{for all } P_e \text{ in } \mathcal{P}_e. \quad (6a)$$

While disturbance response is not considered at this point, there is added, just as in the continuous system and for the same reason, a damping constraint on the loop

$$\left| \frac{L^*}{1 + L^*} \right| \leq \gamma, \quad \text{a constant for all } \omega \text{ and all } P_e \text{ in } \mathcal{P}_e; \quad (6b)$$

$\gamma = 6$ db is used in this example. Also, if conditional stability is not allowed, there is added

$$\text{Arg } L^* \geq -180 + \theta, \quad \theta > 0 \quad \text{for } \omega < \omega_c \quad (6c)$$

where ω_c is the "crossover frequency"; $L^*(j\omega_c) \triangleq 1$, $|L^*| < 1$ for $\omega > \omega_c$.

It is assumed that $a(\omega)$, $b(\omega)$, γ , θ have been apriori given or chosen as part of the problem statement. It is seen from (3) that $\Delta \log T(z)$ can be made arbitrarily small by choosing $L(z)$ sufficiently large. In quantitative design (QFT), the designer finds the minimum $L(z)$ or more conveniently $L(jv)$, to satisfy (6).

Note that $V = \lim_{v \rightarrow \infty} [\text{Lmq}_{11}(jv)]$ for the UHFB.

C-2.2 Plant templates

The first step is to find the complex number sets $\mathcal{T}p_e(\omega) = \{P_e^*(j\omega)\}$ at a "sufficient" number of discrete ω values. In sampled-data, these "plant templates" are functions of the sampling frequency $\omega_s = 2\pi/T$, but this becomes apparent only for $\omega > .25\omega_s$ approximately (see Sec. 2.6). At lower ω , the continuous plant template $\mathcal{T}p(\omega) = \{P(j\omega)\}$ is closely the same as $\mathcal{T}p_e(\omega) = \{P_e^*(j\omega)\}$. Since ω_s has yet to be chosen, it is therefore reasonable to tentatively use the continuous $\mathcal{T}p(\omega)$ templates. A number of plant templates (for $v = \tan \pi\omega/\omega_s$) are shown in Fig. 3 for the plant (4) calculated at $\omega_s = 60$. These templates are very similar to the continuous ones and those at $\omega_s = 120, 240$.

¹In Chap. 2 the symbol M_L is used instead of γ .

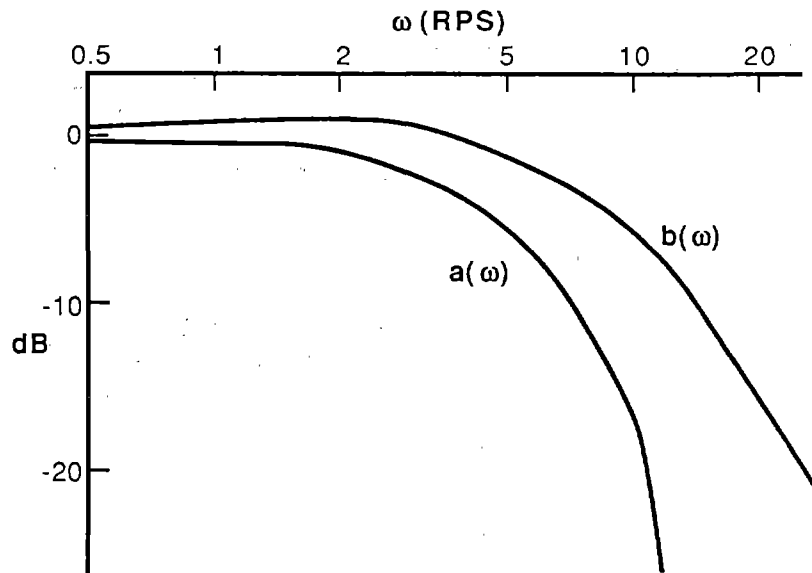


Fig. 2 Bounds on $|T(j\omega)|$ used for T^* up to -20 db point.

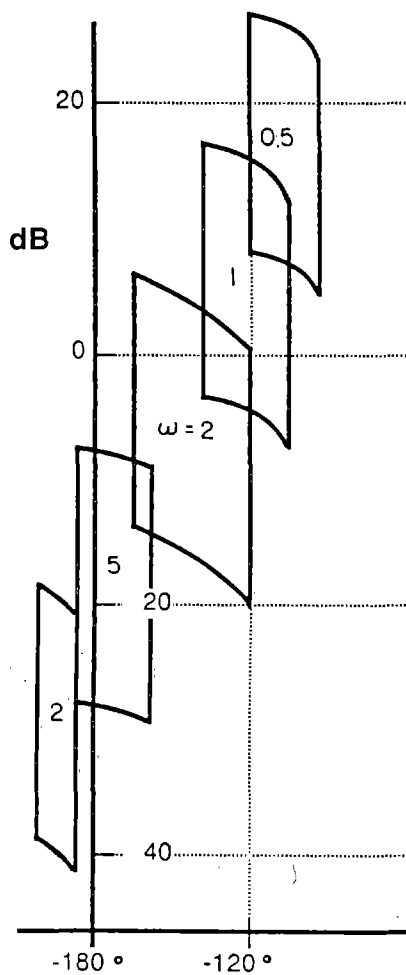


Fig. 3 Plant templates at $\omega = .5, 1, 2, 5, 10$ calculated for $\omega_s = 60$ rps, but almost identical for $\omega_s = 120, 240$ and continuous plant.

C-2.3 Bounds $B(v)$ on $L_0(jv)$

The actual design of F, G_1 in Fig. 1, is best done in the jv domain, $w = u + jv$ of (1), for $v \in [0, \infty)$, which corresponds to the upper half unit circle in the z domain, and to $[0, j\omega_s/2]$ of the $j\omega$ axis. The reason is that eventually one must shape L to satisfy certain bounds, and such shaping is difficult to do on the unit circle in the z domain. It is much easier to do so on the positive $[0, \infty)v$ imaginary axis of w , for then precisely the same techniques as $j\omega$ shaping may be used (Horowitz 1963, Sec. 11.11). But the appropriate v values of the templates are available from (1), only after ω_s has been chosen. Hence, a tentative $\omega_s = 120$ was chosen. The choice was made deliberately low, in order to emphatically display the relation between $(\omega_s)_{\min}$, the plant uncertainty and the performance tolerances. It will be seen that after one understands the QFT technique, a good guess can be apriori made of $(\omega_s)_{\min}$.

Choose a nominal plant $P_{e0}(w)$, ($a = k = 1$). The corresponding nominal $L_0(w)$ is from (1,2b), $L_0(w) = G(w)P_{e0}(w) = \frac{2w}{1+w}G_1(w)P_{e0}(w)$,

$$P_{e0}(w) = \frac{.0131(1 - w^2)(1 + .00436w)}{2w^2(1 + 76.4w)} \quad (7)$$

Then (3a,6a) impose the requirement that

$$\left| \frac{L_0 \frac{P_e}{P_{e0}}(jv)}{1 + \frac{L_0 P_e}{P_{e0}(jv)}} \right|_{\text{db}} \leq b(v)_{\text{db}} - a(v)_{\text{db}} \quad (8)$$

for all P_e in $\mathcal{P}_e$, with the usual notation $a(v) \neq a(\omega)$, but v, ω related by (1). The constraint imposed on $L_0(jv)$ by (8) is conveniently displayed on the logarithmic complex plane (Nichols chart), as a bound $B(v)$ on $L_0(jv)$. The Nichols chart is convenient because it contains loci of constant $|L/(1 + L)|$ — see (Horowitz and Sidi 1972) for more details. For example, in Fig. 4, $L_0(j.013)$ must be on or above $B(.013)$, in order to satisfy (8) for all P_e in $\mathcal{P}_e$.

C-2.4 Nonminimum-phase $L_0(w)$

It is important to note that in the w domain any practical $L(w)$ is nonminimum-phase (non-mp) with a zero at $w = 1$ (Horowitz 1963). The reason is that any practical $L(z)$ has an excess of at least one pole over zeros which leads via (1c) to the zero at $w = 1$. The modified design technique described in (Horowitz and Sidi 1978) is therefore used: Let the nominal $L_0(w) = L_{m0}(w)A(w)$, $A(w) = (1 - w)/(1 + w)$ all-pass ($|A(jv)| = 1$ for all v in $[0, \infty)$), and $L_{m0}(w)$ minimum-phase (mp). Since $|L_0(jv)| = |L_{m0}(jv)|$ and

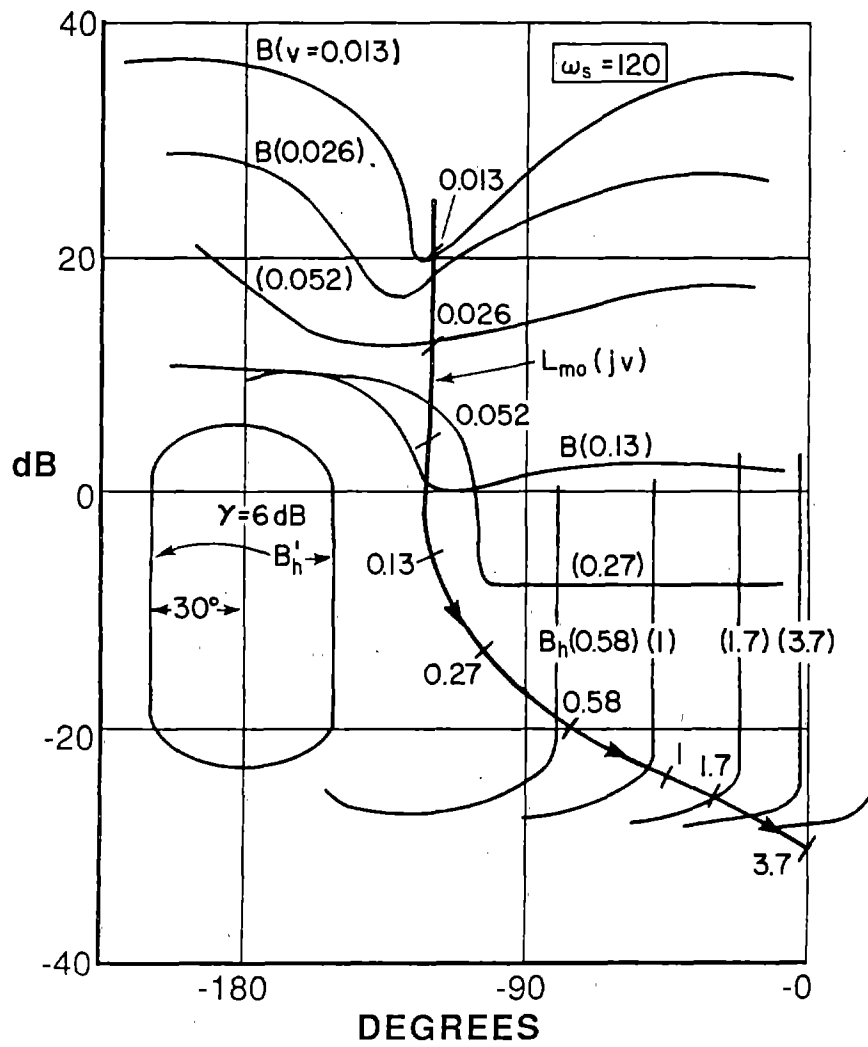


Fig. 4 Unsuccessful design at $\omega_s = 120$.

ω	.5	1	5	10	50
v	.013	.026	.13	.27	3.7

$\arg L_0(jv) = \arg L_{m0}(jv) + \arg A(jv)$, a bound $B'(v)$ on $L_0(jv)$ becomes $B(v)$ on $L_{m0}(jv)$, by shifting $B'(v)$ positively (to the right on the Nichols chart) by $-\arg(jv) = 2 \arctan v > 0$. The amount of shift is 90° at $v = 1$, 152° at $v = 4$, and $\rightarrow 180^\circ$ as $v \rightarrow \infty$.

The inherent limitations on the feedback capabilities of a non-mp system are thereby conveniently revealed, as follows. At larger v (or ω), τp_e tends to be a vertical line and (6b) tends to dominate, leading in mp systems to a single universal high-frequency bound B'_h basically effective for all $\omega > \text{some } \omega_h$ (Horowitz and Sidi 1972) (B'_h is shown in Fig. 4 with width $2(30)$ degrees for $\gamma = 6$ db — see any Nichols chart). However, in the non-mp system, B'_h is shifted in the Nichols chart by $2 \arctan v$, to become $B_h(v)$ a function of v whose right boundary crosses the vertical line $\arg L = 0$ at v_x , where $2 \tan v_x = 180^\circ - 30^\circ = 150^\circ$, so $v_x = 3.73$. The Nyquist stability criterion dictates that the $L_{m0}(jv)$ locus is on the right side of the $B_h(v)$ bounds. It has been shown in (Horowitz and Sidi 1978), that $L_{m0}(jv)$ must reach the right hand bottom of $B_h(v_x)$, i.e. approximately point X in Fig. 4, at a value of $v \leq v_x$, in order that there exist a practical L_{m0} which satisfies the bounds $B(v)$. This condition is very useful for determining the minimum ω_s needed for any specific uncertainty problem, as follows.

C-2.5 $\omega_s = 120$ is too small

The bounds on $L_{m0}(jv)$ are shown in Fig. 4. Does there exist a practical $L_{m0}(jv)$ which can satisfy these bounds? i.e. lies on or outside the $B(v)$ at small v , but then decreases vs v fast enough on the right side of the $B_h(v)$, in order to reach the bottom at $v \leq v_x$? The difficulty is that $d \log |L_{m0}(jv)| / d \log v$ is a function of $\arg L_{m0}(jv)$: on the average 20μ db per decade decrease for $\arg L_{m0}(jv) = -90\mu$ degrees. Thus, the more phase lag available for L_{m0} , the faster $|L_{m0}(jv)|$ may decrease vs v . However, the $B_h(v)$ steadily move to the right vs v , because of the non-mp $A(\omega)$ of Sec. 2.4, until at v_x there is no phase lag at all left for decrease of $L_{m0}(jv)$. Hence, it is necessary that $L_{m0}(jv)$ reach the bottom of this moving barrier, before v_x .

The following numerical procedure is convenient. Work backwards: Assign $\sim 0^\circ$ to $\arg L_{m0}(jv_x)$ and the corresponding right side phase values of $B_h(v)$ to $L_{m0}(jv)$ for $v < v_x$: -29° at $v = 1.7$, -45° at $v = 1.0$, -76.5° at $.58$, see Fig. 4. For $v < .58$ in Fig. 4, "reasonable" $\arg v$ values are tried: -103.5° at $v = .27$, and -125° at $v = .13, .052, .026, .013$. A little experience with loop shaping is helpful here. (Since the loop transmission $L(j\omega)$ or $L(jv)$ provides the benefits of feedback, loop shaping is as important to the feedback designer, as the lathe to the machinist.) As is well known, $\arg L_{m0}(jv)$ determines $|L_{m0}(jv)|$ within an arbitrary multiplier, and an algorithm due to (Bode 1945, Ch. 15; used in Gera and Horowitz 1980), is especially efficient in calculating numerically $|L_{m0}(jv)|$

from given numerical $\arg(jv)$ data.

The resulting $|L_{m0}(jv)|$ so obtained, is shifted vertically on the Nichols chart in Fig. 4, such that $L_{m0}(jv_x)$ is at point X , and checked whether it satisfies the other $B(v)$. It was impossible to do so in this example, as seen in Fig. 4. If L_{m0} satisfies the bounds at higher v , it violates them at most of the lower v (.026, .052, .13, .27). If $|L_{m0}(jv)|$ is shifted upward so that it satisfies all the bounds, then it is above the critical point X at $v_x = 3.7$. Then, for $v > v_x$, $\arg L_{m0} > 0$ in order to satisfy the $B(v)$ for $v > v_x$, so $d|L_{m0}|/dv > 0$ and $|L_{m0}|$ must monotonically increase, which is impractical. Hence, $\omega_s = 120$ is too small a sampling frequency for this uncertainty problem; $\omega_s = 200$ was similarly unsatisfactory.

If the original plant is mp, it is guaranteed that a satisfactory ω_s exists, because the sampled system $\rightarrow$ the continuous system as $\omega_s \rightarrow \infty$. The challenge is to use minimum ω_s value; $\omega_s = 240$ was satisfactory — see L_{m01} in Fig. 5. The mechanism whereby larger ω_s does this is seen in Figs. 4, 5. The B_h high-frequency (v) bounds are independent of ω_s . They are the same in Figs. 4, 5 (at $v = .58, 1, 1.7, 3.7, \dots$). However, the bound at $v = .013$ in Fig. 4 for $\omega_s = 120$, becomes from (1) the bound at $.013/2 = .0065$ for $\omega_s = 240$, because this low-frequency (ω) bound is basically invariant with respect to ω . Now the same decrease from about 20 db to -30 db is to be achieved in the v range $[\ .0065, 3.7]$ in Fig. 5, instead of the $[\ .013, 3.7]$ range in Fig. 4, i.e. an extra octave is available for the purpose. The Bode algorithm provides a satisfactory numerical $L_{m0}(jv)$, labelled L_{m01} in Fig. 5.

Note that $L_{m01}(jv)$ numerically obtained in this manner, can be achieved as accurately as desired by a rational function $L_{m0}(w)$ (Bode 1945), which may however have to be of high degree. Hence, there is some trade-off between ω_s and the degree of $L_{m0}(w)$, which can be made at this numerical stage by allowing some overdesign. Here an apparently satisfactory L_{01} is

$$L_{01}(w) = \frac{.0316(1-w)(w+.17)(w+.6)}{w(w+.04)(w+1)} \quad (9)$$

sketched in Fig. 5. Note that $G_1(w) = L_{01}(w)/P_0(w)$. Refer to page E-3 for the determination of the proper sign of the gain term of $G_1(w)$.

C- 2.6 Error in the design

There is always the problem of choosing enough discrete ω (in continuous design), or v values here, especially at large ω or v . The poles and zeros of $P(s)$ or $P_e(w)$ are important, because the plant templates do not remain almost vertical lines (giving the B_h type of bounds for all $\omega > \text{some } \omega_h$), until a frequency several times larger than the largest pole or zero of P_e . It is usually easy to detect the largest pole or zero in the continuous plant; also the largest pole in $P_e(w)$,

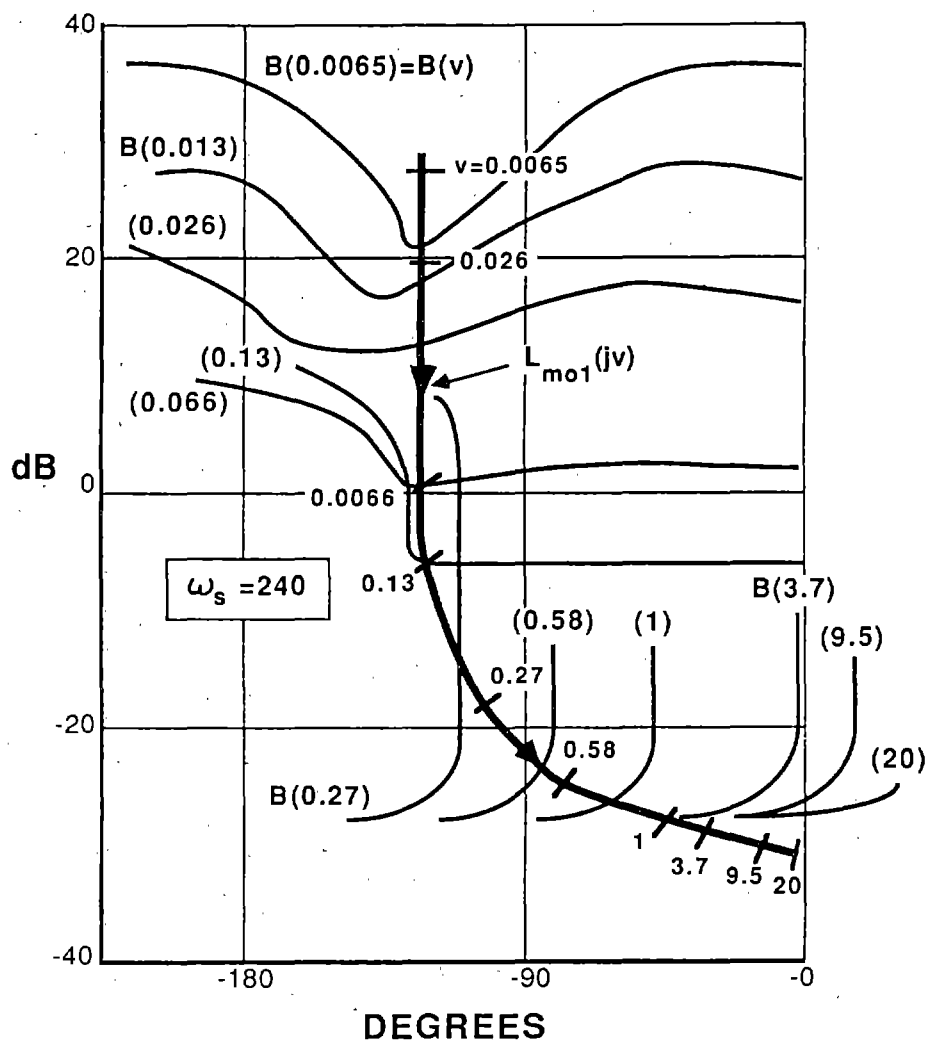


Fig. 5 Apparently successful design L_{m01} at $\omega_s = 240$.

ω	.5	1	5	10	50
v	.0065	.013	.066	.13	1.85

because the poles of P map directly via (1) into $P_e(w)$. However, there may be large shifts in the zeros of $P_e(w)$, as was the case here, with far-off zero at $w = -1/.00436 = -230$ at the nominal $P_0(w)$ of (7).

The result is almost vertical templates and hence B_h type of bounds from $v \sim 20/76.4 = .26$ to $v \sim 230/20 = 12$, (allowing a factor of 20 for the effect of a pole or zero on the phase). But the $\mathcal{T}p_e(v)$ broaden out again for $v > 12$, as shown in Fig. 6a giving the more stringent bounds in Fig. 6b for $v = 20, 200, 2000, \dots$. This was overlooked at first, with the "universal" type B_h assumed in Fig. 5 for all $v > .27$. Design 1 stops at point X in Fig. 6b, definitely violating the bounds for $v \geq 200$: moderately at $v = 200$, but by ~ 12 db at $v = 2000$, enough for instability. A high frequency pole-zero pair sufficed to correct the situation, with

$$L_{m02}(w) = \frac{(1 + \frac{w}{1840})}{(1 + \frac{w}{115})} L_{m01}(w) \quad (10)$$

which satisfies the correct $B(v)$, as seen in Fig. 6b.

C-2.7 Design of prefilter $F(w)$

The design of $F(w)$ is exactly as in the continuous system (Horowitz and Sidi 1972): The above choice of $L_0(w)$ only guarantees that the variation $\Delta \log |T(jv)|$ does not exceed that permitted by (6). For example if at some v_0 , $a(jv_0) = -6$ db, $b(jv_0) = 0$ db, then the permitted $\Delta \log |T(jv_0)| = 6$ db. An acceptable $L_0(jv)$ design may result in $|T'(jv_0)|$ (of $T = FT'$) extremes of -4 db, 1 db, with $\Delta \log |T'(jv_0)|$ only 5 db. It is then necessary that, -2 db $\leq |F(jv_0)| \leq -1$ db. The 1 db overdesign of $|T'(jv_0)|$ allows 1 db tolerance for $|F(jv_0)|$. $F(w)$ is chosen to satisfy the resulting tolerances on $|F(jv)|$. In this problem, a satisfactory

$$F(w) = .033(w + 3)/(w + .1). \quad (11)$$

In the z domain, which is more suitable for synthesis

$$G_1(z) \text{ of Fig. 1} = \frac{166.8(z - .71)(z - .25)(z - .974)}{z(z - .923)(z + .99)} \quad (12)$$

$$F(z) = .121(z + .5)/(z - .82). \quad (13)$$

Step responses for five extreme cases are shown in Fig. 7, corresponding to

Case	1	3	5	7	9
a	1	4	2	1	4
k	1	1	3	10	10

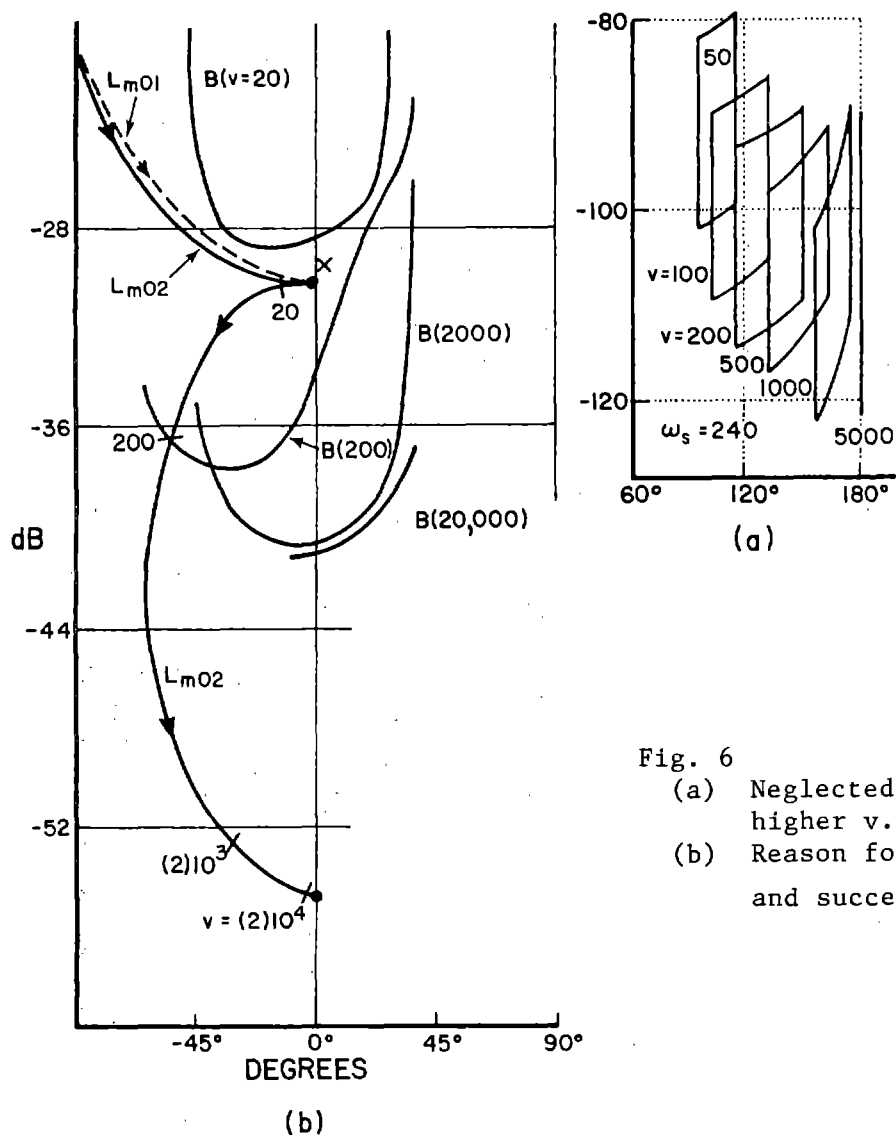


Fig. 6
 (a) Neglected plant templates at higher v .
 (b) Reason for unstable L_{m01} design and successful L_{m02} design.

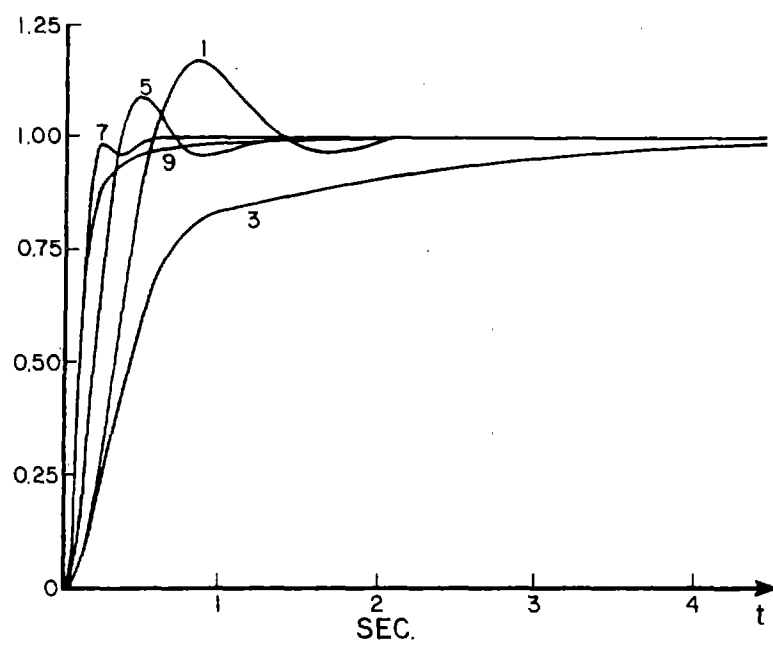


Fig. 7 Step responses.

Case	1	3	5	7	9
a	1	4	2	1	4
k	1	1	3	10	10

C-3. Applicability of Design Technique to Other Plants

Multiple input-multiple output (mimo) plants

In the $n \times n$ mimo plant, in QFT the mimo problem is converted into a number of siso problems, whose solutions are guaranteed to satisfy the original mimo problem. Some familiarity with this technique is assumed (Horowitz and Breiner 1981, Betzold et al. 1984). In the execution, one loop function ℓ_i handles n siso loop problems associated with the n T_{ij} (output i due to command input j , for $j = 1, \dots, n$). The technique described here can be used giving ω_{si} , the sampling frequency associated with loop i . It is quite conceivable that ω_{sj} ($j \neq i$) can be different from ω_{si} , and this would emerge naturally in the course of the design.

Nonlinear plants

The QFT approach is to convert the nonlinear plant set $\mathcal{W} = \{w\}$ into a lti set $\mathcal{P}_{eq} = \{P_{eq}\}$ which is equivalent to $\mathcal{W}$ with respect to the set $\mathcal{A}$ of acceptable outputs. In the continuous case, the plant inputs normally emerge continuous. In the sampled system with zero order hold, the actual input is discontinuous. However, P_{eq} obtained by the continuous approach can still be used if a step-wise (polygonal for first order hold) approximation of x , achieves closely the same output as the continuous x . If this is not so and no simple plant prefilter is allowed to make it so, then one must find the nature of the constraints imposed on the plant output y , due to x being a staircase (or polygonal) function. The set of acceptable plant outputs $\mathcal{A} = \{a\}$ is then formulated such that $w^{-1}(a)$ emerges staircase (or polygonal). This gives an accurate linear time invariant $\mathcal{P}_{eq}$, for which the technique herein described may be applied.

C-4. Discussion and Conclusions

C-4.1 Minimum-phase, non-minimum phase and unstable $P(s)$

It is seen that the minimum required sampling frequency ω_s is very strongly a function of the uncertainty problem — the extent of the plant uncertainty and of the performance tolerances. If the continuous plant $P(s)$ is minimum-phase, then the sampled loop transmission $L(w)$ need have only one right half-plane zero, at $w = 1$. This can be asserted even though the zeros of $P(s)$, unlike the poles, do not map directly via (1), into the zeros of $P_{eq}(w)$. The reason is that the range of the s -plane over which the mapping is almost directly so, can be made as large as desired by increasing ω_s . The larger ω_s , the smaller the importance of the surplus phase lag of the allpass $A(w) = (1 - w)/(1 + w)$, because the bandwidth

of importance recedes into lower v values (compare the low v bounds of Figs. 4, 5).

However, if $P(s)$ is non-mp, then it may be impossible to achieve the desired performance for all P in $\mathcal{P}$, precisely as in continuous systems (Horowitz and Sidi 1978). One should then first check if the problem is solvable in a continuous setting. If so, it can be done in a sampled setting by use of large enough ω_s . If it is barely solvable in the former, ω_s would have to be very large. If it is not solvable in the continuous design, it is certainly not solvable in a sampled design. Unstable but mp (no rhp zeros) plants pose no problem, even though the sampled $L(w)$ then has rhp poles and the rhp zero at $w = 1$. The reason is that with large enough ω_s , the rhp poles are arbitrarily close to the origin in the w -plane (see Horowitz 1986). Unstable, non-mp $P(s)$ do pose a great problem (Horowitz 1979b), but it has recently been shown how time-varying compensation may be used for such plants to achieve stability (but not small system sensitivity) over large plant parameter uncertainty (Horowitz 1986).

C-4.2 Disturbance attenuation

In continuous systems, plant disturbance attenuation imposes its own bounds on the loop transmission $L(j\omega)$. The procedure for finding these bounds in the continuous design is quite similar to the procedure for the $T(j\omega)$ problem ($a(\omega)$, $b(\omega)$ of Fig. 2) — see (Horowitz and Sidi 1972). The procedure for obtaining the equivalent disturbance sampled-data bounds is identical to the procedure herein described, and therefore need not be repeated. The stronger of the two bounds (disturbance and $T(jv)$) must, of course, be used, just as in the continuous system.

C-4.3 Conclusions

The basic feedback mechanism and quantitative design procedures are the same in sampled-data systems as in continuous systems, if design is executed in the v domain for the former. The price paid is in the loop bandwidth for both, achievable in the former by making ω_s large enough. Any practical sampled system is non-mp, but the harmful effect of the rhp zero at $w = 1$ can be made arbitrarily small, by making ω_s large enough. The main difference in quantitative design is the need for some experimentation to find $(\omega_s)_{\min}$.

C-4.4 Application

A w -plane QFT design was applied to the AFTI F-16 aircraft with surface failures (Schneider 1986).

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Appendix D

Longitudinal Handling Qualities Approximations and Bandwidth Minimization for Robust Automatic Flight Control System Design Using Quantitative Feedback Theory

Capt Bruce T. Clough USAF AFWAL/FIGX

D-1. INTRODUCTION

In designing automatic flight control systems for modern aircraft several concerns confront the design engineer. How to include MIL-F-8785 specifications in the design? Will the design be robust enough to handle the uncertainties, not only in the approximations used to realize a design, but also in the conditions the system will see once placed into the "real" world? How large will the required system bandwidths be, and what are the chances of exciting spurious modes(unmodelled high frequency dynamics)?

One of the areas of current interest that faces these issues is the design of fixed-compensation controllers for reconfigurable flight control systems[47,48]. A desired design will stabilize the aircraft in case of failure until the source can be determined and corrected for. The salient features of this control law are:

Robust Fixed Compensation
MIL-F-8785 HQ for Healthy Aircraft
Stability for Large Failure Set

Satisfaction of these requirements can lead to large system bandwidths, leading to noise corruption, actuator saturation, and spurious mode excitation, any of which can be serious in this era of reduced static stability aircraft.

One of the control synthesis techniques being investigated for the design process is Quantitative Feedback Theory(QFT)[1-3,7-17]. This paper addresses factoring of HQ and bandwidth considerations into QFT to formulate a more reasonable point to start off practical design work. It is assumed that the reader has some experience in designing MIMO control systems using QFT and is familiar with the terminology developed by Dr. Isaac Horowitz.

D-2. BACKGROUND

QFT relies on shaping the loop transmission L to meet certain restrictions(bounds) placed upon it by desired system responses and disturbance rejection levels. QFT is used extensively at the Air Force Institute of Technology for investigations into robust automatic flight control system(AFCS) design resulting in fixed compensation without extensive failure detection/identification (FDI) routines. From the flying qualities specifications[52,53] Level 1 A/C are the "good" planes that all

designers desire, Level 2 have deficiencies which should be corrected, and Level 3 are barely controllable. In this presentation the term "flying qualities" FQ is assumed the same as "handling qualities" HQ. This isn't quite true since FQ is a subset of HQ, but for this paper it will suffice. During AFCS synthesis using the minimum-phase QFT design technique it is hard to guarantee Level 1 (MIL-F-8785) HQ for healthy aircraft(A/C) while trying to control the same A/C with failures within reasonable loop bandwidths. This stems from two basic assumptions/steps in design. The inclusion of the no-fail plant into the same set as all the failed plants, and assuming the output(s) are controllable, but not Level 1(it is normal to relax the tracking specifications a bit for failures to reduce actuator demands). The design become concerned with the response of the set rather than that one individual plant lost in its midst. Some of those plants may meet the stricter Level 1 demands, or they may not. Reducing the size of the plant set will decrease the "distance"(in a frequency response sense) between plants, but it also reduces the total number of situations the controller can handle. The acceptable outputs could be limited a priori Level 1; however, this would lead to controllers having unrealistic gains and bandwidths trying to force a crippled A/C to fly like a healthy one. Compensation could be scheduled, but now some sort of FDI routine is required(which is the reason for fixed compensation in the first place). For Reduced Static Stability aircraft(RSS) the problem is compounded further since any actuator rate/position saturation resulting from over zealous compensation can lead to loss of control[49-51].

What is required is a logical method by which the designer can factor in HQ criteria while loop shaping and at the same time keep loop bandwidths reasonable. From an engineering standpoint it should be simple as possible and provide "feedback" to the designer during synthesis. Some way should be devised such that the particular military specifications can be included into the design process to begin with, thus avoiding control "kluges" when the design fails to work properly. Closed-loop checks of the actuator states can provide insight into saturation.

D-3. DESIRED HANDLING QUALITIES

In order to set the proper frequency responses for control synthesis they must first be defined from the requisite specifications[52,53]. A typical desired longitudinal short-period response to a pilot stick input is a second order system with a closed-loop transfer function of

$$TF(s) = \frac{\omega_{sp}^2}{s^2 + 2\zeta_{sp}\omega_{sp}s + \omega_{sp}^2} \quad (1)$$

ω_{sp} is the short period natural frequency

ζ_{sp} is the short period damping ratio

The damping ratio and natural frequency vary for the type of A/C, type of

task, and the n/α factor. In this paper it is assumed that the A/C is an advanced fighter configuration in a landing scenario, with n/α as the independent variable. With the task, aircraft, and n/α fixed, an approximate range of ω_{sp} and ζ_{sp} can be determined from MIL-F-8785C. Background documents to the spec give actual experimental data on ranges of ω_{sp} and ζ_{sp} from various A/C simulations and test flights. Figure D-1 shows such a chart. The points A-H denote the boundary of Level 1 flight. The ω_{sp} and ζ_{sp} for these points are averaged over numerous charts of similar A/C doing similar tasks. The result is a representative group of short-period frequencies and damping ratios which denote the borders of Level 1 flight. From these a representative group of transfer functions are developed. The set of transfer functions developed for the conditions above are in Table D-1. From these transfer functions, and from those developed for Level 3 flying qualities, frequency bound can be plotted to define the range of each. Figure D-2 shows a magnitude verses frequency plot of these bounds for Level 1 and Level 3 HQ. This plot is required later on when the synthesis procedure is modified.

D-4. MODIFICATION OF QFT DESIGN PROCEEDURE

This section addresses the problem of insuring handling qualities for a given subset of the total plant set, that subset assumed a small part of the plant set. To do this, the QFT design process is slightly altered to take advantage of the difference in stability and tracking criteria for the plant set in question. Before this is done, however, the whole concept of robust control for reconfigurable flight control systems requires examination so the "task" of the controller is well defined.

What are the causes of unrealistic, high bandwidths in the QFT design process? Large plant uncertainty and tight tracking/disturbance rejection tolerances. Does that uncertainty in the plant lead to large plant templates even at "high" frequencies? Does the strict tolerances place too much of a burden on compensation for a plant with failures? Normally, given a problem with strict tolerances and large plant uncertainty one would expect "healthy" loop bandwidths. To get around this one could limit the amount of uncertainty dealt with, leaving some situations uncontrollable. Or, the system specifications could be relaxed for the total plant set, ending up with a sluggish and/or noisy system. Our goal is to do a little of both, compromising to insure stability and tracking performance.

Obviously, a pilot wants to fly a Level 1 A/C all the time, but most pilots are practical and will accept HQ degradation for failures as a fact of life. To take advantage of this in the design process relax the idea of initial acceptable tracking responses from Level 1 to Level 2 or minimum control conditions, Level 3. This meets one of the goals:

FOR ROBUST CONTROL IN THE FACE OF SURFACE FAILURES WE SIMPLY WANT TO KEEP THE PLANE IN THE AIR LONG ENOUGH FOR THE FDI SYSTEM TO EFFECT AN OPTIMAL SOLUTION(or get the pilot out)!

For this Level 1 HQ is not required, Level 3 qualities are acceptable. Relaxing tracking requirements as shown in Figure D-3 can significantly

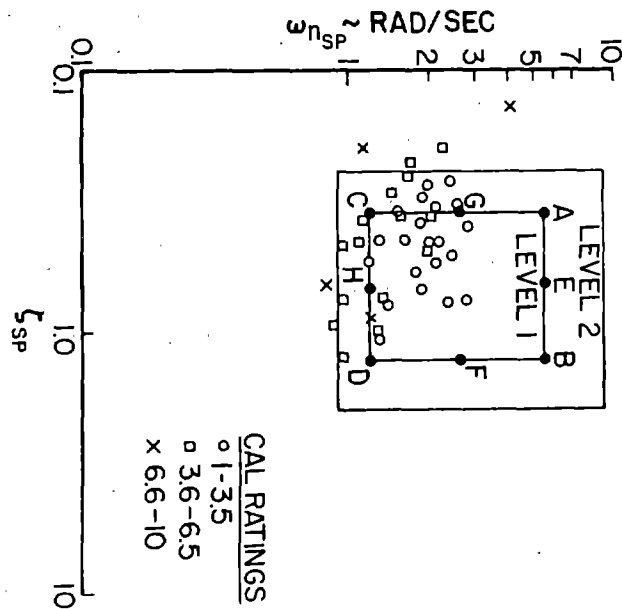


Figure 23 (3.2.2.1.1)
LANDING APPROACH, $n/\alpha = 8.5$
(T-33, "FRONT-SIDE" DATA ONLY, REFERENCE E12)

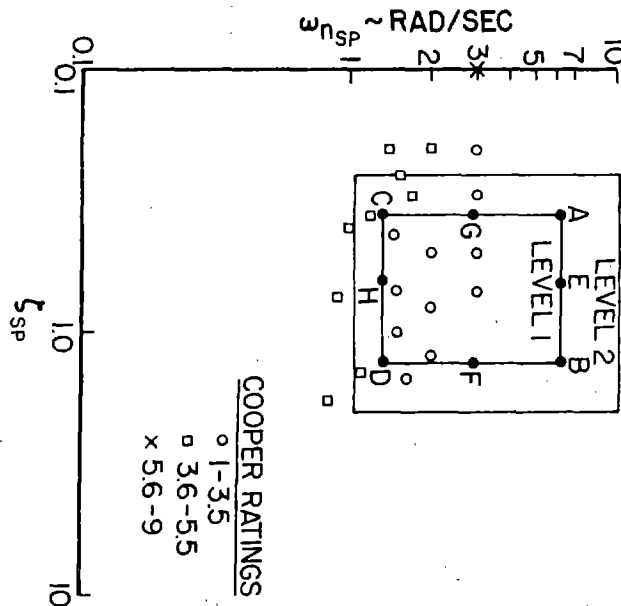


Figure 24 (3.2.2.1.1)
LANDING APPROACH, $n/\alpha = 11.0$
(NAVION, OPTIMUM F_s/n , REFERENCE E4)

Fig.D-1 Example of Ranges of Short Period Frequency and Damping Ratio Used to Develop the Acceptable Set of Closed Loop Transfer Functions.(MIL-F-8785C)

Point	Transfer Function
A	$\frac{27}{s^2 + 3.54s + 27}$
B	$\frac{27}{s^2 + 12.5s + 27}$
C	$\frac{1.44}{s^2 + 0.82s + 1.44}$
D	$\frac{1.44}{s^2 + 2.88s + 1.44}$
E	$\frac{27}{s^2 + 6.76s + 27}$
F	$\frac{14.4}{s^2 + 2.58s + 14.4}$
G	$\frac{14.4}{s^2 + 9.12s + 14.4}$
H	$\frac{1.44}{s^2 + 1.56s + 1.44}$

Table D-1 Closed-Loop Transfer Functions Representing the Boundaries for Level 1 HQ.

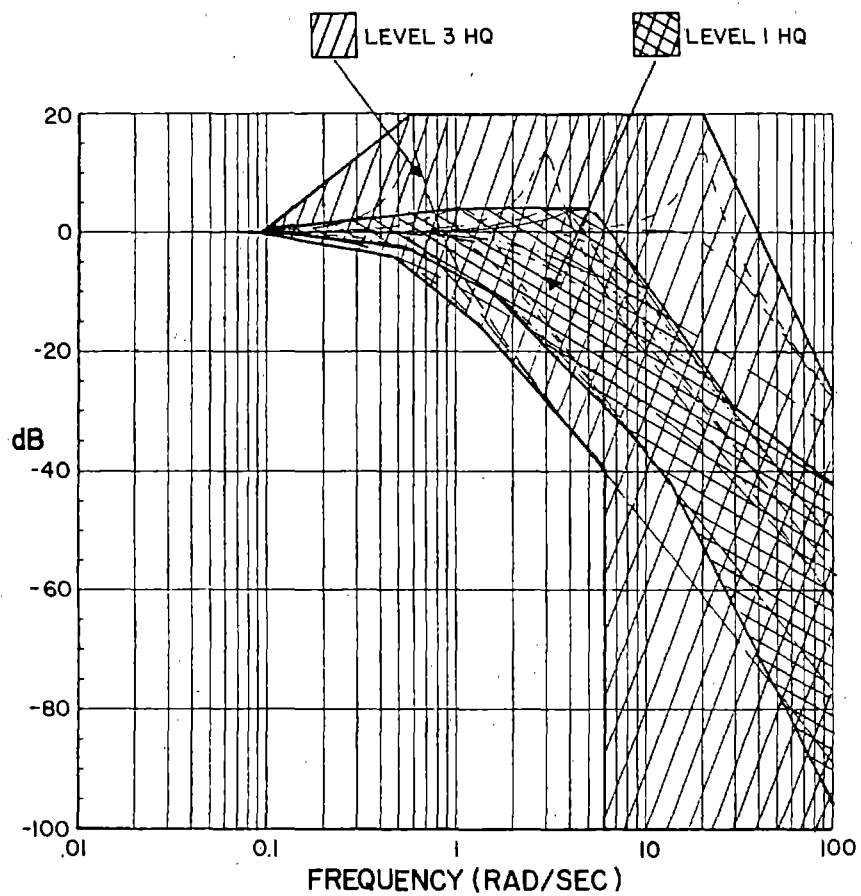


Fig.D-2 Frequency Bounds(Approximate) on Level 1 and Level 3
Handling Qualities(MIL-F-8785C).
Fighter Aircraft
Landing Task

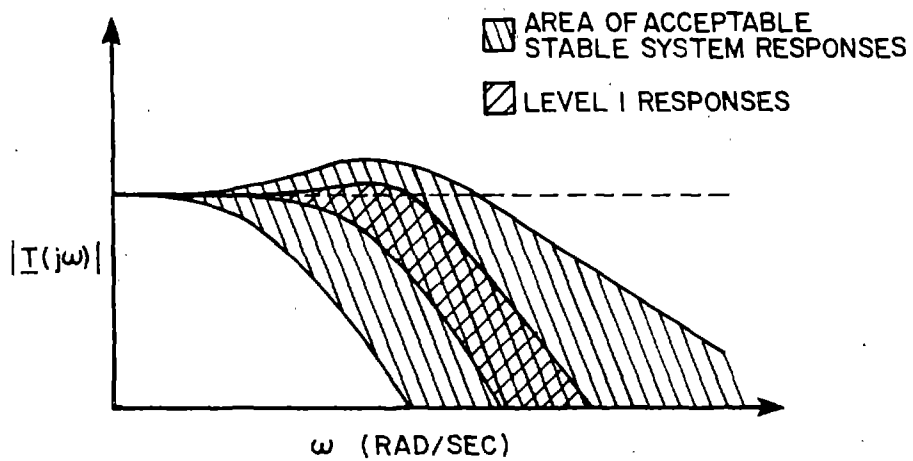


Fig.D-3 Acceptable Frequency Response Areas for Flight Stability and Controllability, and Level 1 Flying Qualities.

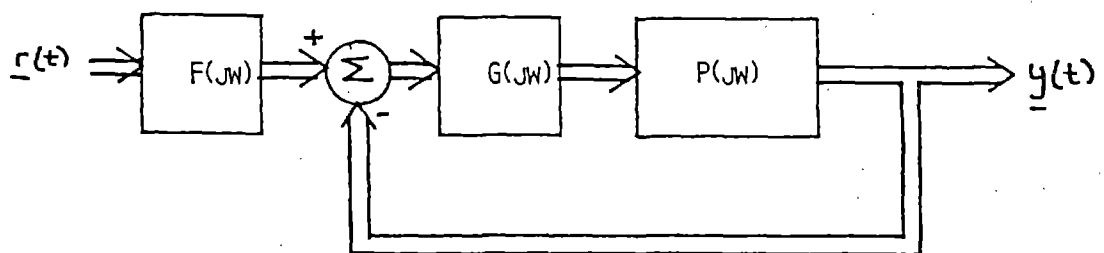


Fig.D-4 QFT System Configuration

increase the area of acceptable tracking responses, likewise pushing the tracking bounds lower on the N.C., reducing loop bandwidth. Now the reader is saying at this point, "Sure you've lowered the loop bandwidth(ignoring disturbances), but the tracking response could be terrible for healthy plants". True; however, the desired tracking response for those plants in our subset of interest will be recovered later in the design process, the loop compensation is insuring stability while the prefiltering will give the required tracking response. In addition to using the lowest acceptable HQ for failed A/C as initial tracking bounds, the disturbance rejection should be moved from "desired" to "acceptable" levels to arrive at the lowest practical disturbance bounds.

The approach taken here is to use QFT almost exactly as before, but relaxing a few tolerances, and making others stricter. Figure D-4 shows the system configuration used in the QFT design process. P is the plant we're attempting to control, G is the loop compensation, and F is the prefilter. The input and output dimensions are assumed the same, the plant being observable and controllable.

The first step is to break the entire plant set $\mathcal{P}$ into two subsets, the plants which should fly Level 1, P' , and those that can fly a worse HQ level, P'' . Two sets of plant templates are required, those for $\mathcal{P}$, and those for P' . Using the tracking specifications for Level 3 and the required disturbance rejection, develop loop bounds on the Nichols Chart with the templates for $\mathcal{P}$. Once the bounds are on the N.C. a loop transfer is shaped in the normal way. The nominal plant used to plot the bounds and to shape the loop is a member of P' , preferably the plant which flies the most. Calculate G in the normal manner. Note that the loop has been designed for stability: the compensation G has now resulted in a range of stable plants that may, or may not, meet the tracking requirements for Level 1 -- no guarantees, all that is known is that they are stable and, if the design process is continued, they will fly Level 3. Normally at this point in the design the prefilter is derived from the original tracking bounds. All this does is guarantee possible bad responses for P' , stable, but not Level 1. Going back to the original premise, only Level 1 responses are desired for the plants in P' (not caring much about P'' as long as it's stable and controllable). Thus the prefilter is synthesized for just that, agreeing to accept whatever responses for the other (non P') plants arise.

One has to insure that the prefilter modifies the system such that a chosen subset of the original plant set have closed-loop responses within a specified range on a Bode plot. The modification results in Level 1 HQ for the chosen set, but can't guarantee good HQ for the rest of the plant set (but it is reasonable to assume some performance degradation for A/C with failures). This method insures stability and a level of controllability with failures (which is what we'd want for a damaged A/C). Since the prefilter is placed ahead of the closed loop, the problem now is a simple cascade compensation, or how best to pick the prefilter. The prefilter is chosen using just the data from P' , recognizing that we really only care about the loop transfers of P' . So working just with those plants:

- a. Derive templates using only the plants in P' .
These have smaller area than the entire plant set's

template, and thus contain less uncertainty.

- b. "Run" the new templates along $L_0(j\omega)$, noting the max and min values (as would be done normally with the entire plant template Φ). Use the stricter Level 1 boundaries (Figure D-2) to calculate the prefilter limits.
- c. Shape the prefilter the normal way.

The prefilter(s) thus designed insure that the particular plants of interest P' have Level 1 HQ while at the same time the loop compensation(s) G guarantee stability for the entire plant set. Note that the (se) prefilter(s) have larger bandwidth(s) than ones designed using QFT and the original, looser tracking bounds for Level 3. This increase should not cause problems since all uncertainty, instability, and disturbances are contained in the compensated loop previously designed, besides they are lower in bandwidth than prefilters designed assuming Level 1 for all plants in Φ . We can't say what the actual time responses are for the non- P' plants, but then again we don't care as long as they are stable and controllable.

D-5. SUMMARY

This paper has presented a method by which compensation that guarantees specific (Level 1) HQ for a subset of a larger plant set containing both healthy and failed plants can be derived using QFT. Since only a few plants must exhibit good HQ, the bandwidth penalties for forcing A/C with failed actuators to fly the same as healthy A/C are diminished, leading to more realistic compensation (or a larger amount of plant uncertainty controllable given a specific bandwidth and actuation). Thus healthy A/C can be included into the design as before, but now with the guarantee that they will fly Level 1, while at the same time insuring robust behavior of the control system. The modifications to QFT are minimal: relax the initial acceptable tracking and disturbance responses to the bare minimum for controllable flight, use the loop compensation to guarantee stability, and develop the prefilter to give Level 1 HQ for the specific plants chosen. This method has yet to be tried out in earnest, but synthesis/simulation work is underway.

D-6. REMARKS BY DR. ISAAC HOROWITZ

I. The idea of using Level 3 performance specifications is certainly sound and worth pursuing in any tough problem where physical limitations such as when practical sampling frequency ω_s (see Appendix C) and loop bandwidths are in conflict. Apriori, one can (at the very onset) define several sets of bounds on $|T(j\omega)|$ (or even more). There are a variety of possibilities:

1. Determine only one set of bounds on L_0 , i.e., such that for any set P_1 bounds B_1 [on $|T(j\omega)|$] are satisfied and for set P_2 bounds B_2 [on $|T(j\omega)|$] are satisfied.
2. For all P (i.e., $P_1 \cup P_2$) the bounds B_1 are satisfied giving one set of bounds on L_0 . Likewise for all $\bar{P}$, B_2 are satisfied, etc.

Possibility 1 appears to be more reasonable.

It is a good idea to see the templates before spending a great deal

of time on formulating B_1 and B_2 . The templates pretty well push you in a certain direction in formulating the lesser desirable B_i .

For difficult problems, one should economize as much as possible on the bounds $\underline{B}(j\omega)$ on $|\underline{T}(j\omega)|$ as follows:

(a) For any $\underline{T}(j\omega)$ [t_{ii}, t_{ij} ($i \neq j$)] consider the actual typical time response characteristics (see MIL specifications 8785 and 9490). For example, a pitch-rate response usually allows for large overshoot with a reasonable settling time, i.e., Fig. D-5(a) is okay but not Fig. D-5(b). A second-order $T_{ii}(s)$ results in the $q_3(t)$ response which is not satisfactory for a $q_1(t)$ time response. A pole-zero pattern for an acceptable $T_{ii}(s)$ is shown in Fig. D-6 (see Chap. 12 of Reference 4, Appendix R). If the constraints on the parameters of Fig. D-7 are specified, then synthesize a pole-zero pattern which will yield this desired response and use it to find the $|\underline{T}_{ij}(j\omega)|$ bounds. Thus the $|\underline{T}(j\omega)|$ bounds are being tuned to the natural system response which can result in a maximum economy in bandwidth. See the example in Chap. 9 of Reference 3, Appendix R which shows a tremendous saving in bandwidth by this tuning technique.

(b) In loop shaping there is often a dominant bound, e.g., $\underline{B}_{L_0}(j\omega)$ in Fig. D-8 (solid lines) dominates. By analyzing the plant templates and the bounds on $|\underline{T}(j\omega)|$, shown in Fig. D-9 (a), an improvement in bandwidth may be achieved, if it is possible, by widening the bounds at $\omega = 2$ and narrowing them at $\omega = 4$ as shown in Fig. D-9 (b). What is the resulting effect on the time response range? The time-frequency relation is via an integral. If Fig. D-9 (b) gives the dashed bounds in Fig. D-8 then an improvement has been achieved. This approach is worth trying where an improvement in bandwidth is desired.

II. The possibility of scheduling the high-frequency gain factor through air data measurements (aerodynamic data for each flight condition mach vs altitude, which is apparently quite reliable, should be considered. Even if the air data measurements are quite inaccurate, say 1 ± 0.25 , then the uncertainty factor (which is the length of the UHFB) is now only $1.25/0.75$ ($< 6\text{dB}$) instead of possibly 30dB . However, this scheduling is the same as for the failure cases (see Reference 54, Appendix R).

III. More advanced approaches should be considered such as:

1. Using a "nonminimum-phase MIMO plant and allowing one row of t_{ij} to suffer. If available, select an output variable which can be sacrificed in order that the others are m.p.
2. Doing a preliminary design with smaller bandwidth, which only stabilizes the system, i.e., the bounds on only $L_m[L/(1 + L)] < 3$ or 4dB . Then doing a second design with the resulting new effective plant (which includes the compensation of the preliminary design) for achieving the desired performance.
3. An oscillating adaptive system was used by Minneapolis-Honeywell on the X-15 but they never developed an analytic design technique. This was done by some PHD students at Weizmann Institute of Science.
4. Nonlinear compensation (non-identification -- strictly passive nonlinear compensation) for loop bandwidth economization should be investigated.

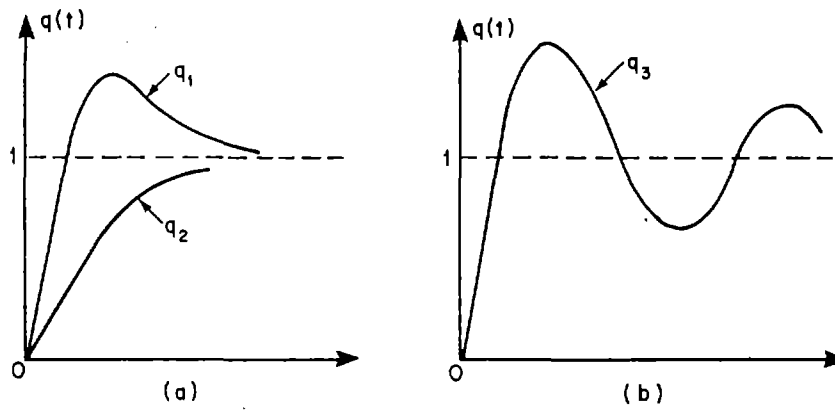


Fig D-5 Pitch-rate time responses: (a) acceptable; (b) unacceptable

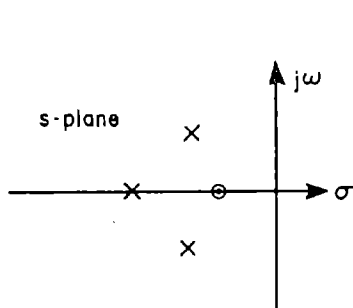


Fig D-6 Pole-zero pattern for an acceptable T 's

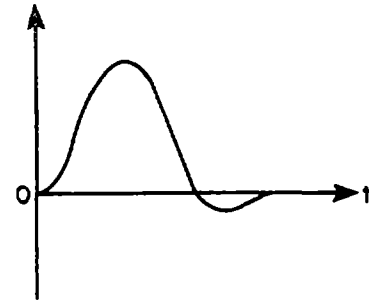


Fig D-7 An interaction response

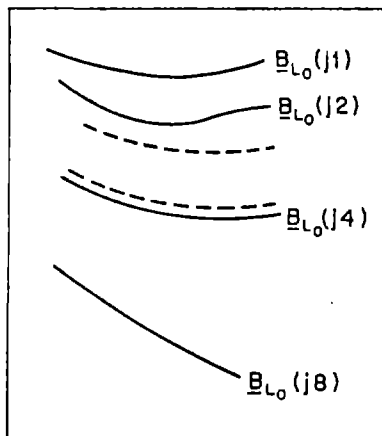


Fig D- 8 L_O bounds

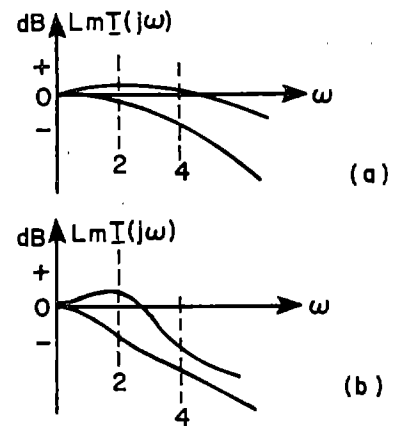


Fig D-9 Upper and lower bounds on $\underline{T}(j\omega)$

Appendix E

Important Observations Regarding The Form Of The Plant Data

(Most Important For A Practical Control System Design)

Given an ℓ -output vector $\underline{y}$, an m -input vector $\underline{\delta}$, the LTI plant transform equations can be written in the form (see Sec. 5-8)

$$\underline{D}(s)\underline{y} = \underline{N}(s)\underline{\delta} \quad (E-1)$$

with $\underline{D}(s) = [d_{ij}(s)]$ and $\underline{N}(s) = [n_{ij}(s)]$ being $\ell \times \ell$ polynomial matrices in s (e.g., see "Automatic Control of Aircraft and Missiles" by J. H. Blakelock, Wiley, pg 36, Eq, (1-113)). The resulting plant matrix ($\underline{y} = \underline{P}\underline{\delta}$) is

$$\underline{P}^{-1} = \underline{D}^{-1}\underline{N} = \frac{(\text{Adj}\underline{D})\underline{N}}{\det \underline{D}} = [p_{ij}(s)] \quad (E-2)$$

Suppose that

$$\underline{P}^{-1} = [p_{ij}^*] = [1/q_{ij}] \quad (E-3)$$

is needed, for example, as in some of the QFT techniques, then the designer can obtain it either from Eq. (E-1), i.e.,

$$\underline{\delta} = \underline{P}^{-1}\underline{y} \quad \text{where} \quad \underline{P}^{-1} = \underline{N}^{-1}\underline{D} = \frac{(\text{Adj}\underline{N})\underline{D}}{\det \underline{N}} \quad (E-4)$$

or from Eq. (E-3), i.e.,

$$\underline{P}^{-1} = [p_{ij}]^{-1} \quad (E-5)$$

For numerical calculations it is much better to use Eq. (E-4) rather than Eq. (E-5). The reason why is best illustrated by considering the following 2×2 ($m = 2$) plant:

$$\begin{aligned} \underline{P} = \underline{D}^{-1}\underline{N} &= \frac{\begin{bmatrix} d_{22} & -d_{12} \\ -d_{21} & d_{11} \end{bmatrix} \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix}}{\det \underline{D}} \\ &= \frac{\begin{bmatrix} d_{22}n_{11} - d_{12}n_{21} & d_{22}n_{12} - d_{12}n_{22} \\ -d_{21}n_{11} + d_{11}n_{21} & -d_{21}n_{12} + d_{11}n_{22} \end{bmatrix}}{d_{11}d_{22} - d_{12}d_{21}} = \frac{\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}}{\det \underline{D}} \end{aligned} \quad (E-6)$$

Therefore

$$\underline{P}^{-1} = \frac{\begin{bmatrix} -d_{21}n_{12} + d_{11}n_{22} & d_{12}n_{22} - d_{22}n_{12} \\ d_{21}n_{11} - d_{11}n_{21} & d_{22}n_{11} - d_{12}n_{21} \end{bmatrix} \cancel{(\det \underline{D})}}{N_{11}N_{22} - N_{12}N_{21}} \quad (\text{E-7})$$

Note that $\det \underline{D}$ cancels out in Eq. (E-7), i.e.,

$$\Delta_N = \cancel{(\det \underline{D})}(\det \underline{N}) = N_{11}N_{22} - N_{12}N_{21} \quad (\text{E-8})$$

Since it is readily found (see Sec. 5-8) that

$$\Delta_N = (d_{11}d_{22} - d_{12}d_{21})(n_{11}n_{22} - n_{12}n_{21}) = \det \underline{D} \det \underline{N} \quad (\text{E-9})$$

This result should not be surprising because from Eq. (E-1)

$$\underline{P}^{-1} = \underline{D}^{-1} \underline{P} = \frac{\begin{bmatrix} d_{22} & -d_{12} \\ -d_{21} & d_{11} \end{bmatrix} \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix}}{\det \underline{N}} \quad (\text{E-10})$$

But, if $\underline{P}^{-1}$ is obtained from $\underline{P}$ rather than directly from Eq. (E-1) by means of $\underline{D}^{-1} \underline{P}$, then the cancellations of $\det \underline{D}$ in Eq. (E-7) will be required. It will be necessary therefore to factor out $\det \underline{D}$ from the numerator elements in Eq. (E-7), in order to do the cancellations. This means factoring the elements of the numerator matrix in Eq. (E-7) and of the denominator. The cancellations will of course not be exact, because of the inevitable computer round-off.

In the general case, $m - 1$ such cancellations of each zero of the $\det \underline{D}$ must occur. Why? Because from Eq. (E-1)

$$\underline{P}^{-1} = \underline{D}^{-1} \underline{P} = \frac{(\text{adj} \underline{D}) \underline{N}}{\det \underline{D}} \quad (\text{E-11})$$

the denominator of $\underline{P}^{-1}$ is the polynomial $\det \underline{N}(s)$. But if $\underline{P}$ with each p_{ij} having the denominator $\det \underline{D}$ is inverted, then from Eq. (E-2)

$$\underline{P}^{-1} = [p_{ij}]^{-1} = \frac{(\det \underline{D})^m [\dots]}{\det(\text{Adj} \underline{D}) \det \underline{N}} = \frac{(\det \underline{D})^m [\dots]}{(\det \underline{D})^{m-1} \det \underline{N}} \quad (\text{E-12})$$

$m - 1$ cancellations must exist. One must factor all the elements of $\underline{P}^{-1}$ and then check out the inevitable inexact cancellations.

Conclusion -- Evaluate $\det \underline{P}$ and $\underline{P}^{-1}$ directly from Eq. (E-1), i.e.,

$$\det \underline{P} = \frac{\det \underline{N}}{\det \underline{D}} \quad (\text{E-13}) \quad \text{and} \quad \underline{P}^{-1} = \underline{D}^{-1} \underline{N} \quad (\text{E-14})$$

To do this, one needs the plant equations in the form of Eq. (E-1), which may not be readily available from the state space form. Therefore, the designer is very strongly advised at the very onset of the design process to obtain the data in the form of Eq. (E-1) -- (see Chap. 6 of lecture notes by Prof. J. J. D'Azzo for his EENG 640 course).

On pages E-4 through E-7 are forms that can facilitate the design of a 3x3 control system design. The number of table entries depends on the scenario for the plant parameter uncertainty.

Due to the characteristics of the bilinear transformation (a gain sign change may occur and a resulting n.m.p. function) the Nyquist stability criterion must be applied, via a polar plot analysis, to determine the proper sign to be associated with the gain term of $L_0(w)$ and in-turn $G_1(w)$.

In determining q_{ii_e} and its corresponding templates and the bounds on L_{i_e} , for $m \geq 2$, must use the data for L_{i-1} for $\underline{P} \in \mathcal{P}$.

Design Checklist

In performing a OFT design it is very important to check the following items very carefully and often as one proceeds with the design.

(1) $\det \underline{P}$ is needed if $\underline{P}^{-1}$ is derived from the plant matrix $\underline{P}$. Usually the elements of $\underline{P}$ have a common denominator (or can be forced to have one). For this situation let $\underline{P} = \underline{V}/d(s)$ where $\underline{V}$ is an $m \times m$ matrix of polynomials in s and $d(s)$ is the common denominator. It may seem that $\det \underline{P}$ has $[d(s)]^m$ polynomials of $d(s)$ in its denominator. This is not the case; $d(s)$ appears only once in the denominator, so there must be $(m - 1)$ cancellations of the zeros of $d(s)$ with $(m - 1)$ zeros of $\det \underline{P}$ (see Sec. 5-8). To see this, note that the $m \times m$ MIMO plant equations can be written as $\underline{Dy} = \underline{Nu}$ where $\underline{D}$ and $\underline{N}$ are $m \times m$ polynomial matrices in s ; $\underline{y}$ and $\underline{u}$ are the plant output and input vectors, respectively. Hence

LOOP 1: DISTURBANCE FORM 1

Frequency: $\omega =$

$\tau_{d11} =$ $\tau_{d12} =$ $\tau_{d13} =$									
Plant No.	$\frac{d_{11}}{b_{21}}$	$\frac{d_{12}}{b_{22}}$	$\frac{d_{13}}{b_{23}}$	$ \underline{q}_{12} $	$\frac{d_{11}}{b_{31}}$	$\frac{d_{12}}{b_{32}}$	$\frac{d_{13}}{b_{33}}$	$ \underline{q}_{13} $	$ \underline{q}_{11} $
1									
2									
3									
4									
5									
6									
7									
8									
9									
10									
11									
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27									
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LOOP 1: DISTURBANCE FORM 2

Frequency: $\omega =$ _____

Plant No.	d_{11}	d_{12}	d_{13}
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2			
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			
13			
14			
15			
16			
17			
18			
19			
20			
21			
22			
23			
24			
25			
26			
27			
28			
29			
30			
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LOOP 3 q_{33}^e - FORM 1A

Frequency: $\omega =$ _____

Plant No.	$\underline{g}_1$	$\underline{q}_{11}$	$\underline{p}_{11}^*$	$(1 + \underline{L}_2)$	$\underline{g}_2$	$\underline{q}_{22}$	$\underline{p}_{22}^*$	$(1 + \underline{L}_2)$
1								
2								
3								
4								
5								
6								
7								
8								
9								
10								
11								
12								
13								
14								
15								
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LOOP 3: $\underline{q}_{33}^e$ - FORM 1B

Frequency $\omega =$ _____

Plant No.	$\underline{q}_{33}$	$\underline{p}_{33}^*$	$\underline{p}_{12}^*$	$\underline{p}_{13}^*$	$\underline{p}_{21}^*$	$\underline{p}_{23}^*$	$\underline{p}_{31}^*$	$\underline{p}_{32}^*$
1								
2								
3								
4								
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$\underline{y} = \underline{D}^{-1}\underline{N}\underline{u}$, so $\underline{P} = \underline{D}^{-1}\underline{N}$ and $\det \underline{P} = \det \underline{N} / \det \underline{D}$ where $\det \underline{D} \equiv d(s)$. Hence $\det \underline{D} = d(s)$ appears only once in the denominator of $\det \underline{P}$. In practice, the cancellations will not be exact. So this can act as a check on the accuracy of the computer-aided-design program being used.

(2) The q_{ij} elements needed in the usual QFT method are desirable to be minimum-phase (m.p.) in order (in the continuous system) that arbitrarily small sensitivity (a.s.s.) may be achievable. These are the zeros of $\det \underline{P}$. Thus check these zeros very carefully; after cancelling out the spurious zeros, as noted in (1), if there are right-half-plane (r.h.p.) zeros stop and check your work for computational accuracy, select other values of δ_{ij} , etc. For the sampled-data control system, the continuous plant determinant may be m.p. but it is conceivable that the sampled plant (with the zero-order-hold unit) may have r.h.p. zeros in the w - (or w' -) plane in addition to those resulting from the bilinear z to w (w') transformation. If these additional r.h.p. zeros exist stop and check your work for computational accuracy, select other values of δ_{ij} , etc.

(3) As $s \rightarrow \infty$, $\det \underline{P} \rightarrow K/s^\lambda$, $\lambda > 0$, in vast majority of cases where λ is an interger. Check that the interger λ is the same over the entire plant set. Stop and recheck your work if this condition is not satisfied. Also, check to verify that the sign of K is the same over the entire plant set. Stop and recheck your work if this is not the case.

(4) In the m.p. continuous system, a.s.s. is achievable and in the QFT design one proceeds with finding the bounds on the loop transmissions, L_1 , in order to make them as economical as possible. But the sampled systems are inherently non-m.p., so the benefits of feedback are restricted, especially if the sampling frequency ω_s is apriori assigned. Nevertheless, it may still be possible to satisfy all of the specifications and have room for economy, i.e., solve the design problem with a smaller value for ω_s . In such cases, the bounds should be obtained. Otherwise, the best one can do is to squeeze in the maximum possible loop gain for the given ω_s . If it seems obvious that the second case applies or

if you have little time, you can save yourself much time and effort by simply following the latter case: squeeze in the maximum possible $|L_{i_0}(j\omega)|$. Bounds are still necessary, but they are only for all $P \in \mathcal{P}$, where the constraint

$$\left| \frac{L_i}{1 + L_i} \right| \leq \gamma \quad (\gamma = 3, 4 \text{ or higher if advisable})$$

is satisfied. Thus the plant templates are needed for this purpose since $L_i = g_i q_{ii}$. Be sure to find the bounds for sufficient values of ω_i and be sure that you shift each bound to the right by $2 \tan^{-1} v/v_0$ where v_0 is the location of the mandatory r.h.p. zero ($v_0 = 1$ in the w -domain or $2/T$ in the w' -domain). How to squeeze in maximum $|L_{i_0}|$? A good test is that the designed L_{i_0} satisfies the bounds very closely, especially in the higher frequency range.

(5) If the nominal plant has any zeros or poles in the r.h.p. then the nominal loop transmission L_{i_0} must also have them. One way to do this, if you prefer to shape a stable L_{i_0} , is to derive the bounds on the g_i in $L_{i_0} = g_i q_{ii_0}$: the bound on g_i is the bound on L_{i_0} divided by q_{ii_0} . E.g., if at v_1 , $q_{ii_0} = -3\text{dB}/-20^\circ$, then the bound is shifted 3dB upwards and 20° to the right. Or, you can write $L_{i_0} = L_{\text{imso}} A_z A_p$ with $A_z = (1 - \tau s)/(1 + \tau s)$ for each r.h.p. zero at $1/\tau$, and $A_p = (s + \mu)/(s - \mu)$ for each pole at μ , and find the resulting bounds on the m.p. and stable L_{imso} .

(6) If your nominal plant q_{ii_0} has β r.h.p. poles and it is Type 0, then the nominal loop transmission L_{i_0} must start at $-180^\circ \beta$ at $\omega(\text{or } v) = 0$ and move eventually to the right at least by $180^\circ(\beta - 1)$ plus some phase margin in order to have a stable design. If the plant is Type 1, it must start at $-(180^\circ \beta + 90^\circ)$ and move to the right at least by $180^\circ(\beta - 1) + 90^\circ$. If it is Type 2, change 90° to 180° .

(7) Note that $L_{i_0} = g_i q_{ii_0}$ thus $g_i = L_{i_0}/q_{ii_0}$. If $L_{i_0}(0) > 0$ but $q_{ii_0}(0) < 0$, then $g_i(0) < 0$. Suppose $q_{ii_0} = K \Pi(s + z_i)/\Pi(s + p_j)$. Note that $q_{ii_0} \neq K$; that is, $K \Pi z_i / \Pi p_j$. So the sign of $q_{ii_0}(0)$ is not necessarily the sign of K .

(8) The final loop (say number m) designed is the most important one, in the improved method. If the design is done properly, then the entire system will be stable and the tolerances on all t_{mi} ($i = 1, 2, \dots, m$ in an $m \times m$ system) will be satisfied even if mistakes are made on the other loop designs; the tolerances on some or all of the other t_{ji} ($j \neq m$) may then (but not necessarily) not be satisfied. It is very important to check for r.h.p. poles and zeros of the equivalent plant q_{mme} . This step is, of course, important for each loop that is designed because the nominal loop should have all the r.h.p. poles and zeros of the nominal plant.

To check for r.h.p. poles and zeros factor q_{mme} for each case. It is advisable to do this factoring if the number of cases are not many. Usually it suffices to do so only for the nominal case and check that the templates do not in any ω range become 180° wide (possible indication that a zero or pole may for some cases be in the r.h.p. and for other cases be in the l.h.p. or a sign change in the gain term). Especially, the templates should narrow towards a vertical line at high enough frequencies and remain so. In the sampled case, often the templates narrow in this manner over a range of frequencies but then at still higher frequency values may widen again (see Appendix C). Watch out for this --- always go to large enough ω values to be sure that the templates thereafter remain vertical lines.

If you factor the nominal plant and make sure the design is stable for the nominal case, and satisfies the tolerances for all $P \in \mathcal{P}$, then you are okay for the vast majority of problems. This is so even if the templates do widen out, even as $\omega \rightarrow \infty$. (it will be then more difficult or even impossible to satisfy the tolerances and have a stable nominal design.) But there are theoretically unusual cases (so far Horowitz has not encountered any) where the design can be unstable for these unusual cases. So the safest approach is to factor all plant cases. If any of these cases have more r.h.p. poles and zeros than the nominal case stop and check your design.

----- Constraint 1 & 2 Amplification For Method 2 Design -----

Constraint 1 -- If this constraint is not satisfied, the theoretically attainable benefits of feedback are limited. They may nevertheless suffice for the specific system being designed (see Reference 9, 56 and 57).

Constraint 2 -- A good way to apply this constraint is to obtain

$$\lim_{\omega \rightarrow \infty} \left[\frac{\det \underline{P}}{\det \underline{P}_0} \right]$$

where $\underline{P}_0$ represents the nominal plant metrix. This ratio must not change sign over the range of uncertainty, i.e., $P \in \mathcal{P}$. This automatically takes care of all loops. If this condition is not satisfied then it is impossible, by the usual LTI design techniques, to achieve significant sensitivity reduction.

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